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Algebraic Statistical Mechanics.

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To be or not to be integrable ...

First past the post ...

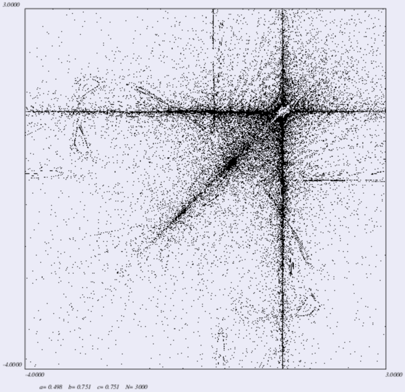
At least two concepts: Yang-Baxter integrability versus integrability of the birational symmetries

• YBE (and their higher dimensional generalizations) necessarily yield commuting transfer matrices T_N **for any size** N, which necessarily yield **algebraic varieties** which are preserved by **birational automorphisms** generated by the **inversion relations**. **Generically** the composition of two inversion relations yield **infinite order** generators of these **birational symmetries**. YBE $\rightarrow$ canonical parametrization in terms of **algebraic varieties** with an **infinite set of birational automorphisms**. The algebraic varieties are not of the "general type", they are highly selected algebraic varieties: elliptic curves, Enriques surfaces, K3 surfaces, Abelian varieties, etc ...

• Discrete dynamical systems corresponding to the iteration of these infinite order birational symmetries.

At least two concepts: Yang-Baxter integrability versus integrability of the birational symmetries

To be Yang-Baxter-integrable you need the **infinite order birational symmetries to be integrable**: the growth of the degree of the infinite order birational generators has to be a **polynomial growth**. Conversely the **integrability of the birational symmetries is a necessary, but not sufficient condition to be YB-integrable**. For instance the sixteen vertex model corresponds to an integrable foliation of its CP_{15} parameter space in terms of ellipic curves, it is not (generically) YB-integrable. An **exponential growth** means that the model **cannot be Yang-Baxter-integrable**. At least, we know what **non-integrable** is ...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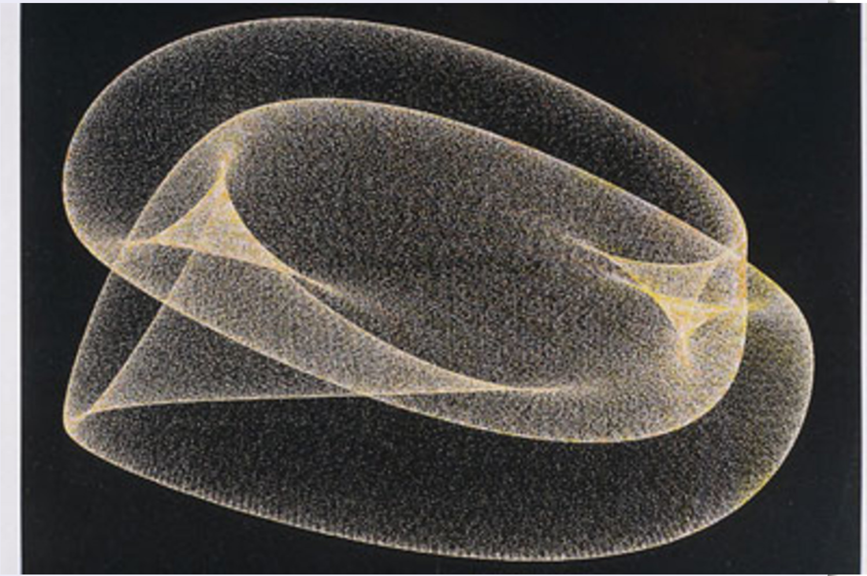
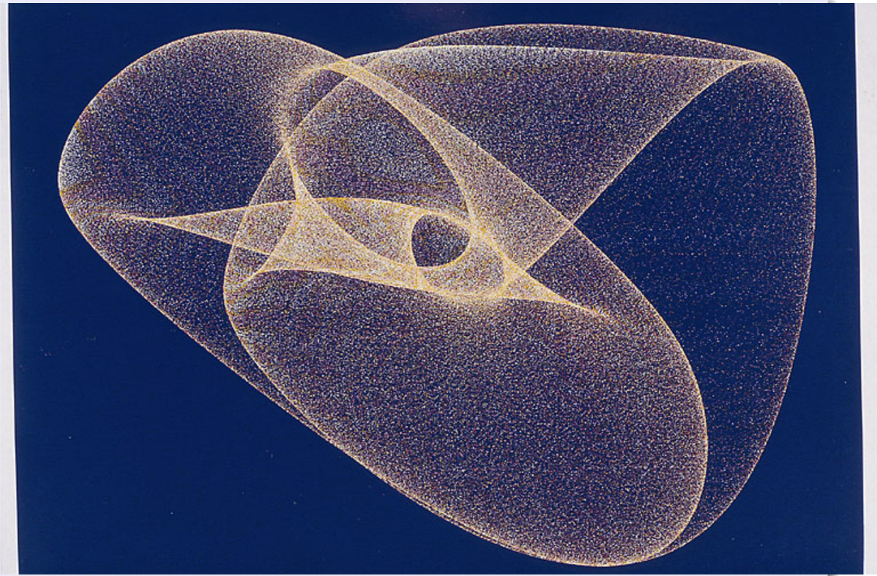
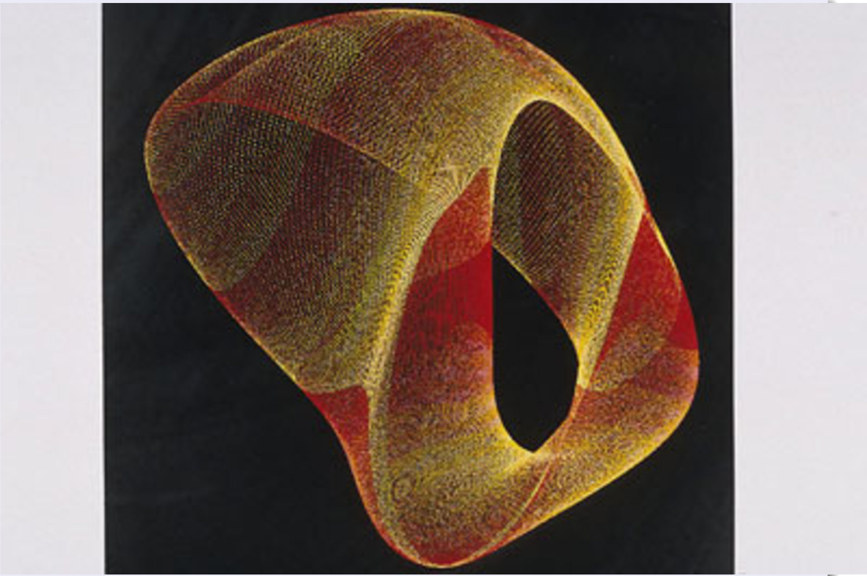
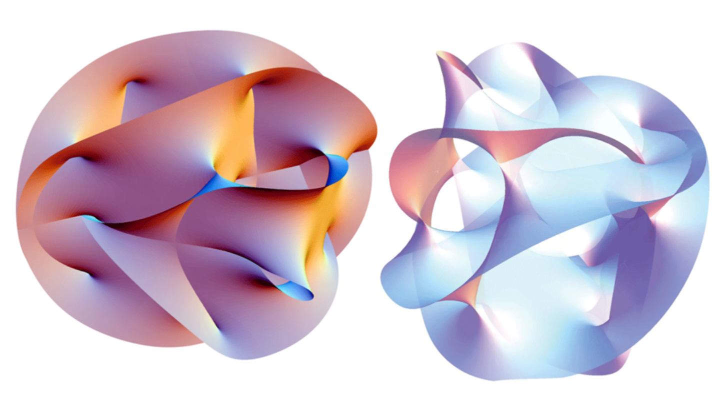






Yang-Baxter integrability versus integrability of the birational symmetries

We have properties of more *arithmetic* and *algebraic geometry* nature. The series expansions of these holonomic functions can be recast into series expansions with **integer** coefficients. This raises the question of the "**modularity**" in these problems: beyond the occurrence of many *modular forms*, we also see the emergence of **Calabi-Yau ODEs**. Calabi-Yau manifolds are, after K3 surfaces, the **"next" generalization of elliptic curves**. We have a natural emergence (in lattice stat. mech.) of **algebraic varieties with an infinite set of birational symmetries**. These algebraic varieties have thus zero canonical class, **Kodaira dimension zero** (zero canonical class, corresponding to admitting flat metrics and Ricci flat metrics, respectively.). We, now, **understand the emergence of Calabi-Yau manifolds**: Abelian varieties and Calabi-Yau manifolds (in dimension one, elliptic curves; in dimension two, complex tori and K3 surfaces) have **Kodaira dimension zero**.



Ising n-fold integrals : the $\chi^{(n)}$'s

The magnetic susceptibility of the two-dimensional Ising model can be written as an **infinite sum** of n-folds integrals **holonomic functions**:

$$\chi(w) \;=\; \sum_{n=1}^{\infty} \chi^{(n)}(w).$$

The **magnetic susceptibility** χ is **not a holonomic function**, it is **not D-finite**: χ is not solution of a linear differential equation. It is much more involved.

We are even going to see that the full susceptibility χ has a (unit circle) **natural boundary**, in the complex k-plane.

$|k| \;=\; 1$ is a natural boundary of $\chi(k)$.

Ising n-fold integrals : the $\chi^{(n)}$'s

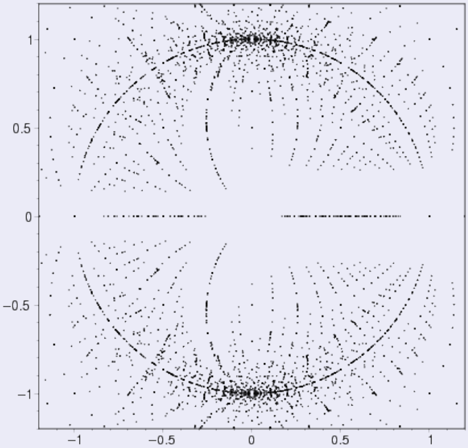
As far as series expansion are concerned, the holonomic $\tilde{\chi}^{(n)}$'s expand as a series with **integer coefficients**:

$$\tilde{\chi}^{(n)}(w) \;=\; 2^n \cdot w^{n^2} \cdot \kappa_n(w),$$

where:

$$\begin{aligned}\kappa_n(w) \;=\; & 1 \;+\; 4\,n^2 \cdot w^2 \;+\; 2 \cdot (4\,n^4 + 13\,n^2 + 1) \cdot w^4 \\ & + \frac{p_6(n)}{3} \cdot w^6 \;+\; \frac{p_8(n)}{3} \cdot w^8 \;+\; \frac{p_{10}(n)}{15} \cdot w^{10} \;+\; \cdots\end{aligned}$$

$$\begin{aligned}p_6(n) \;&=\; 8 \cdot (n^2+4)\,(4\,n^4 + 23\,n^2 + 3),\\ p_8(n) \;&=\; \cdot (32\,n^8 + 624\,n^6 + 4006\,n^4 + 8643\,n^2 + 1404),\\ p_{10}(n) \;&=\; 4 \cdot (n^2+8) \cdot (32\,n^8 + 784\,n^6 + 6238\,n^4 \\ &\qquad + 16271\,n^2 + 3180).\end{aligned}$$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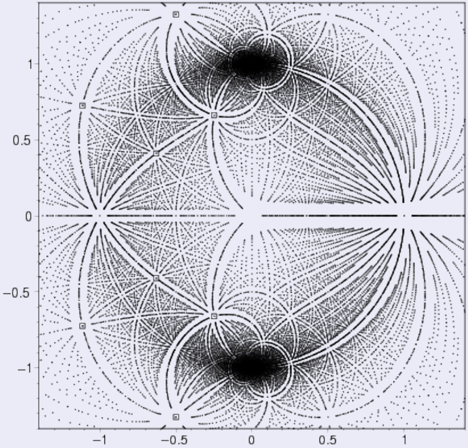
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Ising n-fold integrals : $\chi^{(5)}$

The five-particle contribution $\tilde{\chi}^{(5)}$ of the susceptibility of the Ising model is solution of an order-33 linear differential operator which has a **direct-sum** factorization (DFactorLCLM in Maple): the selected linear combination

$$\tilde{\chi}^{(5)} \; - \frac{1}{2}\,\tilde{\chi}^{(3)} \; + \frac{1}{120}\,\tilde{\chi}^{(1)},$$

is solution of an order-29 (globally nilpotent) linear differential operator

$$L_{29} \; = \quad L_5 \cdot L_{12} \cdot \tilde{L}_1 \cdot L_{11},$$

where:

$$L_{11} \; = \quad (Z_2 \cdot N_1) \oplus V_2 \oplus (F_3 \cdot F_2 \cdot L_1^s).$$

Z_2 in $\chi^{(2)}$: a modular form

The solution of the linear differential operator Z_2 can be expressed in terms of the ${}_2F_1$ hypergeometric function **up to a modular invariant pull-back**:

$$\mathcal{S} = \Big(\Omega \cdot \mathcal{M}_x\Big)^{1/12} \times {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}]; [1]; \mathcal{M}_x\Big), \qquad \text{where:}$$

$$\Omega = \frac{1}{1728} \frac{(1-4x)^6 (1-x)^6}{x \cdot (1+3x+4x^2)^2 (1+2x)^6},$$

$$\mathcal{M}_x = 1728 \frac{x \cdot (1+3x+4x^2)^2 (1+2x)^6 (1-4x)^6 (1-x)^6}{(1+7x+4x^2)^3 \cdot P^3},$$

$$P = 1 + 237x + 1455x^2 + 4183x^3 + 5820x^4 + 3792x^5 + 64x^6.$$

It is a **modular form**.

Ising n-fold integrals : $\chi^{(6)}$

Similarly $\tilde{\chi}^{(6)}$ is solution of an order-52 linear differential operator which has a **direct-sum** factorization: the selected linear combination

$$\tilde{\chi}^{(6)} \, - \frac{2}{3}\tilde{\chi}^{(4)} \, + \frac{2}{45}\tilde{\chi}^{(2)},$$

is solution of an order-46 (globally nilpotent) linear differential operator

$$L_{46} \; = \; L_6 \cdot L_{23} \cdot L_{17},$$

where: $$L_{17} \; = \; \tilde{L}_5 \oplus L_3 \oplus (L_4 \cdot \tilde{L}_3 \cdot L_2),$$

$$\tilde{L}_5 \; = \; \left(\frac{d}{dx} - \frac{1}{x}\right) \oplus \Big(L_{1,3} \cdot (L_{1,2} \oplus L_{1,1} \oplus D_x)\Big).$$

The "Quarks" in $\chi^{(5)}$ and $\chi^{(6)}$

Quasi-trivial order-one (globally nilpotent) linear differential operators: $\tilde{L}_1, \; N_1, \; L_1^s, \; L_{1,n} \quad \longrightarrow \quad D_x \; - \frac{1}{N} \cdot \frac{d\ln(R(x))}{dx}$

$V_2, \; L_2, \; L_3, \; L_5$ and L_6 are respectively equivalent (homomorphic) to L_E, to the symmetric square of L_E and to the *symmetric fourth and fifth power* of L_E.

Remain to understand the "very nature" of:

$F_2, \, F_3, \, \tilde{L}_3, \quad L_4$ and: $L_{12}, \, L_{23}$

The order-12 operator L_{12} has been shown to be irreducible and not equivalent to symmetric product of differential operators of smaller orders.

L_{23}: beyond current computational resources ?

The puzzling L_4: preliminary results on L_4

Preliminary calculations show that L_4 **cannot be reduced to elliptic functions**, modular forms, and it is not ${}_4F_3$-solvable *if one restricts to rational pull-backs*.
Is this operator going to be a **counter-example** to our favourite "mantra" that the Ising model is nothing but the theory of elliptic curves and other modular forms ?

Computing the exterior square of the linear differential operator L_4, one finds an order-six linear differential operator with the *direct sum decomposition*

$$\mathrm{ext}^{(2)}(L_4) \;\; = \;\; \tilde{N}_1 \oplus N_5,$$

where $\tilde{N}_1$ has a **rational function solution**. Along this line L_4 has a symplectic differential Galois group $SP(4, \mathbb{C})$.

The puzzling L_4: preliminary results on L_4

Along a globally nilpotent line, the L_4 operator is "more" than a *G-operator*, with its associated G-series. The series solution (analytical at $x = 0$) $sol(L_4)$ is a **series with integer coefficients** in the variable $y = x/2$:

$$\begin{aligned} sol(L_4) \,=\, & 175 + 34398\, y + 4017125\, y^2 + 362935156\, y^3 \\ & + 28020752579\, y^4 + 1943802285620\, y^5 + 124761498220195\, y^6 \\ & + 7549851868859190\, y^7 + 436341703365296321\, y^8 \\ & + 24309515324321362986\, y^9 + 1314618756208478845353\, y^{10} \\ & + 69377289961823319909960\, y^{11} \\ & + 3588051829563766082490527\, y^{12} \\ & + 182471551181260556637299032\, y^{13} \\ & + 9150139649421210256395488775\, y^{14} + \cdots \end{aligned}$$

L_4 is a Hadamard product of two elliptic curves:

it is a Calabi-Yau operator !

Seeking for ${}_4F_3$ hypergeometric functions up to homomorphisms, and assuming an **algebraic pull-back** with the *square root extension*, $(1 - 16 \cdot w^2)^{1/2}$, we actually found that the solution of L_4 *can be expressed in terms of a selected* ${}_4F_3$

$$
\begin{aligned}
&{}_4F_3\Big([1/2,\, 1/2,\, 1/2,\, 1/2], [1,1,1];\ z\Big) \\
&\quad = \quad {}_2F_1\Big([1/2,\, 1/2], [1];\ z\Big) \star {}_2F_1\Big([1/2,\, 1/2], [1];\ z\Big),
\end{aligned}
$$

where: $$z \;=\; \Big(\frac{1 \,+\, (1 \,-\, 16 \cdot w^2)^{1/2}}{1 \,-\, (1 \,-\, 16 \cdot w^2)^{1/2}}\Big)^4$$

where the pull-back z *is nothing but* the fourth power of the modulus k of the elliptic functions !

Differential algebra viewpoint: the differential Galois group

$$S_R(Ext^2(L_{12}^{(\text{left})})) \;=\; \frac{P_{312}(x)}{A_{131}(\tilde{L}_1 \cdot L_{11}) \cdot \; D_{211}(x)}, \qquad \text{with:}$$

$$D_{211}(x) \;=\; x^{18} \cdot (2x-1)^2\,(x-1)^{12}\,(x+1)^2\,(2x+1)^{13}\,(4x+1)^{22}$$
$$(4x-1)^{24}(4x^2-2x-1)^2\,(4x^2+3x+1)^{14}\,(x^2-3x+1)^2$$
$$(8x^2+4x+1)^8(4x^3-3x^2-x+1)^6\,(4x^3-5x^2+7x-1)^8$$
$$(4x^4+15x^3+20x^2+8x+1)^6,$$

where $P_{312}(x)$ is a polynomial of degree 312, and where $A_{131}(\tilde{L}_1 \cdot L_{11})$ is the apparent polynomial of the product $\tilde{L}_1 \cdot L_{11}$.

The **differential Galois group** of $L_{12}^{(\text{left})}$ is included in the **symplectic group** $Sp(12, \mathbb{C})$.

Differential algebra viewpoint: the differential Galois group

$$L_{23} \;=\; L_{21} \cdot \tilde{L}_2.$$

$$S_R(Sym^2(L_{21})) \;=\; \frac{P_{714}(x)}{D_{529}(x)}, \qquad \text{where:}$$

$$D_{529}(x) \;=\; x^{13} \cdot (1-16x)^{56}\,(1-4x)^{63}\,(1-9x)^{47}\,(1-25x)^{63} \times (1-x)^{47}\,(1-10x+29x^2)^{57}\,(1-x+16x^2)^{63},$$

where P_{714} is a polynomial of degree 714.

The **differential Galois group** of L_{21} is included in the **orthogonal group** $SO(21, \mathbb{C})$.

The $\chi^{(n)}$'s are diagonal of rational functions.

Let us consider the series of $\tilde{\chi}^{(3)}/8/w^9$

$$1 \; + \; 36\,w^2 \; + \; 4\,w^3 \; + \; 884\,w^{13} \; + \; 196\,w^5 \; + \; 18532\,w^6 \; + \; \cdots$$

Let us now consider this very series *modulo the prime* $p \; = \; 2$. It reads the quite lacunary series

$$1 \; + w^8 \; + w^{24} \; + w^{56} \; + w^{120} \; + w^{248} \; + w^{504} \; + w^{1016} \; + \cdots \, ,$$

In fact,*modulo the prime* $p \; = \; 2, \; H(w) \; = \; \tilde{\chi}^{(3)}/8$ is, actually, an **algebraic function**, solution of the quadratic equation:

$$H(w)^2 \; + w \cdot \; H(w) \; + w^{10} \quad = \quad 0 \qquad \mod 2.$$

The $\chi^{(n)}$'s are diagonal of rational functions.

In fact, the series for $\tilde{\chi}^{(3)}$, or for **any** $\tilde{\chi}^{(n)}$, modulo **any prime**, reduces to an algebraic function (the complexity of the algebraic functions growing with p).
This is, in fact, the consequence of the fact that the $\chi^{(n)}$'s are **diagonal of rational functions**.

Definition of the diagonal of series of several complex variables:

$$\mathcal{F}\Big(z_1, z_2, \ldots, z_n\Big) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} F_{m_1, m_2, \ldots, m_n} \cdot z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n},$$

$$Diag\Big(\mathcal{F}\Big(z_1, z_2, \ldots, z_n\Big)\Big) = \sum_{m=0}^{\infty} F_{m, m, \ldots, m} \cdot z^m.$$

Pedagogical examples of diagonal of rational functions.

Let us consider the rational function of three complex variables $\mathcal{F} = 1/(1 - z_2 - z_3 - z_1 z_2 - z_1 z_3)$. Its diagonal reads:

$$1 \;\; + 4z \;\; + 36z^2 \;\; + 400z^3 \;\; + 4900z^4 \;\; + 63504z^5 \;\; + \;\; \cdots$$

which is nothing but the complete elliptic integral of the first kind

$$\sum_{m \geq 0} \binom{2m}{m}^2 \cdot z^m \quad = \quad {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], 16\, z\Big)$$

Such diagonals of rational functions are **highly selected functions**: they are solutions of G-operators. They are also functions that are always algebraic mod. **any prime** p. They fill the gap between algebraic functions and G-series: they can be seen as **generalisations of algebraic functions**.

Mathematical examples of diagonal of rational functions.

Rational functions of three, or four variables: $R = 1/(1-P)$, $deg(P)_{x,y,z,w} \le 1$, coefficients of the monomials in $\{0, 1\}$. For $P = x + y + z + x\,y + x\,z + y\,z$, the diagonal reads:

$$Diag(R) = 1 + 12\,x + 366\,x^2 + 13800\,x^3 + 574650\,x^4 + \cdots$$

$$= Q(x)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{P(x)}{Q(x)^3}\Big), \qquad \text{where:}$$

$$Q(x) = 1 - 48\,x - 24\,x^2,$$

$$P(x) = 1728 \cdot x^3 \cdot (x+2)^3 \cdot (1 - 54x - 28x^2),$$

Four variables: 876 cases, 1 of order 1, 2 of order 2, 20 of order 3, 128 of order 4, 240 of order 5, 231 of order 6, 155 of order 7, 41 of order 9, 7 of order 10, **all correspond to $SO(n, C)$ differential Galois groups !!!** For $P = x\,y\,z + w\,x + y\,z + w + x + y + z$, one has a $SO(6, C)$ decomposition: $(A_1\,B_1\,C_1\,D_3 + A_1\,B_1 + A_1\,D_3 + C_1\,D_3 + 1) \cdot r(x)$.

Towards Modularity: far beyond modular forms

The linear differential operators are **globally nilpotent**, which means that the operators are not only **Fuchsian**, they are such that their p-curvatures are nilpotent, and all their **critical exponents are rational numbers**, ... This is a consequence of the fact that the holonomic functions are **diagonal of rational functions**, which yields (globally bounded) series that can be recast into series with **integer coefficients**. Together with these properties of **algebraic-geometry and arithmetic nature**, one also has properties of more **differential algebra and differential geometry nature**, as can be seen with the emergence of **selected differential Galois groups**, consequence of **homomorphisms of the operators with their adjoint**.

Differential algebra viewpoint: the differential Galois group

$$
\begin{aligned}
L_{[2]} &= (U_2 \cdot U_1 + 1) \cdot r(x), \\
L_{[3]} &= (U_3 \cdot U_2 \cdot U_1 + U_1 + U_3) \cdot r(x), \\
L_{[4]} &= (U_4 \cdot U_3 \cdot U_2 \cdot U_1 \\
&\quad + U_4 \cdot U_1 + U_2 \cdot U_1 + U_4 \cdot U_3 + 1) \cdot r(x), \\
L_{[5]} &= (U_5 \cdot U_4 \cdot U_3 \cdot U_2 \cdot U_1 + U_5 \cdot U_4 \cdot U_1 + U_5 \cdot U_2 \cdot U_1 \\
&\quad + U_5 \cdot U_2 \cdot U_1 + U_5 \cdot U_4 \cdot U_3 + U_3 \cdot U_2 \cdot U_1 \\
&\quad + U_1 + U_3 + U_5) \cdot r(x), \qquad \cdots
\end{aligned}
$$

$$
L_{[N]} = U_N \cdot L_{[N-1]} + L_{[N-2]}.
$$

$$
adjoint(L_{[N]}) \cdot L_{[N-1]} = adjoint(L_{[N-1]}) \cdot L_{[N]}.
$$

Differential algebra viewpoint: the differential Galois group

Using a criterion of Namikawa, Batyrev and Kreuzer found 30241 **reflexive 4-polytopes** such that the corresponding Calabi-Yau hypersurfaces are smoothable by a flat deformation. In particular, they found 210 reflexive 4-polytopes defining 68 topologically different **Calabi-Yau 3-folds** with $h_{11} = 1$, P. Lairez obtained recently, in a systematic analysis, a set of 210 explicit linear differential operators annihilating periods arising from **mirror symmetries** (associated with reflexive 4-polytopes defining 68 topologically different Calabi-Yau 3-folds). These periods are also **diagonals of rational functions**. We found the decomposition of these linear differential operators, for instance

$$\mathcal{L}_{12} \, = \, (M_2 \cdot N_2 \cdot P_2 \cdot Q_2 \cdot R_4 \, + \, M_2 \cdot N_2 \cdot R_4 \, + \, M_2 \cdot Q_2 \cdot R_4 \\ + \, M_2 \cdot N_2 \cdot P_2 \, + \, P_2 \cdot Q_2 \cdot R_4 + \, M_2 + \, P_2 + \, R_4) \cdot r(x).$$

Differential Galois group for lattice Green functions ODEs

We have been able to find the linear differential operator of the **seven-dimensional fcc lattice Green function**. It is an order-11 operator.

$$G_{11}^{7Dfcc} = (U_5 \cdot U_4 \cdot U_3 \cdot U_2 \cdot U_1 + U_5 \cdot U_4 \cdot U_1 + U_5 \cdot U_2 \cdot U_1 + U_5 \cdot U_4 \cdot U_3 + U_3 \cdot U_2 \cdot U_1 + U_1 + U_3 + U_5) \cdot r(x),$$

where $r(x)$ is a rational function, where U_2, U_3, U_4 and U_5 are *order-one self-adjoint* operators, and where U_1 is an *order-seven self-adjoint* operator. G_{11}^{7Dfcc} is **non-trivially homomorphic to its adjoint**

$$adjoint(L_{10}) \cdot G_{11}^{7Dfcc} = adjoint\left(G_{11}^{7Dfcc}\right) \cdot L_{10}.$$

The 11-dimensional fcc operator is of order 27 (2464 coeff. are necessary), the 12-dimensional fcc operator is of order 32 (3618 coeff. are necessary).

SECOND PART of the TALK: SPECULATIONS

A LONG WAY TO GENERALIZE MODULAR FORMS and other CALABI-YAU

MODULARITY: A WORK IN PROGRESS

FROM LINEAR ODEs to NON-LINEAR ODEs

FROM THE MODULUS k, TO THE NOME q (mirror maps)

TEASING: Towards a deeper understanding of the full susceptibility χ

The elliptic parametrization of the Ising model must play a fundamental role. Along this line two different types of transformations should be considered:

• the **isogenies** of the elliptic curves $\tau \rightarrow N \cdot \tau$, the simplest being the Landen transformation $k \rightarrow 2\sqrt{k}/(1+k)$; they **do correspond to exact generators of the renormalization group**:

$$k \longrightarrow \frac{2\sqrt{k}}{1+k}, \qquad \tau \rightarrow N \cdot \tau$$

• the **modular group**

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1,$$

How do the $\chi^{(n)}$ transform under the isogenies (i.e. the renormalization group) and the modular group ?

Let us recall that $\chi^{(2)}$ reads

$$\chi^{(2)} \; = \; \frac{k^4}{64} \cdot {}_2F_1\Big([\frac{3}{2}, \, \frac{5}{2}], \, [3], \, k^2\Big),$$

and that its Landen transform reads:

$$\chi_L^{(2)} \; = \; \frac{1}{64} \cdot \Big(\frac{4\,k}{(1\,+k)^2}\Big)^2 \cdot {}_2F_1\Big([\frac{3}{2}, \, \frac{5}{2}], \, [3], \, \frac{4\,k}{(1\,+k)^2}\Big).$$

Remarkably one finds that the two corresponding linear differential operators ($\Omega(\chi^{(2)}) \, = \, 0, \; \Omega_L(\chi_L^{(2)}) \, = \, 0$) are **(non-trivially) homomorphic !!!** :

$$\Big(\frac{k+1}{k}\Big)^2 \cdot (k\,\frac{d}{dk} \, - 1) \cdot \Omega \; = \; \Omega_L \cdot \frac{x+1}{x} \cdot \frac{d}{dk}.$$

One needs to rephrase the question: how the well-suited $\tilde{\chi}^{(6)} - \frac{2}{3}\tilde{\chi}^{(4)} + \frac{2}{45}\tilde{\chi}^{(2)}$, etc ... transform ?

It is solution of an order-46 linear diff. operator

$$L_{46} \;=\; L_6 \cdot L_{23} \cdot \Big(\tilde{L}_5 \oplus L_3 \oplus (L_4 \cdot \tilde{L}_3 \cdot L_2)\Big),$$

Most of the operators have polynomial solutions in E and K: one **can expect some nice representation of the modular group as well as the isogenies on these operators**. However, we also have operators with selected differential Galois groups, that **cannot be reduced to operators associated with elliptic curves**: for instance L_4 corresponds to a Calabi-Yau manifold. How do the **isogenies of the elliptic curve of the Ising model act on this Calabi-Yau manifold** ?

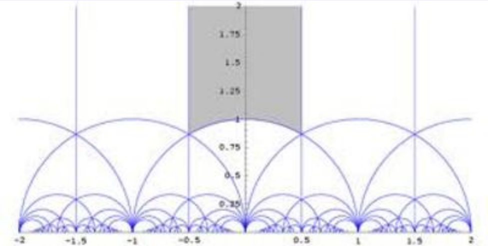
A few scenarii

• Nice representation of the modular group (but not of the isogenies) on $\tilde{\chi}^{(6)} - \frac{2}{3}\tilde{\chi}^{(4)} + \frac{2}{45}\tilde{\chi}^{(2)}$, etc ...

• Nice representation of the modular group and the isogenies on χ, but the decomposition of χ in the holonomic $\chi^{(n)}$'s **is not the good way to see it**.

• The $\chi^{(n)}$'s being too involved composite objects, one only has nice representation of the modular group (and possibly the isogenies) on the form factors $f_{N,M}^{(j)}$.

Crash course on modular forms, modular curves, modular group, ...



The maths textbook are hopeless and useless for our needs ... One never finds the remarkable/magic/amazing equations one badly needs ...

Modular Forms

Let us consider the second order linear differential operator

$$\frac{d^2}{dz^2} \; + \frac{(z^2+56\,z+1024)}{z\cdot(z+16)\,(z+64)}\cdot\frac{d}{dz} \; - \; \frac{240}{z\cdot(z+16)^2\,(z+64)}.$$

which has the (modular form) solution:

$${}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1];\,1728\,\frac{z}{(z+16)^3}\Big)$$
$$= \;\; 2\cdot\Big(\frac{z+256}{z+16}\Big)^{-1/4}\cdot\;{}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1];\,1728\,\frac{z^2}{(z+256)^3}\Big).$$

Fundamental modular curve

The **two pull-backs** in the previous modular form

$$u \; = \; u(z) \; = \; \frac{1728 \; z}{(z+16)^3}, \qquad v \; = \; \frac{1728 \; z^2}{(z+256)^3} \; = \; u\Big(\frac{2^{12}}{z}\Big).$$

are related by a *Atkin-Lehner involution* $z \quad \leftrightarrow \quad 2^{12}/z$, and correspond to a rational parametrization of the **fundamental modular curve** $X_0(2)$:

$$5^9 \, v^3 \, u^3 \; - 12 \cdot 5^6 \, u^2 \, v^2 \cdot (u+v)$$
$$+375 \; u \, v \cdot (16 \, u^2 \; + 16 \, v^2 \; - 4027 \, v \, u)$$
$$-64 \, (v+u) \cdot (v^2 + 1487 \, v \, u \; + u^2) \; + 2^{12} \cdot 3^3 \cdot u \, v \; = \; 0.$$

relating two **Hauptmoduls** u and v.

Dedekind η function

Getting rid of this $(2\pi)^{12}$ factor, (q is the nome of the elliptic curve) $\Delta(q) = q \cdot \prod_{n=1}^{\infty} (1 - q^n)^{24}$, one can now introduce a "second layer" of parametrization identifying the previous z with the (well-known) j-function and writing it as a ratio of **Dedekind eta function**

$$z = j(q) = \Delta(q)/\Delta(q^2),$$

The *Atkin-Lehner involutive* transformation $j \rightarrow 2^{12}/j$ and transformation $q \rightarrow q^2$ are *actually compatible* thanks to the remarkable "Ramanujan-like" functional identity on Dedekind η functions

$$\begin{aligned} 4096 \cdot \Delta(q) \cdot \Delta(q^4)^2 - \Delta(q^2)^3 + (\Delta(q) \\ + 48 \cdot \Delta(q^2)) \cdot \Delta(q) \cdot \Delta(q^4) = 0. \end{aligned}$$

Isogenies, Landen transformation, Renormalization Group

The *exact* generators of the *renormalization group* must necessarily identify with various isogenies which amounts to multiplying, or dividing, τ the ratio of the two periods of the elliptic curves, by an integer. The simplest example is the *Landen transformation*:

$$k \quad \longleftrightarrow \quad k_L \, = \, \frac{2\sqrt{k}}{1+k}, \qquad\qquad \tau \, \longleftrightarrow \, 2\,\tau.$$

which corresponds to the previous *genus zero fundamental modular curve* two **Hauptmoduls** $u \, = \, 12^3/j$ and $v \, = \, 12^3/j'$, and relating the two j-functions

$$j(k) \, = \, 256 \cdot \frac{(1-k^2+k^4)^3}{(1-k^2)^2 \cdot k^4}, \qquad j(k_L) \, = \, 16 \cdot \frac{(1+14\,k^2+k^4)^3}{(1-k^2)^4 \cdot k^2}.$$

Isogenies, Landen transformations, Modular curve

The Landen transformation corresponds to the *genus zero fundamental modular curve*

$$j^2 \cdot j'^2 \; - (j + j') \cdot (j^2 + 1487 \cdot j\, j' \; + j'^2) \\ +3 \cdot 15^3 \cdot (16\, j^2 \; - 4027\, j\, j' \; + 16\, j'^2) \\ -12 \cdot 30^6 \cdot (j + j') \;\; + 8 \cdot 30^9 \;\; = \;\; 0,$$

which relates the two j-functions

$$j(k) \; = \; 256 \cdot \frac{(1-k^2+k^4)^3}{(1-k^2)^2 \cdot k^4}, \qquad j(k_L) \; = \; 16 \cdot \frac{(1+14\,k^2+k^4)^3}{(1-k^2)^4 \cdot k^2}.$$

Isogenies are exact generators of the RG

An **exact generator of the renormalization group must preserve** the three "points" (actually algebraic varieties): $k = 0, 1, \infty$, namely the zero and infinite temperature fixed points and the critical temperature fixed point. **The Landen transformation has these three points as fixed points**.

Such an exact generator must also be compatible with all the exact symmetries of the model: gauge-like (linear) symmetries, the **set of birational** (non-linear) symmetries, the **lattice of periods** of the elliptic parametrization.

The **Landen transformation** and the other **isogenies** actually satisfy **all** these constraints. They are **the only transformations satisfying these drastic constraints**.

Isogenies, Landen transformations on EllipticK

Landen transformation [1775]:

$$K\Big(\frac{2\sqrt{k}}{1+k}\Big) \;=\; (1+k)\cdot\, K(k). \tag{1}$$

EllipticModulus versus EllipticNome:

$$m \;=\; \frac{\lambda}{16} \;=\; \frac{k^2}{16} \;=\; q\cdot\,\Big(\prod_{n=0}^{\infty}\frac{1\,+q^{2n}}{1\,+q^{2n-1}}\Big)^8.$$

$$q \;=\; k^2/16 + k^4/32 + 21/1024\cdot\, k^6 \; + 31/2048\cdot\, k^8 \; + \;\cdots$$

$$= \; m \; + 8\,m^2 \; + 84\,m^3 \; + 992\,x^4 \; + 12514\,m^5 \; + \;\cdots$$

Let us introduce the eulerian product:

$$F(q) \;=\; \Big(\prod_{n=0}^{\infty}\frac{1\,-q^{2n}}{1\,+q^{2n}}\Big)^2 \;=\; \frac{2}{\pi}\cdot\,(1\,-k^2)^{1/4}\cdot\, K(k). \tag{2}$$

Isogenies, Landen transformations on EllipticK

$$F(q) \,=\, 1 \,-k^4/64 \,-k^6/64 \,-231/16384\cdot\, k^8 \,+\, \cdots$$
$$=\, 1 \,-4\,m^2 \,-64\,m^3 \,-924\,m^4 \,-13184\,m^5 \,+\cdots$$
$$F(q^{1/2}) \,=\, 1 \,-k^2/4 \,-7/64\cdot\, k^4 \,-17/256\cdot\, k^6 \,+\, \cdots$$
$$=\, 1 \,-4\,m \,-28\,m^2 \,-272\,m^3 \,-3036\,m^4 \,-36624\,m^5 \,+\cdots$$

The Landen transformation corresponds to $q \,\rightarrow\, q^{1/2}$, Eq. (2) becoming:

$$F(q^{1/2}) \,=\, \frac{2}{\pi}\cdot\Big(1\,-\Big(\frac{2\sqrt{k}}{1+k}\Big)^2\Big)^{1/4}\cdot K\Big(\frac{2\sqrt{k}}{1+k}\Big). \qquad (3)$$

Equations (2), (3) together with the Landen relation (1) gives:

$$\frac{F(q^{1/2})}{F(q)} \,=\, (1\,-k^2)^{1/4}. \qquad (4)$$

Landen transformation, inverseLanden transformation, isogenies on EllipticK

The same calculations for the inverse Landen transformation

$$K\Big(\frac{1-(1-k^2)^{1/2}}{1+(1-k^2)^{1/2}}\Big) = \frac{1+(1-k^2)^{1/2}}{2}\cdot K(k). \qquad (5)$$

yield

$$\frac{F(q^2)}{F(q)} = \frac{(1-k)^{1/2}+(1+k)^{1/2}}{2\cdot(1-k^2)^{1/8}}. \qquad (6)$$

solution of the **linear differential operator**

$$16\cdot(1-k^2)\cdot D_k^2 + 24\cdot(k^2-1)\cdot k\cdot D_k - 3\,k^2. \qquad (7)$$

Similarly, all the ratio $F(q^N)/F(q)$ corresponding to all the isogenies $q \to q^N$, are, not only **solutions of linear differential operators**, but, in fact (quite involved) **algebraic expressions**.

Modular invariance, isogeny covariance, Schwarzian non-linear ODEs

The **Schwarzian equation** reads:

$$\{\tau, \ \lambda\} \ = \ \frac{1}{2} \cdot \frac{(k^4 \ - \ k^2 \ + 1)}{k^4 \cdot (k^2 \ - 1)^2}.$$

The j-function, seen as a function of the nome, expands as:

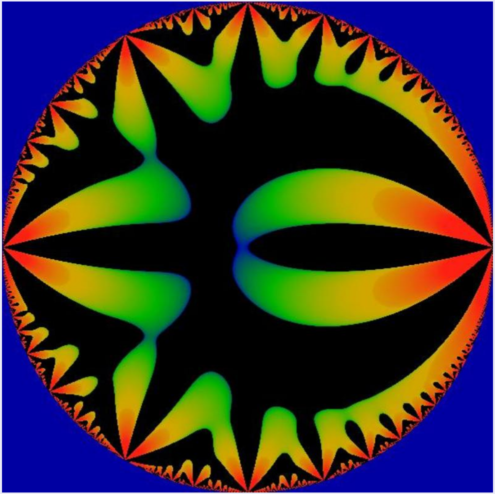
$$j(q) \ = \ \frac{1}{q} \ + 744 \ + 196884 \, q \ + 21493760 \, q^2 \ + \ \cdots$$

and satisfies the **replicable non-linear Schwarzian ODE** corresponding to the equality of two *weight four* modular forms:

$$\{j, \tau\} \ = \ -\frac{1}{2} \cdot \frac{j^2 - 1968 \, j \ + 2654208}{(j \ - 1728)^2} \cdot \left(\frac{1}{j} \cdot \frac{dj}{d\tau}\right)^2. \qquad (8)$$

$j \ = \ j(q^N)$ does verify (8) for **any values of the integer** N

Eisenstein series: from modular forms to Chazy III ...



Modular forms and non-linear ODEs; Eisenstein series.

Let us introduce the Eisenstein series

$$E_4(q) \;=\; 1 \,+\, 240 \cdot \sum_{n=1}^{\infty} n^3 \cdot \frac{q^n}{1-q^n},$$

satisfies the following **non-linear functional equation**:

$$33\, E_4(q^2)^2 \;+ E_4(q)^2 \;- 18 \cdot (16\, E_4(q^4) \,+ E_4(q)) \cdot E_4(q^2)$$
$$+16 \cdot (16\, E_4(q^4) \,+ E_4(q)) \cdot E_4(q^4) \;=\; 0.$$

For $E_4(q^n)$ the **non-linear ODE** reads ($N_n \;=\; d^N N/d\tau^n$):

$$20\, N^2\, N_3^2 \;- 180\, N\, N_1\, N_2\, N_3 \;+ 144\, N\, N_2^3 \;+ 150\, N_1^3\, N_3$$
$$-135\, N_1^2\, N_2^2 \;=\; \frac{5\, n^2}{4} \cdot N \cdot (4\, N\, N_2 \,- 5\, N_1^2)^2.$$

Quasi-modular forms and non-linear ODE of the Painlevé type: Chazy III

$$N \, = \, E_2(q), \quad N_1 \, = \, q \cdot \frac{dN}{dq}, \quad N_2 \, = \, q \cdot \frac{dN_1}{dq}, \quad N_3 \, = \, q \cdot \frac{dN_2}{dq},$$

Quasi-modular form:

$$N(x) \quad \longrightarrow \quad \frac{a\,d \, - \, b\,c}{(c\,x \, + d)^2} \cdot N\Bigl(\frac{a\,x \, + \, b}{c\,x \, + d}\Bigr) \quad - A \cdot \frac{c}{c\,x \, + d},$$

$$N_1(x) \quad \longrightarrow \quad \frac{(a\,d \, - \, b\,c)^2}{(c\,x \, + d)^4} \cdot N_1\Bigl(\frac{a\,x \, + \, b}{c\,x \, + d}\Bigr)$$

$$-2\,c \cdot \frac{a\,d \, - \, b\,c}{(c\,x \, + d)^3} \cdot N\Bigl(\frac{a\,x \, + \, b}{c\,x \, + d}\Bigr) \quad + \, A \cdot \frac{c^2}{(c\,x \, + d)^2},$$

$$
\begin{aligned}
N_2(x) \quad \longrightarrow \quad & \frac{(a\,d\,-\,b\,c)^3}{(c\,x\,+d)^6}\cdot N_2\Big(\frac{a\,x\,+\,b}{c\,x\,+d}\Big) \\
& -6\,c\cdot\,\frac{(a\,d\,-\,b\,c)^2}{(c\,x\,+d)^5}\cdot N_1\Big(\frac{a\,x\,+\,b}{c\,x\,+d}\Big) \\
& +6\,c^2\cdot\,\frac{a\,d\,-\,b\,c}{(c\,x\,+d)^4}\cdot N\Big(\frac{a\,x\,+\,b}{c\,x\,+d}\Big)\;-2\,A\cdot\,\frac{c^3}{(c\,x\,+d)^3}, \\
N_3(x) \quad \longrightarrow \quad & \frac{(a\,d\,-\,b\,c)^4}{(c\,x\,+d)^8}\cdot N_3\Big(\frac{a\,x\,+\,b}{c\,x\,+d}\Big) \\
& -12\,c\cdot\,\frac{(a\,d\,-\,b\,c)^3}{(c\,x\,+d)^7}\cdot N_2\Big(\frac{a\,x\,+\,b}{c\,x\,+d}\Big) \\
& +36\,c^2\cdot\,\frac{(a\,d\,-\,b\,c)^2}{(c\,x\,+d)^6}\cdot N_1\Big(\frac{a\,x\,+\,b}{c\,x\,+d}\Big) \\
& -24\,c^3\cdot\,\frac{a\,d\,-\,b\,c}{(c\,x\,+d)^5}\cdot N\Big(\frac{a\,x\,+\,b}{c\,x\,+d}\Big)\;+6\,A\cdot\,\frac{c^4}{(c\,x\,+d)^4},
\end{aligned}
$$

Quasi-modular forms and non-linear ODE of the Painlevé type: Chazy III

Let us introduce the **Eisenstein series**

$$E_2(q) \;=\; 1 \;-\; 24 \cdot \sum_{n=1}^{\infty} n \cdot \frac{q^n}{1 - q^n}.$$

It is a **quasi-modular form** (previous formula with $A \,=\, 12$), and verifies

$$2\,N\,N_2 \,-3\,N_1^2 \,-2\,N_3 \;=\; 0, \qquad \text{where:}$$

$$N \;=\; E_2(q),\;\; N_1 \;=\; q \cdot \frac{dN}{dq},\;\; N_2 \;=\; q \cdot \frac{dN_1}{dq},\;\; N_3 \;=\; q \cdot \frac{dN_2}{dq},$$

This is nothing but the Chazy III equation:

$$\frac{d^3y}{dx^3} \;=\; 2\,y\frac{d^2y}{dx^2} \,-3\left(\frac{dy}{dx}\right)^2.$$

Schwarzian derivative and natural boundary

It can be rewritten in terms of a Schwarzian derivative:

$$f^{(4)} \;=\; 2\, f'^2 \cdot \{f,\, x\} \;=\; 2\, f'\, f''' \,-\, 3\, f"^{\,2} \qquad \text{with:} \quad y \;=\; \frac{df}{dx}.$$

It was introduced by Jean Chazy (1909, 1911) as an example of a third-order differential equation with a movable singularity that is a **natural boundary** for its solutions. It is also worth recalling the **Halphen-Ramanujan differential system**:

$$P' \;=\; \frac{P^2 \,-\, Q}{12}, \qquad Q' \;=\; \frac{P\,Q \,-\, R}{3}, \qquad R' \;=\; \frac{P\,R \,-\, Q^2}{2},$$

where $P \,=\, E_2,\; Q \,=\, E_4,\; R \,=\, E_6$ and X' denotes here the homogeneous derivative $q \cdot \frac{dX}{dq}$.

Non-holonomic functions ratio of holonomic functions

In fact (see arXiv0902.3861v1[nlin.SI]) y in Chazy III is nothing but a log-derivative of a modular form Δ:

$$y \,=\, \frac{1}{2} \cdot \frac{\Delta'}{\Delta} \,=\, \frac{1}{2} \cdot P$$

Log-derivative of modular forms are quasi-modular forms.
The modular discriminant Δ satisfies the non-linear ODE:

$$2\,(\Delta^3 \,-\, 5\,\Delta^2\,\Delta')\,\Delta''' \,-\, 3\,\Delta''^3\,\Delta^2 \,+\, 24\,\Delta''\,\Delta'^2\,\Delta \,-\, 13\,\Delta'^4 \,=\, 0$$

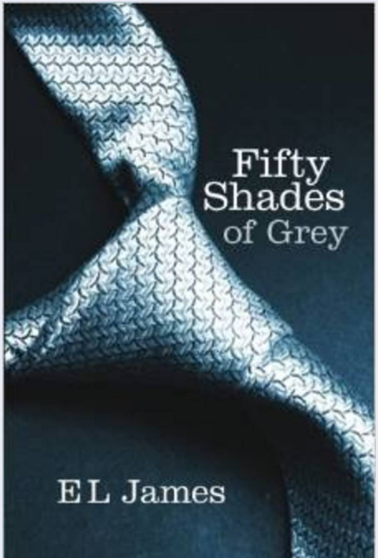
Along this line it is fundamental to recall that the **ratio** (not the product !) of **two holonomic** functions is **non-holonomic**

$$\frac{d^2y}{dx} \,+\, R(x) \cdot y \,=\, 0, \qquad \tau(x) \,=\, \frac{y_1}{y_2},$$

$$\{x,\, \tau\} \,+\, 2\,R(x) \cdot \left(\frac{dx}{d\tau}\right)^2 \,=\, 0.$$

Integrability versus non-integrability ...

Not black or white, but rather fifty shades of grey ...



A grey conclusion

Integrability: Holonomic functions.

Non-integrability: Non-holonomic functions.

Non-holonomic functions like **Chazy III**, and also the susceptibility of the square Ising model are non-holonomic but they **do belong to the "Integrability world"**. The $\chi^{(n)}$ decomposition of the χ susceptibility yields Calabi-Yau ODE (and manifolds) and highly selected linear differential operators (special differential Galois groups, etc ...). The $\chi^{(n)}$'s are diagonal of rational functions: they are the **class of transcendental functions which is the "closest" to algebraic functions** (modulo a prime they do reduce to algebraic functions). As far as the algorithmic complexity of the calculations of the χ series, these calculations are polynomial (in N^4, consequence of J.H.H. Perk's finite difference equations (which can be viewed as a finite difference generalization of Painlevé equations). **Natural boundary** is not even characteristic of non-integrability: think of Chazy **III**.

Separating the wheat from the chaff

The Riemann zeta function is a transcendental non-holonomic function. The p.f. of the hard-square model is, quite certainly, **not even solution of a non-linear ODE**. In contrast, we encountered:

• **Diagonal of rational functions are transcendental holonomic functions that are the closest transcendental functions to algebraic functions**.

• **Non-holonomic functions that are ratio of holonomic functions, solutions on non-linear ODEs of Painlevé type**.

In Dante's inferno these various ""functions"" are not at the same "level" (circle ...).

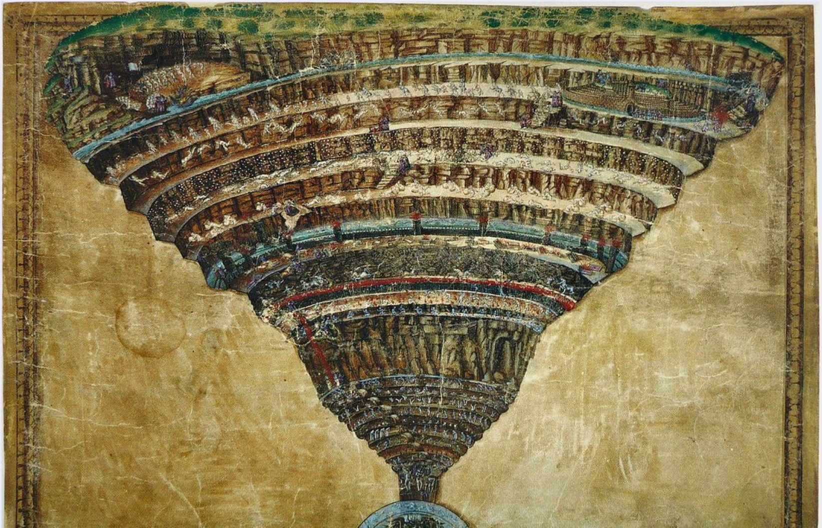
Another grey conclusion (different shade)

Interplay between different domains of physics (field theory, enumerative combinatorics, lattice statistical mechanics, condensed matter, particle physics, ...) and different domains of mathematics: **Algebraic Geometry, Differential Algebra, Differential Geometry**, (differential Galois groups), **Arithmetics, Number Theory**.

Not surpringly for Yang-Baxter integrability experts, the deepest ideas do not come from continuous symmetries but do emerge with infinite discrete symmetries (birational symmetries, isogenies, ...). Doing physics is not doing less mathematics. Paradoxically, doing (good) physics is (without knowing it ...) doing quite fundamental mathematics, working, in a quite deep way, precisely at the crossroad of different domains of mathematics,

as Monsieur Jourdain talked prose, without knowing it.

THE END. Chaos versus Integrability, Inferno or Paradise, a transcendence problem: Dante's Inferno ...



THE END