From Baxter's model to modular correspondences and differentially algebraic globally bounded series.

Life and achievements of R.J. Baxter conference 2025, MSI, ANU Canberra

Dedicated to Rodney Baxter

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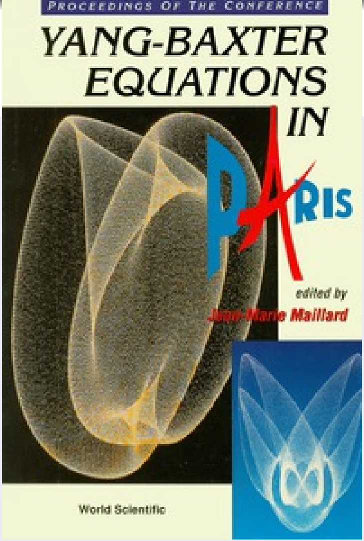
I first met Rodney in 1980.
Fundamental Problems in Statistical Mechanics. Proceedings, 5th International Summer School, Enschede, Netherlands, June 23 - July 5, 1980 E.G.D. Cohen editor.
Two years later Rodney sent me a three months (dream ...) invitation in Canberra !! This was my first visit to Canberra, but not the last !
The last time I saw him was in London in 2023 at the Royal Society.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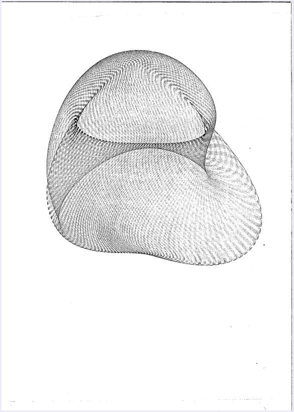
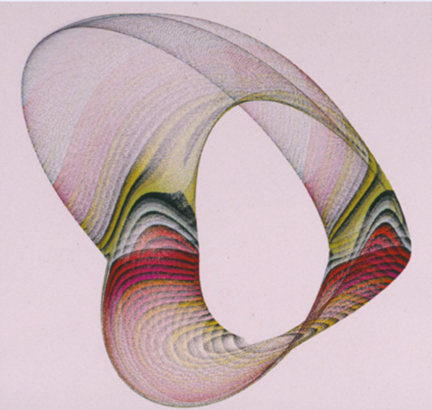


YBE are necessarily parametrized in terms of **algebraic varieties** with a (generically) infinite set of **birational automorphisms**. In the case of curves, this is the reason of the emergence of **elliptic curves** (i.e. genus one), in so many YB integrable models, like the Baxter model.



You can actually obtain the algebraic variety from the iteration of a birational transformation: this corresponds to the so-called "baxterization".

Baxter model

Let us consider the parametrization of the Baxter model:

$$a = \rho \cdot sn(v+\eta,\, k),\;\; b = \rho \cdot sn(\eta - v,\, k),\;\; c = \rho \cdot sn(2\,\eta,\, k),$$
$$d = \rho \cdot k \cdot sn(2\,\eta,\, k) \cdot sn(v+\eta,\, k) \cdot sn(\eta - v,\, k).$$

The iteration of a birational transformation (baxterization) corresponds, in the spectral parameter, to $v \longrightarrow v + 2\,n \cdot \eta$ where the **modulus** of the elliptic functions **remains fixed**. In this talk we are not going to consider these **birational** symmetries, but, rather, to infinite discrete (algebraic) transformations, **in the modulus** k**,** and we will see that they correspond to exact generators of the renormalization group, to modular forms and to modular correspondences.

Before the Baxter model: 399 solutions of the Ising model

The partition function (per site) of the (isotropic) Onsager model can be written in terms of a ${}_4F_3$ hypergeometric function

$$\ln(Z) \; = \; \ln(2\cosh(2K)) \; - \frac{k_L^2}{16} \cdot {}_4F_3\Big([1,\, 1,\, \frac{3}{2},\, \frac{3}{2}],\, [2,\, 2,\, 2],\, k_L^2\Big),$$

where: $$k_L \; = \; \frac{2\sqrt{k}}{1 + k}$$ with:

$$k \; = \; \sinh(2K_1) \cdot \sinh(2K_2) \; = \; \sinh(2K)^2.$$

k is the modulus of the elliptic functions parametrizing the 2-D Ising model, and $k \; \rightarrow \; k_L$ is the **Landen transformation**.
See for instance: *The hypergeometric series for the partition function of the Ising model*, by G. M. Viswanathan, 2015 https://arxiv.org/pdf/1411.2495

Before the Baxter model: 399 solutions of the Ising model

On this ${}_4F_3$ hypergeometric form it is crystal clear that the partition function is **D-finite** (i.e. solution of a linear differential operator with polynomial coefficients).
See also *399-th solution of the Ising model*, by R.J. Baxter 1978 J. Phys. A: Math. Gen. **11** 2463

$$\ln(Z) \, = \, \ln(2\,\cosh(2\,K)) + \frac{1}{2\,\pi^2} \cdot \int_0^{\pi} \int_0^{\pi} \ln\Big(1 \, - k_L \cdot \cos(q_1)\,\cos(q_2)\Big) \cdot dq_1\, dq_2$$

Let us perform a derivative of the partition function in order to get rid of the log, this Onsager's double integral form becomes similar to Lattice Green Functions (LGF), like the hyper-cubic LGF:

$$G(t) \, = \, \frac{1}{(2\pi)^n} \int_0^{\pi} \cdots \int_0^{\pi} \frac{dq_1 \cdot dq_2 \, \cdots \, dq_n}{1 \, - t \cdot \lambda},$$

$$\lambda \, = \, c_1 + c_2 \, + \cdots + c_n, \qquad c_j \, = \, \cos q_j$$

LGF are D-finite: they are **diagonal of rational functions**.

Is the partition function per-site of the Baxter model D-finite ? or $D - D$ finite or differentially algebraic ?

Recalls on the Baxter model.

$$T(v) \simeq A(q\,z/x) \cdot A(q/z/x) \cdot \frac{\mathcal{F}(x^2\,z)\,\mathcal{F}(x^2/z)}{\mathcal{F}(q/z)\,\mathcal{F}(q\,z)},$$

$$\mathcal{F}(z) = \prod_{m=0}^{\infty} \frac{A(x^{4m+1}\,z)}{A(x^{4m+3}\,z)}, \qquad A(z) = \prod_{n=0}^{\infty} (1 - q^n\,z)^N,$$

where q denotes the nome of the elliptic functions parametrizing the model, where x and z are exponentials of the shift η and of the spectral parameter u along the elliptic curve (see (10.7.9) page 225 and (10.7.19) page 226 in Baxter's book):

$$q = \exp\Big(-\frac{\pi\,I'}{I}\Big), \quad x = \exp\Big(-\frac{\pi\,\eta}{2\,I}\Big), \quad z = \exp\Big(-\frac{\pi\,u}{2\,I}\Big).$$

Infinite products of infinite products. Elliptic beta functions, Elliptic gamma functions, Barnes functions $G(z+1) = \Gamma(z) \cdot G(z)$.

Almost like having three periods

.

The three variables q, x and z are "almost" on the same footing. At least q and x are "quite" on the "same footing".

It is like the function had three periods (half-periods) I, I' and η.

Jacobi (1835) proved that a **single-valued univariate** function cannot have distinct periods (Boyer and Merzbach 1991, p. 525), thus showing that **elliptic functions are the most general multiply periodic single-valued functions possible in a single variable**.

The ""third period "" η, is a quasi-period, it corresponds to a covariance of the function (see the functional equation associated with the **inversion relation**, together with the crossing relation $Z(z) = Z(1/z)$):

$$Z(z) \cdot Z(x^2/z) \;=\; \lambda \cdot A(x\,z) \cdot A\Big(\frac{x^3}{z}\Big) \cdot A\Big(\frac{q\,z}{x^3}\Big) \cdot A\Big(\frac{q}{x\,z}\Big)$$

$$Z(x^4 \cdot z) \;=\; \Big(A(\,) \cdot A(\,) \cdots A(\,)\Big) \cdot Z(z)$$

Selected subcases of the Baxter model: RSOS models

$$\tau \; = \; \frac{2\pi\, K}{K'}, \quad \lambda \; = \; \frac{2\pi\, \eta}{K'}, \quad u \; = \; \frac{\pi \cdot \, (\eta \, + v)}{K'}, \quad q \; = \; \exp(-\tau).$$

The **Ising** model is a subcase of the **Baxter** model corresponding in these variables, to the condition (see (10.9.7) page 238 in Baxter's Book and Appendix 2):

$$q \; = \; x^4.$$

The other subcases of the Baxter model are, in the RSOS family (see G.E. Andrews, R. Baxter and P. Forrester) which corresponds to $\eta \, = \, K/r$ (r is an integer, K is the complete elliptic integral of the first kind), the hard hexagon model which corresponds to $\eta \, = \, K/5$, i.e. to the condition:

$$q \; = \; x^5,$$

when the other RSOS conditions read $q \, = \, x^r$ (r is an integer).

Magnetization of the Baxter model

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The **magnetization** of the Baxter model reads (see (10.9.10) page 238 in Baxter's book):

$$M(q) \;=\; \prod_{n=1}^{\infty} \frac{1\,-\,q^{2n-1}}{1\,+\,q^{2n-1}},$$

which is independent of x and or course z.
Note, however, that written in terms of the modulus k of elliptic functions, instead of the nome q of elliptic functions, the spontaneous magnetization has the remarkably simple **algebraic form**

$$M(q) \;=\; (1\,-\,k^2)^{1/8}.$$

Calabi-Yau manifolds, Calabi-Yau equations, mirror symmetries, series with integer coefficients.

With the elliptic functions and modular forms, and the two previous k versus q, modulus versus nome descriptions, we have the simplest illustration of the mirror symmetry. In the "q-world" the automorphy properties (infinite products), the modular group symmetries are obvious, we have diffentially algebraic equations, but the D-finite (holonomic) **linear** structures are hidden. In the "k-world" we have (or we may have ...) linear differential equations, D-finite functions, but the automorphy properties (infinite products), the modular group symmetries are hidden.

We need the two complementary descriptions.

Polarization of the Baxter model.

On the other hand, the **polarization** of the Baxter model (see (10.10.24) page 245 in Baxter's book) reads:

$$P(q, x) \;=\; \prod_{n=1}^{\infty} \Big(\frac{1+q^n}{1-q^n} \cdot \frac{1-x^{2n}}{1+x^{2n}} \Big)^2.$$

At first sight, it seems difficult to imagine that, similarly to the spontaneous magnetization, this last infinite product exact expression could also reduce, possibly in the isotropic case, to an algebraic expression, or just a D-finite function in k. In fact, it is algebraic, when q and x are **"commensurate"**, i.e. if there exist two positive integers N and M such that:

$$x^N \;=\; q^M.$$

To be D-finite in the modulus k or (differentially algebraic) in the nome q ?

The spontaneous magnetization is clearly D-finite in the **modulus** k (it is algebraic ...), but it is **not** D-finite in the nome q: it is differentially algebraic, which means solution of a **non-linear** differential equation. The $q \leftrightarrow k$ transformation is **differentially algebraic**.

Corresponds to the concept of mirror-map.

$$\tilde{X}(q) \; = \; q \; - 744\, q^2 \; + 356652\, q^3 \; - 140361152\, q^4 \; + \; \cdots$$

and the *nome* which is its *compositional inverse*:

$$\tilde{Q}(x) \; = \; x \; + 744\, x^2 \; + 750420\, x^3 \; + 872769632\, x^4 \; + \; \cdots$$

We will come to this in a few slides ...

Calabi-Yau manifolds, Calabi-Yau equations, mirror symmetries, series with integer coefficients.

With the elliptic functions and the modular forms, and the two previous k versus q, modulus versus nome descriptions, we have the simplest illustration of the mirror symmetry. In the "q-world" the automorphy properties (infinite products), the modular group symmetries are obvious, we have diffentially algebraic equations, but the D-finite (holonomic) **linear** structures are hidden. In the "k-world" we have (or we may have ...) linear differential equations, D-finite functions, but the automorphy properties (infinite products), the modular group symmetries are hidden. **We need the two complementary descriptions**.

Ising n-fold integrals : the $\chi^{(n)}$'s

The full susceptibility of the two-dimensional Ising model can be written as an **infinite sum** of n-folds integrals which are **holonomic functions** ($w = s/2/(1+s^2)$, $k = s^2$, $s = \sinh(2\,K)$):

$$\chi(w) \;=\; \sum_{n=1}^{\infty} \chi^{(n)}(w).$$

All these n-fold integrals $\chi^{(n)}$ are actually **diagonals of rational functions** !!

In the contrast, the **magnetic susceptibility**, χ, is **not a holonomic function**, it is **not D-finite**: χ is **not solution of a linear differential equation**. It is much more involved: **is it differentially algebraic** ?

The full susceptibility χ has a (unit circle) **natural boundary**, in the complex k-plane: $|k| = 1$ is a natural boundary of $\chi(k)$.

The well-suited framework: diagonal of rational functions

We also found in enumerative combinatorics, lattice statistical mechanics, many other solutions of selected **linear** differential operators, which have **special differential Galois groups**. All these linear differential operators are **globally nilpotent**: they are not only **Fuchsian**, they are such that their p-curvatures are nilpotent, and all their **critical exponents are rational numbers**, ... They are "**Derived from Geometry**": they annihilate n-fold integrals of **algebraic integrands** (in mathematician's wording "**Periods**"). These n-fold integrals are (or can be recast into) series with **integer coefficients** (globally bounded series). These two set of properties are, in fact, the consequence of the fact that these holonomic functions are actually **diagonal of rational functions**. *As Monsieur Jourdain talked prose, without knowing it*, n-fold integrals in physics are, without knowing it, **diagonal of rational functions**, which corresponds to a **quite remarkable set**.

((Diagonal of rat. func.) solutions of high order linear differential operators

As seen in "*Experimental mathematics on the magnetic susceptibility of the square lattice Ising model*", or "*High order Fuchsian equations for the square lattice Ising model: $\chi^{(5)}$*", with A J Guttmann, the $\chi^{(n)}$'s are solutions of linear differential operators of quite large order, which factorize into **products** and **direct sums** of **many** factors:

$$\Big(\big(\,\big)\cdot\big(\,\big)\cdots\big(\,\big)\Big) \oplus \Big(\big(\,\big)\cdot\big(\,\big)\cdots\big(\,\big)\Big) \oplus \cdots$$

where each factor has highly selected function solutions: **elliptic functions**, **modular forms**, derivatives of modular forms, and other remarkable functions with **modularity properties** (Calabi-Yau but that's another story, ...).

At this step: Modular forms are just “involved” elliptic functions.

In the following, we will focus on modular forms, modular curves, modular correspondences ... At this step, just see a modular form as an “automorphic function” $\Phi(x)$ for a “**symmetry**” $x \rightarrow y(x)$:

$$\Phi\Big(y(x)\Big) \; = \; \mathcal{A}(x) \cdot \Phi(x).$$

Spoiler: in physics the symmetry $x \rightarrow y(x)$ will be a generator of the renormalization group.

Z_2 in $\chi^{(3)}$ or $\chi^{(5)}$: a modular form

The solution of the linear differential operator Z_2 can be expressed in terms of the ${}_2F_1$ hypergeometric function **up to a modular invariant** **pull-back** $\mathcal{M}_x$:

$$\mathcal{S} \,=\, \Big(\Omega \cdot \mathcal{M}_x\Big)^{1/12} \times \,\, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}]; [1]; \, \mathcal{M}_{\mathbf{x}}\Big), \qquad \text{where:}$$

$$\Omega \,=\, \frac{1}{1728} \, \frac{(1-4\,x)^6\,(1-x)^6}{x \cdot (1\,+3\,x\,+4\,x^2)^2\,(1+2\,x)^6},$$

$$\mathcal{M}_{\mathbf{x}} \,=\, 1728\, \frac{x \cdot (1\,+3\,x\,+4\,x^2)^2\,(1\,+2\,x)^6\,(1-4\,x)^6\,(1-x)^6}{(1\,+7\,x+4\,x^2)^3 \cdot P^3},$$

$$P \,=\, 1\,+237\,x\,+1455\,x^2+4183\,x^3+5820\,x^4+3792\,x^5+64\,x^6.$$

It is a **modular form**.

Be careful not any ${}_2F_1([\alpha, \beta], [1], p(x))$ corresponds to a modular form ...

Simple (automorphy) covariance is too simple: the full susceptibility of the Ising model

Remarkably long series expansion (5000 **coefficients !!!**) were obtained for the **low-temp. full susceptibility** of the Ising model (here $w = s/(1+s^2)/2$):

$$\tilde{\chi}_L(w) = 4\,w^4 + 80\,w^6 + 1400\,w^8 + 23520\,w^{10} + 388080\,w^{12} + 6342336\,w^{14} + 103062976\,w^{16} + 1668639424\,w^{18} + 26948549680\,w^{20} + \cdots$$

to be compared with the series for $\tilde{\chi}^{(2)}(w)$ namely :

$$\tilde{\chi}_L^{(2)} = 4\,w^4 \cdot {}_2F_1\Big([\frac{3}{2}, \frac{5}{2}], [3], 16\,w^2\Big)$$
$$= 4\,w^4 + 80\,w^6 + 1400\,w^8 + 23520\,w^{10} + 388080\,w^{12} + 6342336\,w^{14} + 103062960\,w^{16} + 1668638400\,w^{18} + 26948510160\,w^{20} + \cdots$$

From modular forms to derivatives of modular forms.

The hypergeometric function $\tilde{\chi}_L^{(2)}$ is "not exactly" of the automorphic form $\Phi\Big(y(x)\Big) \, = \, \mathcal{A}(x) \cdot \Phi(x)$ but rather

$$\Phi\Big(y(x)\Big) \;=\; \mathcal{L}(x) \cdot \Phi(x),$$

where $\mathcal{L}(x)$ is a linear differential operator. The hypergeometric function $\tilde{\chi}_L^{(2)}$ can, in fact, be written as an order-one linear diff. operator $\mathcal{L}_1$ acting on a **modular form**:

$$\tilde{\chi}_L^{(2)} \;=\; -\frac{1}{12} \cdot \mathcal{L}_1 \cdot {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], 16\, w^2\Big).$$

$$\mathcal{L}_1 \;=\; w \cdot (8\, w^2 \, - 1) \cdot \frac{d}{dw} \; + 8\, w^2.$$

and we have a rather simple generalization of the previous automorphy relation:

Simple (automorphy) covariance is too simple: Renormalization group

$$\tilde{\chi}^{(2)}\Big(\frac{2\sqrt{k}}{1\,+\,k}\Big) \;=\; 4 \cdot \frac{1\,+\,k}{k} \cdot \frac{d\,\tilde{\chi}^{(2)}(k)}{dk},$$

where: $$\tilde{\chi}^{(2)}(k) \;=\; \frac{k^4}{4^3} \cdot {}_2F_1\Big([\frac{3}{2},\, \frac{5}{2}],\; [3],\; k^2\Big).$$

Conversely, this relation can also be written as

$$\tilde{\chi}^{(2)}(k) \;=\; \frac{1}{4} \cdot \Big(k \cdot (k-1) \cdot \frac{d}{dk} \,+\, \frac{k^2\,+k\,+2}{k\,+1}\Big) \cdot \tilde{\chi}^{(2)}\Big(\frac{2\sqrt{k}}{1\,+\,k}\Big),$$

or, introducing the *inverse (descending)* **Landen transformation**:

$$\tilde{\chi}^{(2)}\Big(\frac{1\,-\sqrt{1\,-k^2}}{1\,+\sqrt{1\,-k^2}}\Big) \;=\; \Big(\frac{(k^2\,-2)\cdot\sqrt{1\,-k^2}\,+\,2}{4\,k^2}\Big) \cdot \tilde{\chi}^{(2)}(k)$$
$$+\; \frac{k^2\,-1}{4\,k} \cdot \Big(1\,-\,\sqrt{1\,-k^2}\Big) \cdot \frac{d\tilde{\chi}^{(2)}(k)}{dk}.$$

Landen transformation and renormalization group.

The Landen transformation, or the inverse Landen transformation, is an exact generator of the renormalization group. An **exact generator** of the renormalization group **must be compatible** with the elliptic parametrization of the Ising (resp. Baxter) model. It must have the critical point, $k = 1$, as a fixed point, but, beyond, one must have $k = 0, 1, \infty$ preserved by such a generator. At this step, an infinite number of functions can be generator of the renormalization group. However one must impose that **the lattice of periods is actually compatible with such generator** of the renormalization group. The only such transformations are **isogenies of the elliptic curves**: they are algebraic transformations, corresponding to modular correspondences. We are going to study these **modular correspondences** in detail, in the following.

Landen transformation.

The Landen transformation is the simplest example of such a transformation. Naively one expects simple covariance

$$\Phi\Big(\frac{2\sqrt{k}}{1\,+\,k}\Big) \;=\; \mathcal{A}(k)\cdot\,\Phi(k),$$

like, for instance, in the simplest example of elliptic function (and modular form ...), namely the (complete elliptic integral) EllipticK function ($2/\pi\cdot\,K(k) \,=\, {}_2F_1([\frac{1}{2},\,\frac{1}{2}],\,[1],\,k^2))$:

$$K\Big(\frac{2\sqrt{k}}{1\,+\,k}\Big) \;=\; (1\,+k)\cdot\,K(k).$$

With $\tilde{\chi}^{(2)}$ we see that we have a "slight" generalization of these automorphy relations (derivative of modular form).

Modular forms

The Ising model seems to be nothing but the theory of **elliptic curves** and other modular forms, and also **derivatives** of modular forms, **what else ?**
Let us focus on **modular forms**, modular curves, modular equations, **modular correspondences**.
We need to understand modular forms, modular correspondences.

Modular Forms. Crash course.

Let us consider the second order linear differential operator

$$\frac{d^2}{dt^2} + \frac{(t^2 + 56\,t + 1024)}{t \cdot (t+16)\,(t+64)} \cdot \frac{d}{dt} - \frac{240}{t \cdot (t+16)^2\,(t+64)}.$$

which has the (modular form) solution:

$$\begin{aligned} &{}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1],\ 1728\,\frac{t}{(t+16)^3}\Big) \\ &= \ 2 \cdot \Big(\frac{t+256}{t+16}\Big)^{-1/4} \cdot \ {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1],\ 1728\,\frac{t^2}{(t+256)^3}\Big). \end{aligned}$$

This looks like **one** identity: in fact it is an **infinite number of identities**.

Fundamental modular curve. Symmetry $x \longrightarrow y = y(x)$.

The **two pull-backs** in the previous modular form

$$x = \frac{t}{(t+16)^3}, \qquad y = \frac{t^2}{(t+256)^3} = x\Big(\frac{2^{12}}{t}\Big),$$

are related by a simple *involution* $t \longleftrightarrow 2^{12}/t$, and correspond to a *rational parametrization* of the (fundamental) **modular curve** $P_2(x, y) = P_2(y, x) = 0$:

$$\begin{aligned} &157464000000000 \cdot x^3 y^3 - 8748000000 \cdot x^2 y^2 \cdot (x+y) \\ &\quad +10125 \cdot x y \cdot (16 x^2 - 4027 x y + 16 y^2) \\ &\qquad -(x+y) \cdot (x^2 + 1487 x y + y^2) + x y = 0. \end{aligned}$$

Modular Forms. Crash course.

$$ {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \ 1728\ y\Big) $$
$$ = \ \mathcal{A}(x) \cdot \ {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \ 1728\ x\Big) $$

This looks like **one** identity: in fact it is an **infinite number of identities**. $x \ \longrightarrow \ y(x) \ \longrightarrow \ y(y(x)) \ \longrightarrow \ \cdots$
Let us introduce **another** *rational parametrization* where the elliptic function parametrization of the Ising (resp. Baxter model) plays a crucial role, thus underlining the Landen transformation as an **exact generator of the renormalization group**.

Isogenies, Landen transformations, modular curve.

We will denote k the modulus of the elliptic functions in the parametrization of the Ising (resp. Baxter model), and $j(k)$ the j-**invariant** of the corresponding elliptic curve.
The previous modular curve has **another rational parametrization**

$$x \;=\; \frac{1}{j(k)}, \qquad y \;=\; \frac{1}{j(k_L)} \qquad \text{where} \qquad k_L \;=\; \frac{2\sqrt{k}}{1+k}$$

$$j(k) = 256 \cdot \frac{(1-k^2+k^4)^3}{(1-k^2)^2 \cdot k^4}, \quad j\Big(\frac{2\sqrt{k}}{1+k}\Big) = 16 \cdot \frac{(1+14\,k^2+k^4)^3}{(1-k^2)^4 \cdot k^2}$$

These two *rational parametrizations* are actually related by the following change of variables:

$$t \;=\; 256 \cdot \frac{k^2}{(k^2-1)^2} \quad \text{or:} \quad 16 \cdot \frac{(k^2-1)^2}{k^2} \quad \text{i.e:} \quad k \rightarrow \frac{1-k}{1+k}$$

Isogenies, Landen transformations, Modular curve.

The **modular curve** is thus an **algebraic representation** of the **Landen transformation** $k \rightarrow 2\sqrt{k}/(1+k)$, **and in the same time**, since this curve is $x \leftrightarrow y$ symmetric, of its **compositional inverse**, the **inverse Landen transformation**. The algebraic function $y = y(x)$ is a "**multivalued** function", but we can single out the **series expansions**:

$$y_2 = x^2 + 1488\,x^3 + 2053632\,x^4 + 2859950080\,x^5 + \cdots$$

and its **compositional inverse** series (with $\omega^2 = 1$):

$$y_{1/2} = \omega \cdot x^{1/2} - 744 \cdot x^{2/2} + 357024 \cdot \omega \cdot x^{3/2} + \cdots$$

More correspondences.

$$ {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \; 1728 \; \frac{t}{(t+27)\cdot(t+3)^3}\Big) $$
$$ = \; A(t) \cdot \; {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \; 1728 \; \frac{t^3}{(t+27)\cdot(t+243)^3}\Big), $$

where $A(t)$ is an involved algebraic function. The elimination of t between the two pullbacks

$$ x \, = \, \frac{t}{(t+27)\cdot(t+3)^3}, \quad y \, = \, \frac{t^3}{(t+27)\cdot(t+243)^3} \, = \, x\Big(\frac{3^6}{t}\Big), $$

gives **another modular curve** $P_3(x, y) \, = \, P_3(y, x) \, = \, 0$

$$ y_3 \, = \, x^3 \; + 2232\, x^4 \; + 3911868\, x^5 \; + 6380013816\, x^6 \; + \; \cdots $$

and its **compositional inverse** series (with $\omega^3 = 1$):

$$ y_{1/3}(\omega, x) \, = \, \omega \cdot x^{1/3} - 744 \cdot \omega^2 \cdot x^{2/3} + 356652 \cdot x^{3/3} \; + \; \cdots $$

More generally: N prime.

Another example:

$$
{}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \ 1728 \ \frac{t}{(t^2+10\,t+5)^3}\Big)
$$
$$
= \ A(t)\cdot \ {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \ 1728 \ \frac{t^5}{(t^2+250\,t+3125)^3}\Big),
$$

where $A(t)$ is an involved algebraic function.
More generally, we have an **infinite number of modular curves** $P_N(x, y) = P_N(y, x) = 0$ with modular correspondences $x \longrightarrow y(x)$:
$y = x^N + \cdots$ and: $y = x^{1/N} + \cdots$
In the nome this just amount to writing $q \longrightarrow q^N$ and $q \longrightarrow \omega \cdot q^{1/N}$ with $\omega^N = 1$.

N not prime: $N = 4$.

$$ {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \frac{t \cdot (t+16)}{(t^2 + 16t + 16)^3}\Big) $$
$$ = A(t) \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \frac{t^4 \cdot (t+16)}{(t^2 + 256t + 4096)^3}\Big), $$

where $A(t)$ is an involved algebraic function. The elimination of t between the two pullbacks

$$ x = \frac{t \cdot (t+16)}{(t^2 + 16t + 16)^3}, \quad y = \frac{t^4 \cdot (t+16)}{(t^2 + 256t + 4096)^3}, $$

gives **another modular curve** $P_4(x, y) = P_4(y, x) = 0$.

N not prime: $N = 4$.

$$y_4 \;=\; x^4 \;+ 2975\, x^5 \;+ 6322896\, x^6 \;+ 118381511424\, x^7 \;+\; \cdots$$

and its **compositional inverse** series (with $\omega^4 = 1$):

$$y_{1/4}(\omega,\, x) \;=\; \omega \cdot x^{1/4} \;- 744 \cdot \omega^2 \cdot x^{2/4} \;+ 356652 \cdot x^{3/4} \;+\; \cdots$$

and also the (involutive series):

$$y_1 \;=\; -x \;- 1488\, x^2 \;- 2214144\, x^3 \;- 3337633792\, x^4 \;+\; \cdots$$

.

N not prime: $N = 4$ as composition of $N = 2$

The modular curve $P_4(x, y) = P_4(y, x) = 0$ can be obtained "composing" the modular curve $P_2(x, y) = P_2(y, x) = 0$ with itself. Composition of algebraic functions is a "slippery terrain" (taking resultant), but there is no problem **composing the algebraic series** solutions of the modular curve $P_2(x, y) = P_2(y, x) = 0$:

$$y_2 = x^2 + 1488\, x^3 + 2053632\, x^4 + 2859950080\, x^5 + \cdots$$

and its **compositional inverse** series (with $\omega^2 = 1$):

$$y_{1/2} = \omega \cdot x^{1/2} - 744 \cdot x^{2/2} + 357024 \cdot \omega \cdot x^{3/2} + \cdots$$

One can see that $y_4(x) = y_2(y_2(x))$, $y_{1/4}(x) = y_{1/2}(y_{1/2}(x))$ and

$$y_1(x) = y_{1/2}(y_2(x)), \qquad y_2(y_{1/2}(x)) = x.$$

.

Simple covariance: modular form.

Revisiting the previous ${}_2F_1$ identities, corresponding to modular correspondence series, one can write:

$$ {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728\, y\Big) \; = \; \mathcal{A}(x) \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728\, x\Big), $$

where $\mathcal{A}(x)$ is an *algebraic function*. The relation $P(y, x) = 0$ is one of the previous **modular equations**. Introducing

$$ F(x) \; = \; x \cdot (1 - 1728 \cdot x)^{1/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot x\Big)^2, $$

the previous covariance relation on ${}_2F_1$ can, in fact, be written

$$ \lambda \cdot F(y) = F(x) \cdot \frac{dy}{dx} \qquad \text{or:} \qquad \lambda \cdot \frac{dx}{F(x)} = \frac{dy}{F(y)}. $$

Modular form: Schwarzian condition.

Eliminating $\mathcal{A}(x)$ (on the corresp. ODE's) one gets the following **Schwarzian differentially algebraic equation** condition:

$$W(x) \quad - W(y(x)) \cdot y'(x)^2 \quad + \{y(x),\, x\} \quad = \quad 0,$$

where $W(x)$ is the **rational function**:

$$\frac{F''(x)}{F(x)} \, - \frac{1}{2} \cdot \Big(\frac{F'(x)}{F(x)}\Big)^2 \; = \quad -\frac{1}{2} \cdot \frac{1 \, - 1968\, x \, + 2654208\, x^2}{x^2 \cdot (1 \, - 1728\, x)^2},$$

and where $\{y(x),\, x\}$ denotes the **Schwarzian derivative**:

$$\{y(x),\, x\} \; = \quad \frac{y'''(x)}{y'(x)} \; - \; \frac{3}{2} \cdot \Big(\frac{y''(x)}{y'(x)}\Big)^2$$

This non-trivial condition coincides exactly with one of the conditions G. Casale obtained in a classification of Malgrange's $\mathcal{D}$-envelope and $\mathcal{D}$-groupoids on $\mathbb{P}_1$.

The Schwarzian equation encapsulates all the modular equations of the theory of elliptic curves: the infinite number of correspondences, $x \to x^n + \cdots$, are actually solutions of the same Schwarzian equation.
Are the solutions of the Schwarzian equations only modular correspondences ?

Beyond the modular correspondence $x \to x^n + \cdots$, a one-parameter series. Replicable functions.

A **one-parameter** series is actually solution of the previous Schwarzian equation:

$$y(a, x) \;=\; a \cdot x \quad - 744 \cdot a \cdot (a-1) \cdot x^2$$
$$+36 \cdot a \cdot (a-1) \cdot (9907\, a - 20845) \cdot x^3$$

$$-32 \cdot a \cdot (a-1) \cdot (4386286\, a^2 - 20490191\, a + 27274051) \cdot x^4$$
$$+6 \cdot a \cdot (a-1) \cdot (8222780365\, a^3 - 61396351027\, a^2$$
$$+171132906629 \;\, a \; -183775457147) \cdot x^5 \quad + \quad \cdots$$

We have a one-parameter family of **commuting** series:

$$y\Big(a, y(a', x)\Big) \;=\; y(a\, a', \; x).$$

This one-parameter series is (at first sight ...) a bit "mysterious" ...

In the $a \to 0$ limit

$$\tilde{Q}(x) = \lim_{a \to 0} \frac{y(a,\ x)}{a} = x + 744\, x^2 + 750420\, x^3 + 872769632\, x^4 + 1102652742882\, x^5 + \cdots$$

In the $a \to \infty$ limit

$$\tilde{X} = \lim_{a \to \infty} y\Big(a,\ \frac{x}{a}\Big) = x - 744\, x^2 + 356652\, x^3 - 140361152\, x^4 + 49336682190\, x^5 + \cdots$$

$$y(a,\ x) = \tilde{X}\Big(a \cdot \tilde{Q}(x)\Big) \quad \text{or:} \quad \tilde{Q}\Big(y(a,\ x)\Big) = a \cdot \tilde{Q}(x).$$

Since $y(1,\ x) = x$, one deduces that $\tilde{X}$ **must be the compositional inverse** of $\tilde{Q}$. $\tilde{X}$ and $\tilde{Q}$ are differentially algebraic: they are also solutions of some **Schwarzian** equations.

Similarly the two previous algebraic series, y_2 and $y_{1/2}$, can be written respectively ($\omega^2 = 1$):

$$\tilde{X}\Big(\tilde{Q}(x)^2\Big) \qquad \text{and:} \qquad \tilde{X}\Big(\omega \cdot \tilde{Q}(x)^{1/2}\Big).$$

In other words $\quad q \longrightarrow q^2, \quad q \longrightarrow q^{1/2}.$

$$2 \cdot y_2 \cdot (1 - 1728 \cdot y_2)^{1/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot y_2\Big)^2$$
$$= \quad x \cdot (1 - 1728 \cdot x)^{1/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot x\Big)^2 \cdot \frac{dy_2}{dx}.$$

$$2 \cdot F(y_2) \; = \; F(x) \cdot \frac{dy_2}{dx}.$$

More generally, for N prime, the modular correspondence series read ($\omega^N = 1$):

$$\tilde{X}\Big(\tilde{Q}(x)^N\Big) \qquad \text{and:} \qquad \tilde{X}\Big(\omega \cdot \tilde{Q}(x)^{1/N}\Big).$$

N-th root values of the parameter

Note that $y(a, x)$ for $a^N = 1$ is such that its N-th compositional iterate is the identity. Such a series must be "special". Let us consider the modular curve Γ_N having $\tilde{X}\Big(\tilde{Q}(x)^N\Big)$ and $\tilde{X}\Big(\omega \cdot \tilde{Q}(x)^{1/N}\Big)$ as solution series. In the nome $\tilde{Q}(x)$ Γ_N amounts to writing in the same time $\tilde{Q} \rightarrow \tilde{Q}^N$ and $\tilde{Q} \rightarrow \omega \cdot \tilde{Q}^{1/N}$. Performing the resultant of Γ_N with itself, in order to get Γ_{N^2}, amounts to performing $\tilde{Q} \rightarrow \tilde{Q}^N \rightarrow \tilde{Q}^{N^2}$, $\tilde{Q} \rightarrow \omega \cdot \tilde{Q}^{1/N} \rightarrow \tilde{Q}^{1/N^2}$ **but also**

$$\tilde{Q} \longrightarrow \quad \tilde{Q}^N \quad \longrightarrow \quad \omega \cdot (\tilde{Q}^N)^{1/N},$$

namely $\tilde{Q} \rightarrow \omega \cdot \tilde{Q}$ with $\omega^N = 1$.

N-th root values of the parameter

In other words $y(a, x)$ for $a^N = 1$ is not only a series of order-N with respect to the composition of function, it is an algebraic series, solution of a modular curve: it is a correspondence. We thus have an **infinite number of algebraic series** solutions of the **Schwarzian equation**.

The one-parameter $y(a, x)$ series encapsulates an infinite number of modular correspondence series, namely the infinite number of correspondences, $x \rightarrow \omega \cdot x + \cdots$, $\omega^N = 1$, which are actually subcases of the one-parameter $y(a, x)$ series.

Remark: when the parameter "a" is an integer $y(a, x)$ is a **differentially algebraic series with integer coefficients !!**

More generally

In the $a \to 1$ limit, let us denote $\epsilon = a - 1$. The one-parameter series $y(x) = y(a, x)$ can, thus, be seen as an ϵ-expansion:

$$y(a, x) = x + \sum_{n=1}^{\infty} \epsilon^n \cdot B_n(x),$$

where $B_1(x) = F(x)$

$$B_2(x) = \frac{1}{2} \cdot F(x) \cdot \left(\frac{dB_1(x)}{dx} - 1\right),$$

$$B_3(x) = \frac{1}{3} \cdot F(x) \cdot \left(\frac{dB_2(x)}{dx} - \frac{dB_1(x)}{dx} + 1\right),$$

$$B_4(x) = \frac{1}{4} \cdot F(x) \cdot \left(\frac{dB_3(x)}{dx} - \frac{dB_2(x)}{dx} + \frac{dB_1(x)}{dx} - 1\right),$$

More generally.

$$(n+1) \cdot \ B_{n+1} \ \ + n \cdot \ B_n \ = \ \ F(x) \cdot \ \frac{dB_n(x)}{dx},$$

$$\frac{\partial \sum_n \ B_{n+1} \ \cdot \ \epsilon^{n+1}}{\partial \epsilon} \ + \epsilon \cdot \ \frac{\partial \sum_n \ B_n \ \cdot \ \epsilon^n}{\partial \epsilon} \\ = \ \ F(x) \cdot \ \Big(\frac{\partial \sum_n \ B_n(x) \ \cdot \ \epsilon^n}{\partial x} \Big),$$

$$a \cdot \ \frac{\partial y(a, x)}{\partial a} \ = \ F(x) \cdot \ \frac{\partial y(a, \ x)}{\partial x}.$$

The series $y(a, \, x)$ is solution of the Schwarzian equation with:

$$W(x) \ = \ \ \frac{F''(x)}{F(x)} \ - \frac{1}{2} \cdot \ \Big(\frac{F'(x)}{F(x)} \Big)^2.$$

This remains valid **for any** function $F(x)$.

Polynomial example

$$F(x) \;=\; x \cdot (1 - 373 \cdot x) \cdot (1 - 371 \cdot x).$$

$$W(x) \;=\; -\frac{1}{2} \cdot \frac{1 - 830298\, x^2 + 411827808\, x^3 - 57449564067\, x^4}{x^2 \cdot (1 - 373\, x)^2 \cdot (1 - 371\, x)^2}.$$

$$y(a,\, x) \;=\; a \cdot x - 744 \cdot a \cdot (a-1) \cdot x^2 + \frac{1}{2} \cdot a \cdot (1245455\, a - 968689) \cdot (a-1) \cdot x^3 + \cdots$$

$$\tilde{Q}(x) \;=\; x \cdot \frac{(1 - 371\, x)^{371/2}}{(1 - 373\, x)^{373/2}} \;=\; \exp\Big(\int \frac{dx}{F(x)}\Big).$$

$$a \cdot \tilde{Q}(x) \;=\; \tilde{Q}\Big(y(a,\, x)\Big), \qquad y(a,\, x) \;=\; \tilde{X}\Big(a \cdot \tilde{Q}(x)\Big).$$

Finding the (simple) **algebraic expressions** of $\tilde{Q}(x)$ and $\tilde{X}(x)$ from large series expansions is quite hard !

Polynomial truncation of the hypergeometric function.

$$F(x) = x - 744\,x^2 - 393768\,x^3 = x \cdot \Big(1 - p \cdot x\Big) \cdot \Big(1 - q \cdot x\Big),$$

with:

$$p = 372 + 6 \cdot 14782^{1/2}, \qquad q = 372 - 6 \cdot 14782^{1/2},$$

$$\tilde{Q}(x) = \frac{x \cdot (1 - p \cdot x)^{p/(q-p)}}{(1 - q \cdot x)^{q/(q-p)}} = x + 744\,x^2 + 750420\,x^3$$

$$+753621408\,x^4 + 782312864472\,x^5 + \frac{4097211834177216}{5}\,x^6 + \cdots$$

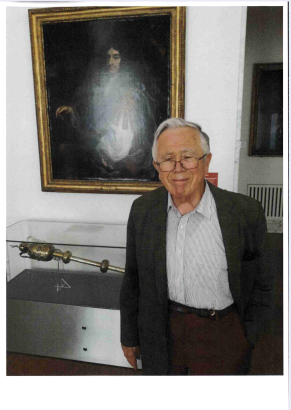
$\tilde{Q}(x)$ is *D-finite*, but the linear differential operator is **not globally nilpotent** and the series for $\tilde{Q}(x)$ is **not globally bounded**.
With this simple example we see that the integer character of the coefficients of the modular correspondence series is not automatic.

Differentially algebraic series.

With $y(a, x)$ associated with canonical correspondences, we had an infinite number of **algebraic functions** for $y(a, x)$ with $a^N = 1$, and an infinite number of differentially algebraic series with integer coefficients for $y(a, x)$ with $a \in \mathbb{Z}$.
The λ**-extensions** of the two-point correlation functions of the square Ising model have very similar properties. These series are solutions of **(sigma-form of) Painlevé equations**, they are, thus, differentially algebraic. For selected values ($\lambda = \cos(\pi\, m/n)$, which can also be written as N-th root of unity) these series actually become **algebraic series**, and for integer values of λ we have differentially algebraic series with integer coefficients.
We thus have the **same remarkable properties** with **different kinds** of differentially algebraic series (Schwarz versus Painlevé, Replicable functions versus isomonodromy).
A lot remains to be understood on such selected differentially algebraic series !!

So many people have a defeatist attitude towards non-linear differential equations: they think nothing can be done on non-linear differential equations.

As far as **globally bounded differentially algebraic series** are concerned:
this is defeatist nonsense.



Fourier believed the main goal of Mathematics is being used to the public and explain natural phenomena, but a philosopher like him should know that the only goal of Science is the honor of the human spirit and that a question about numbers is as important as a question about the world system.
Letter of Gustav Jacobi to Adrien-Marie Legendre.

Additional slides to answer the questions the public did not ask.

One finds, more frequently in the literature, the exact expression of the *free-energy* (instead of the partitition function) in terms of these three q, x, z variables, namely (see (10.8.47) page 237 in Baxter's book):

$$- \beta f \;=\; \ln(c) \; + Z(q, x, z),$$

where:

$$Z(q, x, z) \;=\; \sum_{m=1}^{\infty} \frac{\left(x^{2m} - q^{m}\right)^{2} \left(x^{m} + \frac{1}{x^{m}} - z^{m} - \frac{1}{x^{m}}\right)}{m\, x^{m}\, (1 - q^{2m})\, (1 + x^{2m})}$$

The double infinite product form is more illuminating as far as the symmetries and structures are concerned.

First infinite product $\longrightarrow$ elliptic functions.

The other **infinite product** $\longrightarrow$ covariance emerging from the inversion relation.

$$F(q) = \prod_{n=0}^{\infty} \Big(\frac{1 - q^{2n}}{1 + q^{2n}}\Big)^2 = (1 - k^2)^{1/4} \cdot \frac{2}{\pi} \cdot K(k).$$

The Landen transformation $q \to q^{1/2}$ amounts to introducing the other eulerian product $F(q^{1/2})$ which reads:

$$F(q^{1/2}) = \prod_{n=0}^{\infty} \Big(\frac{1 - q^{n}}{1 + q^{n}}\Big)^2 = \Big(1 - \Big(\frac{2\sqrt{k}}{1+k}\Big)^2\Big)^{1/4} \cdot \frac{2}{\pi} \cdot K\Big(\frac{2\sqrt{k}}{1+k}\Big).$$

$$\frac{F(q^{1/2})}{F(q)} = (1 - k^2)^{1/4}.$$

The same calculations for the inverse Landen transformation yield

$$\frac{F(q^2)}{F(q)} = \frac{(1 - k)^{1/2} + (1 + k)^{1/2}}{2 \cdot (1 - k^2)^{1/8}}.$$

D-finite polarization.

The expression of the **polarization** of the Baxter model, can be written in terms of the function $F(q)$:

$$P(q, x) \;=\; \frac{F(x)}{F(q^{1/2})}.$$

Recalling that all the ratio $F(q^{1/2})/F(q)$, $F(q^2)/F(q)$ and more generally $F(q^N)/F(q)$, $F(q^{1/N})/F(q)$, $F(q^N)/F(q^M)$, ... are not only **solutions of linear ODEs** but are actually **algebraic functions** (associated with modular curves), one easily sees that the polarization $P(q, x) = F(x)/F(q^{1/2})$, is an algebraic function, not only in the Ising case $q = x^4$, but more generally when q and x are **"commensurate"** namely, there exist two positive integers N and M such that:

$$x^N \;=\; q^M.$$

Remark on D-finite functions.

$$\ln(Z_{red}) \, = \, {}_4F_3\Big([1, \, 1, \, \frac{3}{2}, \, \frac{3}{2}], \, [2, \, 2, \, 2], \, x\Big),$$

is D-finite. **Its exponential is also D-finite** (the exponentional function $\exp(x)$ is D-finite and the **composition** of two D-finite is D-finite). For instance, the function $Z \, = \, \exp(\ln(Z_{red}))$ is actually solution of an order-four linear differential operator $L_4 \, = \, L_2 \cdot \, (\theta \, + 2) \cdot \, (\theta \, + 2)$, where $\theta \, = \, x \cdot \, \frac{d}{dx}$ is the *homogeneous derivative*, and where the order-two linear differential operator L_2 has the ${}_2F_1$ hypergeometric solution:

$$\mathcal{S}ol(L_2) \, = \, {}_2F_1\Big([\frac{5}{2}, \, \frac{5}{2}], \, [3], \, x\Big).$$

Definition of the diagonal of series of several complex variables

Definition:

$$\mathcal{F}\Big(z_1,\, z_2,\, \ldots,\, z_n\Big) \;=\; \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} F_{m_1, m_2, \ldots, m_n} \cdot z_1^{m_1}\, z_2^{m_2} \,\cdots\, z_n^{m_n},$$

$$Diag\Big(\mathcal{F}\Big(z_1,\, z_2,\, \ldots,\, z_n\Big)\Big) \;=\; \sum_{m=0}^{\infty} F_{m, m, \ldots, m} \cdot z^m.$$

Extracting the diagonal terms is like extracting the constant terms in a Laurent/Taylor series. This amounts to residue calculations in several variables:

$$\longrightarrow \quad \int_C \cdots \int_C \mathcal{R}(z_1,\, z_2,\, \cdots,\, z_n;\, t) \cdot \frac{dz_1}{z_1} \cdot \frac{dz_2}{z_2} \cdots \frac{dz_n}{z_n}$$

The result: if the **algebraic, or rational, integrand** of a n-fold integral has a **multi-Taylor expansion**, then this n-fold integral **is the diagonal of a rational function**.

Two by-products: Diagonal of rational functions are (or can be recast into) series with **integer coefficients**, which **actually reduce modulo any prime to algebraic functions !!**

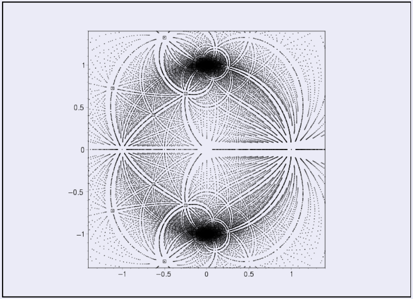
Lattice Green functions are obviously diagonal of rational functions and, thus, are D-finite.

Let us introduce the complex variables $z_j = \exp(2\, i\, \pi \cdot q_j)$. The lattice Green function can be rewritten

$$\frac{1}{(2\pi)^n} \int \cdots \int \, R(c_1,\, c_2,\, \cdots,c_n;\, t) \cdot dq_1 \cdot dq_2 \cdots dq_n$$
$$\longrightarrow \int_C \cdots \int_C \, \mathcal{R}(z_1,\, z_2,\, \cdots,z_n;\, t) \cdot \frac{dz_1}{z_1} \cdot \frac{dz_2}{z_2} \cdots \frac{dz_n}{z_n}$$

where it is crystal clear that the lattice Green function is a diagonal of a rational function, and thus is D-finite, solution of a linear differential operator with polynomial coefficients (in t).

Accumulation of the singularities of the linear ODEs for the $\chi^{(n)}$ in the k complex plane



the full susceptibility is clearly a quite involved function !
Remark: for a holonomic function, there is a difference between the **singularities of that function**, and the **singularities of the linear differential operator** annihilating the function !!
But this is another story ...

Is the full susceptibility of the Ising model differentially algebraic ?

We also considered the full susceptibility of the square Ising model, in order to see if it could be **differentially algebraic**:

Automata and the susceptibility of the square lattice Ising model modulo powers of primes, A.J. Guttmann and JMM, 2015 J. Phys. A: Math. Theor. 48 474001

Lacunary series mod. 32, 64 and thus reduce to algebraic series mod. 2^5, 2^6: $L(u^2) + u = L(u) \longrightarrow L(u)^2 + u = L(u)$.

$$L(u) = 1 + u + u^2 + u^4 + u^8 + u^{16} + u^{32} + u^{64} + u^{128} + u^{256} + u^{512} + u^{1024} + \cdots$$

More generally, the full susceptibility series reduces to algebraic series mod. 2^r.

However the full susceptibility series does not seem to reduce to algebraic series modulo other integers. We have **only some** properties of the diagonal of rational functions.

Modular form, Eisenstein series

$$E_4 = 1 + 240 \sum_{n=1}^{\infty} n^3 \cdot \frac{q(\tau)^n}{1 - q(\tau)^n} = {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{1728}{j(\tau)}\Big)^4$$

In terms of k the modulus of the elliptic functions, the E_4 **Eisenstein series** can also be written as:

$$\begin{aligned} &{}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{27}{4} \frac{k^4 \cdot (1 - k^2)^2}{(k^4 - k^2 + 1)^3}\Big)^4 \\ &\qquad = (1 - k^2 + k^4) \cdot {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], k^2\Big)^4. \end{aligned}$$

$$\begin{aligned} E_6 &= (1 + k^2) \cdot (1 - 2\,k^2) \cdot \Big(1 - \frac{k^2}{2}\Big) \cdot {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], k^2\Big)^6 \\ &= (1 + k^2) \cdot (1 - 2\,k^2) \cdot \Big(1 - \frac{k^2}{2}\Big) \\ &\quad \times (1 - k^2 + k^4)^{-3/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{27}{4} \frac{k^4 \cdot (1 - k^2)^2}{(k^4 - k^2 + 1)^3}\Big)^6. \end{aligned}$$

$$ {}_2F_1\left([\frac{1}{2}, \frac{1}{2}], [1], 16\ x\right), \quad {}_2F_1\left([\frac{1}{2}, \frac{1}{3}], [1], 36\ x\right), $$

$$ {}_2F_1\left([\frac{1}{3}, \frac{1}{3}], [1], 27\ x\right), \quad {}_2F_1\left([\frac{1}{3}, \frac{2}{3}], [1], 27\ x\right), $$

$$ {}_2F_1\left([\frac{1}{6}, \frac{1}{2}], [1], 432\ x\right), \quad {}_2F_1\left([\frac{1}{6}, \frac{1}{3}], [1], 108\ x\right), $$

$$ {}_2F_1\left([\frac{1}{6}, \frac{2}{3}], [1], 108\ x\right), \quad {}_2F_1\left([\frac{1}{6}, \frac{1}{6}], [1], 432\ x\right), $$

$$ {}_2F_1\left([\frac{1}{6}, \frac{5}{6}], [1], 432\ x\right), \quad {}_2F_1\left([\frac{1}{4}, \frac{1}{4}], [1], 64\ x\right), $$

$$ {}_2F_1\left([\frac{1}{4}, \frac{1}{2}], [1], 32\ x\right), \quad {}_2F_1\left([\frac{1}{4}, \frac{3}{4}], [1], 64\ x\right), $$

$$ {}_2F_1\left([\frac{1}{8}, \frac{3}{8}], [1], 256\ x\right), \quad {}_2F_1\left([\frac{1}{8}, \frac{5}{8}], [1], 256\ x\right), $$

$$ {}_2F_1\left([\frac{3}{8}, \frac{7}{8}], [1], 256\ x\right), \quad {}_2F_1\left([\frac{2}{3}, \frac{5}{6}], [1], 108\ x\right), $$

$$
\begin{aligned}
&{}_2F_1\left([\tfrac{1}{3}, \tfrac{5}{6}], [1], 108\ x\right), \quad {}_2F_1\left([\tfrac{1}{2}, \tfrac{3}{4}], [1], 32\ x\right), \\
&{}_2F_1\left([\tfrac{3}{4}, \tfrac{3}{4}], [1], 64\ x\right), \quad {}_2F_1\left([\tfrac{5}{8}, \tfrac{7}{8}], [1], 256\ x\right), \\
&{}_2F_1\left([\tfrac{2}{3}, \tfrac{2}{3}], [1], 27\ x\right), \quad {}_2F_1\left([\tfrac{5}{6}, \tfrac{5}{6}], [1], 432\ x\right), \\
&{}_2F_1\left([\tfrac{1}{2}, \tfrac{5}{6}], [1], 144\ x\right), \quad {}_2F_1\left([\tfrac{1}{2}, \tfrac{2}{3}], [1], 36\ x\right), \\
&{}_2F_1\left([\tfrac{1}{12}, \tfrac{7}{12}], [1], 1728\ x\right), \quad {}_2F_1\left([\tfrac{1}{12}, \tfrac{5}{12}], [1], 1728\ x\right), \\
&{}_2F_1\left([\tfrac{5}{12}, \tfrac{11}{12}], [1], 1728\ x\right), \quad {}_2F_1\left([\tfrac{7}{12}, \tfrac{11}{12}], [1], 1728\ x\right).
\end{aligned}
$$

A pedagogical example of diagonal of rational functions.

Let us consider the **rational function of three complex variables** $\mathcal{F} = 1/(1 - z_2 - z_3 - z_1 z_2 - z_1 z_3)$. Its diagonal reads:

$$1 \ + 4z \ + 36z^2 \ + 400z^3 \ + 4900z^4 + 63504z^5 \ + \ \cdots$$

which is nothing but the **complete elliptic integral** (first kind):

$$\sum_{m \geq 0} \binom{2m}{m}^2 \cdot z^m \quad = \quad {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], 16\, z\Big)$$

This diagonal **modulo any prime** reduces to an **algebraic function**, for instance:

$$Diag(\mathcal{F}) \quad \bmod 7 \ =$$
$$= \quad 1 \ + 4\, z \ + z^2 + z^3 + 4z^7 + 2z^8 + 4z^9 \ + \cdots$$
$$= \quad \frac{1}{\sqrt[6]{1 + 4z + z^2 + z^3}} \quad \bmod 7.$$

Another example of diagonal of rational functions.

A less obvious example corresponds to the **modular form**:

$$\Big(\frac{1}{1 - z_1 - z_2 - z_3 - z_1 z_2 - z_2 z_3 - z_3 z_1 - z_1 z_2 z_3}\Big) = \frac{1}{1 - z} \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1]; \frac{54\, z}{(1 - z)^3}\Big).$$

Such **diagonals of rational functions** are **highly selected functions**: modulo **any prime** they reduce to **algebraic functions**.

They can be seen as the **simplest (transcendental) generalisations of algebraic functions**.

The integrands of the $\chi^{(n)}$ n-fold integral of the Ising model have a **multi-Taylor expansion** and are, thus, **diagonals of a rational function**.

Ising n-fold integrals : $\chi^{(5)}$

The five-particle contribution $\tilde{\chi}^{(5)}$ of the susceptibility of the Ising model is solution of an order-33 linear differential operator which has a **direct-sum** factorization (DFactorLCLM in Maple): the selected linear combination

$$\tilde{\chi}^{(5)} \; - \frac{1}{2}\, \tilde{\chi}^{(3)} \; + \frac{1}{120}\, \tilde{\chi}^{(1)},$$

is solution of an order-29 (globally nilpotent) linear differential operator

$$L_{29} \; = \quad L_5 \cdot L_{12} \cdot \tilde{L}_1 \cdot L_{11},$$

where:

$$L_{11} \; = \quad (Z_2 \cdot N_1) \oplus V_2 \oplus (F_3 \cdot F_2 \cdot L_1^s).$$

Ising n-fold integrals : $\chi^{(6)}$

Similarly $\tilde{\chi}^{(6)}$ is solution of an order-52 linear differential operator which has a **direct-sum** factorization: the selected linear combination

$$\tilde{\chi}^{(6)} \; - \frac{2}{3}\tilde{\chi}^{(4)} \; + \frac{2}{45}\tilde{\chi}^{(2)},$$

is solution of an order-46 (globally nilpotent) linear differential operator

$$L_{46} \;=\; L_6 \cdot L_{23} \cdot L_{17},$$

where: $L_{17} \;=\; \tilde{L}_5 \oplus L_3 \oplus (L_4 \cdot \tilde{L}_3 \cdot L_2),$

$$\tilde{L}_5 \;=\; \left(\frac{d}{dx} - \frac{1}{x}\right) \oplus \Big(L_{1,3} \cdot (L_{1,2} \oplus L_{1,1} \oplus D_x)\Big).$$

The “Quarks” in $\chi^{(5)}$ and $\chi^{(6)}$

Quasi-trivial order-one (globally nilpotent) linear differential operators: $\tilde{L}_1, \; N_1, \; L_1^s, \; L_{1,n} \quad \longrightarrow \quad D_x \; - \frac{1}{N} \cdot \frac{d \ln(R(x))}{dx}$

$V_2, \; L_2, \; L_3, \; L_5$ and L_6 are respectively equivalent (homomorphic) to L_K, to the symmetric square of L_K and to the *symmetric fourth and fifth power* of L_K, where L_K is the second order linear differential operator annihilating the **complete elliptic integral** $K \; = \; _2F_1([1/2, \, 1/2], \, [1], \, k^2)$.

$F_2, \; F_3, \; \tilde{L}_3$ do correspond to **modular forms**: F_3 and $\tilde{L}_3$ are homomorphic to the symmetric square of order-two operators associated with the (fundamental) **modular curve** $X_0(2)$, and F_2 is related to Z_2 (and thus h_6, Apéry, ...).

Remains to understand the “very nature” of:

L_4 and: $L_{12}, \; L_{23}$

L_4 is a Hadamard product of two elliptic curves:

it is a Calabi-Yau operator !

Seeking for ${}_4F_3$ hypergeometric functions up to homomorphisms, and assuming an **algebraic pull-back** with the *square root extension*, $(1 - 16 \cdot w^2)^{1/2}$, we actually found that the solution of L_4 *can be expressed in terms of a selected* ${}_4F_3$

$$ {}_4F_3\Big([1/2,\, 1/2,\, 1/2,\, 1/2], [1,1,1];\; z\Big) $$

$$ = \quad {}_2F_1\Big([1/2,\, 1/2], [1];\; z\Big) \star {}_2F_1\Big([1/2,\, 1/2], [1];\; z\Big), $$

where: $$ z \; = \; \Big(\frac{1 + \sqrt{1 - 16 \cdot w^2}}{1 - \sqrt{1 - 16 \cdot w^2}} \Big)^4 \; = \; k^4 $$

where the pull-back z *is nothing but* the fourth power of the **modulus** k of the elliptic functions !

The $\chi^{(n)}$'s are diagonal of rational functions.

Let us consider the series of $\tilde{\chi}^{(3)}/8/w^9$

$$1 \ + 36\,w^2 \ + 4\,w^3 \ + 884\,w^{13} \ + 196\,w^5 \ + 18532\,w^6 \ + \ \cdots$$

Let us now consider this very series **modulo the prime** $p \,=\, 2$. It reads the **lacunary** series

$$1 \ + w^8 + w^{24} + w^{56} + w^{120} + w^{248} + w^{504} \ + w^{1016} \ + \cdots ,$$

In fact,*modulo the prime* $p \,=\, 2$, $H(w) \,=\, \tilde{\chi}^{(3)}/8$ is, actually, an **algebraic function**, solution of the quadratic equation:

$$H(w)^2 \ + w \cdot \ H(w) \ + w^{10} \quad = \quad 0 \qquad \mod 2.$$

Modulo $p = 3$. Indeed, $H(w)$ satisfies a polynomial equation of degree nine (the p_n are polynomials of degree less that 63):

$$p_9 \cdot \ H(w)^9 \ + \ w^6 \cdot p_3 \cdot \ H(w)^3 \ + \ w^{10} \cdot p_1 \cdot \ H(w) \ + p_0.$$

Elimination of the automorphic prefactor $\mathcal{A}(x)$

$$\mathcal{A}(x) \cdot {}_2F_1\Big([\alpha, \beta], [\gamma], x\Big) = {}_2F_1\Big([\alpha, \beta], [\gamma], y(x)\Big),$$

The Gauss hypergeometric function ${}_2F_1([\alpha, \beta], [\gamma], x)$ is solution of the second order linear differential operator of wronskian $w(x)$:

$$\Omega = \frac{d^2}{dx^2} + A(x) \cdot \frac{d}{dx} + B(x), \qquad B(x) = \frac{\alpha\, \beta}{x \cdot (x-1)},$$

$$A(x) = \frac{(\alpha + \beta + 1) \cdot x - \gamma}{x \cdot (x-1)} = -\frac{w'(x)}{w(x)},$$

A straightforward calculation gives the algebraic function $\mathcal{A}(x)$ in terms of the **algebraic function pullback** $y(x)$:

$$\mathcal{A}(x) = \left(\frac{w(y(x))}{w(x)} \cdot y'(x)\right)^{-1/2}$$

The set of solutions of the Schwarzian condition has a closure property for composition of functions

$$\mathcal{A}(x) \cdot {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, x\Big) \;=\; {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, y(x)\Big),$$

$$\mathcal{B}(x) \cdot {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, x\Big) \;=\; {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, z(x)\Big),$$

$$\mathcal{B}(y(x)) \cdot {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, y(x)\Big) \;=\; {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, z(y(x))\Big)$$

$$\;=\; \mathcal{B}(y(x)) \cdot \mathcal{A}(x) \cdot {}_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, x\Big)$$

The set of solutions of the Schwarzian condition *must have a closure property for composition of functions*. It works: see the Schwarzian derivative of a composition of function:

$$\{z(y(x)),\, x\} \;=\; \{z(y),\, y\}_{y=y(x)} \cdot y'(x)^2 \;+\; \{y(x),\, x\}$$

Non-holonomic functions ratio of holonomic functions

Along this line it is fundamental to recall that the **ratio** (not the product !) of **two holonomic** functions is **non-holonomic**

$$\frac{d^2y}{dx} + R(x)\cdot y \;=\; 0, \qquad \tau(x) \;=\; \frac{y_1}{y_2}, \qquad \{\tau(x),\, x\} \;=\; 2\, R(x).$$

The **Chazy III equation** is a third-order **non-linear** differential equation (it can also be rewritten using a **Schwarzian derivative**) that has a **natural boundary** for its solutions:

$$\frac{d^3y}{dx^3} \;=\; 2\, y \frac{d^2y}{dx^2} - 3\left(\frac{dy}{dx}\right)^2.$$

It has the **quasi-modular form** Eisenstein series E_2 has a solution

$$y \;=\; \frac{1}{2}\cdot\frac{\Delta'}{\Delta} \;=\; \frac{1}{2}\cdot E_2$$

where Δ is a selected holonomic function: a **modular form**.

Schwarzian derivative and natural boundary

It can be rewritten in terms of a Schwarzian derivative:

$$f^{(4)} \;=\; 2\, f'^2 \cdot \{f,\, x\} \;=\; 2\, f'\, f''' - 3\, f"^2 \qquad \text{with:} \quad y \;=\; \frac{df}{dx}.$$

It was introduced by Jean Chazy (1909, 1911) as an example of a third-order differential equation with a movable singularity that has a **natural boundary** for its solutions. It is also worth recalling the **Halphen-Ramanujan differential system**:

$$P' \;=\; \frac{P^2 - Q}{12}, \qquad Q' \;=\; \frac{P\,Q - R}{3}, \qquad R' \;=\; \frac{P\,R - Q^2}{2},$$

where $P \,=\, E_2$, $Q \,=\, E_4$, $R \,=\, E_6$ and X' denotes here the homogeneous derivative $q \cdot \frac{dX}{dq}$, and E_n the Eisenstein series.