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Inversion relations and disorder solutions on Potts models

M T Jaekel† and J M Maillard‡

† Laboratoire de Physique Théorique de l'Ecole Normale Supérieure§, 24 rue Lhomond, 75231 Paris Cedex 05, France

‡ Laboratoire de Physique Théorique et Hautes Energies||, 4 Place Jussieu, 75230 Paris Cedex 05, France

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Abstract. The inversion symmetry and the automorphy group generated by the latter, when combined with the spatial symmetries, are studied for the anisotropic Potts models on the triangular and checkerboard lattices. The exact expression of the partition function, which is known on some particular disorder subvarieties of the triangular model, is checked and a generalisation is proposed for the checkerboard lattice. The automorphy group is then used to extend the disorder solutions to the infinity of transformed subvarieties.

1. Introduction

Besides its applications to two-dimensional solid state physics, the Potts model also shows interesting features in the realm of exactly solvable models (see Wu (1982) for a review). Not only does its partition function reduce, for particular values of its parameters, to that of solved models (Ising for $q=2$, six-vertex for the critical curve, (Baxter *et al* 1978)), but it also shows remarkable analytic and combinatorial properties: it satisfies the Lee–Yang theorem (Hintermann *et al* 1981); at the critical temperature, the discontinuity of its derivative (and its magnetisation) can be computed as infinite products (Baxter 1973, 1982a) and its critical exponents are known (conjecture of den Nijs 1979); it can also be seen as a limit of the Whitney–Tutte polynomial (Baxter *et al* 1976).

An exact property of the partition function of the anisotropic Potts model is also known to exist for any values of the parameters, the inversion functional relation (Jaekel and Maillard 1982). The latter is derived from a simple geometrical relation on the local Boltzmann weights, and happens to hold for various lattices (square, triangular, checkerboard, ...). Moreover, when combined with the spatial symmetries of the lattice, this inversion symmetry generates an infinite discrete group, which can be seen to play the role of an automorphy group for the partition function: to this group is associated an infinite set of analytic functional relations. However, the additional analytic information, which is required in order to completely determine the partition function from these equations, cannot be obtained, in general, from a mere qualitative study of the diagrammatic expansions (Jaekel and Maillard 1983).

§ Laboratoire propre du Centre National de la Recherche Scientifique associé à l'Ecole Normale Supérieure et à l'Université de Paris Sud.

|| Laboratoire associé au Centre National de la Recherche Scientifique et à l'Université Pierre et Marie Curie.

Recently, Rujan (1984) has obtained an exact expression for the partition function of the anisotropic triangular Potts model, for a subvariety of values of the parameters, called a disorder variety. This result generalises the disorder solution given by Stephenson (1970) and Gibberd (1969) for the anisotropic triangular Ising model. Disorder solutions have also been exhibited on various models of the Ising kind, using different techniques: crystal growth (Welberry and Galbraith 1973), Markov processes (Verhagen 1976), conditional probabilities (Enting 1977a). The common feature of these particular solutions is that they exhibit a remarkable relation between the parameters of the model, for which some decoupling of neighbouring degrees of freedom occurs, resulting in a reduction of the effective dimensionality of the model. In the previous cases, this leads to simple expressions, and even algebraic ones, for the partition function and some n-point correlation functions. Moreover, as was pointed out by Enting (1977b, 1978), these exact solutions impose severe constraints on series expansions.

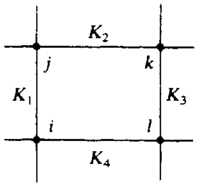
It is natural to try to use these particular algebraic solutions as additional constraints to the functional equations on the partition function, generated by the automorphy group. It is the purpose of this paper, first to show that such a simultaneous use is possible, that is, that these relations share a common domain of validity. This will be done by checking the inversion relation and the disorder solution on a partially resummed diagrammatic expansion, which appears to be compatible with both. Then, we shall derive some analytic consequences, and in particular give a compact expression for an automorphic function, which extends the disorder solutions to the infinity of their transformed varieties, by the automorphy group.

2. Resummed expansions

The simplest anisotropic two-dimensional Potts model is the one on a square lattice, since its partition function depends on two parameters only. However, the disorder solution has been given by Rujan (1984) for the anisotropic two-dimensional Potts model on a triangular lattice, and one can easily see that the disorder variety disappears (more precisely, is sent to infinity), when the limit of the square lattice is taken. Thus, in order to make use of the disorder solution in the framework of the inversion relation, we shall need to work on a triangular lattice, and hence, first to establish the corresponding inversion functional relation. This is expressed by an equation on the partition function and an analytic continuation of the latter. In the absence of an exact solution, both expressions in this equation are best characterised by partially resummed expansions, since at least one partial resummation is needed in order to apply the inversion symmetry. Our first task will be to obtain the high-temperature partially resummed expansion on a triangular lattice and then to verify the inversion relation. On the other hand, the disorder solution can easily be seen, on a high-temperature expansion for instance, to be the correct expression for the partition function in the domain of high temperature. In fact, we shall check that the analytic expression given by Rujan coincides with the analytic continuation entering the inversion relation, on the whole domain defined by the partially resummed expansion (a neighbourhood of a one-dimensional model).

2.1. *Diagrams and inversion relations*

As it will appear, for technical reasons, it is easier to obtain the resummed diagrammatic expansion for an even more general model, that is the checkerboard anisotropic lattice, which depends on four parameters, and then to recover the expression for the triangular case, by an appropriate limit ($K_3 \to \infty$). Denoting



$$a = e^{2K_1}, \qquad b = e^{2K_2}, \qquad c = e^{2K_3}, \qquad d = e^{2K_4}.$$

The partition function per site Z will be defined as

$$Z^N(a, b, c, d) = \sum_{\{\sigma\}} \prod_{\langle ij \rangle} a^{\delta\sigma_i\sigma_j} \prod_{\langle jk \rangle} b^{\delta\sigma_j\sigma_k} \prod_{\langle kl \rangle} c^{\delta\sigma_k\sigma_l} \prod_{\langle li \rangle} d^{\delta\sigma_l\sigma_i}$$

where each of the N spins σ can take q values; (as in the triangular limit $c \to \infty$ the number of sites is divided by two, note that Z must also be replaced by Z^2 in this limit). Let us define the normalised partition function per site Λ:

$$Z(a, b, c, d) = q^{-1}[(a+q-1)(b+q-1)(c+q-1)(d+q-1)]^{1/2}\Lambda(A, B, C, D)$$

where the following high-temperature variables have been introduced

$$A = \frac{a-1}{a+q-1}, \qquad B = \frac{b-1}{b+q-1}, \qquad C = \frac{c-1}{c+q-1}, \qquad D = \frac{d-1}{d+q-1}.$$

The diagrams and their contributions, which correspond to the different terms of the expansion of Λ (up to fourth order in B, D), can be found in appendix 1. They give the following result

$$\left(X = \frac{AC}{1-A^2C^2}(ACB + D); \qquad Y = \frac{AC}{1-A^2C^2}(B + ACD) \right)$$

$$\Lambda = \frac{(q-1)}{2}[B(X+(q-2)X^2) + D(Y+(q-2)Y^2)]$$
$$+ \frac{(q-1)}{2}\left(\frac{A^2+C^2+2A^2C^2}{1-A^2C^2} + (q-2)\frac{AC(A+C+2A^2C^2)}{1-A^3C^3}\right)$$
$$\times[BD + BY + DX]^2 + \frac{(q-1)(q-2)^2}{2}\frac{AC}{1-A^2C^2}$$
$$\times[2BDXY + AC(B^2X^2 + D^2Y^2)] + \frac{(q-1)^2}{2}\frac{A^2C^2}{(1-A^2C^2)^3}$$
$$\times[A^2C^2(-\tfrac{7}{2} + \tfrac{1}{2}A^2C^2)(B^4 + D^4) - 4AC(2 + A^2C^2)(B^3D + BD^3)$$
$$-(5 + A^2C^2)(1 + 2A^2C^2)B^2D^2]. \qquad (1)$$

As this expansion has already been computed in the particular case of the anisotropic square lattice (Jaekel and Maillard 1982), two immediate checks can be made: it is easily seen that both limits $A = C$, $B = D$, and $C = 1$, $D = 0$, allow one to recover the previous result. Let us just remark that a closer examination of the second limit shows (at fourth order in B, D), an interesting cooperation of connected and disconnected diagrams on the checkerboard lattice, which leads to the same connected diagrams on the square lattice. This property might be used, for instance, to put constraints on the coefficients of different topological kinds of diagrams on the checkerboard lattice. Another kind of important constraint has also to be satisfied by these expansions: each one must be invariant under the spatial symmetries of the corresponding lattice. Indeed, it is easily checked that the expression for the checkerboard lattice is invariant under the symmetry group of the square (C_{4v}: generated by ($A\leftrightarrow C$, $B\leftrightarrow D$) and ($A\leftrightarrow B$, $C\leftrightarrow D$) and $A\to B\to C\to D\to A$), and that the expression for the triangular lattice (any of the limits $A=1$, $C=1$ of the previous one), is invariant under the symmetry group of the triangle (S_3 generated by ($A\leftrightarrow B$), ($A\leftrightarrow D$)).

Although only geometrically derived in the cases of the triangular and checkerboard lattices, the inversion relations for the partition function of these models can be explicitly checked, now that the corresponding partially resummed expansions are available. Indeed it is easy, although rather tedious, to verify (up to fourth order) the inversion functional relation on the expansion for the checkerboard lattice:

$$Z(a, b, c, d)Z(2-q-a, 1/b, 2-q-c, 1/d)=[(a+q-1)(a-1)(c+q-1)(c-1)]^{1/2}$$

or else:

$$\ln \Lambda(A, B, C, D)+\ln \Lambda\left(\frac{1}{A}, \frac{-B}{1+(q-2)B}, \frac{1}{C}, \frac{-D}{1+(q-2)D}\right)$$
$$=\tfrac{1}{2}\ln[\lambda(B)\lambda(D)]=\tfrac{1}{2}\ln\left[\frac{(1-B)(1+(q-1)B)}{1+(q-2)B}\,\frac{(1-D)(1+(q-1)D)}{1+(q-2)D}\right]. \qquad (2)$$

As usual, one can note the successive, order by order, cancellations of all the poles (in A and C) in the rational coefficients of the B^nD^m terms, leading to a final expression which depends only on B and D. For instance, at second order

$$\frac{(q-1)}{2}\left[\frac{A^2C^2}{1-A^2C^2}(B^2+D^2)+\frac{2AC}{1-A^2C^2}BD+\frac{-1}{1-A^2C^2}((-B)^2+(-D)^2)-\frac{2AC}{1-A^2C^2}BD\right]$$
$$=-\frac{(q-1)}{2}(B^2+D^2).$$

As the limit to the triangular lattice is preserved by the inversion relation ($C=1$ is invariant under $C\leftrightarrow 1/C$), the inversion relation for the triangular lattice follows directly from the previous one. Those results contribute to establish the actions of different automorphy groups on the partition functions of lattices with different geometries.

2.2. *Disorder solutions*

Rujan's exact solution for the anisotropic triangular Potts model can be stated as follows: once the relation

$$\frac{d-1}{(q-1)d+1}+\frac{a-1}{a+q-1}\,\frac{b-1}{b+q-1}=0 \qquad \text{or} \qquad AB+\frac{D}{1+(q-2)D}=0$$

is satisfied, the partition function takes the simple form:

$$Z(a, b, d) = \frac{(a+q-1)(b+q-1)}{(1/d+q-1)} \qquad \text{or}$$

$$\Lambda(A, B, D) = \frac{(1-D)(1+(q-1)D)}{1+(q-2)D} = \lambda(D). \tag{3}$$

One will note the remarkable form of the normalised partition function Λ, (compare with (2)), which depends only on the distinguished parameter D. Fortunately, the disorder variety does intersect the domain defined by the partially resummed expansion: $B \to 0$, $D \to 0$, A finite. This intersection is defined by

$$B = -A^{-1}(D-(q-2)D^2+(q-2)^2D^3+\ldots).$$

The check thus reduces to introducing this expression for B into the partially resummed expansion (for the triangular lattice) and verifying that the result takes the form (up to fourth order)

$$\ln \Lambda(A, B, D) = -(q-1)D^2+(q-1)(q-2)D^3-[(q-1)(q-2)^2+\tfrac{1}{2}(q-1)^2]D^4+\ldots. \tag{4}$$

In fact, as can be easily seen at second and third order in B, D (by just replacing A by AC, but is more involved at fourth order), an even better verification can be done directly on the checkerboard lattice; once the expansion of B

$$B = -(AC)^{-1}(D-(q-2)D^2+(q-2)^2D^3+\ldots)$$

is inserted into expression (1), the normalised partition function Λ^2 takes the simple form (4). For instance, up to second order:

$$\frac{(q-1)}{2}\left(\frac{A^2C^2}{1-A^2C^2}\left(\frac{D^2}{A^2C^2}+D^2\right)+\frac{2AC}{1-A^2C^2}\left(-\frac{D^2}{AC}\right)\right) = -\frac{(q-1)}{2}D^2.$$

Two conclusions can be drawn from this result: firstly, it establishes on the triangular lattice the agreement between the analytic expression (3) on the disorder variety and the analytic continuation which is needed to apply the inversion relation; secondly, it indicates the possible extension, to the checkerboard Potts model, of a simple expression

$$Z(a, b, c, d) = \left(\frac{(a+q-1)(b+q-1)(c+q-1)}{(1/d+q-1)}\right)^{1/2} \tag{5}$$

for the partition function on a disorder variety:

$$ABC(2-q-1/D) = 1 \qquad \text{or} \qquad \frac{d-1}{(q-1)d+1}+\frac{a-1}{a+q-1}\,\frac{b-1}{b+q-1}\,\frac{c-1}{c+q-1} = 0. \tag{6}$$

Concerning this second point, a further confirmation is obtained when considering the Ising model ($q=2$). The partition function of the checkerboard Ising model has been obtained by Utiyama (1951) for any value of the four parameters K_1, K_2, K_3, K_4. The results can be rewritten by means of complete elliptic integrals of the third and first kinds. The latter reduce to a simple rational expression when the modulus k of the elliptic functions tends to 0 or ∞. It is shown in appendix 2 that this symmetric condition ($k=\infty$) splits into the following relations

$$\tanh K_i + \tanh K_j \tanh K_k \tanh K_l = 0 \qquad (i, j, k, l) = (1, 2, 3, 4)$$

which are nothing other than equation (6), and its images under spatial symmetries. The computation of the double integral which expresses the partition function has also been performed in appendix 2, leading to the respective symmetry breaking results

$$Z = [(e^{2K_1}+1)(e^{2K_2}+1)(e^{2K_3}+1)(e^{2K_4}+1)]^{1/2}\, e^{K_i}/(e^{2K_i}+1)$$

which are just particular cases of (5) for $q=2$.

The combined results of this section, that is, that expression (5) coincides with the partition function on the disorder variety up to fourth order in the resummed expansion, for any value of q of the checkerboard Potts model, and that it also coincides exactly with the partition function of the checkerboard Ising model ($q=2$), suggest that this expression (5) should be the partition function of the general checkerboard Potts model on a disorder variety (6).

2.3. *Order by order determination*

The compatibility, with the partially resummed expansion, of the exact expression for the partition function on the disorder variety, leads naturally to the question whether one could use this information, together with the automorphy group, in order to determine the partition function, order by order in the resummed expansion. Let us consider the triangular Potts model. The second and third orders are quite simple and will be assumed to be known. Let us try to determine the fourth order, with the help of the previous constraints. A qualitative examination of the diagrams allows one to suppose that only $(1-A^2)^n$ and $(1-A^3)$ $(n=1,2,3)$ singularities enter the exact expression at these orders. Its general form can then be written as

$$\frac{1}{(1-A^2)^3(1-A^3)}[P(A)(B^4+D^4)+Q(A)(B^3D+BD^3)+R(A)B^2D^2]$$

$$P(A)=\sum_{i=0}^{9} p_iA^i \qquad Q(A)=\sum_{i=0}^{9} q_iA^i \qquad R(A)=\sum_{i=0}^{9} r_iA^i.$$

The spatial symmetries imply for these expressions that p_i, $i=0, 1, 2, 3$, q_j, $j=0, 1, 2$, r_k, $k=0, 1$ are known. The inversion symmetry leads to $p_i+p_{9-i}=$ known, $q_i+q_{9-i}=0$, $r_i+r_{9-i}=0$. Finally, the disorder solution results in a constraint of the following form: $(1+A^4)P(A)+A(1+A^2)Q(A)+A^2R(A)=$ known expression, or $p_4-q_3+r_2$, $p_4+q_4-r_3$, $p_3+q_3-q_4-r_4$ are known. Clearly, some coefficients remain undetermined, so that the disorder solutions, while putting new constraints on the expansion, are nonetheless insufficient to allow a complete determination of the partition function. One could then envisage using the exact solution on the critical variety, but it appears that the inconvenience in dealing simultaneously with all the available information, (like large q and partially resummed expansions for instance), calls for a more direct and global analytical approach.

3. Global analytical approach

The local approach, by means of partially resummed expansions, only enabled one to make separate use of the spatial and of the inversion symmetries. On the other hand, in a global approach, one can envisage iterated combinations of these symmetries, thus generating new constraints, which will relate values of the partition function at points lying outside the domain of the partially resummed expansion. Indeed, the inversion and spatial symmetries can easily be seen to generate an infinite discrete group, which we shall first rapidly describe.

3.1. *Automorphy groups*

The action of the inversion symmetry can be written in a general form

$$I:(a, b, c, d)\mapsto(f(a), g(b), f(c), g(d))$$

$$f\circ f=g\circ g=1 \qquad g\circ f=h \qquad f\circ g=h^{-1}$$

where in the case of the Potts model, the functions f, g, h take the form

$$f(x)=2-q-x, \qquad g(x)=1/x, \qquad h(x)=1/(2-q-x).$$

In the triangular case, the spatial symmetries are generated by

$$S:(a, b, d)\mapsto(b, a, d), \qquad S':(a, b, d)\mapsto(d, b, a)$$

and the generated group is easily seen to contain finite subgroups isomorphic to S_3 and Z_2 acting semidirectly on an infinite subgroup isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$ and generated by the two commuting elements:

$$(SI)^2:(a, b, d)\mapsto(h(a), h^{-1}(b), d)$$

$$(S'I)^2:(a, b, d)\mapsto(h(a), b, h^{-1}(d))$$

(this is the same group as that of the three-dimensional cubic model, (Jaekel and Maillard 1983); it can also be recovered as a subgroup of the group generated by S_4 and I, which is studied in appendix 3). In the checkerboard case, the spatial symmetries are generated by

$$S'':(a, b, c, d)\mapsto(c, b, a, d), \qquad \sigma:(a, b, c, d)\mapsto(b, c, d, a)$$

and the group contains finite subgroups isomorphic to C_{4v} and Z_2, acting semi-directly on an infinite subgroup isomorphic to $\mathbb{Z}$ and generated by†

$$\sigma^2(\sigma I)^2:(a, b, c, d)\mapsto(h(a), h^{-1}(b), h(c), h^{-1}(d)).$$

Indeed, one remarks that S'' and σ^2 generate an Abelian subgroup K of C_{4v}, which commutes with I and which is stable under conjugation by σ; thus, every element of G:

$$g=k_n\sigma^{\alpha_n}I\ldots Ik_1\sigma^{\alpha_1}Ik_0\sigma^{\alpha_0}$$

where

$$\alpha_0, \alpha_1, \ldots, \alpha_n=0, 1, \qquad k_0, k_1, \ldots, k_n\in K$$

can also be written as

$$g=k'\sigma^{\alpha_n}I\ldots I\sigma^{\alpha_1}I\sigma^{\alpha_0}, \qquad k'\in K$$

or

$$g=k''\sigma^{\alpha''}(\sigma I)^n, \qquad n\in\mathbb{Z}, \qquad \alpha''=0, 1, \qquad k''\in K$$

† This gives the complete group generated by C_{4v} and I, in contrast to the study in Maillard and Rammal (1983, p 358), where only $K\sim Z_2\oplus Z_2$ (Klein subgroup of C_{4v}) was taken into account.

or else

$$g = k\sigma^{\alpha} I^{\beta}[\sigma^2(\sigma I)^2]^n,$$

with

$$n \in \mathbb{Z}, \qquad \alpha, \beta = 0, 1, \qquad k \in K.$$

As was seen on diagrammatic expansions, the inversion and spatial symmetries are also represented by a simple multiplicative action on the partition function (up to an inversion). By combined iterations, these symmetries extend to a whole group which acts on the partition function as an automorphy group. The description just given, by means of a representation on the parameters of the model, has put into evidence relations which mix the inversion and the other symmetries. In order to identify the group thus defined with the automorphy group acting on the partition function, one still needs to verify that these relations translate into similar relations on the automorphic factors. Indeed, the relations corresponding to the commutation of the inversion (I) with other symmetries (like $SS'S$ for the triangular lattice, or S'' and σ^2 for the checkerboard lattice), are easily checked, as they correspond to the invariance of the inversion factor under such symmetries. Finally, in the triangular case, the Abelian character of the infinite subgroup $\mathbb{Z}\oplus\mathbb{Z}$ results from the form of the automorphic factors associated with $(SI)^2$ and $(S'I)^2$

$$I\colon Z(f(a), g(b), g(d))Z(a, b, d) = \varphi(a) = (a+q-1)(1-a) = \varphi(f(a))$$
$$(SI)^2\colon Z(h(a), h^{-1}(b), d) = [\varphi(g(b))/\varphi(a)]Z(a, b, d)$$
$$(S'I)^2\colon Z(h(a), b, h^{-1}(d)) = [\varphi(g(d))/\varphi(a)]Z(a, b, d)$$

which lead to the following commutative diagram

$$\begin{array}{ccc}
(a, b, d) & \xrightarrow[(SI)^2]{\varphi(h^{-1}(b))/\varphi(a)} & (h(a), h^{-1}(b), d) \\
\frac{\varphi(h^{-1}(d))}{\varphi(a)} \Big\downarrow (S'I)^2 & & (S'I)^2 \Big\downarrow \frac{\varphi(h^{-1}(d))}{\varphi(h(a))} \\
(h(a), b, h^{-1}(d)) & \xrightarrow[\varphi(h^{-1}(b))/\varphi(h(a))]{(SI)^2} & (h\circ h(a), h^{-1}(b), h^{-1}(d)).
\end{array}$$

One should note that the preceding structure for the automorphy group results only from the separative action of the inversion symmetry I, and of the corresponding automorphic factor (φ). Such properties still hold for other models, like for instance, spin models with soluble groups (Zamolodchikov and Monastyrskii 1979).

Let us remark that, as the triangular model is a particular case of the checkerboard model, one might ask about the relationship between their respective automorphy groups. *A priori*, two possibilities can arise. Either the triangular model is a specialisation which has more symmetries than the generic checkerboard model, or its symmetries are the trace of larger and hidden symmetries of the generic checkerboard model. In any case, the group for the checkerboard model at least contains the previously studied one, generated by the inversion I and C_{4v}. The first possibility is the weakest one. However, two remarkable facts can be put into evidence. First, a large q expansion of the partition function of the checkerboard model (up to the sixth order, (Rammal and Maillard 1983)), shows a complete symmetry under the whole group S_4 of permutations of the four parameters. Secondly, the partition function can be computed exactly in the Ising case, and shows the same complete symmetry S_4 (see appendix 2). These two surprising results suggest that the general Potts model on the checkerboard lattice has a partition function which is in fact invariant under the whole symmetry group S_4. Anyway, it is easy to extend the previous study and show that the corresponding automorphy group can be described as semi-direct products of S_4 and Z_2 on an infinite subgroup now isomorphic to $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ (see appendix 3). Such a group enlarges the automorphy group of the triangular model, thus realising the second possibility.

3.2. *Disorder varieties*

For simplicity, the results will from now be made explicit for the triangular case only. In order to give a compact form to the action of the automorphy group on the disorder varieties, it will appear convenient to re-express them in terms of canonical variables

$$a=\frac{1}{t}\frac{u-t^3}{1-tu} \qquad b=\frac{1}{t}\frac{v-t^3}{1-tv} \qquad d=\frac{1}{t}\frac{z-t^3}{1-tz} \qquad \left(c=\frac{1}{t}\frac{w-t^3}{1-tw}\right)$$

on which the group acts multiplicatively

$$I:(u,v,z)\mapsto(1/t^2u,\ t^2/v,\ t^2/z)$$

$$(SI)^2:(u,v,z)\mapsto(t^4u,\ v/t^4,\ z)$$

$$(S'I)^2:(u,v,z)\mapsto(t^4u,\ v,\ z/t^4).$$

The disorder varieties can then be written as

$$(1-t^2)^2(t-uvz)-(tu-1)(tv-1)(z-t^3)=0 \tag{7}$$

or any of the other two analogous expressions one obtains by permuting the three parameters u, v, z. Having the automorphy group act on one of these varieties, one first sees that the infinite subgroup $\mathbb{Z}\oplus\mathbb{Z}$ provides varieties of the following form: $(D_{\alpha\beta\gamma})$

$$(1-t^2)^2(1-t^3\alpha\beta\gamma uvz)-t^2(t\alpha u-1)(t\beta v-1)(t\gamma z-1)=0 \tag{8}$$

with

$$\alpha\beta\gamma=t^{-4},\ \alpha=t^{4n},\ \beta=t^{4m},\ \gamma=t^{4p}, \qquad n,m,p\in\mathbb{Z} \tag{8'}$$

and then that the remaining elements leave this set invariant (S_3 permutes (α,β,γ) and I gives

$$(\alpha,\beta,\gamma)\to(1/\alpha,\ 1/t^4\beta,\ 1/t^4\gamma).$$

The latter is immediately seen to be an infinite set of different varieties, on which the automorphy group acts globally.

One remarks that there is a small degeneracy in the action of the group: for instance, S and $SS'SI$ generate a subgroup of order 12, which leaves the disorder variety (7) invariant. In particular, these are elements which transform a point of a disorder variety into another point which is also on a disorder variety. Hence they enable one to compare the analytical extension given by the automorphy group (with the automorphic factors), with the exact expression provided by the disorder solution. In fact, they reduce to the following equality

$$Z(2-q-a,\ 1/d,\ 1/b)Z(a,b,d)=(a+q-1)(1-a)$$

which is indeed verified by the disorder solution (3).

3.3. *Exact expressions*

The agreement between the disorder solution and the automorphy group, on their intersection domain, and the complementarity they show in extending the disorder varieties to an infinite set of transformed varieties, lead us, as a further step, to introduce an automorphic function one can build by extending the disorder solutions to the transformed varieties, and to compare it with exact expressions for the partition function. First we shall give a compact expression for the values of the function on all these varieties. Introducing the functions F, G:

$$F(x) = \prod_{n=1}^{\infty} \frac{1 - t^{4n-1}x}{1 + t^{4n+1}x}; \qquad G(x) = \frac{F(x)F(1/x)}{1 - t^3 x}$$

so that $\lambda(D) = (1+t^2)^2\, G(z)/G(t^{-4}z)$ (see (2) and (3)) allows one suitably to take account of all the automorphic factors, and to write the required function as:

$$\bar{\Lambda}(u, v, z) = (1+t^2)^2 \frac{G(u)G(v)G(z)}{G(\alpha u)G(\beta v)G(\gamma z)}, \qquad ((u, v, z) \in D_{\alpha\beta\gamma}) \tag{9}$$

(a similar expression can be written for the checkerboard model; in fact, such an expression strictly only applies for $q > 4$, and an appropriate analytic continuation in t must be understood for $q < 4$). On the other hand, the exact expression for the partition function is known on the critical variety ($uvz = t$, Baxter *et al* 1978). As such a variety is invariant under the action of the group, it thus intersects the infinite set of transformed disorder varieties, and hence provides an infinite set of subvarieties where the previous automorphic function can be compared with the known partition function. First, one notices that the intersection of the critical variety with any transformed variety splits into curves:

$$\alpha\beta uv = 1/t^2 \quad \text{and} \quad \gamma z = 1/t \qquad \begin{array}{l} \alpha = t^{4n}, \quad \beta = t^{4m}, \quad \gamma = t^{4p} \\ n, m, p \in \mathbb{Z}, \quad n+m+p = -1 \end{array} \tag{10}$$

and the similar curves obtained by permutation of αu, βv, γz. Then, introducing any of the split equations (10) into expression (9) and using the remarkable functional equality for G for G

$$G(x)G(1/t^2 x) = 1 \qquad (G(1/t) = 1)$$

one verifies that expression (9) actually reduces to

$$\bar{\Lambda}(u, v, z) = (1+t^2)^2 G(u)G(v)G(z).$$

This is nothing other than the exact expression for the normalised partition function on the critical variety (Baxter *et al* 1978). (This verification is easily extended to the checkerboard case (critical variety: $uvwz = 1$), which, in the limit $u = w$ and $v = z$, can also provide another comparison: on the critical antiferromagnetic curve of the square lattice ($uv = -1$); on this curve too, expression (9) coincides, on some appropriate domain, with the known partition function (Baxter 1982b).) Let us notice that expression (9), though taking different values for a point which simultaneously belongs to different $D_{\alpha\beta\gamma}$, remarkably becomes univalued on the critical variety.

It seems tempting to extend the compact expression (9) to non-integer values of n, m, p. However, other conditions, including outside the critical variety, restrictions on the validity domain of (9) for given α, β, γ, should be imposed on the variables, in order to determine the function. There, large q expansions could provide hints.

4. Conclusion

The partial resummations of the diagrammatic expansions for Potts models (in fact, the expansions around one-dimensional models), display an interesting feature of their partition functions for various lattices: the rational character of the coefficients in the expansion. The disorder solutions, which appear to be valid along the same neighbourhoods of the one-dimensional models, are also expressed as rational functions which adapt to the expansion. Moreover, the form of the disorder solutions (on the disorder algebraic varieties) also matches with the automorphic factors. Rational expansions, rational particular expressions, together with the automorphy group functional equations build a coherent framework which could lead to an interesting characterisation and perhaps a determination of these partition functions.

Further information, though more difficult to work out, is also provided by another simplification which occurs in these models: on the disorder varieties, many correlation functions, which have their n points aligned in the same direction, take a rational dependence in the parameters. As these functions are related to the derivatives of the partition function, one could use them in a systematic expansion around a disorder variety.

Of course, disorder solutions have also been established for three-dimensional models (Welberry and Miller 1978), for Ising models with a field (Verhagen 1976), and for general interaction round a face model (Enting 1977a), thus providing an exact expression for the partition function on rather large subvarieties (with codimension one or two). In general, they give fruitful information which can be used in conjunction with the automorphic functional equations, as the latter can also be derived for the same models.

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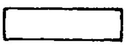
We are grateful to P Rujan for sending us a copy of his work on disorder solutions, prior to publication.

Appendix 1

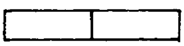
A unique diagram will represent all the diagrams of arbitrary length, and also those which can be deduced by symmetry. Moreover it will be convenient to introduce the following expressions

$$X = \frac{AC}{1 - A^2C^2}(ACB + D) \qquad Y = \frac{AC}{1 - A^2C^2}(B + ACD).$$

The resummed high-temperature expansion for the checkerboard Potts model is given by the following diagrams at second order

 $\frac{1}{2}(q-1)(BX + DY)$

at third order

 $\frac{1}{2}(q-1)(q-2)(BX^2 + DY^2)$

at fourth order

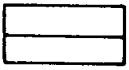
$$\tfrac{1}{2}(q-1)\,\frac{A^2+C^2+2A^2C^2}{1-A^2C^2}\,B^2D^2$$

$$(q-1)\frac{A^2+C^2+2A^2C^2}{1-A^2C^2}\,BD(BY+DX)$$

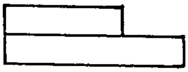
$$(q-1)\frac{1}{1-A^2C^2}[(A^2+C^2)BDXY+A^2C^2(B^2Y^2+D^2X^2)]$$

$$\tfrac{1}{2}(q-1)\frac{1}{1-A^2C^2}[4A^2C^2BDXY+(A^2+C^2)(B^2Y^2+D^2X^2)]$$

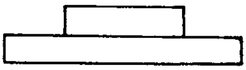
and



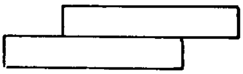
$$\tfrac{1}{2}(q-1)(q-2)\frac{AC(A+C+2A^2C^2)}{1-A^3C^3}\,B^2D^2$$



$$(q-1)(q-2)\frac{AC(A+C+2A^2C^2)}{1-A^3C^3}\,BD(BY+DX)$$

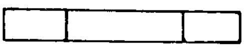


$$(q-1)(q-2)\frac{AC}{1-A^3C^3}[(A+C)BDXY+A^2C^2(B^2Y^2+D^2X^2)]$$



$$\tfrac{1}{2}(q-1)(q-2)\frac{AC}{1-A^3C^3}[4A^2C^2BDXY+(A+C)(B^2Y^2+D^2X^2)]$$

and



$$\tfrac{1}{2}(q-1)(q-2)^2\frac{AC}{1-A^2C^2}[2BDXY+AC(B^2X^2+D^2Y^2)]$$

and, at last, for the disconnected terms

$$\frac{(q-1)^2}{2}\Bigg[\bigg(-\frac{1}{2}-\frac{3}{1-A^2C^2}\bigg)\bigg(\frac{A^2C^2}{1-A^2C^2}\bigg)^2(B^4+D^4)$$
$$+4\bigg(1-\frac{3}{1-A^2C^2}\bigg)AC\bigg(\frac{AC}{1-A^2C^2}\bigg)^2(B^3D+BD^3)$$
$$+2\bigg(\frac{1}{2}-\frac{3}{1-A^2C^2}\bigg)(1+2A^2C^2)\bigg(\frac{AC}{1-A^2C^2}\bigg)^2B^2D^2\Bigg].$$

Appendix 2

A shorthand notation will be used in the following

$$C_i = \cosh 2K_i, \qquad S_i = \sinh 2K_i, \qquad T_i = \tanh K_i,$$
$$\ln Z = \tfrac{1}{2}(K_1 + K_2 + K_3 + K_4) + \ln Z_0(K_1, K_2, K_3, K_4)$$

and the duality relation will also be useful

$$C_i^* = \cosh 2K_i^*, \qquad S_i^* = \sinh 2K_i^*, \qquad (e^{-2K_i^*} = \tanh K_i)$$
$$Z_0(K_1^*, K_2^*, K_3^*, K_4^*) = \frac{Z_0(K_1, K_2, K_3, K_4)}{(S_1 S_2 S_3 S_4)^{1/4}}.$$

The partition function of the checkerboard Ising model (generalised square lattice) has been obtained by Utiyama (1951) under the following form

$$\ln Z_0 = \frac{1}{16\pi^2} \int_{-\pi}^{\pi} d\omega_1 \int_{-\pi}^{\pi} d\omega_2 \ln 8[1 + C_1 C_2 C_3 C_4 + S_1 S_2 S_3 S_4 - S_2 S_4 \cos(\omega_1 + \omega_2)$$
$$- S_1 S_3 \cos(\omega_1 - \omega_2) - (S_1 S_2 + S_3 S_4) \cos \omega_1 - (S_1 S_4 + S_2 S_3) \cos \omega_2]. \qquad (11)$$

The C_{4v} invariance of the partition function is obvious on that expression. In fact, the partition function is invariant under all permutations of the four coupling constants K_1, K_2, K_3, K_4 (S_4 group). To satisfy this complete invariance it is in fact sufficient that the partition function be invariant under a transposition which does not belong to C_{4v}, say the transposition $K_1 \leftrightarrow K_4$. Let us rewrite the argument in the logarithm under the form:

$$\alpha + \beta \cos \omega_1 + \gamma \sin \omega_1$$
$$\alpha = 1 + C_1 C_2 C_3 C_4 + S_1 S_2 S_3 S_4 - (S_1 S_4 + S_2 S_3) \cos \omega_2$$
$$\beta = -(S_1 S_2 + S_3 S_4) - (S_1 S_3 + S_2 S_4) \cos \omega_2$$
$$\gamma = -(S_1 S_3 - S_2 S_4) \sin \omega_2.$$

Integration over the angle ω_1 is easily performed to give

$$\int_{-\pi}^{\pi} \ln(\alpha + \beta \cos \omega_1 + \gamma \sin \omega_1)\, d\omega_1 = 2\pi \ln \tfrac{1}{2}[\alpha + (\alpha^2 - \beta^2 - \gamma^2)^{1/2}] \qquad (12)$$

α is obviously invariant under the transposition $K_1 \leftrightarrow K_4$, and $\beta^2 + \gamma^2$ also

$$\beta^2 + \gamma^2 = (S_1^2 + S_4^2)(S_2^2 + S_3^2) + 2[(S_1^2 + S_4^2) S_2 S_3 + S_1 S_4 (S_2^2 + S_3^2)] \cos \omega_2 + 4 S_1 S_2 S_3 S_4 \cos^2 \omega_2$$

so that the partition function is invariant under that transposition and therefore under the S_4 group. Let us note that another hint for the $K_1 \leftrightarrow K_4$ symmetry is also provided by the exact expression of the diagonal two-point correlation function (Gabay 1980).

It is known (Green and Hurst 1964) that the double integral (11) can be evaluated in terms of elliptic integrals of the first and third kinds. In the case of the triangular lattice, which corresponds to the limit $C_3 \sim S_3 \to \infty$, the partition function on the Stephenson's disorder varieties ($\tanh K_i + \tanh K_j \tanh K_k = 0$, $i, j, k = 1, 2, 4$) reduces to a very simple expression (Gibberd 1969)

$$Z_0 = 2(\cosh K_1 \cosh K_2 \cosh K_4)(\cosh K_i)^{-2}.$$

How can elliptic functions reduce to such a simple expression? The reason is that the disorder varieties correspond to a trivial value of the modulus k that occurs in the elliptic functions: $k=\infty$. It is natural to use this remark and try to find some candidates for disorder varieties in the case of the checkerboard Ising lattice. The exact expression of the modulus k for that model has been obtained by Syozi and Naya (1960). The condition ($k=\infty$) can be written as

$$\begin{aligned} 0 &= 1 + C_1^* C_2^* C_3^* C_4^* + S_1^* S_2^* S_3^* S_4^* + \tfrac{1}{2}(S_1^{*2} + S_2^{*2} + S_3^{*2} + S_4^{*2}) \\ &= \tfrac{1}{8}(T_1 + T_2 T_3 T_4)(T_2 + T_1 T_3 T_4)(T_3 + T_1 T_2 T_4)(T_4 + T_1 T_2 T_3)(T_1 T_2 T_3 T_4)^{-2} \end{aligned} \tag{13}$$

which obviously generalises Stephenson's disorder varieties.

Let us now calculate the partition function restricted to (13). Clearly, it is more convenient to compute its dual image. One remarks that the term $\alpha^{*2} - \beta^{*2} - \gamma^{*2}$ becomes a perfect square:

$$\alpha^{*2} - \beta^{*2} - \gamma^{*2} = [\tfrac{1}{2}(S_1^{*2} + S_4^{*2} - S_2^{*2} - S_3^{*2}) + (S_1^* S_4^* - S_2^* S_3^*) \cos \omega_2]^2.$$

Hence, the integral (12) reduces to

$$\int_{-\pi}^{\pi} d\omega_2 \int_{-\pi}^{\pi} d\omega_1 \ln(\alpha^* + \beta^* \cos \omega_1 + \gamma^* \sin \omega_1) = 2\pi \int_{-\pi}^{\pi} d\omega_2 \ln\left[-\frac{S_1^{*2} + S_4^{*2}}{2} + S_1^* S_4^* \cos \omega_2 \right]$$

or

$$2\pi \int_{-\pi}^{\pi} d\omega_2 \ln\left[-\frac{S_2^{*} + S_3^{*2}}{2} + S_2^* S_3^* \cos \omega_2 \right]$$

which can then take the different forms

$$\begin{aligned} &4\pi^2 \ln(-\tfrac{1}{2}S_1^{*2}) \qquad 4\pi^2 \ln(-\tfrac{1}{2}S_4^{*2}), \\ &4\pi^2 \ln(-\tfrac{1}{2}S_2^{*2}) \qquad 4\pi^2 \ln(-\tfrac{1}{2}S_3^{*2}). \end{aligned} \tag{14}$$

One should notice that the equation of the disorder variety has been introduced under a S_4-invariant form and that the S_4-symmetry breaking in the final result (14) comes from two bifurcations in the determination of the square roots. Taking into account all the multiplicative factors, one gets the result that the partition function per site is given by

$$Z_0 = 2(\cosh K_1 \cosh K_2 \cosh K_3 \cosh K_4)^{1/2}(\cosh K_i)^{-1} \qquad i = 1, 2, 3, 4$$

when restricted to the varieties:

$$\tanh K_i + \tanh K_j \tanh K_k \tanh K_l = 0 \qquad (i, j, k, l) = (1, 2, 3, 4).$$

Appendix 3

The structure of the group G, generated by S_4 and I, can be obtained in two steps. Firstly, any element of G

$$g = s_n I \ldots I s_1 I s_0, \qquad s_0, s_1, \ldots, s_n \in S_4$$

can be rewritten

$$g = I^{\alpha} s[(s'_p s_p)^{-1} I s'_p I s_p] \dots [(s'_1 s_1)^{-1} I s'_1 I s_1][(s'_0 s_0)^{-1} I s'_0 I s_0]$$

where

$$\alpha = 0, 1, \qquad s, s_0, s'_0, s_1, s'_1, \dots, s_p, s'_p \in S_4.$$

This form puts into evidence the infinite subgroup H, which consists of all the elements of G, possessing an even number of I and resulting in a global permutation which is the identity. The generators of H can also be written:

$$s' I (ss')^{-1} I s, \qquad s, s' \in S_4$$

Let us denote by P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34} the elementary transpositions which generate S_4, and by σ a cycle of order four. P_{13} and P_{24} generate an Abelian subgroup K (isomorphic to $Z_2 \oplus Z_2$), which commutes with the inversion I. Furthermore, a complementary set E can be exhibited, which has the following properties: any element s of S_4 can be written in a unique way

$$s = ke = e'k', \qquad k, k' \in K \qquad e, e' \in E.$$

One obtains E as the subset: $(1, P_{12}, P_{14}, P_{23}, P_{34}, \sigma)$. The generators of H then reduce to

$$e' I e'^{-1} e^{-1} I e.$$

If e, e' are both transpositions, such an element is easily rewritten (I is an involution)

$$(e'I)^2 (Ie)^2 = (e'I)^2 (eI)^{-2}$$

If one of them is σ, it can be rewritten

$$\sigma^2 (\sigma I)^2 (eI)^{-2} \qquad \text{or} \qquad (e'I)^2 [\sigma^2 (\sigma I)^2]^{-1}.$$

Therefore H is generated by the elements of the form

$$(P_{12}I)^2, \qquad (P_{14}I)^2, \qquad (P_{23}I)^2, \qquad (P_{34}I)^2, \qquad \sigma^2(\sigma I)^2.$$

Secondly, using the separative form of the action of the inversion, one can show that the generators

$$\begin{aligned}
&(P_{12}I)^2\colon (a, b, c, d) \mapsto (h(a), h^{-1}(b), c, d) \\
&(P_{14}I)^2\colon (a, b, c, d) \mapsto (h(a), b, c, h^{-1}(d)) \\
&(P_{23}I)^2\colon (a, b, c, d) \mapsto (a, h^{-1}(b), h(c), d) \\
&(P_{34}I)^2\colon (a, b, c, d) \mapsto (a, b, h(c), h^{-1}(d)) \\
&\sigma^2(\sigma I)^2\colon (a, b, c, d) \mapsto (h(a), h^{-1}(b), h(c), h^{-1}(d))
\end{aligned} \tag{15}$$

satisfy further relations

$$\begin{aligned}
&(P_{34}I)^2 = (P_{23}I)^2 (P_{12}I)^{-2} (P_{14}I)^2 \\
&\sigma^2(\sigma I)^2 = (P_{23}I)^2 (P_{14}I)^2
\end{aligned}$$

and that only three of them are independent

$$(P_{12}I)^2, \qquad (P_{14}I)^2, \qquad (P_{23}I)^2$$

for instance. Moreover, (15) show that they commute, and thus lead to an infinite subgroup which is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$.

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