

Large q expansion for the anisotropic Potts model

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Abstract. A large q expansion of the partition function of the anisotropic q -state Potts model in two dimensions is given up to the sixth order in $q^{-1/2}$. At the critical temperature, it is in agreement with the exact solutions for the partition function and for the latent heat (the latter does not depend on the anisotropy). It does not show any simple factorisation in the parameters of the model.

1. Introduction

Although much is known about the q -state Potts model (see Wu 1982 for a review), no exact solution for all values of the parameters is available. Nonetheless, information on the exact solution can be obtained through expansions around curves where the solution is known (the critical curve (Baxter 1973) or the one-dimensional model (Jaekel and Maillard 1981b)). The advantage of a large q expansion (Ginsparg *et al* 1980) is to provide information for arbitrary values of the two parameters of the anisotropic model. Moreover, the exact solution on the critical curve, and also the latent heat, can be written as infinite products, with factors behaving like $1 + \alpha t^n$ (where $t + 1/t = q^{1/2}$, so that $t = O(q^{-1/2})$ (Baxter 1971, 1973)). This particular form of the solution on the critical curve can be recovered from functional equations, which are satisfied by the partition function and which remain valid at all temperatures (Jaekel and Maillard 1981a). There also exist 'minimal' solutions to these functional equations, which can be written as similar infinite products and which extend to all temperatures the exact solution on the critical curve. The large q expansion thus provides a way to check if such a solution can correspond to the partition function of the Potts model. Here, the large q expansion is computed up to the sixth order in $q^{-1/2}$ and then compared with the exact solutions on the critical curve, for the partition function and the latent heat, with which it is shown to agree perfectly. But it does not factorise in the parameters of the model in the way that would be required by the 'minimal' solutions, and thus excludes such simple Ansätze for the partition function of the Potts model.

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2. Diagrams and expansion

The partition function per site, Z , of the q -state Potts model on a square lattice will be written

$$Z^N(a, b) = \sum_{\{\sigma\}} \prod_{\langle i,j \rangle} a^{\delta_{\sigma_i \sigma_j}} \prod_{\langle k,l \rangle} b^{\delta_{\sigma_k \sigma_l}} \tag{2.1}$$

conf. horiz. links
vert. links

(where each of the N spins σ_i can take q values: $\sigma_i \in \mathbb{Z}_q$) and the free energy per site

$$-\beta F(a, b) = \ln Z(a, b) = \ln[(a + q - 1)(b + q - 1)\Lambda(a, b)]. \tag{2.2}$$

The large q expansion can be obtained using a diagrammatic method (Ginsparg *et al* 1980)[†] which exploits a high-temperature expansion ($a \rightarrow 1, b \rightarrow 1$, or equivalently

$$\frac{1}{a^*} = \frac{a-1}{a+q-1} \rightarrow 0, \quad \frac{1}{b^*} = \frac{b-1}{b+q-1} \rightarrow 0)$$

and the characters of the spin group \mathbb{Z}_q , so that only closed polygons survive:

$\ln \Lambda(a, b)$

$$\begin{aligned}
 &= \square (q-1) \frac{1}{a^{*2}} \frac{1}{b^{*2}} \quad \left. \vphantom{\square} \right\} (O=2) \\
 &+ \square \square (q-1)(q-2) \frac{1}{a^{*4}} \frac{1}{b^{*3}} \quad \left. \vphantom{\square \square} \right\} (O=3) \\
 &+ \begin{array}{|c|} \hline \square \\ \hline \end{array} (q-1)(q-2) \frac{1}{a^{*3}} \frac{1}{b^{*4}} \\
 &+ \square \square \square (q-1)(q-2)^2 \frac{1}{a^{*6}} \frac{1}{b^{*4}} \\
 &+ \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} (q-1)(q-2)^2 \frac{1}{a^{*4}} \frac{1}{b^{*6}} \\
 &+ \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} 4(q-1)(q-2)^2 \frac{1}{a^{*5}} \frac{1}{b^{*5}} \quad \left. \vphantom{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}} \right\} (O=4) \\
 &+ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} (q-1)(q-2)[(q-3)^2 + (q-2)] \frac{1}{a^{*6}} \frac{1}{b^{*6}} \\
 &+ \square (q-1) \frac{1}{a^{*4}} \frac{1}{b^{*2}} + \square (q-1) \frac{1}{a^{*2}} \frac{1}{b^{*4}} \\
 &+ \square \square - \frac{5}{2}(q-1)^2 \frac{1}{a^{*4}} \frac{1}{b^{*4}}. \tag{2.3}
 \end{aligned}$$

As the model is self-dual, a low-temperature expansion ($a \rightarrow \infty, b \rightarrow \infty$) can be obtained

[†] Another use for the large q expansion for the Potts model can be found in Kim (1981), who considers an isotropic system in a field, instead of an anisotropic system without a field.

with the duality transformation

$$F(a, b) = F(a^*, b^*) + \ln[(a^* - 1)(b^* - 1)/q]. \tag{2.4}$$

In the large q expansion, the parameters of the model are chosen such that

$$\frac{1}{a^*} = t \frac{u-t}{1-t^3u} = O(q^{-1/2}), \quad \frac{1}{b^*} = t \frac{v-t}{1-t^3v} = O(q^{-1/2}), \quad t + \frac{1}{t} = q^{1/2}. \tag{2.5}$$

Hence the order of each diagram is given by the following formula:

$$O = L - 2B \quad \text{with } O = \text{order in } q^{-1/2}, L = \text{number of links}, B = \text{number of loops}, \tag{2.6}$$

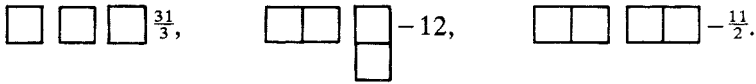
and only a finite number of diagrams contribute to each order. The contribution of each diagram is a product of three factors:

the product of the interaction constants corresponding to the links;

a symmetry factor, counting the diagrams which can be deduced by rotation or reflection;

a polynomial in q , counting the number of ways of colouring the faces with $(q - 1)$ colours and the restriction that two adjacent faces should have different colours.

The main difficulty lies in the counting of the disconnected diagrams, as the complexity of these increases rapidly with the order. For instance, at the sixth order, the following diagrams and symmetry factors appear:



The development has been computed up to the sixth order in $q^{-1/2}$ and, to facilitate further comparison with exact solutions, the result is given in the variables u, v and t^\dagger :

$\ln \Lambda(u, v, t)$

$$\begin{aligned} &= u^2 v^2 t^2 \\ &+ [-2(u^2 v + uv^2) + u^4 v^3 + u^3 v^4] t^3 \\ &+ [u^2 + 4uv + v^2 + u^2 v^2 - 2(u^4 v^2 + 4u^3 v^3 + u^2 v^4) \\ &- \frac{5}{2} u^4 v^4 + u^6 v^4 + 4u^5 v^5 + u^4 v^6 + u^6 v^6] t^4 \\ &+ [-2(u + v) - 2(u^2 v + uv^2) + u^4 v + 16u^3 v^2 + 16u^2 v^3 + uv^4 \\ &+ 11(u^4 v^3 + u^3 v^4) - 2(u^6 v^3 + 11u^5 v^4 + 11u^4 v^5 + u^3 v^6) \\ &- 12(u^6 v^5 + u^5 v^6) + u^8 v^5 + 8u^7 v^6 + 8u^6 v^7 + u^5 v^8 \\ &+ 4(u^8 v^7 + u^7 v^8) + u^9 v^8 + u^8 v^9] t^5 \\ &+ [1 + u^2 + 4uv + v^2 - (12u^3 v + 31u^2 v^2 + 12uv^3) \\ &- (17u^4 v^2 + 48u^3 v^3 + 17u^2 v^4) + u^6 v^2 + 40u^5 v^3 + 96u^4 v^4 \\ &+ 40u^3 v^5 + u^2 v^6 + 39u^6 v^4 + 116u^5 v^5 + 39u^4 v^6 \\ &- (2u^8 v^4 + 40u^7 v^5 + \frac{257}{3} u^6 v^6 + 40u^5 v^7 + 2u^4 v^8) \\ &- (\frac{65}{2} u^8 v^6 + 100u^7 v^7 + \frac{65}{2} u^6 v^8) + u^{10} v^6 + 8u^9 v^7 + 5u^8 v^8 \end{aligned}$$

† This computation was performed with J M Drouffe's Algebraic Manipulation Program.

$$\begin{aligned}
 &+ 8u^7v^9 + u^6v^{10} + 8(u^{10}v^8 + 3u^9v^9 + u^8v^{10}) \\
 &+ 2(2u^{11}v^9 + 7u^{10}v^{10} + 2u^9v^{11}) \\
 &+ u^{12}v^{10} + 4u^{11}v^{11} + u^{10}v^{12} + u^{12}v^{12}]t^6.
 \end{aligned}
 \tag{2.7}$$

3. Comparison with exact solutions

At the critical temperature $((a - 1)(b - 1) = q$ or $uv = 1$), the partition function has been computed exactly (Temperley and Lieb 1971, Baxter 1973), and can be written as an infinite product (Baxter 1971):

$$\Lambda(z, t) = \frac{1 + t^2}{(1 - t^3z)(1 - t^3/z)} \prod_{n=1}^{\infty} \left(\frac{1 - t^{4n-1}z}{1 - t^{4n+1}z} \frac{1 - t^{4n-1}/z}{1 - t^{4n+1}/z} \right)^2
 \tag{3.1}$$

(with $u = z, v = 1/z$). Developing up to the sixth order yields

$$\ln \Lambda(z, t) = t^2 - (z + 1/z)t^3 - \frac{1}{2}t^4 + 2(z + 1/z)t^5 + \left[\frac{1}{3} - \frac{1}{2}(z^2 + 1/z^2) \right]t^6 + \dots
 \tag{3.2}$$

which coincides with the large q expansion. The discontinuity of the first derivative of the free energy (latent heat) has also been computed exactly (Baxter 1973):

$$L = - \left. \frac{\partial}{\partial \beta} \right|_{\geq T_c} \ln Z + \left. \frac{\partial}{\partial \beta} \right|_{\leq T_c} \ln Z = \left(\frac{1}{u} \frac{du}{d\beta} + \frac{1}{v} \frac{dv}{d\beta} \right) \Big|_{uv=1} P
 \tag{3.3}$$

with

$$P = \prod_{n=1}^{\infty} \left(\frac{1 - t^{2n}}{1 + t^{2n}} \right)^2 = 1 + 4 \sum_{n=1}^{\infty} (-)^n \frac{t^{2(2n-1)}}{1 + t^{2(2n-1)}}.
 \tag{3.4}$$

On the other hand, considering the high-temperature and low-temperature versions of the large q expansion, and taking the difference of the corresponding first derivatives, one obtains:

$$\begin{aligned}
 L &= \frac{\partial}{\partial \beta} \left[\ln \Lambda\left(\frac{1}{u}, \frac{1}{v}\right) - \ln \Lambda(u, v) + \ln \left(\frac{1-tu}{u-t} \frac{1-tv}{v-t} \right) \right] \Big|_{uv=1} \\
 &= \left(\frac{1}{u} \frac{du}{d\beta} + \frac{1}{v} \frac{dv}{d\beta} \right) \Big|_{uv=1} (1 - 4t^2 + 4t^4 + O(t^8))
 \end{aligned}
 \tag{3.5}$$

which coincides with the exact solution up to the sixth order. One can thus explicitly verify on the anisotropic expansion, the following remarkable property: the discontinuity P of the free energy does not depend on the anisotropy, all along the critical curve. The agreement between the large q expansion and the exact solution (and its discontinuity) at the critical temperature suggests that the large q limit is a regular one, and legitimates the use of the large q expansion for testing exact solutions that can be proposed for the partition function of the Potts model.

For instance, the partition function satisfies functional equations (Jaekel and Maillard 1981a), associated with transformations S and I :

$$\begin{aligned}
 Z(u, v) &= Z(v, u), & (S) \\
 Z(u, v) Z\left(\frac{1}{t^2u}, \frac{t^2}{v}\right) &= \frac{1+t^2}{t^2} \frac{(1-u/t)(1-t^3u)}{(1-tu)^2} & (I).
 \end{aligned}
 \tag{3.6}$$

In particular, these functional equations allow one to recover the exact solution on the critical curve, into the form of an infinite product produced by the action of the group G generated by the transformations S and I . These functional equations remain valid for any temperature and possess general exact solutions which can be written in the form of infinite products on the group G . There exist, in particular, 'minimal' solutions which factorise in the following form:

$$Z(u, v) = f(u)f(v)g(uv) \quad (3.7)$$

(and which, incidentally, do agree with the exact critical results (3.1) and (3.4), where P can be interpreted as the log derivative of an infinite product). Unfortunately, any factorisation of this kind is in contradiction with the large q expansion. This can also be seen as follows: the group G looks much like an automorphic group for the partition function, which allows one to carry the partition function from one fundamental domain to another. The fixed points of the group are accumulation points of such fundamental domains and the iterations leading to these fixed points do not tend to a regular limit. The same situation should hold in higher dimensions.

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