

ON THE SYMMETRIES OF INTEGRABILITY

M. Bellon, J-M. Maillard, C. Viallet

Abstract.

We show that the Yang-Baxter equations for two dimensional models admit as a group of symmetry the infinite discrete group $A_2^{(1)}$. The existence of this symmetry explains the presence of a spectral parameter in the solutions of the equations. We show that similarly, for three-dimensional vertex models and the associated tetrahedron equations, there also exists an infinite discrete group of symmetry. Although generalizing naturally the previous one, it is a much bigger hyperbolic Coxeter group. We indicate how this symmetry can help to resolve the Yang-Baxter equations and their higher-dimensional generalizations and initiate the study of three-dimensional vertex models. These symmetries are naturally represented as birational projective transformations. They may preserve non trivial algebraic varieties, and lead to proper parametrizations of the models, be they integrable or not. We mention the relation existing between spin models and the Bose-Messner algebras of algebraic combinatorics. Our results also yield the generalization of the condition $q^n = 1$ so often mentioned in the theory of quantum groups, when no q parameter is available.

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1 Introduction

The results presented here appear in a series of papers by M. Bellon, J-M. Maillard and C-M. Viallet [1, 2, 3, 4, 5, 6].

The Yang-Baxter equations, which appeared twenty years ago¹, have acquired a predominant role in the theory of integrable two-dimensional models in statistical mechanics [12, 13] and field theory (quantum or classical). They have actually outpassed the borders of physics and have become fashionable in some parts of the mathematics literature. They in particular support the construction of quantum groups [14, 15].

The Yang-Baxter equations [13] and their higher dimensional generalizations are now considered as the defining relations of integrability. They are the “Deus ex machina” in a number of domains of Mathematics and Physics (Knot Theory [16], Quantum Inverse Scattering [17], S-Matrix Factorization, Exactly Solvable Models in Statistical Mechanics, Bethe Ansatz [18], Quantum Groups [19, 15], Chromatic Polynomials [20] and more awaited deformation theories). The appeal of these equations comes from their ability to *give global results from local ones*. For instance, they are a sufficient and, to some extent, necessary [21] condition for the commutation of families of transfer matrices of arbitrary size and even of corner transfer matrices. From the point of view of topology, one may understand these relations by considering them as the generators of a large set of *discrete deformations of the lattice*. This point of view underlies most studies in knot theory [16] and statistical mechanics (**Z**-invariance [22, 23]).

We want to analyze the Yang-Baxter equations and their higher dimensional generalizations [24, 25, 26, 27] without prejudice about what should be a solution, that is to say proceed by *necessary* conditions, and have as an input the form of the matrix of Boltzmann weights.

We will exhibit an infinite discrete group of transformations acting on the Yang-Baxter equations or their higher dimensional generalizations (tetrahedron, hyper-simplicial equations).

These transformations act as an automorphy group of various quantities of interest in Statistical Mechanics (partition function,...), and are of great help for calculations, even outside the domain of integrability (critical manifolds, phase diagram,...) [28].

We show here is that *they form a group of symmetries of the equations defining integrability*. They consequently appear as a group of automorphisms of the Yang-Baxter or tetrahedron equations. We will denote this group \mathcal{Aut} .

The existence of \mathcal{Aut} drastically constrains the varieties where solutions may be found. In the general case, it has *infinite* orbits and gives *severe constraints* on the algebraic varieties which parametrize the possible solutions (genus zero or one curves, algebraic varieties which are not of the general type [29]). In the non-generic case, when \mathcal{Aut} has finite order orbits, the algebraic varieties can be of general type, but *the very finiteness condition allows for their determination* [1].

In the framework of infinite group representations, it is crucial to recognize the *essential difference* between what these symmetry groups are for the Yang-Baxter equations and what they are for the higher dimensional tetrahedron and hyper-simplicial relations: the number of involutions generating our groups increases from 2 to 2^{d-1} when passing from

¹In fact, fifty years ago, Lars Onsager was totally aware of the key role played by the star-triangle relation in solving the two-dimensional Ising model, but he preferred to give an algebraic solution emphasizing Clifford algebras [7, 8, 9, 10, 11].

two-dimensional to d -dimensional models and the group jumps from the semi-direct product $\mathbf{Z} \times \mathbf{Z}_2$ to a much larger group, i.e., a *group with an exponential growth* with the length of the word.

The existence of $\mathcal{A}ut$ as a symmetry of the Yang-Baxter equations has the following consequence: we may say that solving the Yang-Baxter equation is equivalent to solving all its images by $\mathcal{A}ut$. These images *generically tend to proliferate*, simply because $\mathcal{A}ut$ is infinite. Considering that the equations form an *overdetermined set*, it is easy to believe that the total set of equations is “less overdetermined” when the orbits of $\mathcal{A}ut$ are of finite order. One can therefore imagine that the *best candidates* for the integrability varieties are *precisely* the ones where the symmetry group possesses *finite orbits*: the solutions of Au-Yang et al. [30, 31, 32] seem to confirm this point of view [33, 34].

A contrario, if one gets hold of an apparently isolated solution, the action of $\mathcal{A}ut$ will multiply it until building up, in experimentally not so rare cases, a continuous family of solutions from the original one. This is the solution to the so-called baxterization problem [4].

We first show that the simplest example of Yang-Baxter relation which is the star-triangle relation [13] has an *infinite discrete group* of symmetries generated by three involutions. These involutions are deeply linked with the so-called *inversion relations* [35, 36, 37, 38].

which two-dimensional model.

This analysis can be extended to the “generalized star-triangle relation” for Interaction aRound the Face models without any major difficulties [12, 39].

2 The star-triangle relations

2.1 The setting

We consider a spin model with nearest neighbour interactions on square lattice. The spins σ_i can take q values. The Boltzmann weight for an oriented bond $\langle ij \rangle$ will be denoted hereafter by $w(\sigma_i, \sigma_j)$. The weights $w(\sigma_i, \sigma_j)$ can be seen as the entries of a $q \times q$ matrix. In the following we will introduce a pictorial representation of the star-triangle relation. An arrow is associated to the oriented bond $\langle ij \rangle$. The arrow from i to j indicates that the argument of the Boltzmann weight w is (σ_i, σ_j) rather than (σ_j, σ_i) . This arrow is relevant only for the so-called chiral models [30], that is to say that the $q \times q$ matrix describing w is not symmetric. An interesting class of $q \times q$ matrices has been extensively investigated in the last few years [30, 32, 31]: the general cyclic matrices. It is important to note that we *do not* restrict ourselves to this particular class of matrices.

The finite algebras we are lead to consider contain in particular Bose-Messner algebras, i.e. structures occurring in graph theory, and more precisely algebraic combinatorics [40]. The detailed relationship will not be explained here. The reader could compare paragraph 5 of [1] and the definition of Bose-Messner algebra of association schemes in [41], page 44, and [42].

Let us give the following non cyclic nor symmetric 6×6 matrix as another illustrative

example:

$$\begin{pmatrix} x & y & z & y & z & z \\ z & x & y & z & y & z \\ y & z & x & z & z & y \\ y & z & z & x & z & y \\ z & y & z & y & x & z \\ z & z & y & z & y & x \end{pmatrix} \quad (1)$$

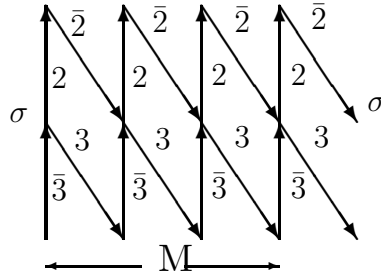
2.2 The relations

We introduce the star-triangle equations both analytically² and pictorially:

$$\sum_{\sigma} w_1(\sigma_1, \sigma) \cdot w_2(\sigma, \sigma_2) \cdot w_3(\sigma, \sigma_3) = \lambda \bar{w}_1(\sigma_2, \sigma_3) \cdot \bar{w}_2(\sigma_1, \sigma_3) \cdot \bar{w}_3(\sigma_1, \sigma_2). \quad (2)$$

$$(1) \begin{matrix} \nearrow \bar{3} & \text{---} & \bar{1} & \searrow 2 \\ & & & \end{matrix} \begin{matrix} (2) \\ (3) \end{matrix} = (1) \begin{matrix} \nearrow 2 & \text{---} & 1 & \searrow 3 \\ & & & \end{matrix} \begin{matrix} (2) \\ (3) \end{matrix} \quad (\text{st1.1})$$

One should note that satisfying equation (2) *together with* the relation (st1.2) obtained by reversing all arrows, is a *sufficient* condition for the commutation of the diagonal transfer matrices of *arbitrary size* M with periodic boundary conditions $\mathbf{T}_M(w_2, \bar{w}_2)$ and $\mathbf{T}_M(\bar{w}_3, w_3)$:



Note that for cyclic matrices ([30, 32, 31]) the star-triangle relations (st1.1) and (st1.2) give the same equations since one exchanges (st1.2) and (st1.1) by spin reversal.

One could obviously imagine many other choices for the arrows on the six bonds, however only three of them lead to the commutation of diagonal transfer matrices. We therefore have three systems of equations to study. For example, if the Boltzmann weights are given by the 6×6 matrix (1), these three systems of equations are respectively made of 20 different equations or 35 or 36.

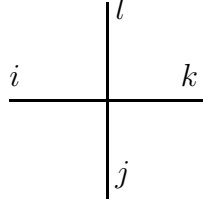
3 The Yang-Baxter relation for vertex models.

We shall not get here into the arcana of this relation, which appears in the theory of integrable models [15], the theory of factorizable S -matrix in two-dimensional field theory,

²Since the w_i and \bar{w}_i are homogeneous variables, there will always be a global multiplicative factor λ floating around in the star-triangle equations.

the quantum inverse scattering method [17], knot theory and has been given a canonical meaning in terms of Hopf algebras [43] (quantum groups [14, 15, 44, 45, 46]) and the list is far from exhaustive. We just want to fix some notations for later use.

We consider a vertex model on a two-dimensional square lattice. To each bond is associated a variable with q possible states and a Boltzmann weight $w(i, j, k, l)$ is assigned to each vertex



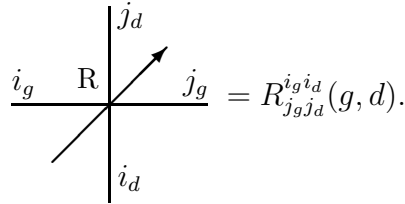
In order to write the Yang-Baxter relation, the q^4 homogeneous weights $w(i, j, k, l)$ are first arranged in a $q^2 \times q^2$ matrix R :

$$R_{kl}^{ij} = w(i, j, k, l). \quad (3)$$

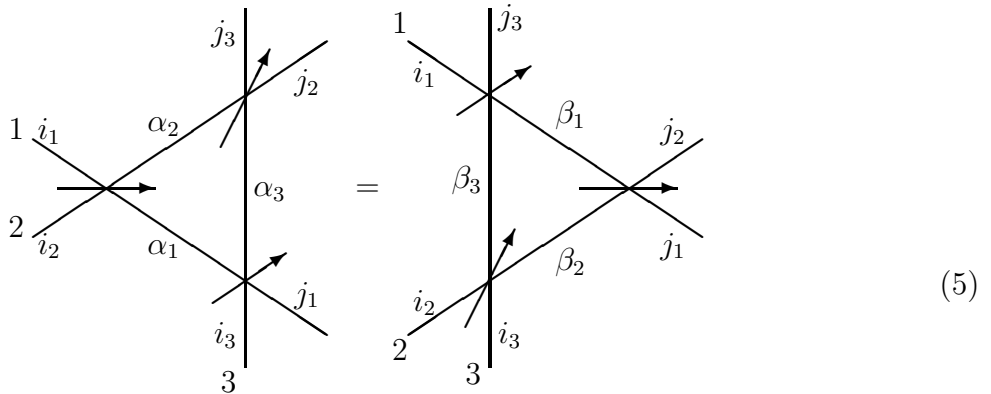
The Yang-Baxter relation is a trilinear relation between three matrices $R(1, 2)$, $R(2, 3)$ and $R(1, 3)$:

$$\sum_{\alpha_1, \alpha_2, \alpha_3} R_{\alpha_1 \alpha_2}^{i_1 i_2}(1, 2) R_{j_1 \alpha_3}^{\alpha_1 i_3}(1, 3) R_{j_2 j_3}^{\alpha_2 \alpha_3}(2, 3) = \sum_{\beta_1, \beta_2, \beta_3} R_{\beta_2 \beta_3}^{i_2 i_3}(2, 3) R_{\beta_1 j_3}^{i_1 \beta_3}(1, 3) R_{j_1 j_2}^{\beta_1 \beta_2}(1, 2). \quad (4)$$

The assignation (3) is arbitrary and we may specify it by complementing the vertex with an arrow and attributing numbers to the lines



With these rules relation (4) has the following graphical representation



The lines carry indices 1,2,3.

Some especially interesting solutions depend on a continuous parameter called the “spectral parameter”. The presence of this parameter is fundamental for many applications in physics, as for example the Bethe Ansatz method [47, 11, 17, 18]. One of the main issues in

the full resolution of (4) is precisely to describe what is this parameter and the algebraic variety on which it lives, although its presence may obscure the algebraic structures underlying the Yang-Baxter equation (*the discovery of quantum groups was allowed by forgetting this parameter* [46, 14, 48, 15]). The problem of building up continuous families of solutions from an isolated one, known as the *baxterization* [16], is made straightforward by our study. Indeed our results *explain the presence of the spectral parameter* in the solution of the equation (see also [1]).

4 The symmetry group of the star-triangle relation

4.1 The inversion relation

Two distinct inverses act on the matrix of nearest neighbour spin interactions: the matrix inverse I and the dyadic (element by element) inverse J . We write down the inversion relations both analytically and pictorially:

$$\sum_{\sigma} w(\sigma_i, \sigma) \cdot I(w)(\sigma, \sigma_j) = \mu \delta_{\sigma_i \sigma_j}, \quad (6)$$

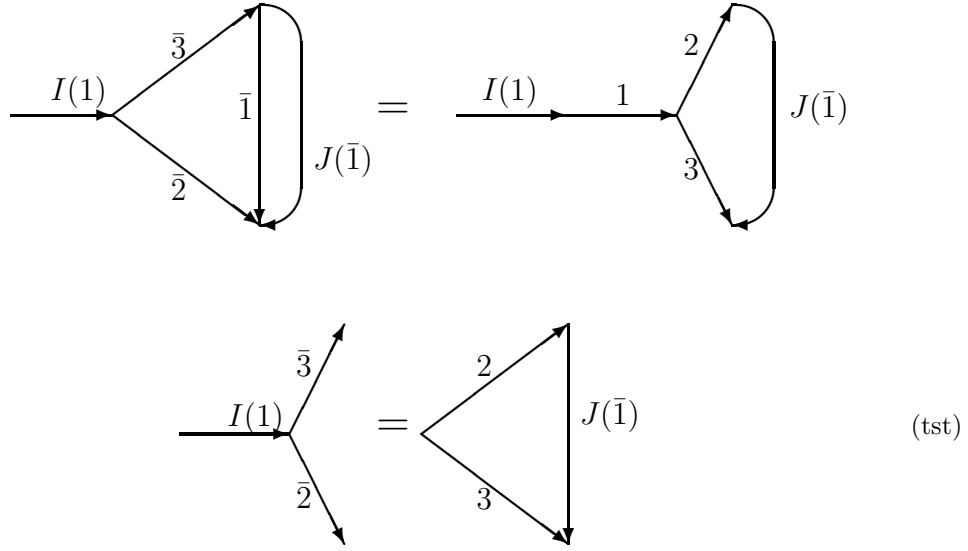
$$w(\sigma_i, \sigma_j) \cdot J(w)(\sigma_i, \sigma_j) = 1. \quad (7)$$

where $\delta_{\sigma_i \sigma_j}$ denotes the usual Kronecker delta.

The two involutions I and J generate an *infinite discrete group* Γ (Coxeter group) isomorphic to the infinite dihedral group $\mathbf{Z}_2 \times \mathbf{Z}$. The \mathbf{Z} part of Γ is generated by IJ . In the parameter space of the model, that is to say some projective space \mathbf{CP}_{n-1} (n homogeneous parameters), I and J are birational involutions. They give a *non-linear representation of this Coxeter group by an infinite set of birational transformations* [1]. It may happen that the action of Γ on specific subvarieties *yields a finite orbit*. This means that the representation of Γ identifies with the p -dihedral group $\mathbf{Z}_2 \times \mathbf{Z}_p$.

4.2 The symmetries of the star-triangle relations

The two inversions I and J act on the star-triangle relation. Let us give a pictorial representation of this action, starting from (st1.1) as an example:



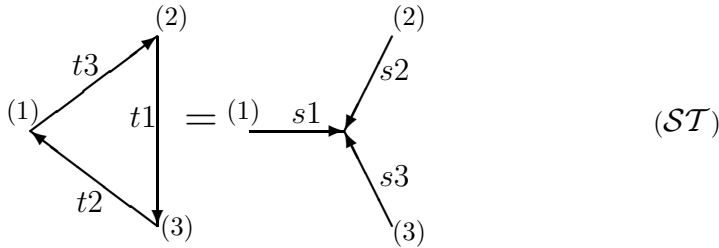
The transformed equation reads:

$$\lambda \sum_{\sigma_1} I(w_1)(\tau, \sigma_1) \cdot \bar{w}_2(\sigma_1, \sigma_3) \cdot \bar{w}_3(\sigma_1, \sigma_2) = w_2(\tau, \sigma_2) \cdot w_3(\tau, \sigma_3) \cdot J(\bar{w}_1)(\sigma_2, \sigma_3). \quad (8)$$

We get an action on the space of solutions of the star-triangle relation.

If $(w_1, w_2, w_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ is a solution of eq(2) (see picture (st1.1) for the specific arrangement of arrows), then $(I(w_1), \bar{w}_3, \bar{w}_2, J(\bar{w}_1), w_3, w_2)$ is also a solution of eq(2), at the price of a permitted redefinition of λ . In this transformation, the weights w_1 and \bar{w}_1 play a special role.

At this point, it is better to formalize this action by introducing some notations. We may choose as a reference star-triangle relation \mathcal{ST} , the symmetric configuration:



Any configuration may be obtained by reversing some arrows and permuting some bonds. With evident notations, we will denote by $R_{s1}, R_{s2}, R_{s3}, R_{t1}, R_{t2}, R_{t3}$ the reversals of arrows, and by $P_{s_i, s_j}, P_{s_i, t_j}, P_{t_i, t_j}$ the permutations of bonds. Moreover I and J act on the bonds as I_{s1}, I_{s2}, \dots . The action of I and J described above (where 1 was playing a special role) identifies with the action of

$$\mathcal{K}_1 = R_{s2} R_{t3} I_{s1} J_{t1} P_{s2, t3} P_{s3, t2}. \quad (9)$$

It is easy to check that \mathcal{K}_1 is an involution.

We may construct two similar involutions \mathcal{K}_2 and \mathcal{K}_3 , obtained by cyclic permutation of the indices 1, 2, 3. The involutions $\mathcal{K}_i (i = 1, 2, 3)$ verify the defining relations of the *Weyl*

group of an affine algebra of type $A_2^{(1)}$ [49]:

$$(\mathcal{K}_1\mathcal{K}_2)^3 = (\mathcal{K}_2\mathcal{K}_3)^3 = (\mathcal{K}_3\mathcal{K}_1)^3 = 1. \quad (10)$$

We denote \mathcal{Aut} the group generated by the three involutions \mathcal{K}_i ($i = 1, 2, 3$).

5 The symmetry group of the Yang-Baxter equation.

5.1 The inversion relations.

The R -matrix appears naturally as a representation of an element of the tensor product $\mathcal{A} \otimes \mathcal{A}$ of some algebra \mathcal{A} with itself. This algebra is a nice Hopf algebra in the context of quantum groups. We shall not dwell on this here but recall some simple operations on R .

In $\mathcal{A} \otimes \mathcal{A}$ we have a product inherited from the product in \mathcal{A} :

$$(a \otimes b)(c \otimes d) = ac \otimes bd. \quad (11)$$

R is an invertible element of $\mathcal{A} \otimes \mathcal{A}$ for this product and we shall denote by $I(R)$ the inverse for this product:

$$R \cdot I(R) = I(R) \cdot R = 1 \otimes 1. \quad (12)$$

In terms of the representative matrix this reads:

$$\sum_{\alpha, \beta} R_{\alpha\beta}^{ij} I(R)_{uv}^{\alpha\beta} = \delta_u^i \delta_v^j = \sum_{\alpha, \beta} I(R)_{\alpha\beta}^{ij} R_{uv}^{\alpha\beta}. \quad (13)$$

This is nothing else but the so-called *inversion relation* for *vertex* models [35, 36, 39, 50, 29]. On $\mathcal{A} \otimes \mathcal{A}$ we have a permutation operator σ :

$$\sigma(a \otimes b) = b \otimes a, \quad (14)$$

$$(\sigma R)_{uv}^{ij} = R_{vu}^{ji}, \quad \text{for the matrix } R. \quad (15)$$

Note that the representation of σ is just the conjugation by the permutation matrix P :

$$P_{kl}^{ij} = \delta_{il}\delta_{jk}, \quad (16)$$

$$\sigma R = PRP. \quad (17)$$

In the language of matrices we have a notion of transposition. Let us define partial transpositions t_g and t_d by:

$$(t_g R)_{uv}^{ij} = R_{iv}^{uj}, \quad (18)$$

$$(t_d R)_{uv}^{ij} = R_{uj}^{iv}, \quad (19)$$

and the full transposition

$$t = t_g t_d = t_d t_g. \quad (20)$$

We shall in the sequel use another inversion J defined by:

$$J = t_g I t_d = t_d I t_g, \quad (21)$$

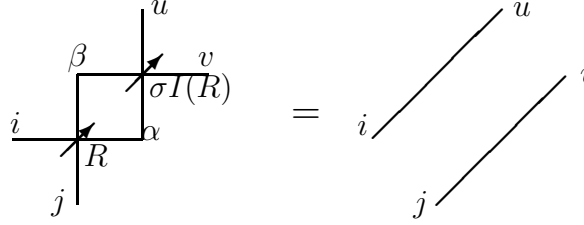
or equivalently:

$$\sum_{\alpha,\beta} R_{v\beta}^{\alpha u} J(R)_{j\beta}^{\alpha i} = \delta_u^i \delta_v^j = \sum_{\alpha,\beta} J(R)_{\alpha j}^{i\beta} R_{\alpha v}^{u\beta} \quad (22)$$

These operators verify straightforwardly:

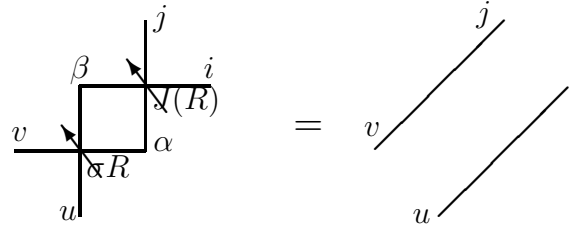
$$\begin{aligned} I^2 &= J^2 = 1, & It &= tI, & Jt &= tJ, \\ \sigma^2 &= t^2 = 1, & \sigma I &= I\sigma, & \sigma J &= J\sigma, \\ (\sigma t_g)^2 &= (\sigma t_d)^2 = t, & \sigma t_g \sigma t_d &= 1. \end{aligned} \quad (23)$$

Each of these operations has a graphical representation. For the inversion I or more precisely for σI it is:



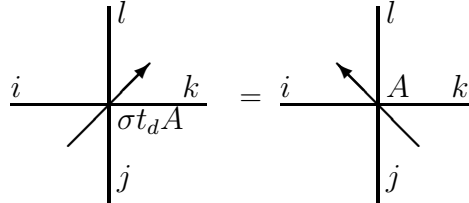
$$\quad (24)$$

the inversion J reads:



$$\quad (25)$$

and the transposition reads:



Note that the two inversions I and J do not commute. They generate an infinite discrete group Γ , the infinite dihedral group, isomorphic to the semi-direct product $\mathbf{Z} \times \mathbf{Z}_2$. This group is represented on the matrix elements by *birational transformations* [1, 51, 52] acting on the projective space of the entries of the matrix R . Remark that for the *vertex models*, the birational transformations associated to the two involutions I and J are naturally related by collineations (see (21): this should be compared with the situation for nearest neighbour interaction spin models [1, 53].

5.2 The symmetries of the Yang-Baxter equations.

At the price of the redefinitions:

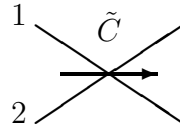
$$A = tR(2, 3), \quad (26)$$

$$B = \sigma t_d R(1, 3), \quad (27)$$

$$C = R(1, 2), \quad (28)$$

we may picture the Yang-Baxter relation in a more symmetric way:

(29)

We may bracket (29) with , where $\tilde{C} = \sigma I(C)$. We get

(30)

that is to say

(31)

This relation is nothing but (29) after the redefinitions

$$\begin{aligned}
 A &\rightarrow t_g A, \\
 B &\rightarrow t_d B, \\
 C &\rightarrow tI C.
 \end{aligned}
 \tag{32}$$

We may denote by K_3 the operation (32). We have two other similar operations K_1 and K_2

$$\begin{aligned}
 K_1 : \quad & A \rightarrow tI A & K_2 : \quad & A \rightarrow t_d A \\
 & B \rightarrow t_g B & & B \rightarrow tI B \\
 & C \rightarrow t_d C & & C \rightarrow t_g C
 \end{aligned}
 .$$

The discrete group \mathcal{Aut} generated by the K_i 's ($i = 1, 2, 3$) is a symmetry group of the Yang-Baxter equations. These generators K_i ($i = 1, 2, 3$) are involutions. The K_i 's satisfy the

relation $(K_1K_2K_3)^2 = 1$. Actually, the operation $K_1K_2K_3$ is just the inversion I on R . Among the elements of the discrete group generated by the K_i 's we have in particular:

$$(K_1K_2)^2 : A \rightarrow It_gIt_gA = tIJA, \quad (33)$$

$$B \rightarrow t_dIt_dIB = tJIB, \quad (34)$$

$$C \rightarrow C. \quad (35)$$

Since IJ is of infinite order, we have generated an *infinite discrete group* of symmetries. This is exactly the phenomenon that we described in section 4.2 for the star-triangle equations.

Under this form it is not so evident to find the actual structure of the group. Let us introduce K_A , K_B and K_C , which are simply related to the K_i 's by the transposition of two vertices:

$$\begin{array}{ccc} K_A : A \rightarrow \sigma tIA & K_B : A \rightarrow \sigma t_gC & K_C : A \rightarrow t_g\sigma B \\ B \rightarrow t_g\sigma C & B \rightarrow \sigma tIB & B \rightarrow \sigma t_gA \\ C \rightarrow \sigma t_gB & C \rightarrow t_g\sigma A & C \rightarrow \sigma tIC \end{array} .$$

It is easily verified that:

$$K_A^2 = K_B^2 = K_C^2 = 1, \quad (36)$$

and

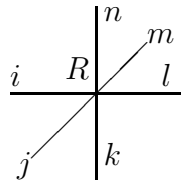
$$(K_AK_B)^3 = (K_BK_C)^3 = (K_CK_A)^3 = 1, \quad (37)$$

with no other relations. We recover the affine Coxeter group $A_2^{(1)}$ we already encountered in section 4.2.

We have here a very powerful instrument: it defines *adequate patterns* for the matrix R [54]. It permits the so-called *baxterization of an isolated solution* just acting with tIJ . Indeed if a set of relations among the entries of R are preserved by IJ (or at least by tIJ), they will stay for every transforms of the initial Yang-Baxter relation. We shall illustrate in section 7.1.1 the baxterization on the Baxter eight-vertex model [55, 22] and show in section 7.1.2 how to introduce a spectral parameter for the solutions of the Yang-Baxter equations associated to $sl(n)$ algebras.

6 The tetrahedron equations and their symmetries.

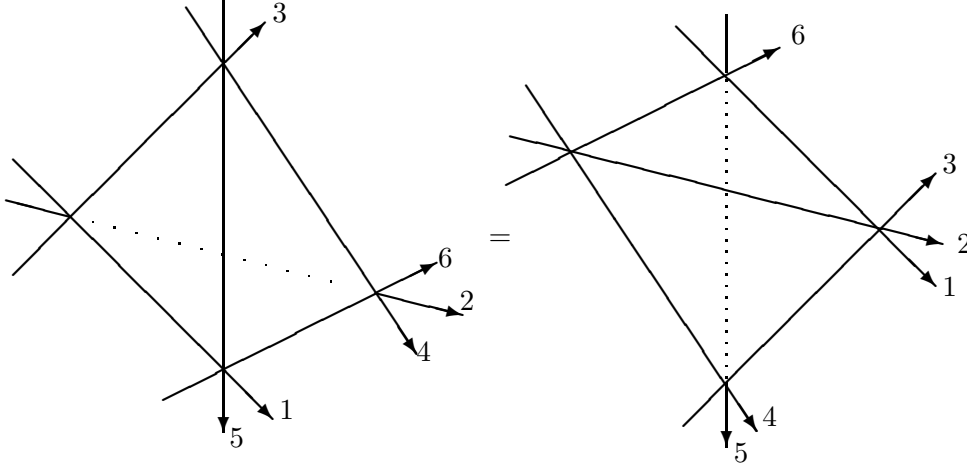
This equation is a generalization of the Yang-Baxter equation to three dimensional vertex models [25, 24, 27]. We give a pictorial representation of the three-dimensional vertex by



The Boltzmann weights of the vertex are denoted $w(i, j, k, l, m, n)$ and may be arranged in a matrix of entries

$$R_{lmn}^{ijk} = w(i, j, k, l, m, n). \quad (38)$$

The tetrahedron equation has a pictorial representation:



The algebraic form is

$$R_{123}R_{543}R_{516}R_{426} = R_{426}R_{516}R_{543}R_{123}. \quad (39)$$

We may here again introduce an inverse I

$$\sum_{\alpha_g, \alpha_m, \alpha_d} (IR)_{\alpha_g \alpha_m \alpha_d}^{i_g i_m i_d} \cdot R_{j_g j_m j_d}^{\alpha_g \alpha_m \alpha_d} = \delta_{j_g}^{i_g} \delta_{j_m}^{i_m} \delta_{j_d}^{i_d}. \quad (40)$$

We also introduce the partial transpositions t_g , t_m and t_d with

$$(t_g R)_{j_g j_m j_d}^{i_g i_m i_d} = R_{i_g j_m j_d}^{j_g i_m i_d}, \quad (41)$$

and similar definitions for t_m and t_d .

We redefine

$$A = R_{123}, \quad B = t_d R_{543}, \quad C = t_g t_m R_{516}, \quad D = t R_{426}, \quad (42)$$

where t is the full transposition $t_g t_m t_d$. Equation (39) then takes the more symmetric form

$$\sum_{s_1, \dots, s_6} A_{s_1 s_2 s_3}^{i_1 i_2 i_3} B_{s_5 s_4 s_3}^{i_5 i_4 j_3} C_{s_5 s_1 s_6}^{j_5 j_1 i_6} D_{s_4 s_2 s_6}^{j_4 j_2 j_6} = \sum_{r_1, \dots, r_6} D_{i_4 i_2 i_6}^{r_4 r_2 r_6} C_{i_5 i_1 j_6}^{r_5 r_1 r_6} B_{j_5 j_4 i_3}^{r_5 r_4 r_3} A_{j_1 j_2 j_3}^{r_1 r_2 r_3}. \quad (43)$$

We may multiply the previous equation by $(IA)_{i_1 i_2 i_3}^{u_1 u_2 u_3}$ and $(tIA)_{j_1 j_2 j_3}^{v_1 v_2 v_3}$ and sum over (i_1, i_2, i_3) and (j_1, j_2, j_3) . This amounts to a bracketing of the tetrahedron equations by two times the same vertex, in a procedure trivially generalizing the one for the Yang-Baxter equation (30). We recover (43) with A , B , C and D transformed by

$$\begin{aligned} K_1 : A &\rightarrow tIA \\ B &\rightarrow t_d B \\ C &\rightarrow t_m C \\ D &\rightarrow t_m D. \end{aligned} \quad (44)$$

We have in a similar way the operations

$$\begin{aligned} K_2 : A &\rightarrow t_d A & K_3 : A &\rightarrow t_g A & K_4 : A &\rightarrow t_m A \\ B &\rightarrow tIB & B &\rightarrow t_g B & B &\rightarrow t_m B \\ C &\rightarrow t_g C & C &\rightarrow tIC & C &\rightarrow t_d C \\ D &\rightarrow t_g D & D &\rightarrow t_d D & D &\rightarrow tID \end{aligned} .$$

Each of these four operations is an involution. They satisfy various relations, for instance $(K_1K_2K_3K_4)^2 = 1$. The K_i 's generate a group \mathcal{Aut}_3 which is a symmetry group of the tetrahedron equations. This group is "monstrous" since the number of elements of length smaller than l is of exponential growth with respect to l , unlike the case of the affine Coxeter groups (as $A_2^{(1)}$ for the Yang-Baxter equation) where this number is of polynomial growth.

The operations playing a role similar to the one of I and J in the two-dimensional Yang-Baxter equations are the *four* involutions

$$I, \quad J = t_g I t_m t_d, \quad K = t_m I t_d t_g, \quad L = t_d I t_g t_m. \quad (45)$$

We call Γ_3 the group generated by these four involutions. Γ_3 is also a symmetry group for the three dimensional vertex model *even if* [37] the model *does not* satisfy the tetrahedron equation.

In order to precise the algebraic structure of the group Γ_3 generated by I, J, K and L , it is simpler to consider as generators two of the partial transpositions t_g and t_d, I and the full transposition t . The third partial transposition can be recovered as the product $t t_g t_d$ and t commutes with all other generators and so contributes a mere \mathbf{Z}_2 factor in the group. We are thus considering the Coxeter group generated by three involutions t_g, t_d and I , with two of them commuting: this is represented by the following Dynkin diagram

$$\begin{array}{c} \infty \qquad \qquad \infty \\ \bullet \text{---} \bullet \text{---} \bullet \\ t_g \qquad \qquad I \qquad \qquad t_d \end{array}$$

For this group again, the number of elements of length smaller than l is greater than $2^{l/2}$. This is in fact a *hyperbolic* Coxeter group [56].

7 Use of the symmetry group.

7.1 The baxterization

The problem of the baxterization is to introduce a spectral parameter into an isolated solution of the Yang-Baxter equations [16]. We have solutions of this problem by acting with the symmetry group Γ .

7.1.1 Baxterization of the Baxter model

Consider the matrix of the symmetric eight vertex model

$$R = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}. \quad (46)$$

Notice that this form is preserved by I and J and that $tR = R$. The action of I is

$$a \rightarrow \frac{a}{a^2 - d^2} \quad b \rightarrow \frac{b}{b^2 - c^2} \quad (47)$$

$$c \rightarrow \frac{-c}{b^2 - c^2} \quad d \rightarrow \frac{-d}{a^2 - d^2} \quad (48)$$

and the action of J is

$$a \rightarrow \frac{a}{a^2 - c^2} \quad b \rightarrow \frac{b}{b^2 - d^2} \quad (49)$$

$$c \rightarrow \frac{-c}{a^2 - c^2} \quad d \rightarrow \frac{-d}{b^2 - d^2} \quad (50)$$

We shall look at the solutions of the Yang-Baxter equations for matrices R of the form (46). The leading idea is that the parametrization of the solutions is just the parametrization of the algebraic varieties preserved by tIJ in the projective space \mathbf{CP}_3 of the homogenous parameters (a, b, c, d) . The remarkable fact is that not only these varieties exist but can be completely described. We use the visualization method we have already used [1, 2] for spin models, that is to say just draw the orbits obtained by numerical iteration and look.

This is best illustrated by figure 1. This figure shows the orbit of point (*), which is a matrix of the form (46). It is drawn by the iteration of IJ acting on the initial point (*). The resulting points densify on the elliptic curve given by the intersection of the two quadrics $\Delta_1 = \text{constant}$ and $\Delta_2 = \text{constant}$ (Clebsch's biquadratic), with Δ_1 and Δ_2 the Γ invariants

$$\begin{aligned} \Delta_1 &= \frac{a^2 + b^2 - c^2 - d^2}{ab + cd}, \\ \Delta_2 &= \frac{ab - cd}{ab + cd}. \end{aligned} \quad (51)$$

Similar calculations can be performed for a general 16-vertex model for which:

$$R = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix} \quad (52)$$

Amazingly the baxterization of the 16-vertex model leads to *curves*. These curves are also *intersection of quadrics* (even in the general case for which there is no solution for the Yang-Baxter equations), and lead to a remarkable elliptic parametrization of the model [6].

7.1.2 Baxterization of the R matrix of $sl_q(n)$

Another example corresponds to the baxterization of solutions associated to $sl(n)$ algebras [19]. There are special solutions generally denoted R_+ and R_- . For the simplest four-dimensional representation of the $sl(2)$ case, we have

$$R_+ = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \quad (53)$$

and a similar expression for R_- [19]. Looking for a family containing both R_+ and R_- our baxterization procedure leads to the well-known [11] six-vertex model R -matrix $R = \lambda R_+ + 1/\lambda R_-$.

We let as an exercise for the reader to treat the $sl(3)$ case. In a forthcoming publication we will show that these ideas can be generalized to all the universal R -matrices [15] for every representation [57]. This group appears in field theory, in the analysis of classical R -matrices [58].

7.1.3 q root of unity

One of the most studied cases of quantum group is obtained when the parameter q is a root of unity. The structure of the representation theory is then extremely rich and differs from the generic one.

Proposition: $q^n = 1$ is equivalent to: the orbit of Γ is finite.

This applies *even if the parameter q is not defined* (e.g. in the elliptic case). See [57].

7.2 Three dimensional models

Our strategy for finding solutions of the tetrahedron equations is to seek for patterns of the Boltzmann weights of the three dimensional vertex *compatible with the symmetry group* Γ_3 . By this we mean that its form should be preserved by Γ_3 .

7.2.1 A first model

We will therefore consider a simple model where i, j, k, l, m and n take only two values $+1$ and -1 . The matrix (38) is an 8×8 matrix. We will require that its pattern is invariant under the inverse I [54] and the various partial transpositions t_g, t_m and t_d . We aim at having a generalization of the Baxter eight-vertex model and we impose the following restrictions:

$$w(i, j, k, l, m, n) = w(-i, -j, -k, -l, -m, -n), \quad (54)$$

$$w(i, j, k, l, m, n) = 0 \quad \text{if } ijklmn = -1. \quad (55)$$

These constraints amount to saying that the 8×8 matrix splits into two times the same 4×4 matrix. It is further possible to impose that this matrix is symmetric since, in this case, $t_g R$ (and any other partial transpose) is also symmetric. Let us introduce the following notations for the entries of the 4×4 block of the R matrix

$$\begin{pmatrix} a & d_1 & d_2 & d_3 \\ d_1 & b_1 & c_3 & c_2 \\ d_2 & c_3 & b_2 & c_1 \\ d_3 & c_2 & c_1 & b_3 \end{pmatrix}. \quad (56)$$

The four rows and columns of this matrix correspond to the states $(+, +, +)$, $(+, -, -)$, $(-, +, -)$ and $(-, -, +)$ for the triplets (i, j, k) or (l, m, n) . The R -matrix can be completed by spin reversal, according to the rule (54). t_g simply exchanges c_2 with d_2 and c_3 with d_3 , t_m and t_d can be similarly defined and I acts as the inversion of this 4×4 matrix.

For this three dimensional model, the coefficients of the characteristic polynomial of the 4×4 matrix (56) give a good hint for invariants under Γ_3 . They are

$$\sigma_1^{(3d)} = a + b_1 + b_2 + b_3, \quad (57)$$

$$\sigma_2^{(3d)} = a(b_1 + b_2 + b_3) + b_1 b_2 + b_2 b_3 + b_3 b_1 - (c_1^2 + c_2^2 + c_3^2 + d_1^2 + d_2^2 + d_3^2), \quad (58)$$

....

Since $\sigma_2^{(3d)}$ is invariant by t_g, t_m and t_d and takes a simple factor (the inverse of the determinant) under the action of I , the variety $\sigma_2^{(3d)} = 0$ is invariant under Γ_3 . Given the hugeness of the group Γ_3 , it is already an astonishing fact to have such a covariant expression. In fact

we can exhibit *five linearly independent polynomials* with the same covariance, which give *four invariants*, as follows:

$$ab_1 + b_2b_3 - c_1^2 - d_1^2, \quad c_2d_2 - c_3d_3, \quad (59)$$

and the ones deduced by permutations of 1, 2 and 3. They form a five dimensional space of polynomials. Any ratio of the five independant polynomials is invariant under all the four generating involutions. In other words \mathbf{CP}_9 is foliated by five dimensional algebraic varieties invariant under Γ_3 .

To have some flavour of the possible (integrable ?) algebraic varieties invariant under Γ_3 , we study its orbits [1, 2]. We start with the study of the subgroup generated by some infinite order element namely IJ . This element gives a special role to axis 1. The transformation IJ *does preserve the symmetry under the exchange of 2 and 3*. If the initial point is symmetric under the exchange of 2 and 3, the orbit under IJ is thus a *curve*. Other starting points lead to orbits lying on a *two dimensional* variety given by the intersection of seven quadrics (see figure 2,3,4). However, what we are interested in are the orbits of the *whole* Γ_3 group. The size of this group prevent us from studying exhaustively the full set of group elements of a given length even for quite small values of this length. We have nevertheless explored the group by a random construction of typical elements of increasingly large length [4]. This confirms that we generically only have the four invariants described previously.

7.2.2 A second model

We also consider a simple model where i, j, k, l, m and n take only two values $+1$ and -1 and which is also a generalization of the Baxter eight vertex model. The Boltzmann weights $w(i, j, k, l, m, n)$ are given by:

$$w(i, j, k, l, m, n) = f(i, j, k) \delta_l^i \delta_m^j \delta_n^k + g(i, j, k) \delta_{-l}^i \delta_{-m}^j \delta_{-n}^k \quad (60)$$

$$f(i, j, k) = f(-i, -j, -k) \text{ and } g(i, j, k) = g(-i, -j, -k) \quad (61)$$

Equations (61) are symmetry conditions reducing the numbers of homogeneous parameters from 16 to 8.

As for the previous model, there exists an invariant of the action of the whole group Γ_3 :

$$\frac{f(+, +, +)f(+, -, -)f(-, +, -)f(-, -, +)}{g(+, +, +)g(+, -, -)g(-, +, -)g(-, -, +)} \quad (62)$$

Considering the subgroup of Γ_3 generated by the infinite order element IJ , one can easily find other invariants, namely

$$\frac{f(+, +, +)f(+, -, -)}{g(+, +, +)g(+, -, -)} \quad (63)$$

and

$$\frac{f(+, +, +)^2 + f(+, -, -)^2 - g(+, +, +)^2 - g(+, -, -)^2}{g(+, +, +)g(+, -, -)} \quad (64)$$

For this model [5], the trajectories under IJ are *curves* in \mathbf{CP}_7 .

8 Conclusion

An important problem in statistical mechanics and field theory, is the understanding of the role of the dimension of the lattice on both the algebraic aspects and the topological aspects. All this touches various fields of mathematics and physics: algebraic geometry, algebraic topology, quantum algebra. Indeed the Coxeter groups we use are at the same time groups of automorphisms of algebraic varieties, symmetries of quantum Yang-Baxter equations (and their higher dimensional avatars). They also provide an *extension to several complex variables functions of the notion of the fundamental group Π_1* of a Riemann surface, with of course a much more involved covering structure [37, 2].

We believe moreover that the *space of parameters* seen as a projective space is the appropriate place to look at, if one wants to substantiate the deep topological notion embodied in the notion of \mathbf{Z} -invariance [22] and free the models from the details of the lattice shape.

Actually, we have exhibited an infinite discrete symmetry group for the Yang-Baxter equations and their higher dimensional generalization acting on this parameter space. This group is the Coxeter group $A_2^{(1)}$ (semi-direct product of $\mathbf{Z} \times \mathbf{Z}$ by some finite group). We have shown that this symmetry is responsible for the presence of the spectral parameter. In other words, the *discrete* symmetry gives rise to a *continuous* one (see [3]). A similar study for the generalized star-triangle relation of the Interaction aRound a Face model, sketched in [39], can be performed rigorously along the same lines, leading to the same result. Also note that the same groups generated by involutions appear in the study of semi-classical

r -matrices [58]. An interesting point will be to exhibit the action of our symmetry group on the underlying quantum group for the Yang-Baxter equations [57].

Our symmetry group is a group of automorphisms of the integrability varieties. This should give precious informations on these varieties. In particular one should decide if, up to Lie groups factors (which cannot be excluded because of the existence of “gauge” symmetries, weak graph duality [59], . . .), these varieties can be anything else than abelian varieties, or even product of curves: can they be for example K_3 surfaces, are they homological obstructions to the occurrence of anything but curves ?

For three-dimensional vertex models, the symmetry group, though generalizing very naturally the previous group (generated by four involutions with similar relations) is drastically different: it is so “large”³ that the chances are quite small that it leaves enough room for any invariant integrability varieties. It is not useless to recall the unique non-trivial known solution of the tetrahedron equations (Zamolodchikov’s solution) [25, 24, 27]. For this model the three axes are not on the same footing, so that we do not have a “true” three dimensional symmetry for the model (two-dimensional checkerboard models coupled together). Is there still any hope for a three-dimensional exactly solvable model with *genuine three-dimensional symmetry*? We think that the group of symmetries we have described gives the best line of attack to this problem. We will show that Γ_3 and even more $\mathcal{A}ut_3$ are generically too “large” to allow any non-trivial solution of the tetrahedron equations with genuine three dimensional symmetry [61].

References

- [1] M.P. Bellon, J-M. Maillard, and C-M. Viallet, *Integrable Coxeter Groups*. Physics Letters **A 159** (1991), pp. 221–232.
- [2] M.P. Bellon, J-M. Maillard, and C-M. Viallet, *Higher dimensional mappings*. Physics Letters **A 159** (1991), pp. 233–244.
- [3] M.P. Bellon, J-M. Maillard, and C-M. Viallet, *Infinite Discrete Symmetry Group for the Yang-Baxter Equations: Spin models*. Physics Letters **A 157** (1991), pp. 343–353.
- [4] M.P. Bellon, J-M. Maillard, and C-M. Viallet, *Infinite Discrete Symmetry Group for the Yang-Baxter Equations: Vertex Models*. Phys. Lett. B **260** (1991), pp. 87–100.
- [5] M.P. Bellon, J-M. Maillard, and C-M. Viallet, *Rational Mappings, Arborescent Iterations, and the Symmetries of Integrability*. Physical Review Letters **67** (1991), pp. 1373–1376.
- [6] M.P. Bellon, J-M. Maillard, and C-M. Viallet. *Quasi integrability of the sixteen-vertex model*. Preprint LPTHE 91 – 56, (1991).
- [7] L. Onsager. The Ising model in two dimensions. In R.E. Mills, E. Ascher, and R.I. Jaffee, editors, *Critical Phenomena in Alloys, Magnets and Superconductors*, New York, (1971). MacGraw-Hill.

³One should keep in mind that very “large” sets of rational transformations may preserve algebraic curve of genus zero or one. Just think of the transformations on the circle generated by $\{\theta \rightarrow \theta + \lambda, \theta \rightarrow 2\theta, \theta \rightarrow 3\theta\}$ [60]

- [8] L. Onsager. Phys. Rev. **65** (1944), pp. 117–149.
- [9] M.J. Stephen and L. Mittag. J. Math. Phys. **13** (1972), pp. 1944–1951.
- [10] R.J. Baxter and Enting, 399th *Solution of the Ising Model*. J. Phys. **A11** (1978), pp. 2463–2473.
- [11] P.W. Kasteleyn. Exactly solvable lattice models. In *Proc. of the 1974 Wageningen Summer School: Fundamental problems in statistical mechanics III*, Amsterdam, (1974). North–Holland.
- [12] R. J. Baxter. In *Proc. of the 1980 Enschede Summer School: Fundamental problems in statistical mechanics V*, Amsterdam, (1981). North–Holland.
- [13] R.J. Baxter. *Exactly solved models in statistical mechanics*. London Acad. Press, (1981).
- [14] V.G. Drinfel’d. Quantum groups. In *Proceedings of the International Congress of Mathematicians*. Berkeley, (1986).
- [15] O. Babelon and C-M. Viallet. *Integrable Models, Yang-Baxter Equation, and Quantum Groups, Part I*. preprint SISSA 54 EP, (1989).
- [16] V.F.R Jones. *Baxterization*. preprint CMA-R23-89, (1989).
- [17] L.D. Faddeev. Integrable models in 1 + 1 dimensional quantum field theory. In *Les Houches Lectures (1982)*, Amsterdam, (1984). Elsevier.
- [18] M. Gaudin. *La fonction d’onde de Bethe*. Collection du C.E.A. Série Scientifique. Paris, (1983).
- [19] L.D. Faddeev. Quantum groups. In *Tel-Aviv Landau conference*, (1990).
- [20] J. M. Maillard and R. Rammal, *Some analytical consequences of the inverse relation for the Potts model*. J. Phys. **A16** (1983), p. 353.
- [21] P. Lochak and J. M. Maillard, *Necessary versus sufficient conditions for exact solubility*. Journ. Math. Phys. **27** (1986), p. 593.
- [22] R.J. Baxter, *Solvable eight-vertex model on an arbitrary planar lattice*. Phil. Trans. R. Soc. London **289** (1978), p. 315.
- [23] H. Au-Yang and J.H.H Perk, *Critical correlations in a \mathbf{Z} invariant inhomogeneous Ising model*. Physica **A144** (1987), pp. 44–104.
- [24] R.J. Baxter, *On Zamolodchikov’s Solution of the Tetrahedron Equations*. Comm. Math. Phys. **88** (1983), pp. 185–205.
- [25] A.B. Zamolodchikov, *Tetrahedron equations and the relativistic S-matrix of straight-strings in 2+1 dimensions*. Comm. Math. Phys **79** (1981), p. 489.
- [26] M. T. Jaekel and J. M. Maillard, *Symmetry relations in exactly soluble models*. J. Phys. **A15** (1982), pp. 1309–1325.

- [27] R.J. Baxter, *The Yang-Baxter equations and the Zamolodchikov model*. Physica **D18** (1986), pp. 321–347.
- [28] D. Hansel, J. M. Maillard, J. Oitmaa, and M. Velgakis, *Analytical properties of the anisotropic cubic Ising model*. J. Stat. Phys. **48** (1987), p. 69.
- [29] J. M. Maillard, *Automorphism of algebraic varieties and Yang–Baxter equations*. Journ. Math. Phys. **27** (1986), p. 2776.
- [30] H. Au-Yang, B.M. Mc Coy, J.H.H. Perk, S. Tang, and M.L. Yan, *Commuting transfer matrices in the chiral Potts models: solutions of the star–triangle equations with genus ≥ 1* . Phys. Lett. **A123** (1987), p. 219.
- [31] B.M. Mc Coy, J.H.H. Perk, S. Tang, and C-H. Sah, *Commuting transfer matrices for the four–state self–dual chiral Potts model with genus three uniformizing Fermat curve*. Phys. Lett. **A125** (1987), p. 9.
- [32] R.J. Baxter, J.H.H. Perk, and H. Au-Yang, *New solutions of the star–triangle relations for the chiral Potts model*. Phys. Lett. **A128** (1988), p. 138.
- [33] D. Hansel and J. M. Maillard, *Symmetries of models with genus > 1* . Phys.Lett. **A133** (1988), p. 11.
- [34] J. Avan, J-M. Maillard, M. Talon, and C-M. Viallet, *Algebraic varieties for the chiral Potts model*. Int. Journ. Mod Phys. **B4** (1990), pp. 1743–1762.
- [35] Y.G. Stroganov. Phys. Lett. **A74** (1979), p. 116.
- [36] R.J. Baxter, *The Inversion Relation Method for Some Two-dimensional Exactly Solved Models in Lattice Statistics*. J. Stat. Phys. **28** (1982), pp. 1–41.
- [37] M. T. Jaekel and J. M. Maillard, *Inverse functional relations on the Potts model*. J. Phys. **A15** (1982), p. 2241.
- [38] J. M. Maillard, *The inversion relation : some simple examples (I) and (II)*. Journal de Physique **46** (1984), p. 329.
- [39] J. M. Maillard and T. Garel, *Towards an exhaustive classification of the star triangle relation (I) and (II)*. J. Phys. **A17** (1984), p. 1251.
- [40] E. Baunai and T. Ito. *Algebraic Combinatorics I: Association Schemes*. Mathematics Lecture Notes. Benjamin / Cummings, (1984).
- [41] A.E. Brouwer, A.M. Cohen, and A. Neumaier. *Distance-Regular Graphs*, volume 18 of *Series of Modern Surveys in Mathematics*. Springer Verlag, (1989).
- [42] F. Jaeger. *Strongly regular graphs and spin models for the Kauffman*. Preprint IMAG, Grenoble, (1991).
- [43] E. Abe. *Hopf Algebras*. Cambridge University Press, Cambridge, (1977).

- [44] O. Babelon. Integrable systems associated to the lattice version of the Virasoro algebra. I: the classical open chain. In *Integrable systems and quantum groups*, March 1990. Pavia.
- [45] J.L. Verdier, *Groupes quantiques [d'après V.G. Drinfel'd]*. Séminaire Bourbaki, 39ème année **685** (1987), pp. 1–15.
- [46] M. Jimbo, *A q -difference analogue of $U(\mathcal{G})$ and the Yang–Baxter equation*. Lett. Math. Phys. **10** (1985), p. 63.
- [47] E.H. Lieb and F.Y. Wu. Two dimensional ferroelectric models. In C. Domb and M.S. Green, editors, *Phase Transitions and Critical Phenomena*, volume 1, pages 331–490, New York, (1972). Academic Press.
- [48] S.L. Woronowicz, *Twisted $SU(2)$ Group. An Example of Noncommutative Differential Calculus*. Publ. Res. Inst. Math. Sci. **23** (1987), pp. 117–181.
- [49] V. G. Kac. *Infinite dimensional Lie algebras*. Cambridge University Press, second edition, (1985).
- [50] J. M. Maillard. The star–triangle relation and the inversion relation in statistical mechanics. In *Brasov International Summer School on Critical Phenomena, Theoretical Aspects*, (1983).
- [51] M.P. Bellon, J-M. Maillard, and C-M. Viallet. *Higher Dimensional Mappings with Exact Properties*. in preparation.
- [52] M.P. Bellon, J-M. Maillard, and C-M. Viallet. *Mappings for the \mathbf{Z}_N chiral Potts Model*. in preparation.
- [53] I.R. Shafarevich. *Basic algebraic geometry*. Springer study. Springer, Berlin, (1977). page 216.
- [54] M.P. Bellon, J-M. Maillard, and C-M. Viallet. *Matrix Patterns for Integrability*. in preparation.
- [55] R.J. Baxter, *Partition Function of the Eight-Vertex Lattice Model*. Ann. Phys. **70** (1972), p. 193.
- [56] J.E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, Cambridge, (1990).
- [57] M.P. Bellon, J-M. Maillard, and C-M. Viallet. *Beyond q -analogs*. in preparation.
- [58] J. Avan. *Classical R -matrices with general automorphism groups*. Preprint Brown HET 781, (1991).
- [59] A. Gaaf and J. Hijmans, *Symmetry relations in the sixteen-vertex model*. Physica **A80** (1975), pp. 149–171.
- [60] M.P. Bellon, J-M. Maillard, and C-M. Viallet. *Mappings In Higher Dimensions*. Preprint PAR-LPTHE-91-31.

- [61] M.P. Bellon, J-M. Maillard, and C-M. Viallet. *A No-Go Theorem for the Tetrahedron Equations.* in preparation.