Progress in Mathematics

Integrable Systems
The Verdier Memorial Conference
Actes du Colloque International de Luminy

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Springer Science+Business Media, LLC

Jean-Louis Verdier (1935–1989)
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Preface

This book constitutes the proceedings of the International Conference on Integrable Systems in memory of J.-L. Verdier. It was held on July 1–5, 1991 at the Centre International de Recherches Mathématiques (C.I.R.M.) at Luminy, near Marseille (France). This collection of articles, covering many aspects of the theory of integrable Hamiltonian systems, both finite- and infinite-dimensional, with an emphasis on the algebro-geometric methods, is published here as a tribute to Verdier who had planned this conference before his death in 1989 and whose active involvement with this topic brought integrable systems to the fore as a subject for active research in France.

The death of Verdier and his wife on August 25, 1989, in a car accident near their country house, was a shock to all of us who were acquainted with them, and was very deeply felt in the mathematics community. We knew of no better way to honor Verdier's memory than to proceed with both the School on Integrable Systems at the C.I.M.P.A. (Centre International de Mathématiques Pures et Appliquées in Nice), and the Conference on the same theme that was to follow it, as he himself had planned them. D. Bennequin, P. Cartier and A. Chenciner agreed to join O. Babelon and Y. Kosmann-Schwarzbach to form a new organizing committee, chaired by P. Cartier. The final list of speakers at the Conference was very close to the original list of invitations discussed with Verdier himself, and the invited participants included ten students chosen from among those who had attended the C.I.M.P.A. School, as originally planned by Verdier.

The refereed articles in this volume represent the advances in the field of complete integrability that were reported at the Luminy conference. In many cases, articles have been updated for publication. In two instances, where the results had been previously published, only summaries with references appear here. The articles represent very diverse methods and report very diverse results. This is a reflection of the complexity and richness of the field of research that is referred to as the theory of completely integrable systems.

In his preliminary text, D. Bennequin takes us into "the garden of integrable systems." He surveys the evolution of the subject, from Abel onwards, explaining the connections with the classical theory of elliptic functions, describing how algebraic curves and the infinite Grassmannian came to play a prominent role in the theory. He then analyzes the important contributions that Verdier made in this area.

The first part of this book contains articles that make essential use of Riemann surfaces and their theta functions in order to construct classes of solutions of integrable systems, and articles dealing with the tau-functions that generalize the classical theta functions. The first three papers in this part exemplify the algebro-geometric methods, while the next four deal more specifically with the tau-functions in their various guises.

A. Treibich, who was a close collaborator of Verdier, studies the family of elliptic solitons of the Kadomtsev–Petviashvili hierarchy associated with a projective curve, showing that if Γ is a tangential cover of an elliptic curve E, then its compactified Jacobian covers a symmetric power of E.

In an article written with B. van Geemen, E. Previato, who also collaborated with Verdier, describes recent work on the space of higher-order nonabelian theta functions over a Riemann surface of genus at least two, whose dimension is given by the Verlinde numbers, arising in the fusion rules of conformal field theory, and some connections of nonabelian theta functions with the Schottky problem.

The lecture of H. Knörrer, describing his work with N. Ercolani and E. Trubowitz, deals with the immersed submanifolds of $\mathbb{R}^3$ with constant mean curvature. The link with integrable systems lies in the fact that such immersions can be found by solving the elliptic sinh–Gordon equation, quasi-periodic solutions of which can be constructed by means of the Riemann theta function of hyperelliptic curves.

In the following articles the tau-functions of various integrable systems play a prominent part.

L. Takhtajan's paper shows that classical modular forms generate a tau-function for several integrable reductions of the self-dual Yang–Mills equations, and therefore general classes of solutions of many equations, in $0+1$, $1+1$, and $2+1$ dimensions.

G. Wilson reviews the generalized Ablowitz–Kaup–Newell–Segur equations associated with a simple Lie algebra, $\mathfrak{g}$, namely the evolution equations equivalent to the zero-curvature equation for a connection obtained from a "bare" connection by the dressing action of the associated loop group. He shows that, when $\mathfrak{g}$ is simply laced, solutions of the $\mathfrak{g}$AKNS equations can be obtained in terms of tau-functions, which he defines by means of the canonical trivialization of the fibration of the central extension of the loop group over the "big cell" in the loop group.

By redefining the tau-function of Segal and Wilson, L. Dickey is able to determine a tau-function for the hierarchies generated by matrix first-order differential operators which generalize the AKNS equations and for the multi-component KP hierarchies.

P. van Moerbeke studies the blowing-up of a solution of the Korteweg–de Vries equation with respect to a complexified time variable, and, more generally, the compactification of isospectral manifolds of differential operators. He then poses and answers analagous questions for the isospectral families of periodic Jacobi matrices.

In the second part of the book, the main emphasis is on the Hamiltonian formalism, and for the last two papers in this part, on the bihamiltonian formalism, while the first one is an illustration of the interplay of the Hamiltonian and the algebro-geometric methods in the study of integrability.

The joint work of N. M. Ercolani, H. Flaschka and S. Singer was presented in Flaschka's lecture on the geometry of the Kostant–Toda lattice, whose Lax matrix has entries below, on and just above the diagonal, the latter being all equal to 1. Using ideas from complex algebraic geometry, they prove the complete integrability of this system, and they determine its constants of motion, which are rational (rather than polynomial) functions on the phase-space.

Studying the vertex set of a Hamiltonian action of a compact commutative Lie group on a compact symplectic manifold, i.e., the image under the moment map of the set of fixed points, V. Guillemin proves a local rigidity theorem for Hamiltonian actions.

In his short contribution, A.T. Fomenko states a conjecture regarding the determination of all the integrable geodesic flows on two-dimensional compact manifolds.

P.J. Olver uses his classification of bihamiltonian structures based on the double Darboux theorem of Turiel in order to draw a list of canonical forms for bihamiltonian systems, and he obtains criteria for both the local and global integrability of such systems.

In the lecture of F. Magri, written in collaboration with P. Casati and M. Pedroni, the theory of soliton equations is presented from the Hamiltonian point of view, the hierarchies of bihamiltonian equations being generated by the Casimir functions of a pencil of Poisson brackets. It is shown that Sato's operator corresponds to the differential of a Casimir function expressed in terms of pseudodifferential operators, and Sato's equations to the vanishing of Poisson brackets on one-forms.

The third part of the book contains two papers that deal with the theory of two-dimensional solvable lattice models in which the quantum Yang–Baxter equation plays a fundamental role.

E. Date's contribution is a summary of his work on the finite-dimensional cyclic representations of the quantum groups at roots of unity and their relation with the two-dimensional lattice models associated with higher genus algebraic curves.

In his lecture, J.-M. Maillard presented his joint work with M. Bellon and C. M. Viallet, showing that the quantum Yang–Baxter equation admits a symmetry group which acts by birational projective transformations on the algebraic varieties which parametrize the solutions of the equation, and he discussed the generalization of these results to the tetrahedron and hyper-simplicial equations that are the analogues of the QYBE for lattices of dimension 3 or more.

In the concluding article, B. Dubrovin shows the interrelations of many aspects of the integrability of the hierarchies of Hamiltonian systems with topological field theory (TFT). He defines the limiting or "averaging" process of a hierarchy and of the tau-function which yields a system of partial equations whose coefficients induce a Frobenius structure on the invariant manifolds of the hierarchy, and hence a solution of the Witten–Dijkgraff–E. Verlinde–H. Verlinde (WDVV) equations from 2-dimensional TFT. The bihamiltonian structure of the integrable hierarchy is used in the calculation of higher genus corrections, and the WDVV equations are shown to specify the periods of the Abelian differentials of Riemann surfaces as functions on moduli spaces of these surfaces, so that both the bihamiltonian approach and the algebro-geometric methods enter the theory.

The Conference was funded by the C.I.R.M. and by the Société Mathématique de France. Additional support was provided by the French Ministry of Foreign Affairs, and grants were offered by the University of Paris VII, the U.F.R. de Mathématiques et Informatique of the University of Paris VII and the C.N.R.S. research unit U.R.A. 212 (Théories Géométriques). Financial support also came from the Fédération Française des Sociétés d'Assurances and is hereby gratefully acknowledged. We thank in particular our friend and colleague, Prof. M. Flato, who helped us with the fundraising even though he was not free to participate in the Conference.

It is a pleasure to thank the C.I.R.M. and its director, G. Lachaud, for a well-organized and very pleasant conference in the beautiful surroundings of Luminy, and the C.I.M.P.A. and its managing director, J.-M. Lemaire, for providing financial support for the students of the School on Integrable Systems who participated in the Conference.

We thank the editors of the series Progress in Mathematics for offering to publish this volume, and the staff of Birkhäuser Publishing Company for their help and efficient work.

The long-term planning and organizational work were performed with the secretarial assistance of Madame Claudine Roussel, from the University of Paris VII, who joined us at Luminy to handle some of the day-to-day needs of the conference. She also assisted us in the editing of this volume and the forthcoming volume of the courses given at the C.I.M.P.A. school. In the name of all the participants in the conference, we thank her for her expert and invariably cheerful collaboration.

Paris, February 1993

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Infinite Discrete Symmetry Group for the Yang–Baxter Equations and their Higher Dimensional Generalizations [1]

M. Bellon, J.-M. Maillard, and C. Viallet

Abstract. We show that the Yang-Baxter equations for two dimensional vertex models admit as a group of symmetry the infinite discrete group $A_2^{(1)}$. The existence of this symmetry explains the presence of a spectral parameter in the solutions of the equations. We show that similarly, for three-dimensional vertex models and the associated tetrahedron equations, there also exists an infinite discrete group of symmetry. Although generalizing naturally the previous one, it is a much bigger hyperbolic Coxeter group. We indicate how this symmetry can help to resolve the Yang-Baxter equations and their higher-dimensional generalizations and initiate the study of three-dimensional vertex models. These symmetries are naturally represented as birational projective transformations. They may preserve non trivial algebraic varieties.

Key-words: Yang-Baxter equations, Star-triangle relations, Tetrahedron equations, Inversion relations, Integrable models, Coxeter groups, Weyl group, Automorphisms of algebraic varieties, Birational transformations, Cremona transformations, Iteration of mappings.

1. Introduction

The Yang-Baxter equations, which appeared twenty years ago[2], have acquired a predominant role in the theory of integrable two-dimensional models in statistical mechanics [6, 7] and field theory (quantum or classical). They have actually outpassed the borders of physics and have become fashionable in some parts of the mathematics literature. They in particular support the construction of quantum groups [8, 9].

The Yang-Baxter equations [7] and their higher dimensional generalizations are now considered as the defining relations of integrability. They are the "Deus ex machina" in a number of domains of Mathematics and Physics

[1]work supported by CNRS

[2]In fact, fifty years ago, Lars Onsager was totally aware of the key role played by the star-triangle relation in solving the two-dimensional Ising model, but he preferred to give an algebraic solution emphasizing Clifford algebras [1, 2, 3, 4, 5].

(Knot Theory [10], Quantum Inverse Scattering [11], S-Matrix Factorization, Exactly Solvable Models in Statistical Mechanics, Bethe Ansatz [12], Quantum Groups [13, 9], Chromatic Polynomials [14] and more awaited deformation theories). The appeal of these equations comes from their ability to *give global results from local ones.* For instance, they are a sufficient and, to some extent, necessary [15] condition for the commutation of families of transfer matrices of arbitrary size and even of corner transfer matrices. From the point of view of topology, one may understand these relations by considering them as the generators of a large set of *discrete deformations of the lattice.* This point of view underlies most studies in knot theory [10] and statistical mechanics ($\mathbb{Z}$-invariance [16, 17]).

We want to analyze the Yang-Baxter equations and their higher dimensional generalizations [18, 19, 20, 21] without prejudice about what should be a solution, that is to say proceed by *necessary* conditions.

We will exhibit an infinite discrete group of transformations acting on the Yang-Baxter equations or their higher dimensional generalizations (tetrahedron, hyper-simplicial equations).

These transformations act as an automorphy group of various quantities of interest in Statistical Mechanics (partition function,...), and are of great help for calculations, even outside the domain of integrability (critical manifolds, phase diagram,...) [22].

We show here is that *they form a group of symmetries of the equations defining integrability.* They consequently appear as a group of automorphisms of the algebraic varieties parametrizing the solutions of the Yang-Baxter or tetrahedron equations. We will denote this group $\mathcal{A}ut$.

The existence of $\mathcal{A}ut$ drastically constrains the varieties where solutions may be found. In the general case, it has *infinite* orbits and gives *severe constraints* on the algebraic varieties which parametrize the possible solutions (genus zero or one curves, algebraic varieties which are not of the general type [23]). In the non-generic case, when $\mathcal{A}ut$ has finite order orbits, the algebraic varieties can be of general type, but *the very finiteness condition allows for their determination* [24].

In the framework of infinite group representations, it is crucial to recognize the *essential difference* between what these symmetry groups are for the Yang-Baxter equations and what they are for the higher dimensional tetrahedron and hyper-simplicial relations: the number of involutions generating our groups increases from 2 to 2^{d-1} when passing from two-dimensional to d-dimensional models and the group jumps from the semi-direct product $\mathbb{Z} \ltimes \mathbb{Z}_2$ to a much larger group, i.e., a *group with an exponential growth* with the length of the word[3].

[3]It is worth recalling that for the Zamolodchikov solution [19, 21] of the tetrahedron relation, the partition function is similar to the one of the two-dimensional checkerboard Ising model. This example seems to indicate that three-dimensional integrability can only occur when the 2^{d-1} generators of the group satisfy additional relations allowing

The existence of $\mathcal{A}ut$ as a symmetry of the Yang-Baxter equations has the following consequence: we may say that solving the Yang-Baxter equation is equivalent to solving all its images by $\mathcal{A}ut$. These images *generically tend to proliferate*, simply because $\mathcal{A}ut$ is infinite. Considering that the equations form an *overdetermined set*, it is easy to believe that the total set of equations is "less overdetermined" when the orbits of $\mathcal{A}ut$ are of finite order. One can therefore imagine that the *best candidates* for the integrability varieties are *precisely* the ones where the symmetry group possesses *finite orbits*: the solutions of Au-Yang et al. [25, 26, 27] seem to confirm this point of view [28, 29].

A contrario, if one gets hold of an apparently isolated solution, the action of $\mathcal{A}ut$ will multiply it until building up, in experimentally not so rare cases, a continuous family of solutions from the original one. This is the solution to the so-called baxterization problem [30].

We first show that the simplest example of Yang-Baxter relation which is the star-triangle relation [7] has an *infinite discrete group* of symmetries generated by three involutions. These involutions are deeply linked with the so-called *inversion relations* [31, 32, 33, 34].

This analysis can be extended to the "generalized star-triangle relation" for Interaction aRound the Face models without any major difficulties [6, 35].

2. The star-triangle relations

2.1 The setting

We consider a spin model with nearest neighbour interactions on square lattice. The spins σ_i can take q values. The Boltzmann weight for an oriented bond $\langle ij\rangle$ will be denoted hereafter by $w(\sigma_i, \sigma_j)$. The weights $w(\sigma_i, \sigma_j)$ can be seen as the entries of a $q\times q$ matrix. In the following we will introduce a pictorial representation of the star-triangle relation. An arrow is associated to the oriented bond $\langle ij\rangle$. The arrow from i to j indicates that the argument of the Boltzmann weight w is (σ_i, σ_j) rather than (σ_j, σ_i). This arrow is relevant only for the so-called chiral models [25], that is to say that the $q \times q$ matrix describing w is not symmetric. An interesting class of $q \times q$ matrices has been extensively investigated in the last few years [25, 27, 26]: the general cyclic matrices. It is important to note that we *do not* restrict ourselves to this particular class of matrices. Let us give the following non cyclic nor symmetric 6×6 matrix as another illustrative

for a mere polynomial growth of the size, and possibly reducing to a semi-direct product of finite groups and $\mathbb{Z}$ factors.

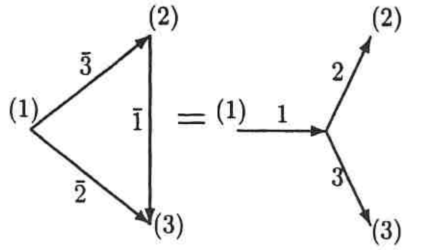
example:

$$\begin{pmatrix} x & y & z & y & z & z \\ z & x & y & z & y & z \\ y & z & x & z & z & y \\ y & z & z & x & z & y \\ z & y & z & y & x & z \\ z & z & y & z & y & x \end{pmatrix} \tag{1}$$

2.2 The relations

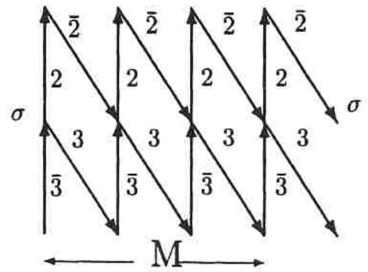
We introduce the star-triangle equations both analytically[4] and pictorially:

$$\sum_\sigma w_1(\sigma_1, \sigma) \cdot w_2(\sigma, \sigma_2) \cdot w_3(\sigma, \sigma_3) = \lambda \, \overline{w}_1(\sigma_2, \sigma_3) \cdot \overline{w}_2(\sigma_1, \sigma_3) \cdot \overline{w}_3(\sigma_1, \sigma_2). \tag{2}$$



(st1.1)

One should note that satisfying equation (2) *together with* the relation (st1.2) obtained by reversing all arrows, is a *sufficient* condition for the commutation of the diagonal transfer matrices of *arbitrary size* M with periodic boundary conditions $\mathbb{T}_M(w_2, \overline{w}_2)$ and $\mathbb{T}_M(\overline{w}_3, w_3)$:



Note that for cyclic matrices ([25, 27, 26]) the star-triangle relations (st1.1) and (st1.2) give the same equations since one exchanges (st1.2) and (st1.1) by spin reversal.

One could obviously imagine many other choices for the arrows on the six bonds, however only three of them lead to the commutation of

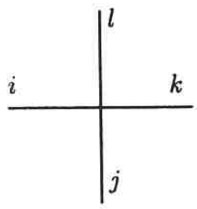
[4]Since the w_i and $\overline{w}_i$ are homogeneous variables, there will always be a global multiplicative factor λ floating around in the star-triangle equations.

diagonal transfer matrices. We therefore have three systems of equations to study. For example, if the Boltzmann weights are given by the 6×6 matrix (1), these three systems of equations are respectively made of 20 different equations or 35 or 36.

3. The Yang-Baxter relation for vertex models

We shall not get here into the arcanes of this relation, which appears in the theory of integrable models [9], the theory of factorizable S-matrix in two-dimensional field theory, the quantum inverse scattering method [11], knot theory and has been given a canonical meaning in terms of Hopf algebras [36] (quantum groups [8, 9, 37, 38, 39]) and the list is far from exhaustive. We just want to fix some notations for later use.

We consider a vertex model on a two-dimensional square lattice. To each bond is associated a variable with q possible states and a Boltzmann weight $w(i,j,k,l)$ is assigned to each vertex



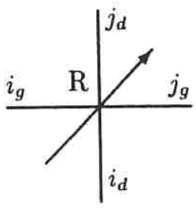
In order to write the Yang-Baxter relation, the q^4 homogeneous weights $w(i,j,k,l)$ are first arranged in a $q^2 \times q^2$ matrix R:

$$R^{ij}_{kl} = w(i,j,k,l). \tag{3}$$

The Yang-Baxter relation is a trilinear relation between three matrices $R(1,2)$, $R(2,3)$ and $R(1,3)$:

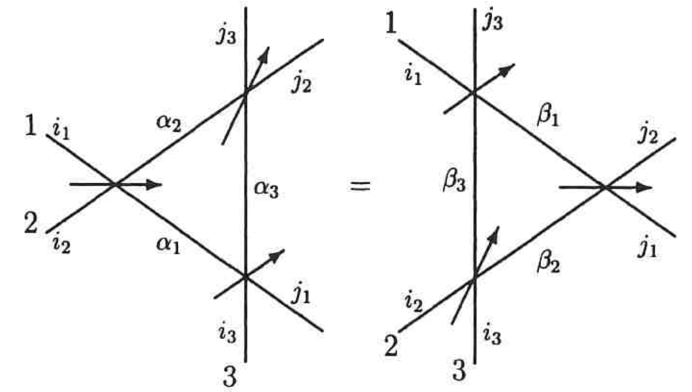
$$\sum_{\alpha_1,\alpha_2,\alpha_3} R^{i_1 i_2}_{\alpha_1 \alpha_2}(1,2) R^{\alpha_1 i_3}_{j_1 \alpha_3}(1,3) R^{\alpha_2 \alpha_3}_{j_2 j_3}(2,3) = \sum_{\beta_1,\beta_2,\beta_3} R^{i_2 i_3}_{\beta_2 \beta_3}(2,3) R^{i_1 \beta_3}_{\beta_1 j_3}(1,3) R^{\beta_1 \beta_2}_{j_1 j_2}(1,2). \tag{4}$$

The assignation (3) is arbitrary and we may specify it by complementing the vertex with an arrow and attributing numbers to the lines



$$= R^{i_g i_d}_{j_g j_d}(g,d).$$

With these rules relation (4) has the following graphical representation



(5)

The lines carry indices 1,2,3.

Some especially interesting solutions depend on a continuous parameter called the "spectral parameter". The presence of this parameter is fundamental for many applications in physics, as for example the Bethe Ansatz method [40, 5, 11, 12]. One of the main issues in the full resolution of (4) is precisely to describe what is this parameter and the algebraic variety on which it lives, although its presence may obscure the algebraic structures underlying the Yang-Baxter equation (*the discovery of quantum groups was allowed by forgetting this parameter* [39, 8, 41, 9]). The problem of building up continuous families of solutions from an isolated one, known as the *baxterization* [10], is made straightforward by our study. Indeed our results *explain the presence of the spectral parameter* in the solution of the equation (see also [24]).

4. Infinite discrete symmetry group for the star-triangle relation

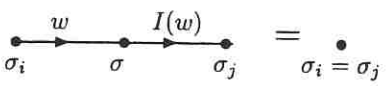
4.1 The inversion relation

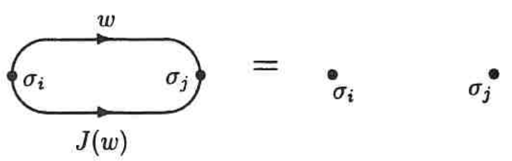
Two distinct inverses act on the matrix of nearest neighbour spin interactions: the matrix inverse I and the dyadic (element by element) inverse J. We write down the inversion relations both analytically and pictorially:

$$\sum_{\sigma} w(\sigma_i, \sigma) \cdot I(w)(\sigma, \sigma_j) = \mu \, \delta_{\sigma_i \sigma_j}, \tag{6}$$

$$w(\sigma_i, \sigma_j) \cdot J(w)(\sigma_i, \sigma_j) = 1. \tag{7}$$

where $\delta_{\sigma_i \sigma_j}$ denotes the usual Kronecker delta.

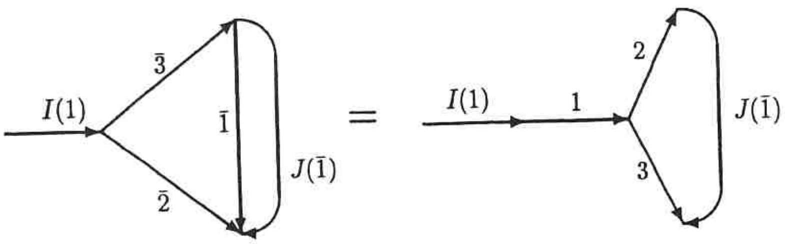


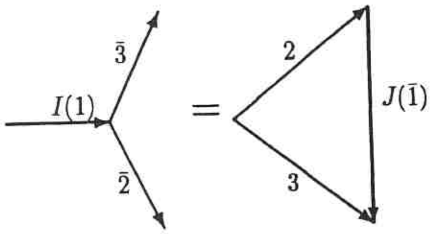
and 

The two involutions I and J generate an *infinite discrete group* Γ (Coxeter group) isomorphic to the infinite dihedral group $\mathbb{Z}_2 \ltimes \mathbb{Z}$. The $\mathbb{Z}$ part of Γ is generated by IJ. In the parameter space of the model, that is to say some projective space $\mathbf{CP}_{n-1}$ (n homogeneous parameters), I and J are birational involutions. They give a *non-linear representation of this Coxeter group by an infinite set of birational transformations* [24]. It may happen that the action of Γ on specific subvarieties *yields a finite orbit.* This means that the representation of Γ identifies with the p-dihedral group $\mathbb{Z}_2 \ltimes \mathbb{Z}_p$.

4.2 The symmetries of the star-triangle relations

The two inversions I and J act on the star-triangle relation. Let us give a pictorial representation of this action, starting from (st1.1) as an example:





(tst)

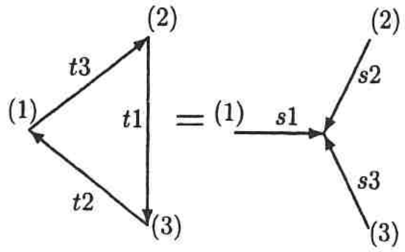
The transformed equation reads:

$$\lambda \sum_{\sigma_1} I(w_1)(\tau, \sigma_1) \cdot \overline{w}_2(\sigma_1, \sigma_3) \cdot \overline{w}_3(\sigma_1, \sigma_2) = w_2(\tau, \sigma_2) \cdot w_3(\tau, \sigma_3) \cdot J(\overline{w}_1)(\sigma_2, \sigma_3). \tag{8}$$

We get an action on the space of solutions of the star-triangle relation.

If $(w_1, w_2, w_3, \overline{w}_1, \overline{w}_2, \overline{w}_3)$ is a solution of eq(2) (see picture (st1.1) for the specific arrangement of arrows), then $(I(w_1), \overline{w}_3, \overline{w}_2, J(\overline{w}_1), w_3, w_2)$ is also a solution of eq(2), at the price of a permitted redefinition of λ. In this transformation, the weights w_1 and $\overline{w}_1$ play a special role.

At this point, it is better to formalize this action by introducing some notations. We may choose as a reference star-triangle relation $\mathcal{ST}$, the symmetric configuration:



$(\mathcal{ST})$

Any configuration may be obtained by reversing some arrows and permuting some bonds. With evident notations, we will denote by $R_{\rm s1}, R_{\rm s2}, R_{\rm s3}, R_{\rm t1}, R_{\rm t2}, R_{\rm t3}$ the reversals of arrows, and by $P_{\rm si,sj}, P_{\rm si,tj}, P_{\rm ti,tj}$ the permutations of bonds. Moreover I and J act on the bonds as $I_{\rm s1}, I_{\rm s2}, \ldots$ The action of I and J described above (where 1 was playing a special role) identifies with the action of

$$\mathcal{K}_1 = R_{\rm s2} R_{\rm t3} I_{\rm s1} J_{\rm t1} P_{\rm s2,t3} P_{\rm s3,t2}. \tag{9}$$

It is easy to check that $\mathcal{K}_1$ is an involution.

We may construct two similar involutions $\mathcal{K}_2$ and $\mathcal{K}_3$, obtained by cyclic permutation of the indices 1, 2, 3. The involutions $\mathcal{K}_i (i = 1,2,3)$ verify the defining relations of the *Weyl group of an affine algebra of type* $A_2^{(1)}$ [42]:

$$(\mathcal{K}_1\mathcal{K}_2)^3 = (\mathcal{K}_2\mathcal{K}_3)^3 = (\mathcal{K}_3\mathcal{K}_1)^3 = 1. \tag{10}$$

We denote $\mathcal{A}ut$ the group generated by the three involutions $\mathcal{K}_i$ $(i = 1,2,3)$.

5. Infinite discrete symmetry group for the Yang-Baxter equation

5.1 The inversion relations.

The R-matrix appears naturally as a representation of an element of the tensor product $\mathcal{A} \otimes \mathcal{A}$ of some algebra $\mathcal{A}$ with itself. This algebra is a nice Hopf algebra in the context of quantum groups. We shall not dwell on this here but recall some simple operations on R.

In $\mathcal{A} \otimes \mathcal{A}$ we have a product inherited from the product in $\mathcal{A}$:

$$(a \otimes b)(c \otimes d) = ac \otimes bd. \tag{11}$$

R is an invertible element of $\mathcal{A} \otimes \mathcal{A}$ for this product and we shall denote by $I(R)$ the inverse for this product:

$$R \cdot I(R) = I(R) \cdot R = 1 \otimes 1. \tag{12}$$

In terms of the representative matrix this reads:

$$\sum_{\alpha,\beta} R^{ij}_{\alpha\beta}\, I(R)^{\alpha\beta}_{uv} = \delta^i_u\, \delta^j_v = \sum_{\alpha,\beta} I(R)^{ij}_{\alpha\beta}\, R^{\alpha\beta}_{uv}. \tag{13}$$

This is nothing else but the so-called *inversion relation* for *vertex* models [31, 32, 35, 43, 23]. On $\mathcal{A} \otimes \mathcal{A}$ we have a permutation operator σ:

$$\begin{aligned} \sigma(a \otimes b) &= b \otimes a, && (14)\\ (\sigma R)^{ij}_{uv} &= R^{ji}_{vu}, \quad \text{for the matrix } R. && (15) \end{aligned}$$

Note that the representation of σ is just the conjugation by the permutation matrix P:

$$\begin{aligned} P^{ij}_{kl} &= \delta_{il}\delta_{jk}, && (16)\\ \sigma R &= PRP. && (17) \end{aligned}$$

In the language of matrices we have a notion of transposition. Let us define partial transpositions t_g and t_d by:

$$\begin{aligned} (t_g R)^{ij}_{uv} &= R^{uj}_{iv}, && (18)\\ (t_d R)^{ij}_{uv} &= R^{iv}_{uj}, && (19) \end{aligned}$$

and the full transposition

$$t = t_g t_d = t_d t_g. \tag{20}$$

We shall in the sequel use another inversion J defined by:

$$J = t_g I t_d = t_d I t_g, \tag{21}$$

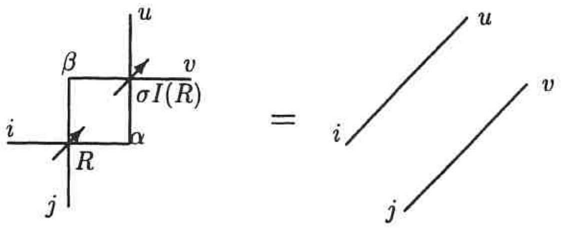
or equivalently:

$$\sum_{\alpha,\beta} R^{\alpha u}_{v\beta}\, J(R)^{\alpha i}_{j\beta} = \delta^i_u\, \delta^j_v = \sum_{\alpha,\beta} J(R)^{i\beta}_{\alpha j}\, R^{u\beta}_{\alpha v} \tag{22}$$

These operators verify straightforwardly:

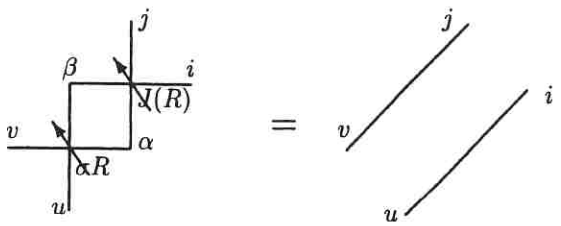
$$\begin{aligned} I^2 &= J^2 = 1, \quad It = tI, \quad Jt = tJ, \\ \sigma^2 &= t^2 = 1, \quad \sigma I = I\sigma, \quad \sigma J = J\sigma, \\ (\sigma t_g)^2 &= (\sigma t_d)^2 = t, \quad \sigma t_g \sigma t_d = 1. \end{aligned} \tag{23}$$

Each of these operations has a graphical representation. For the inversion I or more precisely for σI it is:



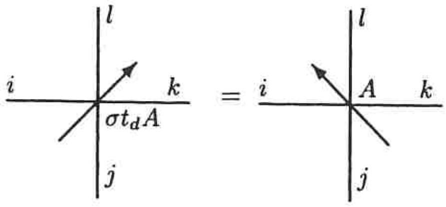
(24)

the inversion J reads:



(25)

and the transposition reads:



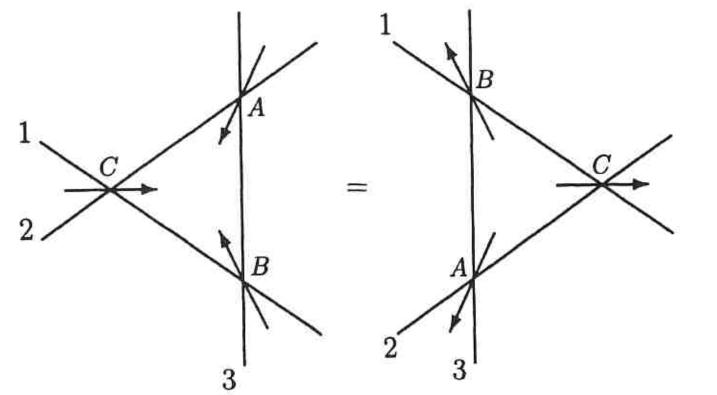
Note that the two inversions I and J do not commute. They generate an infinite discrete group Γ, the infinite dihedral group, isomorphic to the semi-direct product $\mathbb{Z} \ltimes \mathbb{Z}_2$. This group is represented on the matrix elements by *birational transformations* [24, 44, 45] acting on the projective space of the entries of the matrix R. Remark that for the *vertex models*, the birational transformations associated to the two involutions I and J are naturally related by collineations (see (21): this should be compared with the situation for nearest neighbour interaction spin models [24, 46].

5.2 The symmetries of the Yang-Baxter equations.

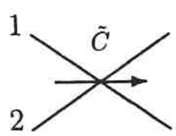
At the price of the redefinitions:

$$
\begin{aligned}
A &= tR(2,3), & (26)\\
B &= \sigma t_d R(1,3), & (27)\\
C &= R(1,2), & (28)
\end{aligned}
$$

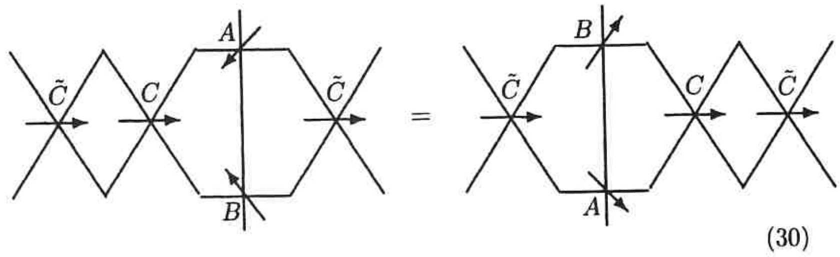
we may picture the Yang-Baxter relation in a more symmetric way:



(29)

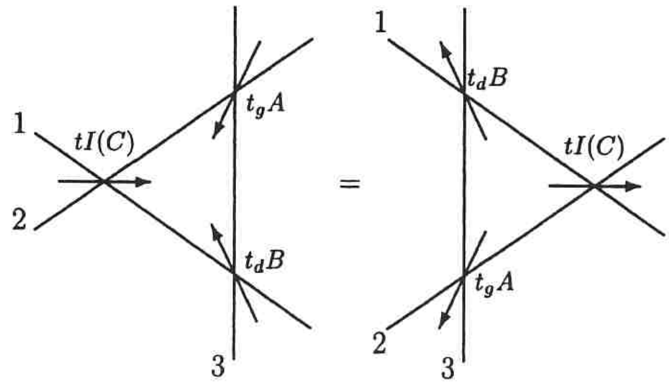
We may bracket (29) with 

, where $\tilde{C} = \sigma I(C)$. We get



(30)

that is to say



(31)

This relation is nothing but (29) after the redefinitions

$$
\begin{aligned}
A &\rightarrow t_g A, \\
B &\rightarrow t_d B, \\
C &\rightarrow tI\,C.
\end{aligned} \tag{32}
$$

We may denote by K_3 the operation (32). We have two other similar operations K_1 and K_2

$$
K_1: \begin{array}{l} A \rightarrow tI\,A \\ B \rightarrow t_g B \\ C \rightarrow t_d C \end{array} \quad , \quad K_2: \begin{array}{l} A \rightarrow t_d A \\ B \rightarrow tI\,B \\ C \rightarrow t_g C \end{array} \quad .
$$

The discrete group $\mathcal{A}ut$ generated by the K_i's ($i = 1, 2, 3$) is a symmetry group of the Yang-Baxter equations. These generators K_i ($i = 1, 2, 3$) are involutions. The K_i's satisfy the relation $(K_1 K_2 K_3)^2 = 1$. Actually, the operation $K_1 K_2 K_3$ is just the inversion I on R. Among the elements of the discrete group generated by the K_i's we have in particular:

$$
(K_1 K_2)^2 : \quad A \rightarrow I t_g I t_g A = tIJA, \tag{33}
$$

$$
B \rightarrow t_d I t_d I B = tJIB, \tag{34}
$$

$$
C \rightarrow C. \tag{35}
$$

Since IJ is of infinite order, we have generated an *infinite discrete group* of symmetries. This is exactly the phenomenon that we described in section 4.2 for the star-triangle equations.

Under this form it is not so evident to find the actual structure of the group. Let us introduce K_A, K_B and K_C, which are simply related to the K_i's by the transposition of two vertices:

$$
K_A: \begin{array}{l} A \rightarrow \sigma tIA \\ B \rightarrow t_g \sigma C \\ C \rightarrow \sigma t_g B \end{array} \quad , \quad K_B: \begin{array}{l} A \rightarrow \sigma t_g C \\ B \rightarrow \sigma tIB \\ C \rightarrow t_g \sigma A \end{array} \quad , \quad K_C: \begin{array}{l} A \rightarrow t_g \sigma B \\ B \rightarrow \sigma t_g A \\ C \rightarrow \sigma tIC \end{array} \quad .
$$

It is easily verified that:

$$K_A^2 = K_B^2 = K_C^2 = 1, \tag{36}$$

and

$$(K_A K_B)^3 = (K_B K_C)^3 = (K_C K_A)^3 = 1, \tag{37}$$

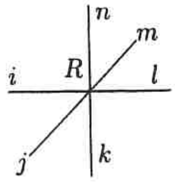
with no other relations. We recover the affine Coxeter group $A_2^{(1)}$ we already encountered in section 4.2.

A fundamental remark: Beware that, due to the different arrangement of indices, the relations we consider are not the Yang-Baxter equations that one considers in the study of quantum groups (shortly $RRR = RRR$) but rather its avatar $ABC = CBA$. The relevance of these relations is detailed in the standard literature on integrable models [9] and quantum groups [37, 38, 39].

We have here a very powerful instrument for two purposes: it defines *adequate patterns* for the matrix R [47]. It permits the so-called *baxterization of an isolated solution* just acting with tIJ. Indeed if a set of relations among the entries of R are preserved by IJ (or at least by tIJ), they will stay for every transforms of the initial Yang-Baxter relation. We shall illustrate in section 7.1.1 the baxterization on the Baxter eight-vertex model [48, 16] and show in section 7.1.2 how to introduce a spectral parameter for the solutions of the Yang-Baxter equations associated to $sl(n)$ algebras.

6. The tetrahedron equations and their symmetries

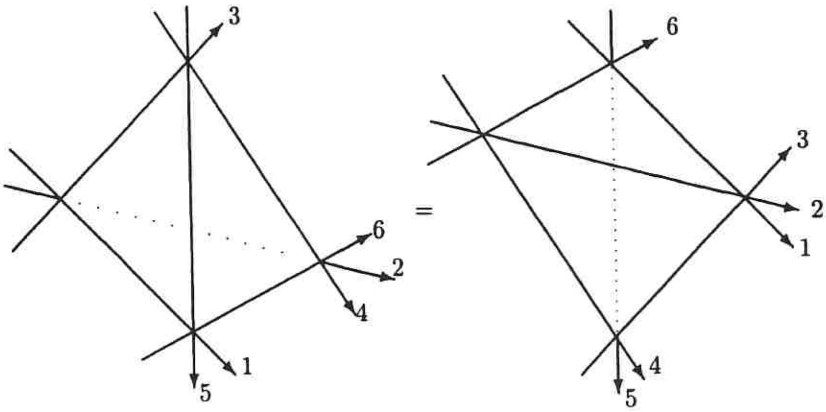
This equation is a generalization of the Yang-Baxter equation to three dimensional vertex models [19, 18, 21]. We give a pictorial representation of the three-dimensional vertex by



The Boltzmann weights of the vertex are denoted $w(i, j, k, l, m, n)$ and may be arranged in a matrix of entries

$$R^{ijk}_{lmn} = w(i, j, k, l, m, n). \tag{38}$$

The tetrahedron equation has a pictorial representation:



The algebraic form is

$$R_{123}R_{543}R_{516}R_{426} = R_{426}R_{516}R_{543}R_{123}. \tag{39}$$

We may here again introduce an inverse I

$$\sum_{\alpha_g,\alpha_m,\alpha_d} (IR)^{i_g i_m i_d}_{\alpha_g \alpha_m \alpha_d} \cdot R^{\alpha_g \alpha_m \alpha_d}_{j_g j_m j_d} = \delta^{i_g}_{j_g}\delta^{i_m}_{j_m}\delta^{i_d}_{j_d}. \tag{40}$$

We also introduce the partial transpositions t_g, t_m and t_d with

$$(t_g R)^{i_g i_m i_d}_{j_g j_m j_d} = R^{j_g i_m i_d}_{i_g j_m j_d}, \tag{41}$$

and similar definitions for t_m and t_d.

We redefine

$$A = R_{123}, \quad B = t_d R_{543}, \quad C = t_g t_m R_{516}, \quad D = t R_{426}, \tag{42}$$

where t is the full transposition $t_g t_m t_d$. Equation (39) then takes the more symmetric form

$$\sum_{s_1,\ldots,s_6} A^{i_1 i_2 i_3}_{s_1 s_2 s_3} B^{i_5 i_4 j_3}_{s_5 s_4 s_3} C^{j_5 j_1 i_6}_{s_5 s_1 s_6} D^{j_4 j_2 j_6}_{s_4 s_2 s_6} = \sum_{r_1,\ldots,r_6} D^{r_4 r_2 r_6}_{i_4 i_2 i_6} C^{r_5 r_1 r_6}_{i_5 i_1 j_6} B^{r_5 r_4 r_3}_{j_5 j_4 i_3} A^{r_1 r_2 r_3}_{j_1 j_2 j_3}. \tag{43}$$

We may multiply the previous equation by $(IA)^{u_1 u_2 u_3}_{i_1 i_2 i_3}$ and $(tIA)^{v_1 v_2 v_3}_{j_1 j_2 j_3}$ and sum over (i_1, i_2, i_3) and (j_1, j_2, j_3). This amounts to a bracketing of the tetrahedron equations by two times the same vertex, in a procedure trivially generalizing the one for the Yang-Baxter equation (30). We recover (43) with A, B, C and D transformed by

$$\begin{aligned} K_1 : A &\rightarrow tIA \\ B &\rightarrow t_d B \\ C &\rightarrow t_m C \\ D &\rightarrow t_m D. \end{aligned} \tag{44}$$

We have in a similar way the operations

$$K_2 : \begin{array}{l} A \rightarrow t_d A \\ B \rightarrow tIB \\ C \rightarrow t_g C \\ D \rightarrow t_g D \end{array}, \qquad K_3 : \begin{array}{l} A \rightarrow t_g A \\ B \rightarrow t_g B \\ C \rightarrow tIC \\ D \rightarrow t_d D \end{array}, \qquad K_4 : \begin{array}{l} A \rightarrow t_m A \\ B \rightarrow t_m B \\ C \rightarrow t_d C \\ D \rightarrow tID \end{array}.$$

Each of these four operations is an involution. They satisfy various relations, for instance $(K_1K_2K_3K_4)^2 = 1$. *The K_i's generate a group Aut_3 which is a symmetry group of the tetrahedron equations.* This group is "monstrous" since the number of elements of length smaller than l is of exponential growth with respect to l, unlike the case of the affine Coxeter groups (as $A_2^{(1)}$ for the Yang-Baxter equation) where this number is of polynomial growth.

The operations playing a role similar to the one of I and J in the two-dimensional Yang-Baxter equations are the *four* involutions

$$I, \quad J = t_g I t_m t_d, \quad K = t_m I t_d t_g, \quad L = t_d I t_g t_m. \tag{45}$$

We call Γ_3 the group generated by these four involutions. Γ_3 is also a symmetry group for the three dimensional vertex model *even if* [33] the model *does not* satisfy the tetrahedron equation.

In order to precise the algebraic structure of the group Γ_3 generated by I, J, K and L, it is simpler to consider as generators two of the partial transpositions t_g and t_d, I and the full transposition t. The third partial transposition can be recovered as the product tt_gt_d and t commutes with all other generators and so contributes a mere $\mathbb{Z}_2$ factor in the group. We are thus considering the Coxeter group generated by three involutions t_g, t_d and I, with two of them commuting: this is represented by the following Dynkin diagram

$$\underset{t_g}{\bullet} \overset{\infty}{\text{———}} \underset{I}{\bullet} \overset{\infty}{\text{———}} \underset{t_d}{\bullet}$$

For this group again, the number of elements of length smaller than l is greater than $2^{l/2}$. This is in fact a *hyperbolic* Coxeter group [49].

7. Consequences of this symmetry group

7.1 The baxterization

The problem of the baxterization is to introduce a spectral parameter into an isolated solution of the Yang-Baxter equations [10]. We have solutions of this problem by acting with the symmetry group Γ.

7.1.1 Baxterization of the Baxter model

Consider the matrix of the symmetric eight vertex model

$$R = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}. \tag{46}$$

Notice that this form is preserved by I and J and that $tR = R$. The action of I is

$$a \rightarrow \frac{a}{a^2 - d^2} \qquad b \rightarrow \frac{b}{b^2 - c^2} \tag{47}$$

$$c \rightarrow \frac{-c}{b^2 - c^2} \qquad d \rightarrow \frac{-d}{a^2 - d^2} \tag{48}$$

and the action of J is

$$a \rightarrow \frac{a}{a^2 - c^2} \qquad b \rightarrow \frac{b}{b^2 - d^2} \tag{49}$$

$$c \rightarrow \frac{-c}{a^2 - c^2} \qquad d \rightarrow \frac{-d}{b^2 - d^2} \tag{50}$$

We shall look at the solutions of the Yang-Baxter equations for matrices R of the form (46). The leading idea is that the parametrization of the solutions is just the parametrization of the algebraic varieties preserved by tIJ in the projective space $\mathbb{C}P_3$ of the homogenous parameters (a, b, c, d). The remarkable fact is that not only these varieties exist but can be completely described. We use the visualization method we have already used [24, 50] for spin models, that is to say just draw the orbits obtained by numerical iteration and look.

This is best illustrated by Figure 1. This figure shows the orbit of point (*), which is a matrix of the form (46). It is drawn by the iteration of IJ acting on the initial point (*). The resulting points densify on the elliptic curve given by the intersection of the two quadrics Δ_1 = constant and Δ_2 = constant (Clebsch's biquadratic), with Δ_1 and Δ_2 the Γ invariants

$$\begin{aligned} \Delta_1 &= \frac{a^2 + b^2 - c^2 - d^2}{ab + cd}, \\ \Delta_2 &= \frac{ab - cd}{ab + cd}. \end{aligned} \tag{51}$$

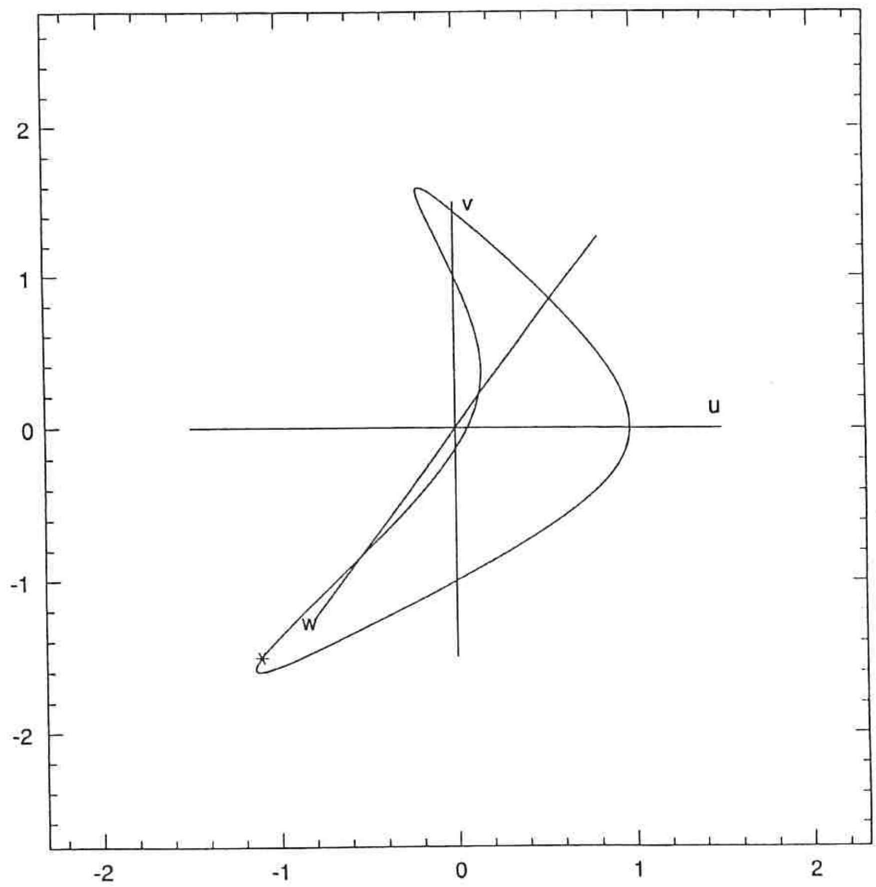


Figure 1. Baxterization of the point *

Similar calculations can be performed for a very general 16-vertex model for which the 4×4 R-matrix is symmetric:

$$R = \begin{pmatrix} a & e & f & d \\ e & b & c & g \\ f & c & b' & h \\ d & g & h & a' \end{pmatrix}. \tag{52}$$

Amazingly the baxterization of this 16-vertex model leads to *curves*. These curves are also *intersection of quadrics* (even in the general case for which their is no solution for the Yang-Baxter equations) [51].

7.1.2 Baxterization of the R matrix of $sl_q(n)$

Another example corresponds to the baxterization of solutions associated to $sl(n)$ algebras [13]. There are special solutions generally denoted R_+ and R_-. For the simplest four-dimensional representation of the $sl(2)$ case, we have

$$R_+ = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{53}$$

and a similar expression for R_- [13]. Looking for a family containing both R_+ and R_- our baxterization procedure leads to the well-known [5] six-vertex model R-matrix $R = \lambda R_+ + 1/\lambda R_-$.

We let as an exercise for the reader to treat the $sl(3)$ case. In a forthcoming publication we will show that these ideas can be generalized to all the universal R-matrices [9] for every representation [52]. This group appears in field theory, in the analysis of classical R-matrices [53].

7.2 A strategy for the resolution of a star-triangle equation

We may use the symmetry group $\mathcal{A}ut$ (or more simply Γ on each Boltzmann weight) to find the integrability varieties. In general, Γ points towards specific algebraic varieties in the parameter space (the varieties where the orbits lie), and $\mathcal{A}ut$ sometimes allows to reduce the number of unknowns in the original equations (sections 4.2 and 5.2). We may in certain instance bring these equations to a handable "isotropic" form ($ABC = CBA \rightarrow RRR = RRR$ form), and find particular isotropic solution of the equations.

This is best exemplified [54] with the chiral model, with five homogeneous parameters $w(k)$, $k = 0, 1, 2, 3, 4$ and weight matrix

$$\begin{pmatrix} w(0) & w(1) & w(2) & w(3) & w(4) \\ w(4) & w(0) & w(1) & w(2) & w(3) \\ w(3) & w(4) & w(0) & w(1) & w(2) \\ w(2) & w(3) & w(4) & w(0) & w(1) \\ w(1) & w(2) & w(3) & w(4) & w(0) \end{pmatrix} \tag{54}$$

For the non-chiral model obtained by setting $w(1) = w(4)$ and $w(2) = w(3)$, the exact result of Fateev-Zamolodchikov [55] is recovered in [54] *without prejudice* on the properties of the solution (e.g. self-duality). One can go further and look at the integrability varieties of the general chiral Potts model [27].

The above (non-chiral) "isotropic" point is a particular solution of the star-triangle relation for this model (isotropic star-triangle). We use the non-homogeneous variables $x(k) = w(k)/w(0)$, $k = 1, \ldots, 4$. We introduce the infinitesimal perturbation $X_i(k)$ of $x_i(k)$, and $\overline{X}_i(k)$ of $\overline{x}_i(k)$, with obvious notations.

The linearized star-triangle relations yield a homogeneous linear system for $X_i(k), \overline{X}_i(k)$. This system is not only compatible, but it has *a four dimensional space of solutions.*

The solutions verify:

$$X_1(k)+X_2(k)+X_3(k) = \overline{X}_1(k)+\overline{X}_2(k)+\overline{X}_3(k) = 0, \quad k = 1, \ldots, 4. \tag{55}$$

If we introduce the symmetric and antisymmetric vectors X^s and X^a (resp. $\overline{X}^s$ and $\overline{X}^a$)

$$X^s = \begin{pmatrix} 1 \\ s \\ s \\ 1 \end{pmatrix} \quad X^a = \begin{pmatrix} 1 \\ a \\ -a \\ -1 \end{pmatrix} \quad \overline{X}^s = \begin{pmatrix} 1 \\ \overline{s} \\ \overline{s} \\ 1 \end{pmatrix} \quad \overline{X}^a = \begin{pmatrix} 1 \\ \overline{a} \\ -\overline{a} \\ -1 \end{pmatrix} \tag{56}$$

with

$$s = \frac{2 - 4c(2) - c(4) + c(6) + 7c(8) - 7c(10) - c(12) - 3c(14)}{-2 - 4c(2) + 8c(4) - c(6) + 2c(8) - 4c(10) + c(12) - 2c(14)}$$

$$a = \frac{-2 + 4c(2) + c(4) - c(6) - 9c(8) + 9c(10) + c(12) - c(14)}{-6 + 2c(2) + 8c(4) + 17c(6) - 42c(8) + 2c(10) + 23c(12) + 22c(14)}$$

$$\overline{s} = \frac{-2 + 10c(2) - 12c(4) + 7c(6) - 7c(8) + 3c(10) + 4c(12) - 2c(14)}{10 - 15c(2) + 11c(4) - 9c(6) + 2c(10) + c(12) + 4c(14)}$$

$$\overline{a} = \frac{10 - 4c(2) - 12c(4) + 18c(6) - 26c(8) + 14c(10) + 8c(12) - 8c(14)}{11 + 20c(2) - 60c(4) + 68c(6) - 90c(8) + 50c(10) + 32c(12) - 50c(14)}$$

with $c(p) = cos(p\theta)$, i.e., numerically:

$s \simeq -58.28463\ldots \quad a \simeq -1.0308189\ldots, \quad \overline{s} \simeq -1.834537\ldots, \quad \overline{a} \simeq 4.543390\ldots$

and set

$$
\begin{aligned}
X_i &= s_i X^s + a_i X^a && (57)\\
\overline{X}_i &= \overline{s}_i \overline{X}^s + \overline{a}_i \overline{X}^a \qquad i = 1, 2, 3, && (58)
\end{aligned}
$$

we get

$$
\begin{aligned}
s_1 + s_2 + s_3 &= a_1 + a_2 + a_3 = 0,\\
\overline{s}_i &= -\alpha\, s_i,\\
\overline{a}_1 &= \beta\,(a_2 - a_3),\\
\overline{a}_2 &= \beta\,(a_3 - a_1),\\
\overline{a}_3 &= \beta\,(a_1 - a_2), && (59)
\end{aligned}
$$

with

$$
\begin{aligned}
\alpha &= -\frac{1}{4\cos^2(\theta)} \simeq -.2527617250... && (60)\\
\beta &= \frac{28}{31} - \frac{38}{31}c(1) + \frac{20}{93}c(3) + \frac{118}{93}c(7) \simeq .0108158287... && (61)
\end{aligned}
$$

This proves[5] the existence of *a four parameter family of solutions* of the star-triangle equations containing the isotropic point. This family contains in particular the previously mentioned Fateev-Zamolodchikov solution [55]. It is remarkable that IJ is of *finite order* on this curve (the order is five). We have the prejudice that the integrability surface is of the same nature, i.e. is a locus of points where $(IJ)^5 = 1$. *Notice that such a locus is automatically invariant by both I and J.* Such a surface is given by the two equations:

$$
\begin{aligned}
A \quad &\Big(w(1)w(3)w(4)^2 - 3w(2)^2w(4)^2 + 2w(1)w(2)w(4)w(0) && (62)\\
+ \quad & w(3)w(2)^2w(0) - 3w(1)w(3)^2w(0) + 2w(4)w(2)w(3)^2 \Big)\\
- \quad & 4w(2)^2w(4)^2 - 4w(1)w(3)^2w(0) + 3w(3)w(2)^2w(0)\\
+ \quad & 3w(1)w(3)w(4)^2 + w(4)w(2)w(3)^2 + w(1)w(2)w(4)w(0) = 0
\end{aligned}
$$

and

$$
A \quad \Big(-2w(3)^2w(0)^2 - 2w(1)w(2)w(4)^2 - w(4)w(3)^2w(1) \qquad (63)
$$

[5] As a consequence of the implicit function theorem and the algebraicity of the solutions of the star-triangle equations

$$
\begin{aligned}
&+ \quad 3w(0)w(3)w(4)^2 + 3w(2)w(1)w(3)w(0) - w(2)w(0)^2w(4)\Big) \\
&- \quad w(1)w(2)w(4)^2 + 2w(2)w(0)^2w(4) - w(0)w(3)w(4)^2 \\
&+ \quad 2w(4)w(3)^2w(1) - w(3)^2w(0)^2 - w(2)w(1)w(3)w(0) = 0
\end{aligned}
$$

with $A = \frac{1}{2}(-1 \pm \sqrt{5})$. The vectors $X^s, \overline{X}^s, X^a, \overline{X}^a$ are tangent to this surface for $A = \frac{1}{2}(-1 + \sqrt{5})$.

The consequences of equation (59) on the commutation of transfer matrices $T_i = \mathbb{T}_M(\overline{w}_i, w_i)$ are the following: locally near the isotropic point T_i depends on $(s_i, a_i, \overline{s}_i, \overline{a}_i)$. The commutation of T_1 and T_2 is obtained by imposing relations (59). We first need $\overline{s}_1 = -\alpha\, s_1$ which allows three parameters for T_1. At this point (59) fixes a_2 and $\overline{a}_2$ and the only free parameter for T_2 is s_2, giving a *one parameter family of commuting transfer matrices.*

The integrability surface is actually the locus of points where $(IJ)^5 = e$ [28, 56, 57, 58, 59, 60]. In general, we believe that the varieties corresponding to *finite dimensional orbits* of the group Γ are *good candidates for integrability* [54].

In support of our inclination for finite orbits, we recall the asymmetric eight-vertex model ((52) with $e = f = g = h = 0$) for which the *free fermion condition* $aa' + bb' - c^2 - d^2 = 0$ implies *both the integrability and the finiteness of the orbits* of Γ (hint: I and J reduce to *linear permutations* up to signs). It is actually easy to find explicitly the subvarieties where Γ is of finite order [24]

When searching for isolated solutions of the Yang-Baxter equations, the group Γ can be used to get necessary conditions on the R-matrices. From the equation $RRR = RRR$, we deduce an infinite set of other relations of the kind $Rg(R)h(R) = h(R)g(R)R$, where g and h belong to Γ, leading to the commutation of the transfer matrices with periodic boundary conditions $T_N(R)$ and $T_N(g(R))$. Even for $N = 1$ and $g = I$, this leads to non-trivial *necessary conditions.* For the 16 vertex model (52), this gives one *quartic* condition with 72 terms.

7.3 Three-dimensional models

Our strategy for finding solutions of the tetrahedron equations is to seek for patterns of the Boltzmann weights of the three dimensional vertex *compatible with the symmetry group* Γ_3. By this we mean that its form should be preserved by Γ_3.

7.3.1 A first model

We will therefore consider a simple model where i, j, k, l, m and n take only two values $+1$ and -1. The matrix (38) is an 8×8 matrix. We will require that its pattern is invariant under the inverse I [47] and the various partial transpositions t_g, t_m and t_d. We aim at having a generalization of the Baxter eight-vertex model and we impose the following restrictions:

$$\begin{aligned} w(i,j,k,l,m,n) &= w(-i,-j,-k,-l,-m,-n), && (64)\\ w(i,j,k,l,m,n) &= 0 \quad \text{if } ijklmn=-1. && (65) \end{aligned}$$

These constraints amount to saying that the 8×8 matrix splits into two times the same 4×4 matrix. It is further possible to impose that this matrix is symmetric since, in this case, t_gR (and any other partial transpose) is also symmetric. Let us introduce the following notations for the entries of the 4×4 block of the R matrix

$$\begin{pmatrix} a & d_1 & d_2 & d_3 \\ d_1 & b_1 & c_3 & c_2 \\ d_2 & c_3 & b_2 & c_1 \\ d_3 & c_2 & c_1 & b_3 \end{pmatrix}. \tag{66}$$

The four rows and columns of this matrix correspond to the states $(+,+,+)$, $(+,-,-)$, $(-,+,-)$ and $(-,-,+)$ for the triplets (i,j,k) or (l,m,n). The R-matrix can be completed by spin reversal, according to the rule (64). t_g simply exchanges c_2 with d_2 and c_3 with d_3, t_m and t_d can be similarly defined and I acts as the inversion of this 4×4 matrix.

For this three dimensional model, the coefficients of the characteristic polynomial of the 4×4 matrix (66) give a good hint for invariants under Γ_3. They are

$$\begin{aligned} \sigma_1^{(3d)} &= a+b_1+b_2+b_3, && (67)\\ \sigma_2^{(3d)} &= a(b_1+b_2+b_3)+b_1b_2+b_2b_3+b_3b_1 && (68)\\ &- (c_1^2+c_2^2+c_3^2+d_1^2+d_2^2+d_3^2), \ldots. \end{aligned}$$

Since $\sigma_2^{(3d)}$ is invariant by t_g, t_m amd t_d and takes a simple factor (the inverse of the determinant) under the action of I, the variety $\sigma_2^{(3d)}=0$ *is invariant under* Γ_3. Given the hugeness of the group Γ_3, it is already an astonishing fact to have such a covariant expression. In fact we can exhibit *five linearly independent polynomials* with the same covariance, which give *four invariants*, as follows:

$$ab_1+b_2b_3-c_1^2-d_1^2, \qquad c_2d_2-c_3d_3, \tag{69}$$

and the ones deduced by permutations of 1, 2 and 3. They form a five dimensional space of polynomials. Any ratio of the five independant polynomials is invariant under all the four generating involutions. In other

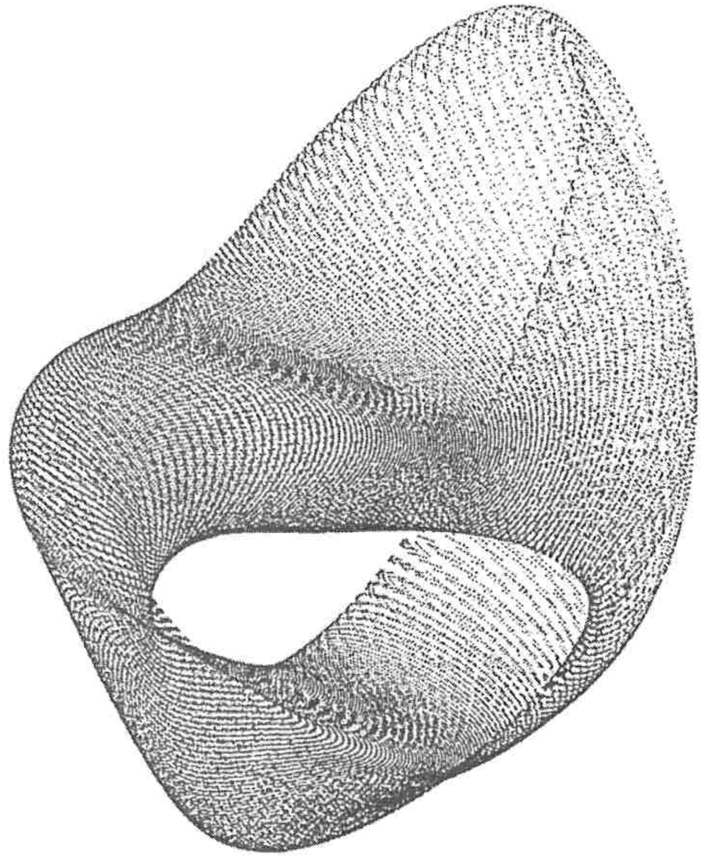


Figure 2.

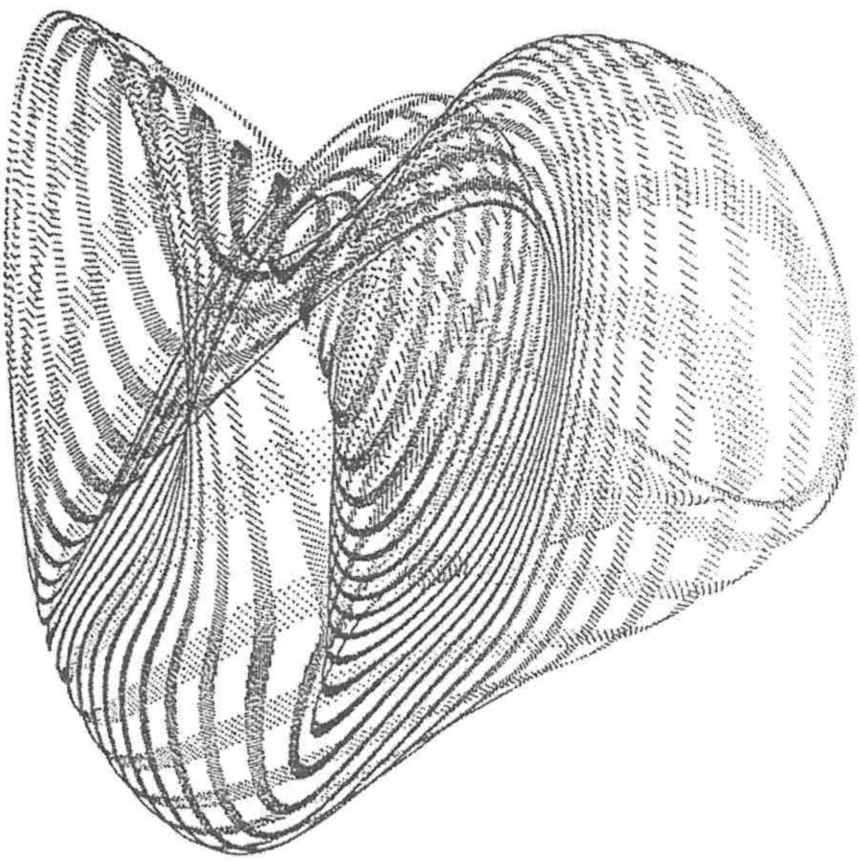


Figure 3.

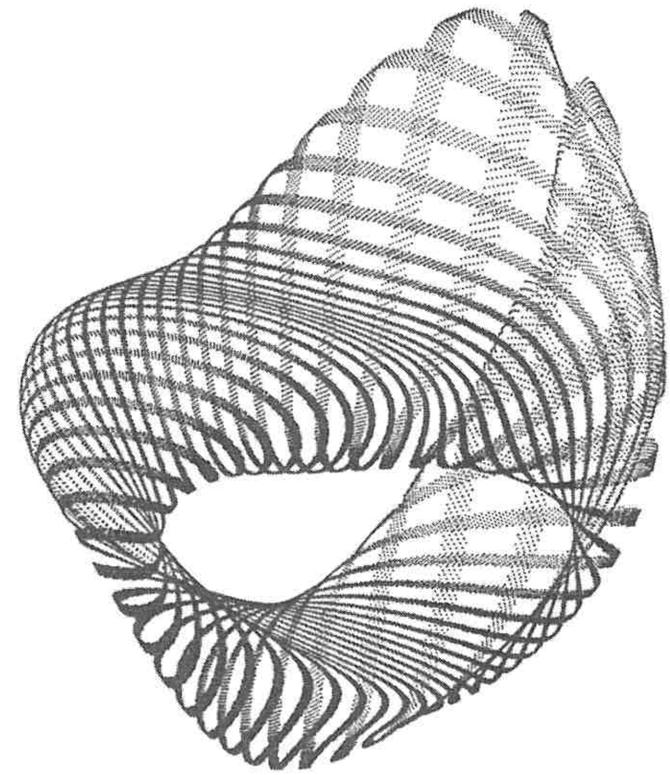


Figure 4.

words $\mathbb{C}P_9$ is foliated by five dimensional algebraic varieties invariant under Γ_3.

To have some flavour of the possible (integrable ?) algebraic varieties invariant under Γ_3, we study its orbits [24, 50]. We start with the study of the subgroup generated by some infinite order element namely IJ. This element gives a special role to axis 1. The transformation IJ *does preserve the symmetry under the exchange of 2 and 3.* If the initial point is symmetric under the exchange of 2 and 3, the orbit under IJ is thus a *curve*. Other starting points lead to orbits lying on a *two dimensional* variety given by the intersection of seven quadrics (see figure 2,3,4). However, what we are interested in are the orbits of the *whole* Γ_3 group. The size of this group prevent us from studying exhaustively the full set of group elements of a given length even for quite small values of this length. We have nevertheless explored the group by a random construction of typical elements of increasingly large length [30]. This confirms that we generically only have the four invariants described previously.

7.3.2 A second model

We also consider a simple model where i, j, k, l, m and n take only two values $+1$ and -1 and which is also a generalization of the Baxter eight vertex model. The Boltzmann weights $w(i,j,k,l,m,n)$ are given by:

$$w(i,j,k,l,m,n) = f(i,j,k)\; \delta^i_l\, \delta^j_m\, \delta^k_n + g(i,j,k)\; \delta^i_{-l}\, \delta^j_{-m}\, \delta^k_{-n} \qquad (70)$$

$$f(i,j,k) = f(-i,-j,-k) \text{ and } g(i,j,k) = g(-i,-j,-k) \qquad (71)$$

Equations (71) are symmetry conditions reducing the numbers of homogeneous parameters from 16 to 8.

As for the previous model, there exists an invariant of the action of the whole group Γ_3:

$$\frac{f(+,+,+)f(+,-,-)f(-,+,-)f(-,-,+)}{g(+,+,+)g(+,-,-)g(-,+,-)g(-,-,+)} \qquad (72)$$

Considering the subgroup of Γ_3 generated by the infinite order element IJ, one can easily find other invariants, namely

$$\frac{f(+,+,+)f(+,-,-)}{g(+,+,+)g(+,-,-)} \qquad (73)$$

and

$$\frac{f(+,+,+)^2 + f(+,-,-)^2 - g(+,+,+)^2 - g(+,-,-)^2}{g(+,+,+)g(+,-,-)} \qquad (74)$$

For this model [61], the trajectories under IJ are *curves* in $\mathbb{C}P_7$.

8. Conclusion

An important problem in statistical mechanics and field theory, is the understanding of the role of the dimension of the lattice on both the algebraic aspects and the topological aspects. All this touches various fields of mathematics and physics: algebraic geometry, algebraic topology, quantum algebra. Indeed the Coxeter groups we use are at the same time groups of automorphisms of algebraic varieties, symmetries of quantum Yang-Baxter equations (and their higher dimensional avatars). They also provide an *extension to several complex variables functions of the notion of the fundamental group* Π_1 of a Riemann surface, with of course a much more involved covering structure [33, 50].

We believe moreover that the *space of parameters* seen as a projective space is the appropriate place to look at, if one wants to substantiate the deep topological notion embodied in the notion of $\mathbb{Z}$-invariance [16] and free the models from the details of the lattice shape.

Actually, we have exhibited an infinite discrete symmetry group for the Yang-Baxter equations and their higher dimensional generalization acting on this parameter space. This group is the Coxeter group $A_2^{(1)}$ (semi-direct product of $\mathbb{Z} \times \mathbb{Z}$ by some finite group). We have shown that this symmetry is responsible for the presence of the spectral parameter. In other words, the *discrete* symmetry gives rise to a *continuous* one (see [54]). A similar study for the generalized star-triangle relation of the Interaction aRound a Face model, sketched in [35], can be performed rigorously along the same lines, leading to the same result. Also note that the same groups generated by involutions appear in the study of semi-classical r-matrices [53]. An interesting point will be to exhibit the action of our symmetry group on the underlying quantum group for the Yang-Baxter equations [52].

Our symmetry group is a group of automorphisms of the integrability varieties. This should give precious informations on these varieties. In particular one should decide if, up to Lie groups factors (which cannot be excluded because of the existence of "gauge" symmetries, weak graph duality [62], ...), these varieties can be anything else than abelian varieties, or even product of curves: can they be for example K_3 surfaces, are they homological obstructions to the occurence of anything but curves ?

For three-dimensional vertex models, the symmetry group, though generalizing very naturally the previous group (generated by four involutions with similar relations) is drastically different: it is so "large"[6] that the chances are quite small that it leaves enough room for any invariant integrability varieties. It is not useless to recall the unique non-trivial known solution of the tetrahedron equations (Zamolodchikov's solution) [19, 18, 21].

[6] One should keep in mind that very "large" sets of rational transformations may preserve algebraic curve of genus zero or one. Just think of the transformations on the circle generated by $\{\theta \rightarrow \theta + \lambda,\ \theta \rightarrow 2\theta,\ \theta \rightarrow 3\theta\}$ [63]

For this model the three axes are not on the same footing, so that we do not have a "true" three dimensional symmetry for the model (two-dimensional checkerboard models coupled together). Is there still any hope for a three-dimensional exactly solvable model with *genuine three-dimensional symmetry*? We think that the group of symmetries we have described gives the best line of attack to this problem. We will show that Γ_3 and even more $\mathcal{A}ut_3$ are generically too "large" to allow any non-trivial solution of the tetrahedron equations with genuine three dimensional symmetry [64].

Acknowledgments: We would like to thank J. Avan, O. Babelon, and M. Talon, for very stimulating discussions and comments.

L'un de nous (JMM), désire rendre hommage à la mémoire de Jean-Louis Verdier. Il y a plus de dix ans de cela, travailler sur les modèles intégrables était plutôt mal vu dans la communauté de la physique théorique française. Alors que je travaillais avec M.-T. Jaekel, Jean-Louis Verdier nous consacra une après-midi de discussion chaque semaine et ce, durant des années. Nous discutions d'équation de Yang-Baxter, de ses avatars (groupes de tresses, algèbre de Hecke, relations d'entrelacement, ...), de relations tetraèdre, de relations d'inverse, de modèle de Potts, toujours dans un esprit de géométrie algébrique.

J'ai un merveilleux souvenir de ces discussions: il y avait de part et d'autre un réel effort pour communiquer et pour montrer à l'autre, au delà des considérations adventices qui trop souvent encombrent, l'idée la plus intrinsèque, le concept, la structure qui font réellement marcher les choses. Il y avait le dialogue profond et honnête de gens qui ont compris que derrière ces idées se trouve quelque chose de passionnant. Je peux témoigner de la patience, de la générosité, du désintéressement de Jean-Louis Verdier: tout ce temps consacré à deux jeunes chercheurs plutôt marginaux ne servait en rien à sa carrière.

A l'heure où l'intégrabilité reçoit une reconnaissance institutionnelle à travers trois médailles Fields, mais où les phénomènes de mode tendent à remplacer les idées par des campagnes d'influence et le savoir-faire par le faire-savoir, à cette heure je tiens à dire que Jean-Louis Verdier me manque profondément.

Appendix: Symmetry group of the star-triangle relation

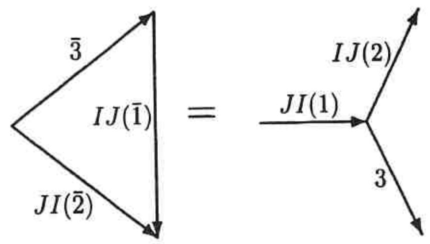
The three involutions generating the symmetry group of the star-triangle relations read:

$$\mathcal{K}_1 = R_{s2}R_{t3}I_{s1}J_{t1}P_{s2,t3}P_{s3,t2}\,, \quad \mathcal{K}_1^2 = 1 \tag{1}$$
$$\mathcal{K}_2 = R_{s3}R_{t1}I_{s2}J_{t2}P_{s3,t1}P_{s1,t3}\,, \quad \mathcal{K}_2^2 = 1 \tag{2}$$
$$\mathcal{K}_3 = R_{s1}R_{t2}I_{s3}J_{t3}P_{s1,t2}P_{s1,t1}\,, \quad \mathcal{K}_3^2 = 1 \tag{3}$$

If σ is the cyclic permutation, $\sigma = \sigma_s \sigma_t$ with $\sigma_s = P_{s_2,s_3} P_{s_1,s_2}$ and $\sigma_t = P_{t_2,t_3} P_{t_1,t_2}$, the involutions $\mathcal{K}_i$ are related by

$$\mathcal{K}_2 = \sigma^2 \mathcal{K}_1 \sigma, \quad \mathcal{K}_3 = \sigma^2 \mathcal{K}_2 \sigma \tag{4}$$

This symmetry group contains an action of IJ . It may be obtained by successively operating with the previous involutions: first act with $\mathcal{K}_1$, then with $\mathcal{K}_3$, then operate with $\mathcal{K}_2$ and finally with $\mathcal{K}_1$. This sequence of operations, when used on relations (st1.1), yields:



(IJst1.1)

This sequence of transformations amounts to acting with the product

$$G_3 = \sigma \mathcal{K}_1 \mathcal{K}_2 \mathcal{K}_3 \mathcal{K}_1 = R_{s1} R_{s2} R_{t1} R_{t2} (JI)_{s1} (IJ)_{s2} (IJ)_{t1} (JI)_{t2}. \tag{5}$$

We may define similarly

$$G_2 = \sigma G_3 \sigma^2 = R_{s3} R_{s1} R_{t3} R_{t1} (JI)_{s3} (IJ)_{s1} (IJ)_{t3} (JI)_{t1} \tag{6}$$
$$G_1 = \sigma G_2 \sigma^2 = R_{s3} R_{s2} R_{t3} R_{t2} (IJ)_{s3} (JI)_{s2} (JI)_{t3} (IJ)_{t2} \tag{7}$$

We have the relations:

$$G_1 G_2 G_3 = 1 \tag{8}$$
$$G_i G_j = G_j G_i \qquad \forall i,j = 1,2,3. \tag{9}$$

The symmetry group $\mathcal{A}ut$ is the semi-direct product of the Weyl group of an A_2 (finite dimensional simple of rank 2) Lie algebra by a bidimensional lattice translation group $\mathbb{Z} \times \mathbb{Z}$.

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