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A STEP-BY-STEP APPROACH TO INTEGRABILITY

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We try to elucidate the role played by different symmetries in simple models of statistical mechanics. Starting with obvious symmetries for the partition function and combining them with duality relations we obtain a set of constraints on the possible algebraic varieties relevant for the integrable manifolds and the phase diagram of the lattice model. When imposing in addition the inversion relation special polynomials are obtained, which are close and sometimes identical to the set of equations defining the parameter subspace of the integrable models. Our procedure is detailed on the q-state chiral Potts models on a square lattice, in particular for $q = 3$ and 4.

1. Introduction

The symmetries of lattice models of statistical mechanics obviously play an important role not only on the singular behaviour near phase transitions (universality) but also on the phase diagram. However, when trying to turn this general statement into a working tool one enters into a domain of great confusion and subtle complexities. Does one consider the symmetries in the parameter space of the model or symmetries in the real space? What are the additional hidden symmetries behind the "miracles" happening at integrability?[1] Is it possible for instance that the Z-invariance of the integrable models[2] might lead to a lattice deformation theory including conformal invariance?[3] One often encounters simplistic views associating self-duality transformations[4] with the possibility of an exact solution, not mentioning here the large number of authors who tried to mimic the Kaufman-Onsager algebra developed for the Ising model.[5] Since the symmetries related to integrability remain quite mysterious, it is not surprising that finding any new solution of the Yang-Baxter[6] (or star-triangle) equations or their three-dimensional equivalent, the tetrahedron equations,[7] requires a great deal of intuition.

In this paper we consider a model which is simple enough for a rather pedagogical presentation of our ideas but complex enough to avoid accidental degeneracy between symmetries of different character. We start from simple symmetries, like the symmetries of the lattice and obvious symmetries of the partition function and successively add more complicated but still linear symmetries like the duality relations. Then we introduce the (non-linear) inversion relations.[8] This leads to a rather large symmetry group acting on the parameter space of the model. We will see that the algebraic varieties invariant under this symmetry group are close to the integrability varieties.

The model we study here is the so-called q-state chiral Potts model on the square lattice and our goal will be to determine all algebraic varieties (homogeneous polynomials of given order) defined on the parameter space of the model (Boltzmann weights) and invariant (or covariant) under the symmetry transformations mentioned above.

There are several reasons for the study of this model. First the model has a very rich phase diagram containing different commensurate phases or floating incommensurate phases.[9] Secondly, the duality transformation D is no longer an involution but a transformation of order four ($D^4 = 1$). Third, special cases of this model, (notably the standard scalar Potts model,[10] the symmetric Ashkin-Teller model,[6] the clock model[11] etc.) have been extensively studied analytically and numerically. Finally, more recently Au-Yang *et al.*[12,13] have discovered integrable subspaces of this model which are the first examples of exactly solvable models for which the correct parameterization involves curves of genus > 1.[14] The symmetries of the model act on these curves of genus > 1.

At this point it is important to mention that our analysis is not restricted in any way to this particular model, nor to two-dimensional models or to models with only nearest neighbour interactions or even to spin models.

2. The Anisotropic q-State Chiral Potts Model

The partition function of the anisotropic q-state chiral Potts model on the square lattice is the following:

$$\mathcal{Z} = \sum_{\{\sigma_i\}} \prod_{\langle ij \rangle} w(\sigma_i - \sigma_j) \prod_{\langle ik \rangle} \bar{w}(\sigma_i - \sigma_k). \tag{1}$$

The first (second) product runs over all horizontally (vertically) oriented bonds edges of the square lattice. Instead of the usual parameterization in terms of the coupling constants we use throughout the paper the equivalent set of corresponding Boltzmann weights $\{w(i), \bar{w}(i)\}_{i=0}^{q-1}$. The spins $\sigma_i = 0, 1, \ldots, q-1$ are q-state variables ($\sigma_i \in Z_q$) and the Boltzmann weights depend on the difference between nearest neighbor spins, not only on its absolute value, as in usual symmetric Z_q models. The set of weights $w(0), w(1), \ldots, w(q-1)$ forms a set of q *homogenous* *variables*: the transformation $w(i) \rightarrow \lambda w(i)$ results on a trivial factor λ^{N_b} multiplying the partition function (N_b is the number of bonds). In what follows we will *always omit multiplicative factors* of this kind. The correspondence between these homogeneous variables and the coupling constants of the Hamiltonian is simple. For the three-state chiral Potts model, for example, $w(n) = [\exp -\beta J\cos(2\pi/3(n+\Delta))]$, where J is the nearest neighbor constant, Δ the chirality of the model and $\beta = 1/kT$.

Spin-group symmetries

The partition function $\mathcal{Z}$ as well as other quantities have some simple "gauge" symmetries related to the fact that the σ_i variables can be seen as dummy variables. For example, the transformation $\sigma_i \rightarrow \sigma_i$, $\sigma_j \rightarrow \sigma_j - 1$ leads to the following cyclic permutation on the $w(i)$'s:

$$C\colon w(0) \rightarrow w(1) \rightarrow w(2) \rightarrow \ldots \rightarrow w(q-1) \rightarrow w(0). \tag{2}$$

The same transformation can be performed independently on the $\bar{w}(i)$ variables. Similarly, the "spin-reversal" symmetry generates an involution ($R^2 = 1$)

$$R\colon w(i) \leftrightarrow w(q-i) \tag{3}$$

(by definition $w(q) = w(0)$). C is a q-cycle and generates a symmetry group Z_q. When the group Z_q has subgroups several other subcycles and involutions exist. To make things clear, consider the $q = 15$ case. One has two subgroups, Z_5 and Z_3. One can associate to the spin $\sigma_i \in Z_{15}$ two spins τ_i and ρ_i belonging to Z_5 and Z_3, respectively, so that with obvious notations, $\sigma_i = 3\tau_i + 5\rho_i$. The spin difference is preserved in the new representation since

$$\sigma_i - \sigma_j = 3(\tau_i - \tau_j) + 5(\rho_i - \rho_j).$$

A new cycle and a new spin reversal can be defined for the variables ρ_i and similarly, for the variables τ_i. The two new cycles are just C^3 and C^5. The two new spin reversal symmetries R_1 and R_2 commute (also with R) and satisfy relations such as $(CR_1)^{30} = (CR_2)^{30} = (R_2C)^{30} = 1$ and $R = R_1 R_2$.

In general one thus associate to Z_q the group generated by C and R as well as R_i corresponding to all representations of Z_q in terms of its subgroups. The symmetry group generated by C and R only is the dihedral group ($C^q = R^2 = (CR)^2 = 1$). For $q = 4$, for instance, this leads to consider the group of the symmetry of the square, C_{4v} generated by the transposition $w(0) \leftrightarrow w(2)$ and the 4-cycle $w(0) \rightarrow w(1) \rightarrow w(2) \rightarrow w(3) \rightarrow w(0)$. When q is a prime number there is no subgroup and one only associates to Z_q the dihedral group generated by C and R.

Duality transformation

Another, less obvious symmetry of the chiral q-state Potts model is the duality transformation[15]:

$$D: \hat{w}(n) = \sum_{m=0}^{q-1} \omega^{nm} w(m), \quad n = 0, 1, \ldots, q-1 \tag{4}$$

where ω is the qth root of unity, $\omega^q = 1$. The normalization factor is omitted here, since we disregard all multiplicative factors of the partition function. The duality transformation can be obtained using several methods. A geometric approach[15] consists of introducing the link variables $\lambda_{ij} = \sigma_i - \sigma_j$, which must satisfy the constraints $\Sigma_{ij} \lambda_{ij} = 0 \pmod{q}$ over each elementary plaquette. These constraints are expressed introducing dummy variables μ_α associated with each plaquette and the Z_q representation of the Kronecker delta (Poisson's formula). Summing over the (now unconstrained) λ_{ij} variables leads to the dual representation of the partition function in terms of the μ_α variables.

Let us remark that D^2 corresponds to the transformation

$$D^2: w(0) \to qw(0), \quad w(i) \to qw(q-i). \tag{5}$$

It is clear that D^2 is an involution identical to R. D is a transformation of order four: $D^4 = 1$. This is different from the usual duality for the standard scalar Potts and Ising model, which is an involution. Hence, a more appropriate name for Eq. (4) would be "quadrality" relation rather than duality. Although D is not a simple permutation on $w(i)$'s it is still a linear transformation of the homogeneous parameters $w(i)$.

The inversion relation

In addition to the linear symmetries discussed so far, there is also a less obvious one, the inversion symmetry, which is nonlinear.[16] Consider the one-dimensional transfer matrix M, whose elements are $M_{\alpha,\beta} = w(\alpha - \beta)$. The inverse of this matrix is also a cyclic matrix, M'. Let us associate to each $w(i)$ the corresponding coefficient of the matrix M':

$$I: w(i) \to w_I(i) = M'_{\alpha+i,\alpha}. \tag{6}$$

We also introduce transformation J, which associates to each of the parameters $w(i)$ its inverse

$$J: w(i) \to \frac{1}{w(i)}. \tag{7}$$

A straightforward generalization of the arguments detailed in Ref. 8 leads to an exact functional equation on the symmetrized $q^L \times q^L$ row-to-row transfer matrices of the chiral Potts model, valid for any finite size L of the system:

$$
\begin{aligned}
&T_L(w(0), \ldots, w(q-1); \bar{w}(0), \ldots, \bar{w}(q-1)) \times \\
&T_L\left(\frac{1}{w(0)}, \ldots, \frac{1}{w(q-1)}; \bar{w}_I(0), \ldots, \bar{w}_I(q-1)\right) = E,
\end{aligned} \tag{8}
$$

where E stands for the $q^L \times q^L$ unit matrix. The inversion relation amounts therefore to applying the transformations I and J to the homogeneous parameters representing the Boltzmann weights associated with the horizontal and vertical bonds, respectively. Although nonlinear transformations, I and J are both involutions. Equation (8) leads to a functional equation on the partition function and its analytical continuation which is also valid outside the parameter subspace corresponding to exact solvability.[8] The inversion relation is actually a (nontrivial) symmetry of the model in the *whole* parameter space and one also expects the critical manifolds of the model to be invariant under such transformations.

When $q = 3$, for instance, the inversion relation reads

$$I\colon w(i) \to w^2(i) - w(j)w(k) \tag{9}$$

with $(i, j, k) = (1, 2, 3)$. The determinant occurring when computing the inverse of M is a multiplicative factor and is again omitted. Similarly, J reads:

$$J\colon w(i) \to w(j)w(k). \tag{10}$$

In the q-state anisotropic model one has also in the thermodynamic limit the obvious symmetry of exchanging vertical and horizontal weights:

$$S\colon w(i) \leftrightarrow \bar{w}(i). \tag{11}$$

Combining together the inversion relation and the symmetry S one gets in general an infinite set of birational transformations. Let us denote by $\mathcal{J}$ the inversion transformation (I, J) in the anisotropic space. Since $\mathcal{J}$ and S are involutions any element of this automorphy group can be written as

$$S^{\alpha}(\mathcal{J}S)^n, \quad \alpha = 0, 1 \text{ and } n \in Z. \tag{12}$$

Note that the transformations I, J, and D are not independent. Since the transformation D "diagonalizes" the cyclic transfer matrix M and since $D^4 = 1$ one has the following relation:

$$I = DJD^{-1} = DJD^3. \tag{13}$$

Let us also note that

$$JD^2 = D^2J. \tag{14}$$

When restricting ourselves to one of the two anisotropic (horizontal or vertical) set of homogeneous parameters, say $\{w(i)\}$, we have only to consider a sequence of the two birational transformations J and D: any element of that group can be written as

$$D^{\alpha}(JD)^n, \quad \alpha = 0, 1, 2, 3 \text{ and } n \in Z, \tag{15}$$

so that the group is a semi-direct product of Z_4 and Z. This group is a quite nontrivial symmetry group of the parameter space. One has also to add the "trivial" permutation symmetries discussed above: J commutes with all these permutation symmetries, while D commutes with R $(= D^2)$. Introducing in addition the transformation

$$\Delta: w(i) \rightarrow \omega^i w(i)$$

one obtains in addition the relations $D\Delta = CD$, $DC = \Delta^{-1}D$, $DC^{-1} = \Delta D$, $\Delta J = J\Delta^{-1}$. From these relations it is simple to see that the symmetry group is the semi-direct product of some simple finite group with Z. For almost all two-dimensional models (q-state vertex models or Interactions a Round a Face = IRF models[6]) one finds basically the same structure: semi-direct product of some simple finite group (dihedral group, . . .) with Z. Z may be replaced by $Z \times Z$ or Z^k in anisotropic triangular models or in staggered models.[17] The situation is drastically different in three dimensions, where one cannot exclude subgroups as large as free groups, generated by transformations without any relations between them.

3. General Considerations and Integrable Varieties of the Chiral Potts Model

One naturally expects this large symmetry group to have important consequences on the phase diagram of the model. For instance, one would expect the critical manifolds, the zeros of the partition function and especially the integrable varieties (which are algebraic varieties) to be invariant under the action of this group of symmetry. In what follows, we will restrict ourselves to the study of the action of these symmetries on *algebraic* varieties (homogeneous polynomials of given order).

One can prove that the integrable manifolds must be invariant under (a subgroup of) the automorphy group: this is basically due to compatibility relations between the Yang-Baxter (or tetrahedron) equations and the automorphy group[8,14]. On the other hand, the integrable manifolds are always algebraic varieties, a consequence of writing the commutation of transfer matrices as a set of algebraic equations on the coefficients of the transfer matrix (and hence in the Boltzmann weights $w(i)$ and $\bar{w}(i)$)[18]:

$$\frac{P_\alpha}{Q_\alpha} = K_\alpha \tag{16}$$

where P_α and Q_α are homogeneous polynomials with integer coefficients of the same degree in the Boltzmann weights $w(i)$ and $\bar{w}(i)$. K_α are eventually not independent constants — see the example of the hard hexagon model.[19]

Hence, a model which is exactly solvable on a given parameter subspace corresponds to a quite pathological situation from the point of view of algebraic geometry: one has an algebraic variety with a large (in general finite) set of birational automorphisms that leave the variety invariant. Actually, an algebraic variety has usually only a very small set of automorphisms. Hence, one can distinguish three different situations:

1. The group is very large, it contains a free subgroup and there is certainly no algebraic variety with such a huge set of automorphisms (except the whole parameter space).
2. The group is infinite but "nice" (semi-direct product of a finite group and the Z or Z^k group, like in our previous examples). There are only very few algebraic varieties having such set of automorphisms: one can even try to classify them as in Ref. 4. When the variety is an algebraic curve, it has to be of genus 0 or 1.
3. The group is finite. The event of a finite group is a rare "accident" that happens only on very restricted subspaces of the parameter space.

In general one gets immediately the generators of this automorphy group for q-state vertex or IRF models even in higher dimensions. We therefore suggest that it is possible to develop a systematic way of searching for exactly solvable models by looking at this automorphy group, which in general is easy to obtain and not too difficult to analyze. Compared with the amount of work and intuition one needs to find new solutions of the Yang-Baxter equations or using the Bethe Ansatz, this method seems advantageous. In addition, this approach clarifies how the symmetries related to integrability emerge from the natural symmetries of the system.

At this point, let us come back to the chiral q-state Potts model and recall some of the results obtained recently by Au-Yang *et al.*[12,13] Baxter *et al.*[20] give a solution of the star-triangle relation for this model in terms of two sets of "rapidities", the two sets of four homogeneous variables (a_p, b_p, c_p, d_p) and (a_q, b_q, c_q, d_q). In fact, these variables occur only in the following particular combinations:

$$x_1 = b_q d_p, \quad x_2 = a_p c_q, \quad x_3 = b_q d_q, \quad x_4 = c_p a_q,$$
$$x_5 = d_q a_p, \quad x_6 = d_p a_q, \quad x_7 = c_p b_q, \quad x_8 = b_p c_q. \tag{17}$$

These homogeneous variables are related to the original homogeneous variables $\{w(i)\}$ and $\{\bar{w}(i)\}$ through the *overdetermined* linear system:

$$w(n)x_1 - w(n)\omega^{n+1}x_2 - w(n+1)x_3 + w(n+1)\omega^{n+1}x_4 = 0$$
$$\bar{w}(n)x_5 - \bar{w}(n)\omega^{n+1}x_6 - \bar{w}(n+1)x_7 + \bar{w}(n+1)\omega^{n+1}x_8 = 0 \tag{18}$$

for every $n = 1, \ldots, q-1$, $\omega^q = 1$. Since the x_i's are products of the rapidities Eqs. (17) reduce to only two equations:

$$x_1 x_4 = x_6 x_7 \quad \text{and} \quad x_2 x_3 = x_5 x_8. \tag{19}$$

However, since both $w(i)$ and $\bar{w}(i)$ — and thus the x_i's are homogenous variables, Eq. (19) can be further reduced to

$$\frac{x_1 x_4}{x_2 x_3} = \frac{x_6 x_7}{x_5 x_8}. \tag{20}$$

When the overdetermined system (18) has non-trivial solutions (the x_i variables exist and are not zero) then the automorphy group is a *finite* group, since the duality and the inversion read in terms of the x_i's:

$$D: x_1 \to x_2\omega, \quad x_2 \to x_4, \quad x_3 \to x_1, \quad x_4 \to x_3,$$
$$J: x_1 \to x_3, \quad x_2 \to x_3, \quad x_3 \to x_1, \quad x_4 \to x_2. \tag{21}$$

The group generated by D and J is in general an infinite discrete group for $q > 3$. However, the genus > 1 solution of the star-triangle relation of Au Yang *et al.*[12] corresponds to a restricted parameter subspace where the group *actually degenerates* into a finite group. In this model exact solvability requires two different kind of conditions: the first set of $(q-3)$ conditions corresponds to the compatibility conditions for the linear system (18) *also implying that the group degenerates into a finite group*; the second set of constraints consists of only one condition specific to integrability Eq. (20). In addition, the first set of conditions do factorize according to the horizontal and vertical Boltzmann weights:

$$F_\alpha(w(0), \ldots, w(q-1)) = 0$$
$$F_\alpha(\bar{w}(0), \ldots, \bar{w}(q-1)) = 0 \tag{22}$$

where $\alpha = 1, 2, \ldots, q-3$ but the second condition, which is specific for integrability, mixes the horizontal and vertical Boltzmann weights:

$$\Phi(w(0), \ldots, w(q-1))\,\Phi(\bar{w}(0), \ldots, \bar{w}(q-1)) = 1. \tag{23}$$

A simple examination of the linear system (18) shows that the *set* of determinantal conditions (22) is invariant under the q-cycle C. However, it is important to note that each F_α *is not invariant* (*or covariant*) *under* C.

One remarks that Eqs. (22) are very special cases of Eq. (16): the polynomials F_α are functions of the horizontal (vertical) Boltzmann factors. In this particular case conditions (22) are compatibility conditions for the overdetermined system (18). Actually the F_α's are polynomials of *order four in only five variables even for* $q \geq 5$. For the three-state chiral Potts model there is no determinantal condition (Eq. (22)) and Eq. (23) reads

$$\begin{aligned} &P(w(0), w(1), w(2))\,Q(\bar{w}(0), \bar{w}(1), \bar{w}(2)) \\ &\quad + Q(w(0), w(1), w(2))\,P(\bar{w}(0), \bar{w}(1), \bar{w}(2)) = 0 \end{aligned} \tag{24}$$

with

$$P(w(0), w(1), w(2)) = \left(\sum_{i=0}^{2} w^3(i)\right)^2 w(0)\,w(1)\,w(2) - 3\,w^2(0)\,w^2(1)\,w^2(2) \tag{25}$$

and

$$\begin{aligned} Q(w(0), w(1), w(2)) = &\left(\sum_{i=0}^{2} w^3(i)\right) w(0)\,w(1)\,w(2) \\ &+ 3w^2(0)\,w^2(1)\,w^2(2) - 2\sum_{i\neq j} w^3(i)\,w^3(j) \end{aligned} \tag{26}$$

which follows from $\Phi = x_1 x_4/(x_2 x_3/\omega)$,

$$x_1 x_4 = Q + 3(\omega - \omega^2)P \tag{27}$$

and

$$\frac{x_2 x_3}{\omega} = Q - 3(\omega - \omega^2)P. \tag{28}$$

In the limit of the three-state scalar Potts model ($w(1) = w(2)$, $\bar{w}(1) = \bar{w}(2)$), Eq. (24) reduces to the known ferromagnetic[21] and antiferromagnetic[22] criticality conditions:

$$\begin{aligned} w(0)\bar{w}(0) + w(0)\bar{w}(1) + w(1)\bar{w}(0) &= 0, \\ w(0)\bar{w}(0) - w(0)\bar{w}(1) - w(1)\bar{w}(0) &= 2. \end{aligned} \tag{29}$$

In the isotropic limit Eq. (24) becomes

$$Q(w(0), w(1), w(2)) = 0$$

while

$$P(w(0), w(1), w(2)) = 0$$

corresponds to trivializations of the model: $w(i) = 0$ or $\hat{w}(i) = 0$, $i = 0$ or 1 or 2.

For $q = 4$ the compatibility condition for the linear system (18), Eq. (22) is

$$\begin{aligned} &[w^2(0) + w^2(2)]w(1)w(3) + [w^2(3) + w^2(1)]w(0)w(2) \\ &- 2w^2(0)w^2(2) - 2w^2(1)w^2(3) = 0. \end{aligned} \tag{30}$$

This condition is invariant under the duality relation, the transformation J, and hence the whole set of transformations (16), which is finite here.

4. Algebraic Varieties Compatible with the Symmetry Group

We now consider how to systematically analyze the homogeneous polynomial expressions compatible with the previously described group of symmetries. The surviving polynomials form a rather restricted set of possible candidates for writing the integrable algebraic varieties and analyzing the model in the whole parameter space. From the previous examples it is clear that we must distinguish between two different kinds of equations: the ones which factorize separately on the horizontal and vertical Boltzmann weights (Eq. (22)) and the ones mixing these weights (i.e. Eq. (23)). According to the integrable solution of the chiral q-state Potts model[12,13,20] there are $q-3$ factorized conditions and only one mixing condition. In what follows we will concentrate therefore on the first, larger set of conditions. This enable us to consider *only one set* of Boltzmann weights (say, the horizontal one). Because of the "factorization" property of this first set of equations, however, it is important to note that *these conditions are valid also for the isotropic model.*

We consider the set of homogeneous polynomials of order N with integer coefficients in the variables $w(0), \ldots, w(q-1)$. At first sight one is interested in the subset of polynomials which transform into themselves (up to a multiplicative factor) under all the generators of the symmetry group. However, as already mentioned before for the cycle C, it may happen that the *set* of polynomials is eventually compatible with the generators of this symmetry group while this is not true for the individual members of the set.

The symmetry transformations can be divided in three different sets: the "trivial" symmetries permuting the homogeneous variables $w(i)$ ($C, R = D^2, R_i$), the linear transformation D, and finally the nonlinear involutions J or I. It is a straightforward but tedious matter to find the homogenous polynomials covariant under a set of linear transformations. This a typical problem of Invariant Theory (see Refs. 23, 24). We give here a "pedestrian" discussion of the problem.

Spin-group and duality symmetry

Each linear transformation can be separately diagonalized. They reduce to a simple multiplicative factor (eigenvalue) when acting on the new variables provided by the eigenvectors. Since all transformations are of finite order (2, 4, q, divisors of q, etc.) the eigenvalues are thus roots of unity. For R, which acts as $w(i) \leftrightarrow w(q-i)$, the new variables are $w(i) \pm w(q-i)$ with eigenvalues ± 1, respectively. For C the new variables are the Fourier series defined as

$$\hat{w}(n) = \sum_{p=0}^{q-1} \omega^{np} w(p) \tag{31}$$

with the corresponding factor ω^{-n}: $C\hat{w}(n) = \omega^{-n}\hat{w}(n)$.

Now consider a monomial expression of order N in the "good" variables $\hat{w}(i)$ such as

$$\prod_{i=0}^{q-1} \hat{w}(i)^{\alpha_i} \tag{32}$$

where the α_i's are non-negative integers obeying the following sum rules:

$$\begin{aligned} \sum_{i=0}^{q-1} \alpha_i &= N \\ \sum_{i=0}^{q-1} i\alpha_i &= -\delta \pmod{q} \end{aligned} \tag{33}$$

$c = \omega^{\delta}$ is the "quantum number" associated with the action of C on polynomials of order N. The general linear combination of monomials (32) associated with the same factor c forms the most general polynomial of order N with quantum number c. Among these linear combinations we have now to find the ones which have also covariant properties with respect to the other symmetry generators. When considering the spin reversal symmetry R, one has to rewrite the previous polynomial expression in terms of the good variables for R, namely $w(i) \pm w(q-i)$. Among all monomial expressions in these new variables some will transform with a quantum number $r = +1$, while others with $r = -1$. The polynomial with quantum number c thus splits into two polynomials: one with quantum number c and $r = +1$ and another one with quantum number c and $r = -1$. Similarly, for D one has a set of linearly independent variables to be obtained from

$$X_{i,\Omega} = \sqrt{q}(\Omega w(i) + \Omega^3 w(q-i)) + (\hat{w}(i) + \Omega^2 \hat{w}(q-i)) \tag{34}$$

where $\Omega^4 = 1$. The set of eigenvectors (34) is redundant: from the $4q$ vectors only q correspond to independent, non-zero eigenvectors. The corresponding multiplicative factor (up to $\sqrt{q}$) is Ω. Iterating successively the procedure of rewriting the polynomial in well suited variables for different symmetries, one can construct polynomials characterized by a set of quantum numbers corresponding to some (eventually all) transformations associated with the symmetries of the model.

Of course, it may happen that for some sequences of quantum numbers there are no polynomials of order N which transform accordingly.

Although we know the set of transformations of finite order defining the symmetry group, the eventual relations between these transformations must also be known. Such relations also apply to the corresponding quantum numbers. For example, since $CRCR = 1$ the respective quantum numbers c and r obey $(cr)^2 = 1$.

As an example, consider the symmetries (C, R, D) for the $q = 3$ model and fix the quantum numbers $c = 1$ and $r = 1$. The well suited variables for D are

$$t = \sqrt{3}\, w(0) + \hat{w}(0)$$

$$u = -\sqrt{3}\, w(0) + \hat{w}(0)$$

and

$$v = w(1) - w(2)$$

with the respective factors $+1, -1, i$. No polynomials can be found for $N = 1, 2$. C and R generate the group of permutation for the three homogeneous variables $w(i)$. We therefore introduce the symmetric polynomials

$$\Sigma = \sum_{i=0}^{2} w^3(i)$$

and

$$\Pi = w(0)w(1)w(2).$$

Denoting by $\hat{\Sigma}$, $\hat{\Pi}$ the dual transforms of Σ and Π, respectively, one finds for $N = 3$ two "good" polynomials, namely

$$(1 \pm \sqrt{3})\,\Sigma + 6\Pi \tag{35}$$

with duality quantum numbers $d = \pm 1$. The polynomial with quantum number $+1$ is a linear combination of t^3, tu^2, uv^2 with the coefficients $\sqrt{3}-1$, $\sqrt{3}+3$, $6(\sqrt{3}+1)$. The polynomial with quantum number -1 is a linear combination of u^3, t^2u, tv^2 with coefficients $\sqrt{3}+1$, $\sqrt{3}-3$, $6(\sqrt{3}-1)$. Although these polynomials do not have integer coefficients like the ones occurring in the integrability condition (16), they are nonetheless algebraically equivalent to integer coefficient polynomials of order six.

For $N = 4$, 5 there are no good polynomials, while for $N = 6$ one obtains four polynomials. Three out of four are obtained as products of the good polynomials of order $N = 3$:

$$\begin{aligned} \Sigma\hat{\Sigma} &= 3(\Sigma^2 + 6\Sigma\Pi) \\ \Pi\hat{\Pi} &= \Sigma\Pi - 3\Pi^2 = P \\ \Pi\hat{\Sigma} - \hat{\Pi}\Sigma &= 6\Pi\Sigma + 18\Pi^2 - \Sigma^2. \end{aligned} \tag{36}$$

Their duality quantum numbers are $d = 1, 1, -1$, respectively.

Note that the obviously good polynomial of order six $\Pi\hat{\Sigma} + \hat{\Pi}\Sigma$ is a linear combination of $\Pi\hat{\Pi}$ and $\Sigma\hat{\Sigma}$ since

$$-18\Pi\hat{\Pi} + \Sigma\hat{\Sigma} - 3(\Pi\hat{\Sigma} + \hat{\Pi}\Sigma) = 0. \tag{37}$$

Other good polynomials of order six deduced from order three ones are linear combinations of the above three polynomials (36). One has, for example

$$\begin{aligned} \hat{\Sigma}\hat{\Pi} + 27\Sigma\Pi &= \Sigma\hat{\Sigma} + 18\Pi\hat{\Pi}, \\ \hat{\Sigma}^2 + 27\Sigma^2 &= 12\Sigma\hat{\Sigma} - 108\Pi\hat{\Pi}, \end{aligned}$$

$$\hat{\Pi}^2 + 27\Sigma^2 = \frac{1}{3}\Sigma\hat{\Sigma} - 12\Pi\hat{\Pi},$$

$$\hat{\Sigma}\hat{\Pi} - 27\Sigma\Pi = 3(\Pi\hat{\Sigma} - \hat{\Pi}\Sigma),$$

$$\hat{\Sigma}^2 - 27\Sigma^2 = 18(\Pi\hat{\Sigma} - \hat{\Pi}\Sigma),$$

$$\hat{\Pi}^2 - 27\Pi^2 = -(\Pi\hat{\Sigma} - \hat{\Pi}\Sigma).$$

There is also a fourth polynomial that cannot be deduced from the previous three: let us introduce the sixth order symmetric polynomial:

$$\Lambda = \sum_{i \neq j} w^3(i)w^3(j).$$

Obviously,

$$\hat{\Lambda} - 27\Lambda$$

is a well suited polynomial with duality quantum number $d = -1$. Remarkably enough,

$$\hat{\Lambda} + 27\Lambda = \frac{1}{3}\Sigma\hat{\Sigma} - 3\Pi\hat{\Pi}$$

and therefore it is not a new independent polynomial. One remarks that the polynomial Q of Eq. (26) is (up to a factor) equal to $-2(\hat{\Lambda} - 27\Lambda) - 6(\Pi\hat{\Sigma} - \hat{\Pi}\Sigma)$. Among these four polynomials, Q and $P = \Pi\hat{\Pi}$ already occur in the exact solvability conditions (24) and in the "integrability" variables x_1, x_2, x_3, x_4 of Eqs. (27), (28).[25]

Inversion relation symmetries

Next, we introduce in this scheme the effect of the inversion relation, which is a non-linear transformation leading to even more severe restrictions. Because I and J (Eqs. (6), (7)) are related by duality, we can restict ourselves to the study of homogeneous polynomials which transform properly under J:

$$P_N(w(0), \ldots, w(q-1)) = P_N\left(\frac{1}{w(0)}, \ldots, \frac{1}{w(q-1)}\right) Q_N(w(0), \ldots, w(q-1)) \tag{38}$$

where Q_N is a polynomial. Since J is an involution Q_N satisfies

$$Q_N\left(\frac{1}{w(0)}, \ldots, \frac{1}{w(q-1)}\right) Q_N(w(0), \ldots, w(q-1)) = 1. \tag{39}$$

A polynomial has no poles, so the only possible form of Q_N is

$$Q_N = \prod_{i=0}^{q-1} w^{\beta_i}(i).$$

Let us first consider the case when P_N *is also covariant under* C: the only solution is $\beta_0 = \beta_1 = \ldots = 2N/q$. For simplicity, consider a monomial expression

$$M_N = \prod_{i=0}^{q-1} w^{\gamma_i}(i)$$

where the intergers γ_i satisfy

$$\gamma_i \geq 0 \quad \text{and} \quad \sum_{i=0}^{q-1} \gamma_i = N. \tag{40}$$

The transformation of M_N under J leads to consider

$$\prod_{i=0}^{q-1} w^{\gamma_i}(i) \pm \prod_{i=0}^{q-1} w^{2N/q - \gamma_i}(i) \tag{41}$$

with the associated quantum numbers $j = \pm 1$. Since the γ_i's are integers constrained to satisfy

$$0 \leq \gamma_i \leq \frac{2N}{q} \tag{42}$$

and since $2N/q - \gamma_i$ must be also an integer, it follows that $2N$ has to be divisible by q. Therefore, for q odd, the order of a polynomial covariant under C can be only $N = qk$, $k = 1, 2, \ldots$. For q even, the order is restricted to $N = q/2k$. This is an interesting result: for odd q chiral Potts models there are no C-covariant polynomials of order N less than q.

One can try to evaluate the number of sequences $\gamma_0, \ldots, \gamma_{q-1}$ satisfying (40) and (42) and also the ratio of this number over the number of sequences satisfying

(40) but not (42). This ratio is a good measure showing how much the inversion relation restricts the number of good polynomials. Introducing $K = N/q$ this ratio behaves for large q as

$$\left(1+\frac{2N}{q}\right)^q / C^q_{N+q} \sim (R(K))^q$$

where

$$R(K) = \frac{(1+2K)}{K}\left(\frac{K}{K+1}\right)^{K+1}$$

$R(1) = 3/4$, $R(2) = 20/27$, $R(3) = 189/256 \ldots R(K) < 3/4, \ldots R(\infty) = 2/e = 0.7357$. Therefore the ratio decreases exponentially with q.

For large values of q the comptability between the inversion relation and the spin group symmetry (covariance under C) becomes increasingly difficult to satisfy, especially since the degree of the polynomials grows with q. Recalling the example of Au-Yang *et al.*[12] one sees that a way out is to have polynomials of *only a restricted set of variables.* This breaks the spin group symmetry (covariance under C) for each individual polynomial but the whole set restores this symmetry. This allows for polynomials of a smaller degree to appear because now instead of $\beta_0 = \ldots = \beta_{q-1} = 2N/q$ for Q_N one has only $\beta_0 + \ldots + \beta_{q-1} = 2N$. The "dimers" Eq. (41) are replaced by

$$\prod_{i=0}^{q-1} w^{\gamma_i}(i) \pm \prod_{i=0}^{q-1} w^{\beta_i-\gamma_i}(i); \quad 0 \le \gamma_i \le \beta_i. \tag{43}$$

Now N can be almost arbitrary. To illustrate this point consider one of the determinantal compatibility conditions (22) for $q \ge 5$:

$$\begin{aligned}&(w(0)w(3)-w(1)w(2))(w(1)w(4)-w(2)w(3))(1+\omega+\omega^2)\\&-(w(0)w(2)-w^2(1))(w(2)w(4)-w^2(3))(1+\omega+\omega^2)(1+\omega)\\&-(w(0)w(4)-w(1)w(3))(w(1)w(3)-w^2(2))(1+\omega)\end{aligned} \tag{44}$$

where $\omega^q = 1$. This is a fourth order polynomial in only five variables (the parameter space has more than five homogeneous variables). Equation (38) is satisfied with Q_N depending on only five homogeneous variables:

$$Q_N = w(0)w^2(1)w^2(2)w^2(3)w(4).$$

The "dimers" of Eq. (41) form a good basis from the point of view of the inversion relation J and they will strongly "prune" the space of all homogeneous polynomials previously discussed. In fact, the non-linear nature of J is very restrictive (Eq. (42)) and hardly fits with constraints coming from the linear transformations. With this last step we add to the string of quantum numbers also the quantum number j. At the same time the number of incompatible sets of quantum numbers is again increased.

After this lengthy but straightforward procedure there is still one step left, namely the search for *anisotropic* polynomials which are compatible with the *anisotropic* duality and inversion relations $\mathcal{J} = (I, J)$ and with the exchange symmetry S between horizontal and vertical variables (for example Eqs. (23), (24)). It is worth recalling that the situation may be very subtle: these polynomials may break the spin-group symmetry (the covariance under C, for instance), which could be restored for the *set* of anisotropic polynomials or even in a more indirect way via the factorized equations (22).

5. Results for Small q

When performing the previous analysis for small values of q one obtains the following results:

$$q = 2 \text{ (Ising model)}.$$

One actually recovers the well-known modulus of the elliptic functions occurring in the model

$$k = \sinh 2K_1 \sinh 2K_2 = \frac{\hat{w}(0)\,\hat{w}(1)\,\hat{\bar{w}}(0)\,\hat{\bar{w}}(1)}{4w(0)\,w(1)\,\bar{w}(0)\,\bar{w}(1)}. \tag{45}$$

Note that the modulus is not invariant but covariant under duality — it becomes its inverse

$$q = 3.$$

The first homogeneous polynomials occur for $N = 6$ and they are given in the previous section. Note that the polynomial Q (26) is not covariant by J. The first anisotropic algebraic variety occurs at order twelve. It is interesting to recall here the expressions of the variables x_i $(i = 1, \ldots, 4)$, well suited for the star-triangle relation $(\omega^3 = 1)$

$$\begin{aligned}
x_1 &= w^2(0)w(2) + \omega w^2(2)w(1) + \omega^2 w^2(1)w(0), \\
x_2 &= w^2(1)w(0) + \omega w^2(2)w(1) + \omega^2 w^2(0)w(2), \\
-x_3 &= w^2(1)w(2) + \omega w^2(0)w(1) + \omega^2 w^2(2)w(0), \\
-x_4 &= w^2(2)w(0) + \omega w^2(0)w(1) + \omega^2 w^2(1)w(2).
\end{aligned} \tag{46}$$

These variables are well suited for transformations J and D but are actually not invariant under C or R. From the point of view of exact solvability the transformations J and D are more important to take into account than the trivial symmetries C or R

$$q = 4.$$

The first homogeneous polynomials are of order four. The set of polynomials well suited for $(J, C, R = D^2, R_i)$ are the following

$$\begin{aligned}
&w^2(0)w^2(2) \pm w^2(1)w^2(3),\\
&(w^2(0) \pm w^2(2))\,w(1)w(3) \pm (w^2(1) \pm w^2(3))\,w(0)w(2),\\
&\Sigma_{i\neq j} w^2(i)w^2(j),\\
&\Sigma_{i\neq j\neq k} w^2(i)w(j)w(k).
\end{aligned} \tag{47}$$

The last two polynomials are not only invariant under the square group C_{4v} but under the whole group of permutation of the four homogeneous parameters $w(i)$. When one imposes invariance under the duality relation the last two polynomials are ruled out and one only gets the polynomial corresponding to the compatibility relation Eq. (30):

$$\begin{aligned}
&w(1)w(3)\hat{w}(1)\hat{w}(3) + w(1)w(3)\hat{w}(0)\hat{w}(2)\\
&\quad + w(0)w(2)\hat{w}(1)\hat{w}(3) - w(0)w(2)\hat{w}(0)\hat{w}(2).
\end{aligned} \tag{48}$$

Recall that this polynomial is actually covariant under J and hence under transformations (17).

6. Conclusions

In this paper we have proposed a step-by-step systematic approach for the search of integrable algebraic varieties in statistical mechanics of lattice models. The idea is to construct a set of homogeneous polynomials covariant with respect to all simple and less simple symmetries of the parameter space, including lattice symmetries, spin group symmetries, duality transformations and finally, the inversion relation. It is always possible to look carefully at some precise examples. However, as illustrated here, a large number of bifurcations may occur: the polynomials may be invariant under some transformations and covariant under others, or, as explained, they can break individually some spin group symmetry which is recovered later as a compatibility property of the set of such polynomials. Such a large number of possible scenarios make the hope that such a scheme can be automatized and eventually implemented on a symbolic manipulating language very dim.

We want to stress the point that the analysis leading to this set of "good" polynomials introduces a set of appropriate variables not only on special manifolds (the varieties of exact solvability) but in the *whole parameter space* of the model. For example, a renormalization group scheme must be compatible with all the symmetries of the model and in particular with the automorphy group considered in this paper. We expect that the homogeneous polynomials discussed above could also be useful in the context of renormalization transformations. In the two-dimensional anisotropic Ising model, for example, the parameterization of the solution is given in terms of an elliptic function of modulus

$$k = \sinh 2K_1 \sinh 2K_2.$$

A given value of k corresponds to a family of diagonal commuting transfer matrices[26] and the corresponding elliptic curve is preserved under the action of the automorphy group described here. The Landen transformation of elliptic functions

$$k \rightarrow \frac{2\sqrt{k}}{1+k}$$

can be identified with a generator of the renormalization group: it is an infinite order transformation with the critical-variety $k = 1$ as fixed point. In general, we would expect any transformation of the *renormalization group to commute with all birational transformations corresponding to the symmetry group of the model.* If one assumes the renormalization group transformation to be an algebraic transformation, we face the following mathematical problem, similar to the one discussed here: find a new algebraic transformation that commutes with a given set of birational transformations.

In this paper we treated in some detail the chiral q-state Potts model. However, the general framework of our paper has a much larger domain of validity and can be applied to multispin interaction models, to higher dimensions, as well as to Boltzmann weights which are not even functions of the difference between pairs of spins. Recalling the general "interaction models" of I. G. Biggs[27] where the spins might belong to any finite group and the Boltzmann weight corresponding to interactions between nearest neighbors σ_i and σ_j is a representation of the group ($w(\sigma_i, \sigma_j) = w(\sigma_i^{-1}\sigma_j)$), one can show that a sizeable part of the previous analysis is still valid. The duality transformation is now associated with the characters of the group. When the group is solvable, duality can be again a simple transformation.[28] Some general arguments on the necessary conditions for the existence of the star-triangle transformation[29] seem to exclude the case of nonsolvable groups (like, for example, the $\mathcal{A}_5$ alternated group of five elements).

We recall that the automorphy group is a group of birational transformations generated from a finite set of rational *involutions* (and some transformations of higher order). Hence, it is not surprising that in the case of finite groups the set of automorphisms identifies with some group of reflections (for example, the finite group of automorphisms of K3 surfaces coincide with finite groups which have certain types of embeddings into Mathieu groups[30]).

From the point of view of statistical mechanics we have seen that it is straightforward to obtain the generators of this set of birational transformations. However, their relations are not *a priori* known.

We stress again that when the group is finite the parameter space is severely reduced, while for infinite groups it is the set of possible algebraic varieties which is strongly restricted.[14] Even in the "bad" case when the symmetry group contains a free group, the orbits of a point in the parameter space cannot be compared to Julia sets, since all the symmetry transformations have here a unique inverse: there exists a drastic difference between models of statistical mechanics on lattices for which a transfer matrix formalism can be introduced (like assumed in this paper) and the ones for which it cannot (non-Euclidean lattices, self similar lattices, etc.)

Even in simple statistical mechanical problems one hits some pure mathematical problems like dealing with the theory of representation of finite groups or recognizing that two different sets of equations define the same algebraic variety. Statistical mechanics allows for a simple way of having a glimpse at some beautiful mathematical structures: automorphisms of algebraic varieties, Coxeter-like groups[31] defined by a set of finite order generators and their relations.

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