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Staggered lattice models

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Abstract. Some staggered models are studied along three different lines: exact solvability (Yang-Baxter equations), the existence of other exact solutions called disorder solutions and their group of symmetry. The analysis, based on the commutation of row to row transfer matrices of arbitrary size N, suggests that the exactly solvable models are either of non-staggered type or of free-fermion staggered type. In contrast, disorder solutions can easily be exhibited for staggered models: the example of the Ashkin-Teller model is analysed. Finally, the symmetry group of these models is seen to be a straightforward generalisation of the one of homogeneous models.

1. Introduction

Most of the exactly integrable models known today in statistical mechanics or in quantum field theory are associated with the existence of the so-called Yang-Baxter equations (or star-triangle relations). A great number of solutions have been found so far (Zamolodchikov and Zamolodchikov 1979, Schultz 1981, Perk and Schultz 1980, Baxter 1982).

Still, an exhaustive classification remains a challenging open problem for the vertex as well as for the interaction-round-a-face (IRF) spin models (Jimbo and Miwa 1985). One should, however, note two papers, by Belavin and Drinfeld (1983) and Jimbo (1986), which constitute important progress in this direction. The staggered models are an interesting class of problems of statistical mechanics on lattices: many important two-dimensional models like the (non-critical) Potts model or the Ashkin-Teller model can be represented as staggered six- or eight-vertex models (Baxter *et al* 1978, Baxter 1982). However, very few exactly solvable staggered (vertex or IRF) models are known. The known cases of integrability correspond either to free-fermion models (Fan and Wu 1970) or to a vanishing condition of the staggering field (critical Potts models or the self-dual Ashkin-Teller model corresponding to some non-staggered six- or eight-vertex models).

This vanishing field condition is reminiscent of the situation for the two-dimensional Ising model in a magnetic field where the presence of the magnetic field destroys the solvability. Recall that among the few models that have been solved in the presence of an appropriate symmetry-breaking field are the spherical and the KDP ferroelectric models (Montroll 1949, Lieb and Wu 1972).

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One could think of basing a classification of staggered models on the analysis of the Yang-Baxter equations. One can, for instance, imagine replacing each of the three vertices (resp IRF Boltzmann weights) occurring in these equations by four vertices (resp four IRF Boltzmann weights) corresponding to the elementary cell of the staggered model (see figure 1). In fact, the Yang-Baxter equations are much more numerous and complicated to deal with than in the non-staggered case. This is why we suggest another way of tackling this problem.

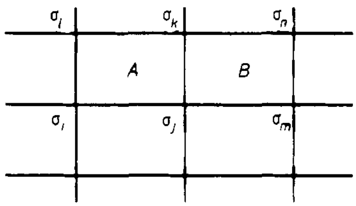


Figure 1. The square lattice showing the typical faces (i, j, k, l) and (j, m, n, k) corresponding to the staggering of the model.

The Yang-Baxter equations are a sufficient and, to some extent, necessary condition for the commutation of transfer matrices of arbitrary size to be satisfied (Lochak and Maillard 1986). The analysis of the integrability of lattice models through the algebraic consequences of the commutation of transfer matrices of small size have been developed previously (Maillard and Garel 1984). This approach also sheds some light on the parametrisation of these models which is a necessary step towards any exact calculation (elliptic or rational uniformisation, etc).

In § 2 of this paper we propose an analysis towards a classification of these staggered models based on these ideas.

Besides the previous exact solutions associated with a Yang-Baxter structure, there exist other exact solutions, the so-called disorder solutions (Stephenson 1970, Enting 1977) that correspond to some trivialisation (some dimensional reduction) of the model on some subvarieties of the parameter space. In § 3 of the paper disorder solutions are considered for staggered models.

Finally, an infinite discrete group of symmetries, the automorphy group, already exhibited on homogeneous models (Jaekel and Maillard 1982), is generalised to the staggered case in § 4.

2. Transfer matrix commutation

2.1. *The staggered IRF model and their transfer matrices*

Staggered models can be introduced on vertex models or interaction-round-a-face (IRF) spin models (Baxter 1980).

Let us recall the IRF model: to each site of the square lattice one associates an Ising spin $\sigma_i = \pm 1$. Let i, j, k, l (resp j, m, n, k) be the four sites round a face of type A (resp type B), as in figure 1. Allow only interactions between spins round a common face. Let us denote by $W_A(\sigma_i, \sigma_j, \sigma_k, \sigma_l)$ (resp $W_B(\sigma_j, \sigma_m, \sigma_n, \sigma_k)$) the Boltzmann weight of the interactions within face (i, j, k, l) (resp (j, m, n, k)).

Then the partition function is

$$Z=\sum_{\sigma}\prod_{A} W_A(\sigma_i,\sigma_j,\sigma_k,\sigma_l)\prod_{B} W_B(\sigma_j,\sigma_m,\sigma_n,\sigma_k). \tag{1}$$

The product is over all faces of type A and also of type B of the square lattice and the sum is over all configurations of all the spins. We restrict ourselves to Boltzmann weights satisfying the spin reversal property

$$W(\sigma_i,\sigma_j,\sigma_k,\sigma_l)=W(-\sigma_i,-\sigma_j,-\sigma_k,-\sigma_l). \tag{2}$$

In this case the model can also be represented as an eight-vertex one (Kadanoff and Wegner 1971). It is more convenient in the framework of the transfer matrix commutation to deal with the IRF spin representation.

Because of this symmetry (2) each of the Boltzmann weights W_A and W_B depends on $2^3=8$ parameters corresponding to the different spin configurations around the faces. These eight homogeneous parameters will be denoted by a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, according to the spin configurations ($i=1$ for type A and 2 for type B)

		(σ_i,σ_j) ++	+−	−+	−−
(σ_l,σ_k)	++	a	b	c	d
	+−	e	f	g	h
	−+	h	g	f	e
	−−	d	c	b	a

Let us now introduce the row to row transfer matrix for a lattice with $2N$ columns and periodic boundary conditions (see figure 2). Let $\sigma_1,\ldots,\sigma_N$ be the spins on the lower rows, and $\sigma'_1,\ldots,\sigma'_N$ the spins on the upper rows. Periodic boundary conditions mean that $\sigma_N=\sigma_0$, $\sigma'_N=\sigma'_0$.

The transfer matrix, which corresponds to N dimers A–B, is a $2^{2N}\times 2^{2N}$ matrix and will be denoted by $T_N(A, B)$. We study at the same time the exact solvability of two different types of staggered models: the models that are staggered in the two directions of the square lattice (denoted type I, like a chessboard, see figure 3(a)) and

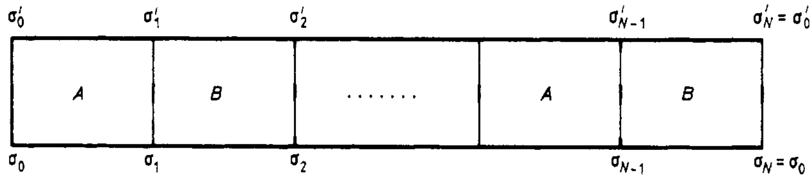
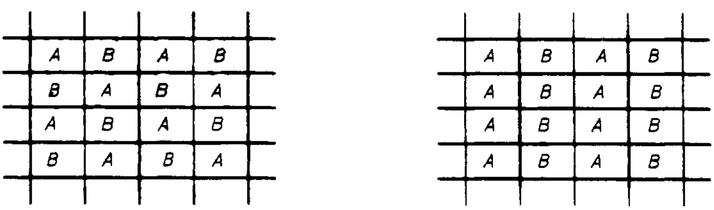


Figure 2. The row to row transfer matrix for a staggered interaction round a face model.



(a) (b)

Figure 3. (a) A chessboard staggering (type I). (b) A one-dimensional staggering (type II).

the models corresponding to a staggering in only one direction (denoted type II, see figure 3(*b*)). The vertex representations of the Potts and Ashkin-Teller models correspond to type I. For the models of type I the transfer matrix equals the product $\mathscr{C}_N(A, B) = T_N(A, B)T_N(B, A)$. For the models of type II one considers directly the row to row transfer matrix $T_N(A, B)$. One should make the following remark for the models of type II: these models are exacty solvable if the column to column transfer matrices $\tilde{T}(A)$ and $\tilde{T}(B)$ belong to the same family of commuting transfer matrices and, in that case, the partition function is just the product of the one for the homogeneous model with Boltzmann weights A only and the one with Boltzmann weights B only. This corresponds to a so-called Z-invariant model (Baxter 1978). This situation will be excluded. In the following we study the possibility that the row to row transfer matrix $T_N(A, B)$ belongs to a family of commuting transfer matrices. Let us also remark that the transfer matrix $T_N(A, B)$ (and therefore the partition function) is left unchanged by the 'gauge' transformations (Gaaf and Hijmans 1975):

$$W_A(\sigma_i, \sigma_j, \sigma_k, \sigma_l)\, W_B(\sigma_j, \sigma_m, \sigma_n, \sigma_k) \to W_A(\sigma_i, \sigma_j, \sigma_k, \sigma_l)\, W_B(\sigma_j, \sigma_m, \sigma_n, \sigma_k)\, \frac{D(\sigma_i, \sigma_l)}{D(\sigma_m, \sigma_n)} \tag{3}$$

and that the partition function (with appropriate periodic boundary conditions) is also left invariant by the transformations

$$W_A(\sigma_i, \sigma_j, \sigma_k, \sigma_l) \to W_A(\sigma_i, \sigma_j, \sigma_k, \sigma_l)\, \frac{\Delta_A(\sigma_l, \sigma_k)}{\Delta_A(\sigma_i, \sigma_j)} \tag{4}$$

or

$$W_B(\sigma_j, \sigma_m, \sigma_n, \sigma_k) \to W_B(\sigma_j, \sigma_m, \sigma_n, \sigma_k)\, \frac{\Delta_B(\sigma_k, \sigma_n)}{\Delta_B(\sigma_j, \sigma_m)}.$$

Here D, Δ_A and Δ_B are arbitrary functions of their arguments. The two transfer matrices we consider from now on depend on two sets of homogeneous parameters.

2.2. *Algebraic invariants*

Basically the idea of our approach is the following: if some (complicated) staggered Yang-Baxter equation is actually satisfied for these models there necessarily exists a family of commuting transfer matrices for arbitrary size N, even for N very small:

$$[\mathscr{C}_N(A, B), \mathscr{C}_N(A', B')] = 0 \qquad N = 1, 2, \ldots \text{ for type I} \tag{5a}$$

$$[T_N(A, B), T_N(A', B')] = 0 \qquad N = 1, 2, \ldots \text{ for type II.} \tag{5b}$$

It can be shown (Lochak and Maillard 1986) that these conditions lead necessarily to a set of algebraic equations in the homogeneous parameters of the model (here $a_i, b_i, c_i, d_i, \ldots, g_i$) of the form $\varphi_\alpha(A, B) = \varphi_\alpha(A', B')$.

The equations $\varphi_\alpha(A, B) = \text{constant}$ thus give a natural foliation of the parameter space of these families of commuting transfer matrices. Let us examine these necessary conditions for small N values. For $N = 1$, both T_1 and $\mathscr{C}_1$ are 4×4 matrices that reduce (because of the spin reversal symmetry) to two 2×2 matrices: the algebraic conditions corresponding to the commutation of 2×2 matrices are easy to write.

For type I we obtain four equations:

$$\frac{a_1a_2+d_1d_2-f_1f_2-g_1g_2}{c_1b_2+b_1c_2}=\text{same expression }(a_1,\ldots,g_i\to a_i',\ldots,g_i') \tag{6}$$

$$\frac{a_1a_2-d_1d_2+f_1f_2-g_1g_2}{e_1h_2+h_1e_2}=\text{same expression }(W\to W') \tag{7}$$

$$\frac{e_1h_2-h_1e_2}{b_1c_2-c_1b_2}=\text{same expression }(W\to W') \tag{8}$$

$$\frac{e_1h_2+h_1e_2}{b_1c_2+c_1b_2}=\text{same expression }(W\to W'). \tag{9}$$

Similarly, for type II we also obtain four equations: equations (6), (8) and (9) are unchanged and (7) is replaced by

$$\frac{a_1a_2-d_1d_2-f_1f_2+g_1g_2}{h_2e_1-h_1e_2}=\text{same expression }(W\to W'). \tag{10}$$

Let us now consider the necessary conditions for $N=2$. The calculations have been performed using the formal language REDUCE 3.1 (Hearn 1984). General results are algebraic expressions that are too large to be written here: from now on we impose the constraints: $g_i=a_i$, $h_i=b_i$, $e_i=c_i$, $f_i=d_i$. This means restricting the general staggered eight-vertex model to the staggered symmetric one. (Do not confuse our parameters a, b, c, d with the canonical parameters of the Baxter model.) Note in this case that equations (6)-(10) are identically satisfied. The two matrices T_2 and $\mathscr{C}_2$ are $2^4\times2^4$ matrices that can be reduced (because of the spin reversal symmetry and the shift invariance of the transfer matrices of two lattice spacing) to six 2×2 matrices and a 4×4 matrix. The algebraic equations corresponding to the commutation of the 2×2 matrices can be written as six equations of the form

$$I_\alpha(a_1,b_1,c_1,d_1,a_2,b_2,c_2,d_2)=I_\alpha(a_1',\ldots)\qquad \alpha=1,\ldots,6. \tag{11}$$

As an example, some of the expressions I_α are given in the appendix for types I and II. All the I_α are homogeneous rational expressions of order two for I_1 and I_2 and of order four for I_3, I_4, I_5 and I_6. In order to deal with the 4×4 matrix we have used the following trick: the commutation $[M,M']=0$ implies the relation $[I+\alpha M+\beta M^2+\gamma M^3, I+\alpha'M'+\beta'M'^2+\gamma'M'^3]=0$ for all α, β, γ, α', β', γ'. One can use these new parameters to obtain zero for all but one coefficients in the same row (or column). The homogeneous algebraic relations $\varphi(a_1,\ldots,d_2)=\varphi(a_1',\ldots,d_2')$ thus obtained are huge (the sum of thousands of terms of degree 24 for the numerator and for the denominator) and will not be given in this paper.

Let us also note that one obtains, *a priori*, an infinite number of algebraic expressions corresponding to the commutation conditions for the infinite number of size N. The integrability corresponds to the remarkable situation where this infinite set of algebraic expressions is redundant and reduces to a finite set of independent algebraic expressions. As remarked in previous papers (Maillard and Garel 1984) the algebraic variety defined by these equations must be invariant under some exact symmetries of the model such as the inversion relation or the duality relation.

In general, the number of equations (11) (and their transforms by these transformations) greatly exceeds the number of parameters of the two transfer matrices (four sets of four homogeneous parameters $a_1,\ldots,d_2,a_1',\ldots,d_2'$). This amounts to saying that the solution is the trivial commutation of a matrix with itself (up to an homogeneous factor): $a_i = a'_i$, $b_i = b'_i$, $c_i = c'_i$ and $d_i = d'_i$. So the question is to impose some conditions on the model (such as symmetries or exclusion of certain spin configurations) to restrict the parameter space to certain subvarieties so that the number of parameters decreases but the number of algebraic expressions I_α, $\varphi_\alpha, \ldots$, that foliate the parameter space decreases even more sharply. The condition of vanishing of the staggering is obviously such a condition. This restricts the parameter space to the four homogeneous parameters a, b, c, d of the Baxter model and there remain only two independent algebraic expressions I_α (I_4 and I_6 previously mentioned).

One checks easily that all these algebraic expressions (11) given in the appendix simplify drastically. In this limit, where the staggering vanishes, the $N = 1$ conditions are automatically satisfied and it is straightforward to verify that for $N = 2$ the algebraic expressions I_α (for type II) simplify to give two independent algebraic expressions, namely:

$$I_4 = \frac{a^2 + d^2 - b^2 - c^2}{2ad} \qquad I_6 = \frac{a^2 + d^2 - b^2 - c^2}{2bc}$$

($1/I_2$ is equal to 0, I_3 and I_5 are equal to 1, I_1 is equal to -1). One recovers (as one should) the well known elliptic uniformisation of the symmetric eight-vertex model (Baxter 1982) as an intersection of two quadrics, $I_4 = \text{constant}$, $I_6 = \text{constant}$, in $\mathbb{P}_3$ (Clebsch's biquadratic).

In view of the algebraic expressions of the appendix it seems at first sight that one is automatically led to ask for a vanishing condition of the staggering. This is actually not true: let us see how some quadratic conditions on the parameters of the model (the free-fermion conditions (Lin and Wang 1977)) drastically simplify the previous algebraic expressions in a quite non-trivial way. Under the quadratic constraints on the two Boltzmann weights

$$a_i^2 + d_i^2 - b_i^2 - c_i^2 = 0 \qquad i = 1, 2 \tag{12}$$

remarkable trivialisations occur. For instance, in the type II model one can verify that

$$2I_5 = 1 \qquad I_4 = 0 \qquad I_6 = 0 \qquad I_3 = 0 \qquad I_1 = -1. \tag{13}$$

Moreover, in order to ensure integrability, it is important that the inverse relation does not generate too many new algebraic invariants. Indeed, one can verify that I_2 is invariant.

Restricted to other quadratic conditions that can be seen as the images of condition (12) through weak graph duality transformations (Baxter 1982), other trivialisations occur; for instance,

$$a_i d_i = b_i c_i \qquad i = 1, 2 \tag{14}$$

leads to

$$I_5 = 1/2I_3 \qquad I_4 = \pm I_6 \qquad 1/I_2 = 0. \tag{15}$$

The inverse relation leaves the set of algebraic invariants globally stable as it should.

One can then see that the type I model with condition (14) corresponds to two different decoupled anisotropic Ising models.

The type II model when condition (14) is satisfied corresponds to two decoupled replicas of the same checkerboard Ising model. Conditions (12) and (14) correspond to exactly solvable free-fermion models. Besides these simple free-fermion cases we have not been able to see any other possibilities of exactly solvable staggered models.

3. Disorder solutions on staggered models

3.1. Disorder solutions

Solutions related to the Yang–Baxter equations are actually not the only exact solutions that have been obtained on lattice models of statistical mechanics; the so-called disorder solutions also exist (Stephenson 1970, Rujàn 1984).

These two sets of exact solutions are very different in nature: the disorder solutions have, because of some effective dimensional reductions, a much more simple analytical structure (Georges *et al* 1986). For many examples such a property is provided by a simple local condition bearing on the Boltzmann weight of the elementary cell generating the lattice. This local condition enables one to get an eigenvector of the transfer matrix that is a simple direct product of factors.

Even if disorder solutions can be formulated in the framework of IRF models (Baxter 1984) we use here the vertex representation which is better suited for the present study. The case of the staggered sixteen-vertex model is described by two 4×4 matrices associated with the two types of Boltzmann weights of the model. Let us denote by Ω_1 and Ω_2 these two 4×4 matrices. We now impose the following local conditions. The image of a pure tensor product by these two matrices is also a pure tensor product, i.e.

$$\Omega_1 U_1 \otimes V_1 = \lambda_1 U_2 \otimes V_2 \tag{16a}$$

$$\Omega_2 U_2 \otimes V_2 = \lambda_2 U_1 \otimes V_1 \tag{16b}$$

with

$$U_1 \otimes V_1 = \begin{bmatrix} 1 \\ p_1 \\ q_1 \\ p_1 q_1 \end{bmatrix} \qquad U_2 \otimes V_2 = \begin{bmatrix} 1 \\ p_2 \\ q_2 \\ p_2 q_2 \end{bmatrix}.$$

Equations (16a) and (16b) are two sets of four homogeneous conditions leading, after elimination of p_1, p_2, q_1, q_2, to a subvariety of codimension two of the parameter space of this staggered model. The two previous conditions imply relations between the diagonal transfer matrices $T(\Omega_1)$ and $\hat{T}(\Omega_2)$, corresponding to the two layers of the staggered model, and two vectors that are simple tensor products of the U_i and V_i:

$$|\psi_i\rangle = U_i \otimes V_i \otimes U_i \otimes V_i \otimes \ldots \qquad i = 1, 2.$$

One has the relations

$$T(\Omega_1)|\psi_1\rangle = \lambda_1^N |\psi_2\rangle$$

$$\hat{T}(\Omega_2)|\psi_2\rangle = \lambda_2^N |\psi_1\rangle$$

where N denotes the number of vertices in the two layers.

The diagonal transfer matrix corresponding to two layers is the product $T = T(\Omega_1)\hat{T}(\Omega_2)$ (see figure 4) and has $|\psi_1\rangle$ for eigenvector with eigenvalue $(\lambda_1\lambda_2)^N$. If $|\psi_1\rangle$ has

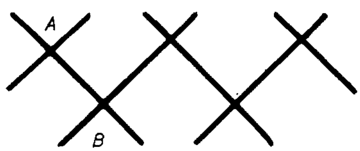


Figure 4. Diagonal transfer matrix corresponding to two layers.

some overlap with the maximum eigenvector of T (this is true in general) we can deduce that the partition function per site will be $(\lambda_1\lambda_2)^{1/2}$. In this way one obtains the disorder solutions for the partition function of these staggered vertex models.

Let us now consider the staggered symmetric eight-vertex model. The matrices Ω_i are equal to

$$\Omega_i = \begin{bmatrix} a_i & 0 & 0 & d_i \\ 0 & b_i & c_i & 0 \\ 0 & c_i & b_i & 0 \\ d_i & 0 & 0 & a_i \end{bmatrix} \qquad i = 1, 2 \tag{17}$$

(here the notation is the canonical one for vertex models).

In this case one has the following results.

(i) For $p_1 = q_2 = +1$, $q_1 = p_2 = +1$, one has $\lambda_1 = a_1 + d_1$ and $\lambda_2 = a_2 + d_2$ when the following two conditions are satisfied:

$$a_1 + d_1 = b_1 + c_1 \tag{18a}$$

$$a_2 + d_2 = b_2 + c_2. \tag{18b}$$

(ii) For $p_1 = q_2 = -1$, $q_1 = p_2 = +1$, one has $\lambda_1 = a_1 - d_1$ and $\lambda_2 = a_2 - d_2$ when the following two conditions are satisfied:

$$a_1 + b_1 = c_1 + d_1 \tag{19a}$$

$$a_2 + b_2 = c_2 + d_2. \tag{19b}$$

Let us consider the important subcase of the Ashkin–Teller model represented as a staggered symmetric eight-vertex model (Baxter 1982):

$$a_1 = a_2 = a \qquad b_1 = b_2 = b \qquad c_1 = d_2 = c \qquad c_2 = d_1 = d.$$

One has the following result for $p_1 = q_2 = -1$ and $q_1 = p_2 = +1$:

$$(\lambda_1\lambda_2)^{1/2} = a - d \tag{20a}$$

when

$$a + b = c + d. \tag{20b}$$

A more specific subcase corresponds to the symmetric Ashkin–Teller model that can also be represented as a staggered six-vertex model (Kohmoto *et al* 1981):

$$a_1 = b_2 = a \qquad a_2 = b_1 = b \qquad c_1 = c_2 = c \qquad d_1 = d_2 = 0.$$

In this case, for $p_1 = q_2 = -1$ and $q_1 = p_2 = +1$,

$$(\lambda_1\lambda_2)^{1/2} = (ab)^{1/2} \tag{21a}$$

when

$$a + b = c. \tag{21b}$$

As far as the partition function is concerned, condition (21b) reduces the symmetric Ashkin–Teller model to a four-state scalar (standard) Potts model. Obviously a trivial expression like $(\lambda_1\lambda_2)^{1/2}$ is different from the partition function of the four-state Potts model (known only for $a = b$ (Baxter *et al* 1978)). This situation corresponds to a case where an eigenvector for the two-layer diagonal transfer matrix can be exhibited but the corresponding eigenvalue is not the largest eigenvalue; $a + b = c$ does not correspond to a disorder solution where the partition function can be calculated exactly. However, for this 'pseudo' disorder solution one sees that a trivialisation of one eigenvalue in the whole spectrum of the two-layer diagonal transfer matrix occurs. Let us also note that on this particular line the symmetry group of the spin $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is enlarged to $\mathbb{Z}_4$ (see also Zamolodchikov and Monarstirskii 1979).

These disorder (or 'pseudo' disorder) solutions also shed some light on the phase diagram of the models (Rujàn 1984, Domany and Gubernatis 1985, Blöte *et al* 1986). For these varieties the partition function (or just one of the eigenvalues) is a perfectly simple analytic expression. If there is an intersection of these varieties with the critical manifold, this is an indication of a multicritical point for which a cancellation of singularities could occur. Actually, for the symmetric Ashkin-Teller model the tricritical point of the phase diagram lies on the intersection of the self-dual line with the 'pseudo' disorder line (21*b*) (see figure 5).

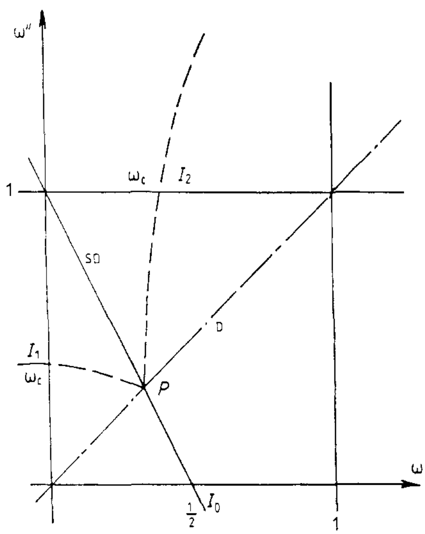


Figure 5. Phase diagram of the symmetric Ashkin-Teller model and the 'pseudo' disorder line. SD: the self-dual line of the Ashkin-Teller model. D: the pseudo disorder line. $PI_0 \cup PI_1 \cup PI_2$: the critical lines. $\omega = b/(a+c)$, $\omega'' = (a-c)/(a+c)$.

3.2. *Disorder varieties and algebraic varieties*

When the model is exactly solvable, it has been remarked (Baxter 1986, Maillard 1986) that a relation between the parametrisation of the model (in terms of algebraic varieties as illustrated in § 2) and the expression of these disorder varieties should exist. The algebraic varieties associated with the disorder varieties must correspond to some trivialisation of the previous foliation of the parameter space in terms of algebraic varieties. For the Baxter model, for instance, the modulus of the elliptic functions that occur is given by the following equation (Baxter 1982, p 246):

$$k+\frac{1}{k}-2=\frac{(a-b-c-d)(a-b+c+d)(a+b-c+d)(a+b+c-d)}{4abcd}. \tag{22}$$

$k=-1$ corresponds to such a trivialisation of the elliptic parametrisation (the image of the modulus k under the Landen transformation $k \to 2\sqrt{k}/(1+k)$ becomes infinite). This $k=-1$ condition is a quartic algebraic condition that reduces to a set of linear conditions on a, b, c and d:

$$a+d=b+c \tag{23}$$

$$a+c=b+d \tag{24}$$

$$a+b=c+d. \tag{25}$$

The first two conditions are just the known disorder conditions on the model. Equation (25) is another one; it corresponds to the previous situation of a simple eigenvector for the two-layer diagonal transfer matrix ($a_1=a_2=a$, $b_1=b_2=b$, $c_1=c_2=c$, $d_1=d_2=d$ and $p_1=q_2=-1$, $p_2=q_1=+1$, $\lambda_1=\lambda_2=(a-d)$).

Similar kinds of trivialisations of the algebraic varieties detailed in § 2 are expected for the staggered models. For instance, in the free-fermion cases previously described and, in particular, in the case where the staggered model reduces to decoupled checkerboard Ising models, the associated modulus k vanishes (Jaekel and Maillard 1984).

4. The symmetry group of lattice models

4.1. The symmetry group of the staggered models

An exact functional relation for the partition function, first introduced in the framework of exactly solvable models, is known to exist for any values of the parameters of the vertex or IRF spin models; it is called the inversion relation (Stroganov 1979, Baxter 1980) and is derived from a simple geometrical relation of the local Boltzmann weights (Jaekel and Maillard 1982).

Let us recall briefly some basic results for a quite general homogeneous sixteen-vertex model. Its sixteen parameters can be seen as the coefficients of a 4×4 matrix Ω:

		(k,l) ++	+−	−+	−−
(i,j)	++	ω_1	ω_2	ω_3	ω_4
	+−	ω_5	ω_6	ω_7	ω_8
	−+	ω_9	ω_{10}	ω_{11}	ω_{12}
	−−	ω_{13}	ω_{14}	ω_{15}	ω_{16}

The functional inversion relation relates the partition function for the parameters corresponding to Ω and its analytic continuation for the parameters corresponding to the inverse matrix Ω^{-1}:

$$Z(\{\Omega\})Z(\{\Omega^{-1}\})\overset{\Omega \xrightarrow{I} \Omega^{-1}}{=}1. \tag{26}$$

To this symmetry one must add some obvious symmetries of the model (see figure 6). For instance, the symmetry with respect to the line D_1 corresponds to the simultaneous

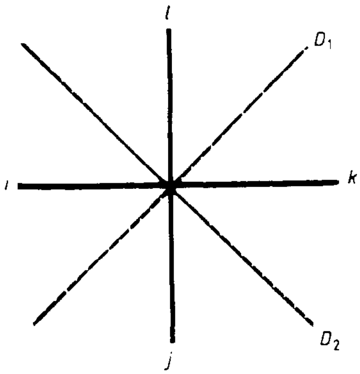


Figure 6. Symmetries of the square vertex model.

permutation $i \leftrightarrow j$ and $k \leftrightarrow l$ that is associated with the following transformation $\Omega \rightarrow^{S_1} P\Omega P$, with

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The partition function is obviously invariant under such a transformation. It is also invariant under a rotation of π of the lattice: this rotation corresponds to the permutation $(i, j) \leftrightarrow (k, l)$, i.e. to transpose the matrix $\Omega \rightarrow^{S_2} {}^t\Omega$. The symmetry with respect to the line D_2 is associated with the transformation $\Omega \rightarrow^{S_3} P^t\Omega P$. Let us denote by S_4 the transformation corresponding to the symmetry of rotation of $\pi/2$ of the lattice. S_4 and the transformations S_1, S_2, S_3 generate the symmetry group of the square. One verifies immediately that the transformations I, S_1, S_2 and S_3 commute. This is not the case for S_4. Because of these properties, it is easy to see that any element of the group generated by S_1, S_2, S_3, S_4 and I can be written as $s(S_4 I)^n$, $n \in \mathbb{Z}$, where s denotes an element of the group of the square. One can also add to these symmetries the 'gauge' symmetries (weak graph duality) that also leaves the partition function invariant (Gaaf and Hijmans 1975). These transformations correspond to a conjugation by a matrix that is a tensor product of two 2×2 matrices $\Omega \rightarrow \rho \otimes \sigma \Omega \rho^{-1} \otimes \sigma^{-1}$. The transformations S_i act simply on (ρ, σ):

$$(\rho, \sigma) \rightarrow (\sigma, \rho)$$
$$(\rho, \sigma) \rightarrow (\rho^{-1}, \sigma^{-1})$$
$$(\rho, \sigma) \rightarrow ({}^t\rho, {}^t\sigma)$$
$$\vdots$$

It is thus a straightforward matter to see that any element of the previous group combined with this gauge group can be written $\mathcal{g}s(S_4 I)^n$ where $\mathcal{g}$ denotes an element of the gauge group. Let us now consider a staggered model: this corresponds to dealing with a larger elementary cell made up of four different Boltzmann weights A, B, C, D (see figure 7) instead of only one elementary vertex. The parameter space represented by a 4×4 matrix Ω is now replaced by four such matrices. It is easy to see that any element of the previous group enlarged by these new symmetries of the square will

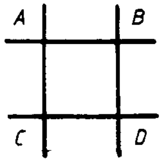


Figure 7. Elementary cell depending on four Boltzmann weights corresponding to a staggered vertex model.

correspond (up to a permutation of the A, B, C and D) to a transformation $g_A s_A (S_4 I)^{n_A}$ on Ω_A where g_A denotes an element of the gauge group (and similarly on Ω_B, Ω_C and Ω_D). In fact one can be easily convinced that $n_A = n_D = n_B = n_C$. Up to a semidirect product by a simple finite group and the gauge groups, the group is isomorphic to $\mathbb{Z}$ (this result should be compared with the group analysis performed on the checkerboard Potts model (Jaekel and Maillard 1984)). Along this line one would have similar results for more general two-dimensional models associated with larger elementary cells.

This is in contrast with the situation for the three-dimensional models. Let us consider, for instance, a Boltzmann weight corresponding to a three-dimensional vertex model. The 4×4 matrix Ω is now replaced by a $2^3\times2^3$ matrix corresponding to the 'in' and 'out' triplets (i, j, k) and (l, m, n) (see figure 8).

As before, the inversion relation corresponds to replacing Ω by its inverse matrix. The partition function is also invariant under the symmetries of the cube. These symmetries are represented by permutations of the coefficients of the matrix. Only some of these transformations on Ω commute with the inversion relation I (for instance, the permutation $i\leftrightarrow j$ and $l\leftrightarrow m$).

Let us denote by R_1 a rotation by $\pi/2$ around the (i, l) axis and by R_2 a rotation by $\pi/2$ around the (j, m) axis. Let us also have $U = R_1 I$ and $V = R_2 I$. In general, the two transformations U and V generate an infinite non-solvable group of transformations. The group is a free group: in general there is no relation at all between U and V. It is not even clear if this group of transformations is discrete or continuous. This is a drastic difference between dimensions two and three.

This remark has very important consequences on the existence of exactly solvable models in three dimensions. As seen in previous papers (Maillard and Garel 1984, Maillard 1986) and also in § 2, the exactly solvable models are necessarily parametrised by algebraic varieties (the expressions $I_1, I_2, \ldots,$ of § 2 for instance). These varieties

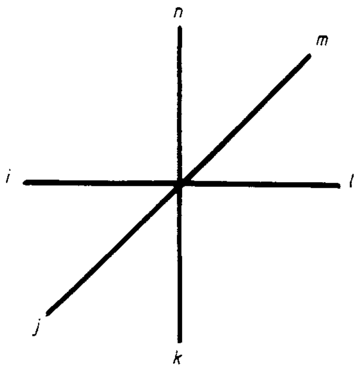


Figure 8. Elementary vertex for a cubic lattice.

have a set of automorphisms corresponding to the previous infinite group. It is very unlikely that any algebraic varieties could have such a large set of automorphisms (like such an infinite set of non-linear transformations). This could confirm a somewhat disappointing situation: the exact solvability could be basically a two-dimensional feature.

5. Conclusion

An analysis based on the commutation of transfer matrices of arbitrary size N for small N has been performed (using formal language calculations) in the case of the staggered eight-vertex model. The necessary conditions corresponding to these finite-size transfer matrix commutations suggest that these commutations correspond either to an absence of staggering or to staggered free-fermion models. This analysis emphasises several remarkable features of the parametrisation of an exactly solvable model: the set of algebraic equations deduced from the commutation of finite-size transfer matrices has to be redundant and the algebraic variety defined by these equations has to be invariant under the inversion relation and the other exact symmetries of the problem. These are extremely drastic constraints and it is enlightening to see how the free-fermion conditions fulfil these constraints. This analysis can of course be generalised in a straightforward manner for more complicated elementary cells (three kinds of Boltzmann weights instead of two) or applied to a diagonal transfer matrix formalism instead of a row to row (or column to column) transfer matrix formalism.

On the other hand, the disorder solutions of the staggered models are not so exceptional. For instance, we have been able to exhibit quite easily a 'pseudo' disorder solution for the symmetric Ashkin–Teller model (we are able to exhibit an eigenvector and the corresponding eigenvalue of the two-layer diagonal transfer matrix of this staggered vertex model).

Finally, the symmetry group of the staggered model (generated by the inversion relation and the other symmetries of the model) is seen to be a simple extension of the symmetry group of the non-staggered model. In addition, one remarks on a fundamental difference between the symmetry groups of the two-dimensional models (even very general) and of the three-dimensional models: for three-dimensional models this group is in general too large to allow any exactly solvable model.

In conclusion, the analysis performed in this paper enables us to see how the generalisation corresponding to the staggering (therefore drastically enlarging the parameter space) affects all these exact structures: only a small number of new solutions of the Yang–Baxter equations are obtained through this generalisation. On the other hand, more simple exact structures like the disorder solutions and the symmetry group support this generalisation very well.

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Appendix

Type I:

$$I_1 = \frac{(a_1^2+d_1^2)(b_2^2+c_2^2)-(b_1^2+c_1^2)(a_2^2+d_2^2)}{2(d_1d_2a_1a_2-b_1b_2c_1c_2)}$$

$$I_2 = \frac{(a_1^2+d_1^2)(a_2^2+d_2^2)-(b_1^2+c_1^2)(a_2^2+d_2^2)}{2(a_1d_1b_2c_2-a_2d_2b_1c_1)}.$$

Type II:

$$I_1 = \frac{b_2^2d_1^2-b_1^2a_2^2+c_2^2a_1^2-c_1^2d_2^2}{b_2^2a_1^2-b_1^2d_2^2+c_2^2d_1^2-c_1^2a_2^2}$$

$$I_2 = \frac{b_1^2b_2^2-b_2^2c_1^2-b_1^2c_2^2+c_2^2c_1^2+d_2^2d_1^2-d_2^2a_1^2-d_1^2a_2^2+a_2^2a_1^2}{b_2c_2a_1d_1-b_1c_1a_2d_2}$$

$$I_3 = \frac{m_{53}m_{36}+m_{57}m_{76}}{m_{57}m_{35}+m_{67}m_{36}} \qquad I_4 = \frac{m_{53}m_{35}-m_{67}m_{76}}{m_{57}m_{35}+m_{67}m_{36}}$$

$$I_5 = \frac{m_{14}m_{42}-m_{12}(m_{18}-m_{11})}{m_{24}m_{12}+m_{14}(m_{18}+m_{11})} \qquad I_6 = \frac{m_{11}^2-m_{18}^2-m_{24}m_{42}}{m_{14}m_{42}-(m_{18}-m_{11})m_{12}}.$$

Matrix elements m_{ij} are given by the following formulae:

$$m_{11} = a_2^2a_1^2-d_2^2d_1^2 \qquad m_{12} = 2(b_2c_1a_2a_1-b_1c_2d_2d_1)$$

$$m_{14} = 2(-b_2c_1d_2d_1+b_1c_2a_2a_1) \qquad m_{18} = b_2^2c_1^2-b_1^2c_2^2$$

$$m_{24} = -b_2^2d_1^2+b_1^2a_2^2+c_2^2a_1^2-c_1^2d_2^2 \qquad m_{35} = -b_2^2d_1^2-b_1^2a_2^2+c_2^2a_1^2+c_1^2d_2^2$$

$$m_{36} = 2(-b_2b_1d_1a_2+c_2c_1d_2a_1) \qquad m_{42} = b_2^2a_1^2-b_1^2d_2^2-c_2^2d_1^2+c_1^2a_2^2$$

$$m_{53} = b_2^2a_1^2+b_1^2d_2^2-c_2^2d_1^2-c_1^2a_2^2 \qquad m_{57} = 2(b_2b_1d_2a_1-c_2c_1d_1a_2)$$

$$m_{67} = b_2^2b_1^2-c_2^2c_1^2+d_2^2a_1^2-d_1^2a_2^2 \qquad m_{76} = -b_2^2b_1^2+c_2^2c_1^2+d_2^2a_1^2-d_1^2a_2^2.$$

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