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Quasi integrability of the sixteen-vertex model

M.P. Bellon, J.-M. Maillard and C.-M. Viallet
Laboratoire de Physique Théorique et des Hautes Energies [1], *Université de Paris 6–Paris 7, Tour 16, 1er étage, boîte 126, 4 Place Jussieu, F-75252 Paris Cedex 05, France*

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We analyze the symmetries of the sixteen-vertex model. We prove the existence of a natural parametrization of the parameter space of the model by elliptic curves, grounding the inversion trick for the exact calculation of the partition function. We proceed with a "pre-Bethe-ansatz" system of equations whose analysis produces an algebraic modular invariant and yields candidates for criticality and disorder conditions.

1. Introduction

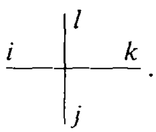
The results we describe here concern statistical mechanics on lattices (vertex models for the case). We emphasize the existence of a large group of symmetry acting on the parameter space of lattice models, and analyze these symmetries in the case of the general sixteen-vertex model on a two-dimensional square lattice. Our results take their roots in the theory of integrable lattice models [1] in statistical mechanics and field theory, but concern very general (integrable or not) models, and in particular extend to higher dimensions.

One of the recurrent features of integrability is the emergence of algebraic curves in the parametrization of the models. These curves are the core of the resolution of the models, be it through the Bethe ansatz [2,3], the description of the solutions of the Yang–Baxter equations, or the implementation of inversion relations [4,5]. We have shown recently how these curves may be *generated* by the action of an infinite group (denoted Γ in the sequel), which happens to be the symmetry group of the Yang–Baxter equations, and we have correspondingly introduced the notion of quasi-integrability (see refs. [6–10]). What is essential is that the group Γ not only acts as a symmetry of integrability but *is a symmetry group of the whole phase diagram whatever the model is*. This is what we use here in the case of the sixteen-vertex model on a square lattice [11], aiming at exact results *beyond the border of integrability*.

We show that the orbits of the group Γ stay on elliptic curves in the parameter space, furnishing the most appropriate parametrization to describe the physics of the model. We give the equations of these curves. We then show how the action of Γ is compatible with weak graph duality (gauge) symmetries [12,11] and give a full set of algebraically independent gauge invariants, with remarkable transformations properties under Γ. We proceed with a "pre-Bethe-ansatz" system of equations whose analysis produces an algebraic *modular invariant*, and candidates for criticality and disorder conditions.

2. The model

The model is a vertex model on a square lattice, with spins taking two values on each bond. The Boltzmann weights are arranged in a 4×4 matrix R of entries r^{ij}_{kl} corresponding to the configuration



We alternatively use for R the notation

[1] Unité associée au CNRS (U.A. 280).

$$R=\begin{pmatrix} r^{11}_{11} & r^{11}_{12} & r^{11}_{21} & r^{11}_{22} \\ r^{12}_{11} & r^{12}_{12} & r^{12}_{21} & r^{12}_{22} \\ r^{21}_{11} & r^{21}_{12} & r^{21}_{21} & r^{21}_{22} \\ r^{22}_{11} & r^{22}_{12} & r^{22}_{21} & r^{22}_{22} \end{pmatrix}=\begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix}.$$

The model is insensitive to a rescaling of all entries by a common factor. The complexified parameter space is thus the projective space CP_{15}. A number of transformations act on R: the group Γ of symmetries of the Yang–Baxter equations, and the group G of gauge transformations. We shall also describe in section 5 a larger group G_{Bethe}, containing G as a subgroup.

2.1. The group Γ

It is generated by involutions represented by *non-linear* (birational) transformations [6–10]: the inversion I (inverse for the 4×4 product in $\mathcal{M}$), the partial transpositions t_1 and t_2, and the full transposition t defined by

$$\sum_{\alpha\beta} (IR)^{ij}_{\alpha\beta}\cdot r^{\alpha\beta}_{kl}=\lambda\delta^i_k\delta^j_l, \tag{1}$$

with λ an arbitrary multiplicative factor, and

$$(t_1R)^{ij}_{kl}=r^{kj}_{il}, \quad (t_2R)^{ij}_{kl}=r^{il}_{kj}, \quad t=t_1t_2. \tag{2}$$

The matrix R is an element in $\mathcal{M}=\mathcal{A}\otimes\mathcal{A}$ with $\mathcal{A}$ the algebra of 2×2 matrices. The indices 1 and 2 recall that we act only on the first (respectively) second factor $\mathcal{A}$ in $\mathcal{M}$. *The inversion I commutes with t but does not commute with t_1 or t_2.*

2.2. The group G

In contrast, the gauge group $G=sl_2\times sl_2$ acts *linearly* on R by similarity transformations (see ref. [11] for details):

$$\text{If } g=g_1\times g_2, \quad g(R)=g_1^{-1}g_2^{-1}\cdot R\cdot g_1g_2. \tag{3}$$

3. Γ-orbits

The pictorial analysis already proposed in refs. [6,9] produces fig. 1 as a typical picture of the orbits of Γ (projection of the orbit on a coordinate two-plane).

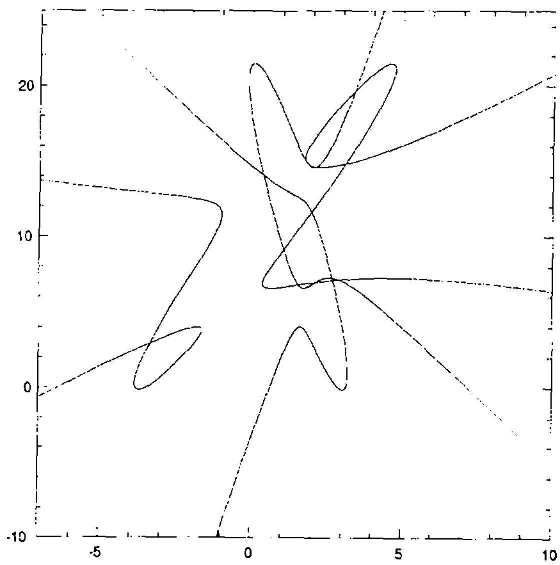


Fig. 1. Orbit of Γ.

The picture demonstrates the existence of *curves* on which the discrete orbit is dense. We have found 18 algebraically related quadratic polynomials (p_1, ..., p_{18}) having simple transformations under Γ. Setting

$$l_1=a_1a_4-a_2a_3, \quad l_2=d_1d_4-d_2d_3,$$
$$l_3=b_1b_4-b_2b_3, \quad l_4=c_1c_4-c_2c_3,$$
$$l_5=a_1d_1-b_1c_1, \quad l_6=a_2d_2-b_2c_2,$$
$$l_7=a_3d_3-b_3c_3, \quad l_8=a_4d_4-b_4c_4,$$
$$l_9=a_1d_4-b_2c_3+a_4d_1-b_3c_2,$$
$$l_{10}=a_1b_4-b_2a_3+a_4b_1-b_3a_2,$$
$$l_{11}=c_1d_4-d_2c_3+c_4d_1-d_3c_2,$$
$$l_{12}=a_1c_4-a_2c_3+a_4c_1-a_3c_2,$$
$$l_{13}=b_1d_4-b_2d_3+b_4d_1-b_3d_2,$$
$$l_{14}=a_1d_2-b_2c_1+a_2d_1-b_1c_2,$$
$$l_{15}=a_1d_3-b_1c_3+a_3d_1-b_3c_1,$$
$$l_{16}=a_3d_4-b_4c_3+a_4d_3-b_3c_4,$$
$$l_{17}=a_2d_4-b_2c_4+a_4d_2-b_4c_2,$$
$$l_{18}=b_4c_1+c_4b_1-a_2d_3-a_3d_2,$$

and

$$p_1=l_1+l_2,\quad p_2=l_1-l_2,\quad p_3=l_3+l_4,$$
$$p_4=l_3-l_4,\quad p_5=l_5+l_8,\quad p_6=l_5-l_8,$$
$$p_7=l_6+l_7,\quad p_8=l_6-l_7,\quad p_9=l_9,\quad p_{18}=l_{18},$$
$$p_{10}=l_{10}+l_{11}+l_{12}+l_{13},\quad p_{11}=l_{10}+l_{11}-l_{12}-l_{13},$$
$$p_{12}=l_{10}-l_{11}+l_{12}-l_{13},\quad p_{13}=l_{10}-l_{11}-l_{12}+l_{13},$$
$$p_{14}=l_{14}+l_{15}+l_{16}+l_{17},\quad p_{15}=l_{14}+l_{15}-l_{16}-l_{17},$$
$$p_{16}=l_{14}-l_{15}+l_{16}-l_{17},\quad p_{17}=l_{14}-l_{15}-l_{16}+l_{17},$$

then

$$I(p_i)=\lambda\cdot\epsilon_i\cdot p_i\,, \tag{4}$$

$$t_1(p_i)=\pm p_i,\quad t_2(p_i)=\pm p_i\,. \tag{5}$$

Appropriate ratios of the p_i's yield 17 algebraically related invariants of Γ. The equations of the curves of fig. 1 are obtained by setting these invariants to constants. They generically have the infinite group Γ as group of automorphisms, and are consequently of genus 0 or 1. The p_i's verify three algebraic identities which may be written as equalities between polynomial expressions $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$ written indifferently in respectively $\mathcal{P}_1=\{p_1, p_2, p_3, p_4, p_{10}, p_{11}, p_{12}, p_{13}, p_9+p_{18}\}$ and $\mathcal{P}_2=\{p_5, p_6, p_7, p_8, p_{14}, p_{16}, p_{15}, p_{17}, p_9-p_{18}\}$:

$$\mathcal{K}_i(\mathcal{P}_1)=\mathcal{K}_i(\mathcal{P}_2),\quad i=2,3,4\,, \tag{6}$$

with

$$\mathcal{K}_2(\mathcal{P}_1)=4(-p_1^2+p_2^2-p_3^2+p_4^2)$$
$$+p_{10}^2+p_{11}^2-p_{12}^2-p_{13}^2-4(p_9+p_{18})^2\,,$$
$$\mathcal{K}_3(\mathcal{P}_1)=\tfrac{1}{2}(p_1-p_3)(p_{10}^2-p_{11}^2)$$
$$+\tfrac{1}{2}(p_1+p_3)(p_{12}^2-p_{13}^2)$$
$$+p_{13}p_{11}p_2-p_{12}p_{11}p_4+p_{13}p_{10}p_4-p_{12}p_{10}p_2$$
$$+2(p_9+p_{18})(p_2^2+p_3^2-p_1^2-p_4^2)\,,$$

$\mathcal{K}_4=$ order four polynomial.

There are 18 more quadratic polynomials $q_1, ..., q_{18}$ with transformation properties like (4), but not invariant by t_1 or t_2. These polynomials can be obtained from the p_i's by exchanging the two middle columns of R.

4. G-orbits

The action of G and Γ do not commute. However G and I do commute and t_1 (respectively t_2) sends orbits of G onto orbits of G. Consequently the action of Γ projects down to the quotient space $\mathcal{Q}=\mathcal{M}/\mathrm{G}$. One should keep in mind that this quotient space may not be a smooth manifold, but rather a stratified space since all gauge orbits do not have the same dimension.

We use the previous results to construct a number of G-invariants. Indeed G acts *linearly* on $p_1, ..., p_{18}$, (respectively $q_1, ..., q_{18}$). Among the G- and Γ-invariants of the representation on the p_i's, we get

$$\mathcal{I}_1=2p_1+p_9+p_{18},\quad \mathcal{I}_2=2p_5+p_9-p_{18}\,,$$
$$\mathcal{I}_3=4p_2^2+p_{10}^2-p_{13}^2,\quad \mathcal{I}_4=4p_6^2+p_{14}^2-p_{17}^2\,,$$
$$\mathcal{I}_5=24t_1^2+8t_3^+t_3^- -2t_5^+t_5^-\,,$$
$$\mathcal{I}_6=24t_2^2+8t_4^+t_4^- +2t_6^+t_6^-\,,$$
$$\mathcal{I}_7=t_3^+(t_5^-)^2+t_3^-(t_5^+)^2$$
$$+16t_1^3-16t_1t_3^+t_3^- -2t_1t_5^+t_5^-\,,$$
$$\mathcal{I}_8=t_4^+(t_6^-)^2+t_4^-(t_6^+)^2$$
$$+16t_2^3-16t_2t_4^+t_4^- +2t_2t_6^+t_6^-\,,$$

with

$$t_1=\tfrac{1}{3}(p_1-p_9-p_{18}),\quad t_2=\tfrac{1}{3}(p_5-p_9+p_{18})\,,$$
$$t_3^\pm=p_3\pm p_4,\quad t_4^\pm=p_7\pm p_8\,,$$
$$t_5^\pm=p_{11}\pm p_{12},\quad t_6^\pm=p_{15}\pm p_{16}\,.$$

The G-invariants $\mathcal{I}_i$ $(i=1,...,8)$ form a system of rank 7, and appropriate ratios yield six G- *and* Γ-invariants.

Similarly the simplest G- *and* I-invariants we get from the q_i's are

$$\mathcal{J}_1=4(q_1^2-q_4^2+q_5^2-q_7^2)$$
$$+8q_9^2-q_{11}^2-q_{12}^2+q_{14}^2+q_{17}^2\,,$$
$$\mathcal{J}_2=4(q_2^2-q_3^2+q_6^2-q_8^2)$$
$$+8q_{18}^2-q_{10}^2-q_{13}^2+q_{15}^2+q_{16}^2\,.$$

We may complement them by any t_1 (and t_2) G-invariant of the form $\mathrm{tr}(R^q)$, the simplest being the trace $\mathrm{tr}(R)$. We thus get nine algebraically independent gauge invariants, which define a finite covering over the quotient space Q.

5. Towards Bethe ansatz

One of the keys to the Bethe ansatz is the existence (see eqs. (B.10), (B11a) in ref [2]) of vectors which are pure tensor products (of the form $v\otimes w$) and which R maps onto the pure tensor product $v'\otimes w'$. If

$$v=\begin{pmatrix}1\\p\end{pmatrix},\quad w=\begin{pmatrix}1\\q\end{pmatrix},\quad v'=\begin{pmatrix}1\\p'\end{pmatrix},\quad w'=\begin{pmatrix}1\\q'\end{pmatrix},$$

then the solution of the "pre-Bethe-ansatz" equation

$$R(v\otimes w)=\mu v'\otimes w' \quad (7)$$

verifies the two biquadratic relations

$$l_4+l_{11}p-l_{12}p'+l_2p^2+l_1p'^2-(l_9+l_{18})pp' -l_{13}p^2p'+l_{10}pp'^2+l_3p^2p'^2 =0\,, \quad (8)$$

$$l_7+l_{16}q-l_{15}q'+l_8q^2+l_5q'^2-(l_9-l_{18})qq' -l_{17}q^2q'+l_{14}qq'^2+l_6q^2q'^2 =0\,. \quad (9)$$

The group $G_{\mathrm{Bethe}}\simeq \mathrm{sl}_2\times\mathrm{sl}_2\times\mathrm{sl}_2\times\mathrm{sl}_2$ acts naturally on (7): the four copies of sl_2 act respectively on v, w, v', w'. This induces a linear action on R,

$$R\to g_{1L}^{-1}\cdot g_{2L}^{-1}\cdot R\cdot g_{1R}\cdot g_{2R}\,,$$

i.e. homographic transformations on p, p', q, q', and linear transformations on the l_i's. This linear representation has three invariants associated to each of its two nine-dimensional irreducible components. The two components reflect the partition of the l_i's from (8), (9). The three invariants are nothing but $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$. Furthermore $\mathcal{K}_3$ is invariant under the large group $\mathrm{sl}_3\times\mathrm{sl}_3$, since it is the determinant of a 3×3 matrix the entries of which are the coefficients in (8).

The parametrization of (8), (9) is obtained as follows: the discriminant Δ of (8), considered as a degree two polynomial in p, is a polynomial of the form

$$\Delta=\alpha p'^4+4\beta p'^3+6\gamma p'^2+4\beta' p'+\alpha'\,. \quad (10)$$

The transformations of Δ under homographic transformations of p' have two fundamental invariants g_2 and g_3 and the classical [13] modular invariant $J=g_3^2/(g_2^3-27g_3^2)$.

$$g_2=\alpha\alpha'-4\beta\beta'+3\gamma^2\,, \quad (11)$$

$$g_3=\alpha\gamma\alpha'+2\beta\gamma\beta'-\alpha\beta'^2-\alpha'\beta^2-\gamma^3\,; \quad (12)$$

g_2 and g_3 are also invariant under the homographic transformations of p (since Δ is).

Exchanging the role of p and p', or, more remarkably, applying the same approach to eq. (9) leads to *identical* g_2 and g_3. The latter is a direct consequence of the constraints (6), since g_2 and g_3 are expressible in terms of $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$:

$$g_2=\tfrac{4}{3}\mathcal{K}_4\,, \quad (13)$$

$$g_3=\tfrac{1}{1728}(4\mathcal{K}_2^3-192\mathcal{K}_2\mathcal{K}_4-27\mathcal{K}_3^2)\,. \quad (14)$$

For $J=\infty$ the elliptic parametrization degenerates into a rational one. This leads to a special variety in the parameter space and yields candidates for algebraic criticality and disorder conditions [1].

There exist a number of models for which $g_2^3-27g_3^2$ factorizes and becomes handable. Such models are obtained by imposing relations between the entries of R. We may consider for instance the following generalizations of the Baxter model:

$$(a_1=d_4,\quad a_2=d_3,\quad a_3=d_2,\quad a_4=d_1\,, \quad b_1=c_4,\quad b_2=c_3,\quad b_3=c_2,\quad b_4=c_1)\,, \quad (15)$$

$$(a_2=a_3=b_1=b_4=c_1=c_4=d_2=d_3\,, \quad a_1=d_4,\quad a_4=d_1,\quad c_2=b_3,\quad b_2=c_3)\,. \quad (16)$$

Condition $J=\infty$ factorizes into simple low degree components. In the instance of (16), these components are of the typical form

$$(a_1+b_3-a_4-b_2)\,,$$

$$(a_1+b_3+a_4+b_2+4a_2)\,,$$

$$(a_1a_4+b_2a_4+b_3a_4-a_4^2-2a_2^2)\,,$$

where one recognizes straightforward generalizations of the disorder and criticality conditions of the Baxter model [1]. Noticeably, these conditions appear on the same level.

The explicit elliptic parametrization shows that t_1It_2I as well as the exchange of p and p' (respectively

q and q') are realized as a shift of the spectral parameter (see for instance ref. [3]).

Remark. We can perform the same symmetry analysis in higher dimension, since all the symmetry groups we have used here have their generalizations to dimension 3, 4, ... [9]. For instance the results of section 5 reproduce themselves, mutatis mutandis, yielding to an intertwinning of three or more curves by matrices R living on higher dimensional varieties.

6. Conclusion

We have proved the existence of a canonical parametrization of the Boltzmann weights of the sixteen-vertex model. This parametrization by elliptic curves yields a framework for the use of the so-called inversion trick [4] in the exact (and straightforward) calculation of the partition function [14].

In addition to this parametrization, the complete analysis of the symmetries of the model brings to the light a remarkable algebraic variety ($J=\infty$). This variety is stable by the *entire* group of symmetries and is a candidate for the criticality and disorder conditions.

We actually believe that getting at the modular invariant J is extracting the very substance of the invariance properties. The foliation of the parameter space by the values of J then becomes a structuring feature for any result concerning the phase space of the model.

References

[1] R.J. Baxter, Exactly solved models in statistical mechanics (London Academic Press, London, 1981).

[2] R.J. Baxter, Ann. Phys. 76 (1973) 25.

[3] P.W. Kasteleyn, Exactly solvable lattice models, in: Proc. 1974 Wageningen Summer School: Fundamental problems in statistical mechanics III (North-Holland, Amsterdam, 1974).

[4] R.J. Baxter, in: Proc. 1980 Enschede Summer School: Fundamental problems in statistical mechanics V (North-Holland, Amsterdam, 1981).

[5] R.J. Baxter, J. Stat. Phys. 28 (1982) 1.

[6] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Phys. Lett. A 159 (1991) 221.

[7] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Phys. Lett. A 159 (1991) 233.

[8] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Phys. Lett. A 157 (1991) 343.

[9] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Phys. Lett. B 260 (1991) 87.

[10] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Phys. Rev. Lett. 67 (1991) 1373.

[11] A. Gaaf and J. Hijmans, Physica A 80 (1975) 149.

[12] J.F. Nagle, J. Math. Phys. 9 (1968) 1007.

[13] J. Dixmier, Gazette des Mathématiciens (Société Mathématique de France) 43 (1990) 39.

[14] J.M. Maillard, J. Phys. (Paris) 46 (1984) 329.