SERIES ANALYSIS OF Q STATE CHECKERBOARD POTTS MODELS :II

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Abstract:

The serie analysis of the low temperature expansion of the checkerboard q state Potts model in a magnetic field initiated in two previous papers is continued. On particular algebraic varieties of the parameter space (corresponding or generalizing the so called disorder solutions) the checkerboard Potts model and its Bethe approximation are indistinguishable as far as one is concerned with the partition function and its first order derivatives. The difference between the two models occurs for higher order derivatives. In particular one gives the exact expression of the (low temperature expansion of) the susceptibility of the checkerboard Ising model in zero magnetic field on one of these varieties.

Introduction.

A great number of models in statistical mechanics on lattices possess some remarkable varieties, called disorder varieties where the partition function of the model can easily be evaluated and is given by a very simple algebraic expression (Stephenson 1970, Enting 1977 and 1980, Jaekel and Maillard 1985 and for reviews on disorder solutions see Rujan 1984 and 1985). These varieties lie in the physical domain of the parameter space. In many case, not only the partition function but also an infinite number of correlation functions and even other quantities like the susceptibility of the Ising model (Dhar and Maillard 1985) can be computed on these disorder varieties. In the particular case of the q state Potts model in zero magnetic field, these disorder solutions map by duality on other solutions, the "order" solutions. They do lie in the non physical domain of the parameter space.

In two previous papers (Hansel and Maillard 1987,1988, hereafter paper 2 and 1), it has been shown that the low temperature expansions of the partition function, but also the magnetization and nearest neighbor correlation functions, of the checkerboard Potts model in a magnetic field are equal to the expansion of some very simple expressions when one restricts the parameter space of the model to an "order" variety which is the generalization of the previous one without magnetic field. The simplification, that occurs on these varieties, are due to some dimensional reduction (through an effective decoupling of the spins). It has also been shown in Georges et al (1987) that, at least for the triangular Ising model in zero magnetic field, one can construct a diagrammatic suitable to the study of the vicinity of disorder varieties and the point of interest is that the coefficients of such a perturbation theory are simple algebraic expressions.

In the following, one will study in a rather systematic way, the information that can be deduced from the existence of "order" and disorder varieties on the checkerboard Potts model on one hand, when considered together with some other exact results which are known to hold by themselves on another hand. It is shown here that the results of papers 1 and 2 lead to a new exact result for the low temperature expansion of the susceptibility of the checkerboard Ising model in zero magnetic field. This will be checked, up to order twelve, on the low temperature expansion of the checkerboard Potts model given in paper 1. One will also show how to get the expressions of some derivatives of order 1, 2, 3 and even 4 of the free energy when restricted to the disorder or "order" varieties. All these results give more insight on the kind of non trivial information one can get in the vicinity of these remarkable varieties.

The low temperature expansion of the isotropic square Ising model in a magnetic field can be seen as the sum of two expressions: the low temperature expansion of the model on a Bethe lattice with the same coordination number, and a correction corresponding to take into account the loops on the square lattice (Bessis et al 1975). The partition function of the model on a Bethe lattice can be computed using a trick first explained by Domb (Domb 1960) (see also the recent paper of Dasgupta and Pandit about the mean field theory on the Potts model (Dasgupta and Pandit 1987)). One generalizes these ideas in the case of the checkerboard Potts model in a magnetic field and study the low temperature expansion of the model as a correction to the exactly known expansion of the Bethe (mean field) approximation. One will show that the Bethe approximation of the checkerboard Potts model also simplifies on the "order" variety of this very model. In fact the previous correction does vanish on this variety. This allows to impose directly some constraints on the correction to the Bethe approximation.

The checkerboard Potts model in a magnetic field.

Let us first recall the results of papers 1, 2. The partition function per site Z of the q-state checkerboard scalar Potts model in a magnetic field is given by:

$$Z^N(a, b, c, d, h) = \sum_{\{\sigma\}} \prod_{<ij>} a^{\delta_{\sigma_i \sigma_j}} \prod_{<jk>} b^{\delta_{\sigma_j \sigma_k}} \prod_{<kl>} c^{\delta_{\sigma_k \sigma_l}} \prod_{<li>} d^{\delta_{\sigma_l \sigma_i}} \prod_{m} h^{\delta_{0,\sigma_m}} \quad (1)$$

a, b, c, d denote the exponential of the four nearest neighbor coupling constants of the model (see figure 1) and h is the exponential of the magnetic field. The spins belong to Z_q. The low-temperature normalized partition function per site Λ is defined by:

$$Z(a, b, c, d, h) = (a\, b\, c\, d\, h^2)^{1/2} .\Lambda(a, b, c, d, h) \quad (2)$$

The low temperature parameters of the model will be denoted A, B, C, D and z (A=1/a, B=1/b, C=1/c, D=1/d, z=1/h). The expansion of ln Λ was given up to twelve order in paper 1. Let us recall its first terms:

$$\ln\Lambda\ (A, B, C, D, z) = (q-1)\ ABCDz + (q-1)(A^2B^2D^2 + A^2B^2C^2 + A^2C^2D^2 + B^2C^2D^2)z^2/2 + \ldots \quad (3)$$

The results of paper 2 are the following: restricted to the algebraic variety

$$D + ABCz + (q-2)\ ABCDz = 0 \quad (4)$$

one has:

$$\ln \Lambda_{I(3)} = 1/2 \ln (1 + (q-1)\ ABCDz) \quad (5)$$

$$\frac{\partial}{\partial A} \ln \Lambda_{I(3)} = 0 \quad (6)$$

$$\frac{\partial}{\partial D} \ln \Lambda_{I(3)} = \frac{1}{2}\frac{\partial}{\partial D} \ln \left(1 - \frac{(q-1)\, D^2}{1+(q-2)D}\right) \quad (7)$$

$$\frac{\partial}{\partial z} \ln \Lambda_{I(3)} = (1-M)\left(\frac{q-1}{q}\right) = 0 \quad (8)$$

M denotes the magnetization of the model. Equations (5) (6) (7) and (8) have to be seen as exact formal constraints on the low temperature expansion of the partition function per site.

Bethe approximation for the checkerboard Potts model.

Equations (5) (6) (7) (8) have been seen to constrain the low temperature expansion of the model in a magnetic field (papers 1 and 2).These constraints can be used to get, or to check, systematically the low temperature expansion order by order. This has been sketched in paper 1 combining these results with other exact results available on the model. Actually a large number of coefficients of this anisotropic expansion (but not all of them) can be obtained this way. At order twelve one remarks, among the number of coefficients, that many of them correspond to diagrams on a Bethe lattice with coordination number four and four coupling constants (see figure 2). Therefore it is tempting to combine these two different kinds of information on the coefficients of the expansion.

The contribution corresponding to this Bethe lattice can be computed in the following way: consider the vertex of fig 3 with four coupling constants, a magnetic field H on the central spin and external spins embedded in different effective magnetic fields H_A, H_B, H_C, H_D. The central and external spins are fixed in order that their mean values are equal.

This self consistent condition leads to the equations:

$$z \left(\frac{A+(q-2)Az_A+z_A}{1+(q-1)Az_A}\right)\left(\frac{B+(q-2)Bz_B+z_B}{1+(q-1)Bz_B}\right)\left(\frac{C+(q-2)Cz_C+z_C}{1+(q-1)Cz_C}\right) = z_D \quad (9)$$

(and three other similar equations where A, B and C are singled out)

z_A, z_B, z_C, z_D denote the fugacities associated to the four different effective magnetic fields. Let :

$$t = \frac{D + (q-2)D\, z_D + z_D}{1 + (q-1)\, D\, z_D}\, z_D$$

The exact expression of the magnetization of the model reads :

$$M = \frac{1-t}{1 + (q-1)\, t} \quad (10)$$

From the four algebraic equations (9) one can (implicitly) get z_A, z_B, z_C and z_D in term of A, B, C, D, and z. When z = 0, z_A, z_B, z_C and z_D are chosen to vanish also and for $z \neq 0$ one must select the corresponding root by continuity.

The expansion of the (low temperature) normalized free energy $\ln \Lambda_{Bethe}(A, B, C, D, z)$ can be obtained integrating the expansion of (10) with respect to z and reads:

$$\ln\Lambda_{Bethe}(A, B, C, D, z) = (q-1)\,ABCDz + (q-1)(A^2B^2D^2+ A^2B^2C^2+ A^2C^2D^2+ B^2C^2D^2)z^2/2$$
$$+(q-1)(q-2)(AB^2C^2D^2+A^2BC^2D^2+A^2B^2CD^2+A^2B^2C^2D)z^2/2 + \ldots\ldots \qquad (11)$$

This expansion has been performed, up to order twelve, using the formal calculation program REDUCE (Hearn 1984) . One has actually verified that (most) of the coefficients with chromatic polynomial $(q-1)^a(q-2)^b$ (a, b integers) of the checkerboard Potts model given in paper 1 are identical to the coefficients of the Bethe approximation (11). The correction to the Bethe approximation gives:

$$\Delta(A,B,C,D,z) = \ln\Lambda(A, B, C, D, z) - \ln\Lambda_{Bethe}(A, B, C, D, z) =$$
$$(q-1)A^2B^2C^2D^2z^4+(q-1)(z^6/2-z^4)(A^2B^2C^2D^4 + \ldots) + (q-1)(q-2)(A^2B^2C^3D^3 + \ldots)$$
$$+ 2\,(q-1)(AB^3C^3D^3 + \ldots)z^5 + \ldots \qquad (12)$$

The diagrams, and their contributions associated to this correction, are given in appendix II. They have to be compared to the one of the checkerboard Potts model given in paper 1.
It has been verified that the logarithm of (10) is equal, up to order twelve, to
$1/2 \ln(1 + (q-1)\,ABCDz)$ when (3) is satisfied.

Moreover the magnetization and nearest neighbor correlation functions can also be calculated exactly on the Bethe lattice and therefore the following partial derivatives of $\Delta(A, B, C, D, z)$ are known exactly when (3) is satisfied: $\partial\Delta/\partial A$, $\partial\Delta/\partial B$, $\partial\Delta/\partial D$, $\partial\Delta/\partial z$, $\partial\Delta/\partial q$. It has been verified up to order eleven or twelve that Δ and its first derivatives vanish when restricted to (3):

$$\Delta_{|(3)} = \frac{\partial\Delta}{\partial A}\Big|_{(3)} = \frac{\partial\Delta}{\partial B}\Big|_{(3)} = \frac{\partial\Delta}{\partial D}\Big|_{(3)} = \frac{\partial\Delta}{\partial z}\Big|_{(3)} = \frac{\partial\Delta}{\partial q}\Big|_{(3)} = 0 \qquad (13)$$

This means that Δ is divisible by δ^2 where δ is equal to $D + ABCz + (q - 2)\,ABCDz$. Indeed on the expansion of the correction Δ obtained up to order twelve in A, B, C, D, one has verified that :

$$\Delta = (q - 1)\,A^2 B^2 C^2 z^4.\, \delta^2 \Big\{ 1 + (z^2/2 -1)(A^2 + B^2+ C^2) + (q-2)(AB + \ldots)$$
$$- (q-2)(A^2B + \ldots) + (q-2)(q-3)ABC +(q-1+z^4)(A^2B^2 + \ldots)$$
$$- (q-2)(q-2+z+z^2)(A^2BC+\ldots) + \ldots \Big\} + O(\delta^3) \qquad (14)$$

Of course the variable D does not play a special role and one can argue that the correction Δ is in fact divisible by π^2 where π is equal to the product $\alpha\,\beta\,\gamma\,\delta$ with
$\alpha = A + BCDz + (q-2)ABCDz$, $\beta = B + ACDz + (q - 2)ABCDz$ and $\gamma = C + ABDz + (q - 2)\,ABCDz$.

On the expansion of the correction Δ, given up to order twelve in A, B, C, D, one verifies actually that Δ is equal to :

$$\Delta = (q - 1)\,\pi^2z^4 \Big\{ 1 + (z^2/2-1)(\alpha^2 + \ldots) + (q-2)(\alpha\beta + \ldots) + (q-2)(q-3)(\alpha\beta\gamma + \ldots)$$
$$- (q-2)(\alpha^2\beta + \ldots) + \ldots \Big\} + O(\pi^3) \qquad (15)$$

These ideas can be straightforwardly generalized to higher dimensional models. For instance, it has been remarked that equations (5) to (8) can be extended to a cubic Potts model with six coupling constants in a magnetic field (paper 2).

Equations (13) mean that, as far as one is concerned with the partition function per site, the magnetization and nearest neighbor correlation functions restricted to (3), there is no difference between the q state Potts model in a magnetic field on a Bethe lattice with four coupling constants and on the checkerboard lattice. These results can be understood in a heuristic way using the decimation procedure detailed in Jaekel and Maillard (1985a) transposed on the Bethe lattice : this procedure needs appropriate boundary conditions but one knows that, in the thermodynamic limit, the number of spins at the boundary is no longer negligeable compared to the total number of spins of the lattice. A priori these boundary conditions cannot be neglected as for the checkerboard lattice: for that reason one has verified directly on the equations (given in appendix I) which define the partition function per site for the Bethe lattice, that equations (5) and (8) are indeed satisfied.

Exact result for the susceptibility of the checkerboard Ising model.

Let us now concentrate on the expansion of the magnetization of the checkerboard Potts model. One has the surprising result that the magnetization is exactly equal to 1 on the algebraic variety (3).

Differentiating the magnetization along this variety, (A, B, C, being fixed), one gets:

$$dM = 0 = \frac{\partial M}{\partial D}\,dD - \chi\,\frac{dz}{z} \qquad (16)$$

where χ denotes the susceptibility of the model. From equation (3) one deduces dD/dz, and therefore χ is obtained straightforwardly from the derivative of the magnetization $\partial M/\partial D$. This quantity can be calculated in the special case of the Ising model without magnetic field restricted to the variety $D + ABC = 0$. Indeed, the spontaneous magnetization of the checkerboard model is known exactly (Syozi and Naya 1960 a,b) for every values of A, B, C, D and is equal to $(1-k^2)^{1/8}$

where k^2 reads:

$$K^2 = \frac{A^*B^*C^*D^*(1-A^{*2})(1-B^{*2})(1-C^{*2})(1-D^{*2})(A+BCD)(B+ACD)(C+ABD)(D+ABC)}{A^2B^2C^2D^2(1-A^2)(1-B^2)(1-C^2)(1-D^2)(A^*+B^*C^*D^*)(B^*+A^*C^*D^*)(C^*+A^*B^*D^*)(D^*+A^*B^*C^*)} \quad (17)$$

A^*, B^*, C^*, D^* denote respectively the high temperature variables $\frac{1-A}{1+A}, \frac{1-B}{1+B}, \frac{1-C}{1+C}, \frac{1-D}{1+D}$.

Let us set $k^2 = \alpha.\,(D+ABC)$

One has

$$\frac{\partial M}{\partial D}(D + ABC = 0, z = 1) = \frac{-1}{8}\frac{\partial k^2}{\partial D}(D + ABC = 0) = \frac{-\alpha}{8} \quad (18)$$

From (3) and (16) one gets immediately an exact expression for the low-temperature expansion of the susceptibility of the checkerboard Ising model in zero magnetic field when restricted to the algebraic variety $D + ABC = 0$:

$$\chi(D + ABC = 0, z=1) = \frac{\alpha}{4} ABC \quad (19)$$

This exact result has been checked up to order twelve in A, B, and in C on the low temperature expansion of paper 1, using the formal calculation program REDUCE (Hearn 1984) .

As the algebraic variety $D + ABC = 0$ does not lie in the physical domain of the parameter space, (19) must be considered as a formal constraint on the low temperature expansion of the model. The susceptibility on (3) is a simple rational expression that should be compared with the exact expression of Dhar and Maillard (1985) for the susceptibility of the checkerboard Ising model in zero magnetic field restricted to the disorder variety, $D^*+A^*B^*C^*=0$.

Generalization to higher derivatives

This exact result for χ is only a consequence of equation (8) (when condition (3) is satisfied) and of the fact that the spontaneous magnetization of the Ising model is known exactly. This approach can be generalized in a straightforward way.

One can take into account the knowledge of not only the magnetization, but also the partition function and the nearest neighbor correlation functions when restricted to (3) (equations (5) (6) (7) (8)). These quantities are also known, for $q = 2$ (Ising model), when the magnetic field of the model is equal to zero for any values of the coupling constants. The calculations are sketched in appendix III where it is shown that the following derivatives can be calculated in zero magnetic field when $D + ABC = 0$ and $q = 2$:

$$\frac{\partial}{\partial q}\ln\Lambda|_{q=2},\ \frac{\partial^2}{\partial q\partial H}\ln\Lambda|_{q=2},\ \frac{\partial^2}{\partial H^2}\ln\Lambda|_{q=2},\ \frac{\partial^2}{\partial q^2}\ln\Lambda|_{q=2},\ \frac{\partial^2}{\partial H^3}\ln\Lambda|_{q=2}$$

The derivatives of first order are equal for the checkerboard lattice and the Bethe lattice (equations (13)). In particular, this is the case for $\partial\ln\Lambda/\partial q$ ($D + ABC = 0$, $H = 0$, $q = 2$). More generally, one can show that one has <u>for any value</u> of q and of the magnetic field that satisfy equation (3) :

$$\frac{\partial}{\partial D}\ln\Lambda_{|(3)} = \frac{1}{2}\frac{\partial}{\partial q}\ln\left(1-\frac{(q-1)D^2}{1+(q-2)D}\right) \quad (20)$$

For higher derivatives the situation is different (even if they can be calculated exactly for both models). This stress the difference and the similarity between the partition function of the two models. For instance equation (16) is the same for the checkerboard Ising model and the Ising model on a Bethe lattice with four coupling constants : however the exact expressions for the susceptibility of both models will be different because their spontaneous magnetization (and therefore $\partial M/\partial D$ restricted to $D + ABC = 0$, $z = 1$) are different . From the equations given in appendix III, one can understand that the partial derivatives of higher orders that can be calculated exactly, are simple (but different) rational expressions for $D + ABC = 0$, $z = 1$. One can verify that these partial derivatives simplify drastically for the two models restricted to the variety $D + ABC = 0$, $q = 2$, $z = 1$. However, up to the order available from our expansion, no difference has been noticed. For instance the following derivatives restricted to (3), have the same expansion on the Bethe and checkerboard lattice up to order eleven:

$$\frac{\partial^2}{\partial q\,\partial A}\ln\Lambda_{|(3)} = -(q-1)A^2B^3C^3z^3 + (q-1)(q-2)A^3B^4C^4z^4 \quad (21a)$$

$$\frac{\partial^2}{\partial q\, \partial D} \ln \Lambda_{I(3)} = ABCz + (q-3/2)A^2B^2C^2z^2 + 2(q-1)A^3B^3C^3z^3 - (q-1)(A^4B^4C^2 + \dots)z^4 - (q-1)(q-2)(A^4B^4C^3 + \dots)z^4 \tag{21b}$$

and up to order twelve:

$$\frac{\partial^2}{\partial q^2} \ln \Lambda_{I(3)} = A^3B^3C^3z^3 - (q-5/2)A^4B^4C^4z^4 \tag{21c}$$

The fact that there is no difference at these orders is reminiscent of a remark made by Baxter (p306 (1982)) about the formal similarity of the magnetization for both models (equations 11.8.29 and 11.8.36). It is a consequence of the equations of appendix III and of the fact that Δ is equal to zero up to order seven.

It should be remarked that these ideas can be applied mutatis mutandis on the disorder (high temperature) solution of the checkerboard Ising model in a magnetic field (Jaekel and Maillard 1985b). It is shown in appendix III that the susceptibility, but also the fourth derivative of the partition function with respect to the magnetic field, can even be calculated .

These ideas can also be applied on the disorder solution of the checkerboard Potts model in a magnetic field (Jaekel and Maillard 1985b). However the disorder variety is defined by two equations in the parameter space and the constraints one gets that way on the partition function are weaker.

Conclusion

The formal constraints of paper 1 and 2 on the low-temperature expansion of the checkerboard Potts model in a magnetic field, are revisited in this paper. It is shown that the equations satisfied by the low-temperature expansion of the partition function and its derivatives of order one with respect to the coupling constants, the number of states, or the magnetic field are also satisfied by the partition function of the Potts model on the Bethe lattice with four coupling constants. This enables to shed some light on the analysis of the coefficients of this low-temperature expansions: many of these coefficients correspond to this Bethe approximation (of course, the relevant part of the expansion, for the critical properties for instance, is "hidden" in the correction to the Bethe approximation). The first order derivatives give soft constraints on the expansion that do not discriminate between the checkerboard and Bethe lattice. It is stressed that the higher order derivatives on the "order" and disorder varieties take partially into account these relevant differences: for instance, using the exact low temperature expansion of the spontaneous magnetization of the checkerboard Ising model, one can get the exact expression of the susceptibility of this model without magnetic field on the variety $D + ABC = 0$. The difference

between the susceptibilities on the "order" and disorder varieties should be traced back to the calculation performed by Wu et al. (1976). Indeed, these authors noted that the high and low temperature susceptibilities are given in the scaling limit by two different kind of expansions.

Appendix I

Let us introduce the variables $\phi_A, \phi_B, \phi_C, \phi_D$ defined by

$$\phi_A = \frac{A+(q-2)Az_A+z_A}{1+(q-1)Az_A} \qquad \text{(and similar definitions for } \phi_B, \phi_C \text{ and } \phi_D\text{)}$$

with these new variables equations (9) read:

$$z\,\phi_A\,\phi_B\,\phi_C = \frac{\phi_D - D}{1+(q-2)D-(q-1)\,\phi_D.D} \qquad (22)$$

(and similar equations for ϕ_A, ϕ_B, ϕ_C). The variable t previously introduced is equal to $z\phi_A\phi_B\phi_C\phi_D$. It is a straightforward matter to see the vanishing condition of ϕ_D corresponds to condition (3) of this paper (and of course to the vanishing condition of variable t): equations (22) become respectively $\phi_A = A$, $\phi_B = B$, $\phi_C = C$ and condition (3), that is to say $D + zABC + (q-2)ABCDz = 0$. This situation corresponds to the following values of the fugacities:

$$z_A = z_B = z_C = 0 \qquad z_D = \frac{-D}{1+(q-2)D} \qquad (23)$$

On these equations one understands that condition (3), which corresponds to a trivialization of the low-temperature expansion of the checkerboard Potts model in a magnetic field, also corresponds to a trivialization of the partition function per site on the Bethe lattice. One gets that the magnetization of the model (equation (10)) restricted to (3) is equal to 1. One can even get the (low temperature) normalized partition function (see for instance equation (4.9.6) of Baxter 1982):

$$\Lambda_{Bethe_{I(3)}} = [(1+(q-1)Az_A)(1+(q-1)Bz_B)(1+(q-1)Cz_C)(1+(q-1)Dz_D)]^{1/2}[\,1 + (q-1)t\,]^{-1}$$

$$= (1+(q-1)Dz_D)^{1/2} = (1+(q-1)ABCDz)^{1/2} \qquad (24)$$

$$= (1 - \frac{(q-1)D^2}{1+(q-2)D})^{1/2}$$

These expressions coincide exactly with equations (5) and (8), and justify the heuristic argument of the decimation procedure on the Bethe lattice.

APPENDIX II

The diagrammatic expansion and the corresponding low temperature coefficients of the checkerboard Potts model in a magnetic field have been given in paper 1 . The diagrammatic expansion of the Potts model in a magnetic field on a Bethe lattice with four coupling constants is very similar up to order twelve in A,B,C,D. Let us give here only the diagrams and the associated low temperature coefficients of the difference between the free energy of the two models.

$\Delta(A,B,C,D,\ z) =$

$+(q-1)z^4\ A^2B^2C^2D^2$ (θ= 8)

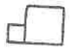
$+(q-1)(q-2)z^4(A^2B^2C^3D^3 + \ldots)$ (θ = 10)

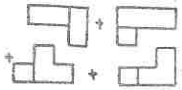
$+1/2(q-1)z^6(A^4B^2C^2D^2 + \ldots)$

$+2(q-1)z^5(A^3B^3C^3D + \ldots)$

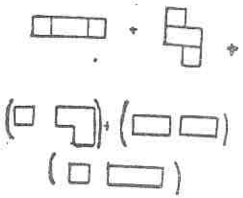
$-(q-1)z^4(A^4B^2C^2D^2 + \ldots)$

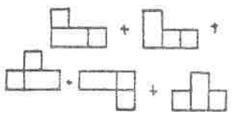
$+(q-1)(q-2)(q-3)z^4(A^3B^3C^3D^2 + \ldots)$

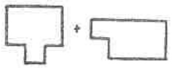
 $+(q-1)(q-2)z^5(A^3B^3C^3D^2 + \ldots)$

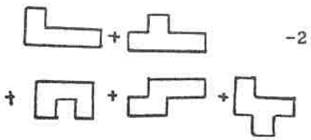
 $-3(q-1)(q-2)z^4(A^4B^3C^2D^2 + \ldots)$

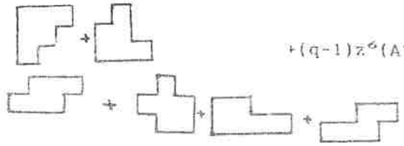
 $+(q-1)(q-2)(q^2-5q+7)z^4A^3B^3C^3D^3$ $(\theta = 12)$

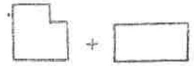
 $+(q-1)^2z^4(A^4B^4C^2D^2 + \ldots)$

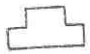
 $-(q-1)(q-2)^2z^4(A^3B^3C^4D^2 + \ldots)$

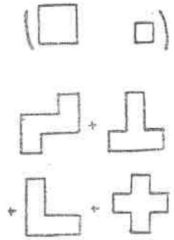
 $+(q-1)z^7(A^5B^3C^3D + \ldots)$

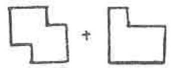
 $-2(q-1)z^5(A^5B^3C^3D + \ldots)$

 $+(q-1)z^6(A^4B^4C^2D^2 + \ldots)$

 $+(q-1)z^8(A^4B^4C^2D^2 + \ldots)$

 $+(q-1)z^6\ (A^4B^4C^4 + \ldots)$

 $-12q(q-1)\ z^5A^3B^3C^3D^3$

 $+10(q-1)z^7\ A^3B^3C^3D^3$

 $+(q-1)z^9\ A^3B^3C^3D^3$

Appendix III

Let us introduce the following notations :

$u = D + ABC, \quad v = z - 1, \quad w = q - 2, \quad \alpha = ABC.$

With these notations, condition (3) reads :

$$u + \alpha . v - \alpha^2 w + \alpha u . w - \alpha^2 vw + \alpha uvw = 0 \qquad (26)$$

Let us Taylor-expand F(u, v, w) the partition function of the q-state checkerboard Potts model in a magnetic field :

$$\begin{aligned} F(u, v, w) = F(0, 0, 0) &+ uF'_u + vF'_v + wF'_w \\ &+ 1/2! \,(u^2F''_{u^2} + v^2F''_{v^2} + w^2F''_{w^2} + 2uvF''_{uv} + 2uwF''_{uw} \\ &+ 2vwF''_{vw}) + \dots \end{aligned} \qquad (27)$$

The Taylor expansion of F, is known exactly on (26). Eliminating v, one gets, order by order in u and w, the following equations :

$$\alpha F'_u - F'_v = C_{1,1} = -2 \frac{\alpha^2}{\alpha^2 - 1}$$

$$\alpha F'_v + F'_w = C_{1,2} = \frac{\alpha^3}{\alpha - 1}$$

$$\alpha^2 F''_{u^2} - 2\alpha F''_{uv} + F''_{v^2} = C_{2,1} = -\alpha \frac{1+\alpha^2}{(1-\alpha^2)^2} \qquad (28)$$

$$\alpha^2 F''_{v^2} + 2\alpha F''_{vw} + F''_{w^2} + 2\alpha^2 F'_v = C_{2,2} = \alpha^4 \frac{\alpha - 2}{(\alpha-1)^2}$$

$$\alpha^2 . F''_{uv} + \alpha F''_{uw} - 2\alpha F'_v - \alpha F''_{v^2} - F''_{vw} = C_{2,3} = -\alpha^2 \frac{\alpha - 2}{(\alpha-1)^2}$$

$$\alpha^3 F'''_{u^3} - F'''_{v^3} - 3\alpha^2 F'''_{u^2 v} + 3\alpha F'''_{v^2 u} = C_{3,1} = -2\alpha^2 \frac{\alpha^2+3}{(\alpha^2-1)^3}$$

....

The expressions $C_{i,j}$ are deduced from the expansion of the right hand term of equation (5). Similar calculations can be performed for the magnetization and nearest neighbor correlation functions of the model. This amounts to replace F by F'_u and F'_v respectively:

$$\alpha F''_{u^2} - F''_{uv} = D_{1,1} = -\frac{1+\alpha^2}{(1-\alpha^2)^2} \qquad (29)$$

$$\alpha F''_{uv} + F''_{uw} = D_{1,2} = -\alpha \frac{\alpha-2}{(\alpha-1)^2}$$

$$\alpha^2 F'''_{u^3} + F'''_{uv^2} - 2\alpha F'''_{u^2 v} = D_{2,1} = -2\alpha \frac{\alpha^2+3}{(\alpha^2-1)^3}$$

.....

and

$$\alpha F''_{uv} - F''_{v^2} = E_{1,1} = 0$$

$$\alpha F''_{v^2} + F''_{vw} = E_{1,2} = 0 \qquad (30)$$

$$\alpha^2 F'''_{u^2 v} + F'''_{v^3} - 2\alpha F'''_{uv^2} = E_{2,1} = 0$$

....

$D_{i,j}$ and $E_{i,j}$ are known from the expansion of the right hand terms of equation (6) and (7) respectively (all the E_{ij} vanish as a consequence of (8)).
One can verify actually on the expressions of C_{ij}, D_{ij} and E_{ij} the following relations which are consequences of (28), (29), (30) :

$$C_{2,1} + E_{1,1} = \alpha D_{1,1} \quad \text{and} \quad C_{3,1} + E_{2,1} = \alpha D_{2,1} \qquad (31)$$

The partition function and the magnetization of the Potts model are known exactly for zero magnetic field (z = 1) in the Ising case q = 2. This means that F(u, 0, 0) and $F'_v(u, 0, 0)$ are known exactly, that is, that F'_u, F''_{u^2}, F'''_{u^3}, F'_v, F''_{uv}, $F'''_{u^2 v}$ are known. From the previous equations one can deduce the expressions of F'_w, $F''_{v}2$, F''_{vw}, F''_{uw}, $F''_{w}2$, $F'''_{uv}2$, $F'''_{v}3$. This means that the second derivatives with respect to q or H and the third derivatives with respect to H of the partition function can be calculated exactly.

One remarks that these calculations are not restricted to partial derivatives for $q = 2$, $z = 1$ and $D + ABC = 0$ but can also be performed to get the same partial derivatives for any point of the variety (3). The only difference is that the coefficients in front of the partial derivatives in the above equations will be slightly modified, that expressions C_{ij}, D_{ij} will also be modified, and that only F_u and F'_v are known exactly. The only exact result one gets, is the following :

$$\frac{\partial}{\partial q} \ln \Lambda_{check} |_{(3)} = \frac{\partial}{\partial q} \ln \Lambda_{Bethe} |_{(3)} = \frac{1}{2} \frac{\partial}{\partial q} \ln (1 - \frac{(q-1) D^2}{1+(q-2)D}) = \frac{-A^2B^2C^2z^2}{1-ABCz} \qquad (32)$$

Similar calculations could have been performed for the disorder solutions of the checkerboard Ising model in a magnetic field. Let us introduce the high-temperature variables $t_i = \mathrm{th}\, K_i$, and set:

$$u = t_4 + t_1t_2t_3 \quad \text{and} \quad v = (z-1)^2/z \qquad (33)$$

The disorder equation reads :

$$u = \alpha \cdot v + \beta v^2 \qquad (34)$$

where α and β are rational functions of the variables t_i.

In this high temperature domain, one has the spin reverse symmetry $H \rightarrow -H$. $F(u, 0)$ and $F'_u(u, 0)$ (and thus F'_u, $F''_{u}2$...) are known exactly and one can also Taylor-expand these functions and get the exact expression of F'_v (this corresponds to the exact expression of the susceptibility on the disorder solution without magnetic field) but also F''_{uv}, $F''_{v}2$. This means that $\partial^4/\partial H^4 \ln \Lambda$ can be calculated on the disorder solution of the checkerboard Ising model without magnetic field.

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Figure captions.

Figure 1 The checkerboard lattice

Figure 2 The Bethe lattice with four coupling constants.

Figure 3 Elementary cell of the checkerboard model and the four effective fields of the mean field approximation.

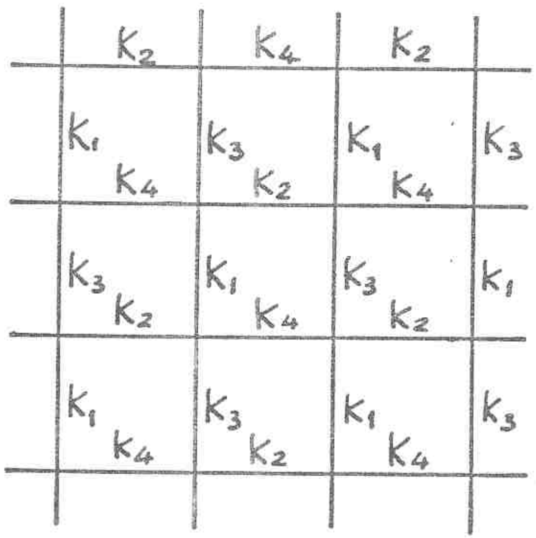

Figure 1

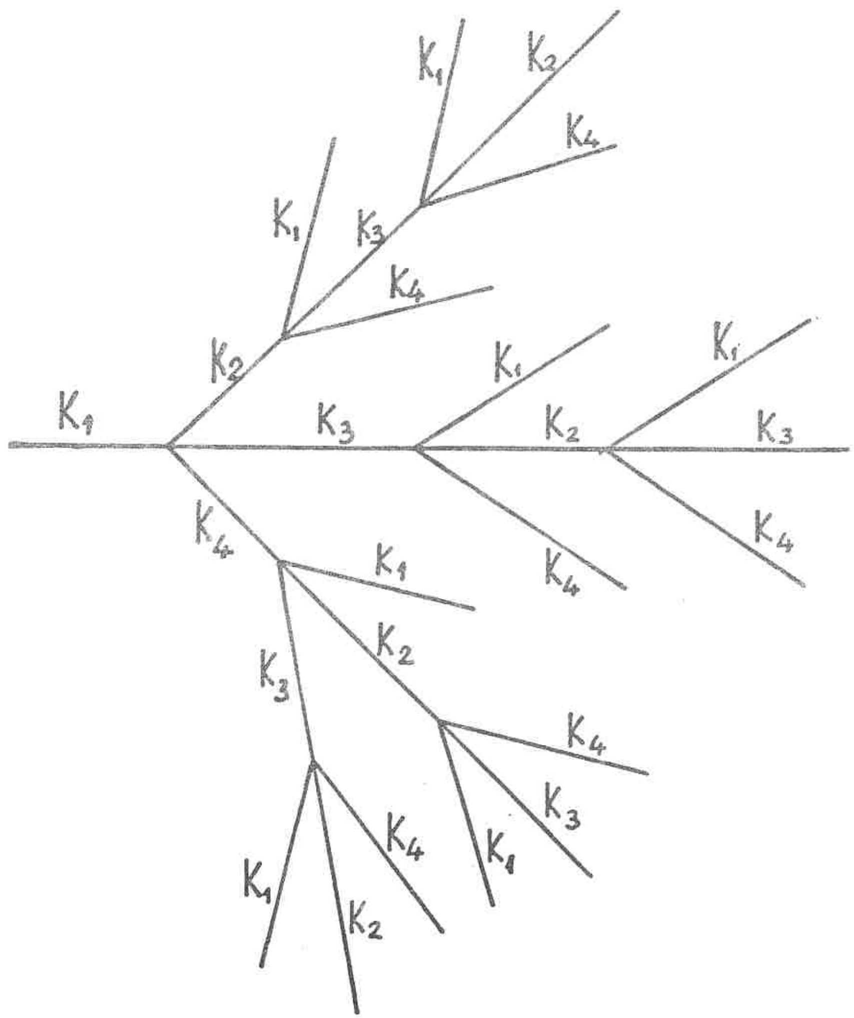

Figure 2

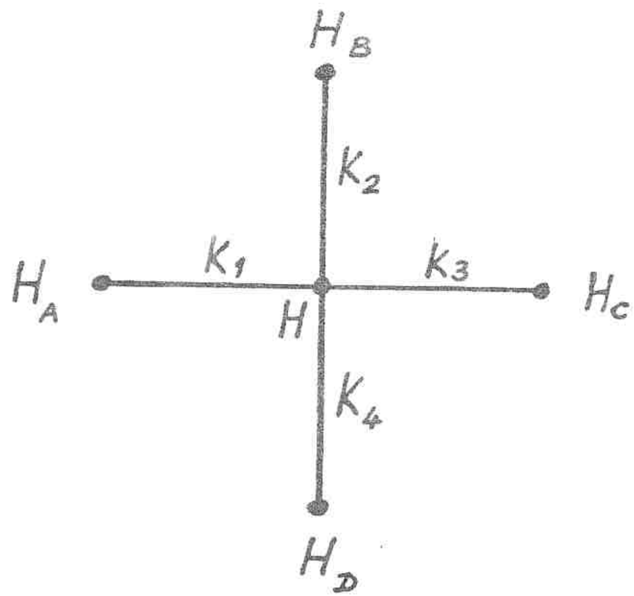

Figure 3