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Selected non-holonomic functions in lattice statistical mechanics and enumerative combinatorics*

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Abstract

We recall that the full susceptibility series of the Ising model, modulo powers of the prime 2, reduce to algebraic functions. We also recall the nonlinear polynomial differential equation obtained by Tutte for the generating function of the q-coloured rooted triangulations by vertices, which is known to have algebraic solutions for all the numbers of the form $2 + 2\cos(j\pi/n)$, the holonomic status of $q = 4$ being unclear. We focus on the analysis of the $q = 4$ case, showing that the corresponding series is quite certainly non-holonomic. Along the line of a previous work on the susceptibility of the Ising model, we consider this $q = 4$ series modulo the first eight primes 2, 3, ... 19, and show that this (probably non-holonomic) function reduces, modulo these primes, to algebraic functions. We conjecture that this probably non-holonomic function reduces to algebraic functions modulo (almost) every prime, or power of prime numbers. This raises the question of whether such remarkable non-holonomic functions can be seen as a ratio of diagonals of rational functions, or even algebraic functions of diagonals of rational functions.

Keywords: non-holonomic functions, nonlinear differential equations, enumeration of coloured maps, modulo prime calculations, susceptibility of the Ising model, diagonals of rational functions
Mathematics Subject Classification: 03D05, 11Yxx, 33Cxx, 34Lxx, 34Mxx, 34M55, 39-04, 68Q70

* Dedicated to A J Guttmann, for his 70th birthday.
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1. Introduction

Our aim in this paper is to study the reduction modulo primes, or power of primes, of certain differentially algebraic power series $F(x) = \sum c_n x^n$, with integer coefficients of interest in physics. Let us first recall that a power series $F(x)$ is called an algebraic series if it satisfies a polynomial relation

$$P(x, F(x)) = 0, \tag{1}$$

that a holonomic series satisfies a finite order linear differential equation (here $P_i(x)$ denotes polynomials with integer coefficients, $F^{(i)}(x)$ denotes the ith derivative of $F(x)$)

$$\sum_{i=0}^{k} P_i(x) F^{(i)}(x) = 0. \tag{2}$$

The series $F(x)$ is called a differentially algebraic series if there exists a polynomial P such that $F(x)$ satisfies a polynomial differential equation

$$P(x, F(x), F'(x), \dots, F^{(k)}(x)) = 0. \tag{3}$$

A series is said to be non-holonomic if it is not a solution of a linear differential equation like (2). We will say that a series is an algebraic function modulo a prime if there is a polynomial P such that the series satisfies equation (1) modulo that prime.

In a previous paper [1] we have shown that the full susceptibility of the Ising model, which is a *non-holonomic* function [2, 3], actually reduces to *algebraic functions modulo any powers of the prime* 2.

Modulo 2^r, one cannot distinguish the full susceptibility from some simple diagonals of rational functions [1] which reduce to *algebraic functions* modulo 2^r. Modulo 2^r these results can, in fact, be seen as being a consequence of the fact that, in the decomposition of the full susceptibility in an infinite sum of n-fold $\tilde{\chi}^{(n)}$ integrals [4], these $\tilde{\chi}^{(n)}$ are actually series with integer coefficients, *with an overall* 2^n *factor*. This may lead to a prejudice that these remarkable reductions to algebraic functions could only take place modulo powers of the prime 2.

It is not clear if such a reduction of the full susceptibility to algebraic functions also takes place for *other primes or powers of primes*. At the present moment, the high or low temperature series of the full susceptibility modulo, for instance, prime 3, are not long enough to confirm, or deny the fact that the associated series could actually correspond to an algebraic function modulo 3.

These exact results shed new light on this iconic function in physics. They provide a strong incentive to *systematically study other non-holonomic series modulo primes* (or powers of primes), in theoretical physics. It is very important to see whether this is an exceptional result, or the first example of a large set of selected non-holonomic functions in theoretical physics.

Remarkably long low-temperature and high-temperature series expansions [5], with respectively 2042 and 2043, coefficients have been obtained for the susceptibility of the square Ising model using an iterative algorithm [6], the *polynomial growth* of that algorithm [6] being a consequence of a *discrete Painlevé quadratic recursion* [7–9]. Sometimes such algorithms with polynomial growth are called 'integrable' algorithms. At the present moment the full susceptibility of the Ising model has only this 'algorithmic integrability': *no nonlinear differential equation*, or even functional equation [10], is known for that very important *non-holonomic* function in physics.

Our aim in the following is to study other *non-holonomic* physical series *modulo primes*, or powers of primes. No nonlinear differential equations are known for non-holonomic functions in lattice statistical mechanics, however, this is not the case in an almost indistinguishable domain of mathematical physics, namely enumerative combinatorics. In that respect, we must recall Tutte's study of triangulations equipped with a proper colouring [11–13], his work culminating in 1982, when he proved that the series $H(w)$ counting q-coloured rooted triangulations by vertices satisfies a *nonlinear polynomial differential equation* [14, 15]:

$$2\,q^2 \cdot (1-q) \cdot w + \left(q\,w + 10\,H(w) - 6\,w\,\frac{dH(w)}{dw}\right) \cdot \frac{d^2H(w)}{dw^2} + q \cdot (4-q) \cdot \left(20\,H(w) - 18\,w\,\frac{dH(w)}{dw} + 9\,w^2\,\frac{d^2H(w)}{dw^2}\right) = 0. \tag{4}$$

This q-family of *nonlinear polynomial differential equations* has a large number of remarkable properties. For instance, the series $H(w)$ reduces to *algebraic functions* for all the well-known Tutte–Beraha numbers, and in fact, for all the numbers of the form[4] $q = 2 + 2\cos(j\,\pi/m)$. This remarkable result first appeared in [20] and was really proved by Bernardi and Bousquet-Mélou in [21]. The Tutte–Beraha numbers accumulate[5] at the integer value $q = 4$. Interestingly, the status of the series $H(w)$, at the integer value $q = 4$, *remains unclear*: if it is not an algebraic function, is it a holonomic function or a non-holonomic function?

Other one-parameter dependent *nonlinear polynomial differential equations* have been found in an enumerative combinatorics framework (see for instance [21–24]). Curiously, few analysis have been performed on the remarkable nonlinear differential equation (4). For instance, one does not know if the nonlinear differential equation (4) fits with some *Painlevé property*.

We will focus, in this paper, on the study of equation (4), because of its historical importance as the first example of exact *nonlinear differential equation* in enumerative combinatorics, and as a toy model for the study of the susceptibility of the Ising model, and, more generally, for the emergence of similar *nonlinear differential equations in lattice statistical mechanics*. More specifically, we will focus on the analysis of the series $H(w)$ *at the integer value* $q = 4$. We will show that even if this series is quite certainly *non-holonomic*, it, however, has a quite remarkable property, totally reminiscent of what we found on the susceptibility of the Ising model [1]: this (probably non-holonomic) function is such that it actually *reduces to algebraic functions modulo the first eight primes*: 2, 3, 5, ... 19, as well as powers of these primes. It is tempting to conjecture that this (probably non-holonomic) function *reduces to algebraic function modulo (almost) every prime* (or every power of prime). This would be compatible with the scenario [1] that this series could be a simple ratio of diagonals of rational functions, or, more generally, an algebraic[6] function of diagonals of rational functions [1]. Such kind of result is clearly a strong incentive to perform similar studies on other nonlinear differential equations emerging in enumerative combinatorics

[4] These selected algebraic values of q have been underlined many times on the standard scalar Potts model on Euclidean lattices (the critical exponents are rational numbers, ...). They are such that a group of birational symmetries of the model, which is generically an infinite discrete group, degenerates into a *finite group* [16–19].

[5] To some extent the study of these remarkable numbers was a strategy in order to make some progress on the four-colour problem.

[6] Note that rational functions of diagonals of rational functions can be reduced to a simple ratio of diagonals of rational functions.

[21–24], or to obtain longer series (modulo some small primes $p = 3, \ldots$) for the susceptibility of the Ising model, or to study systematically (in a first step) the ratio of diagonals of rational functions.

2. A few remarks on the solutions of Tutte's nonlinear differential equation (4)

Let us consider the series $H(w) = \sum h_n\, w^n$, solution of equation (4) which counts the q-coloured rooted triangulations by vertices. Its coefficients are the number h_n of rooted triangulations with n vertices. They satisfy a remarkably simple *quadratic recurrence relation*[7]:

$$q \cdot (n+1)(n+2) \cdot h_{n+2} \,=\, q \cdot (q-4) \cdot (3n-1)(3n-2) \cdot h_{n+1} + 2 \sum_{i=1}^{n} i \cdot (i+1) \cdot (3n-3i+1) \cdot h_{i+1}\, h_{n-i+2}, \qquad (5)$$

with the initial conditions $h_0 = 0$, $h_1 = 0$, $h_2 = q\,(q-1)$. The number of proper q-colourings of a triangle is $h_3 = q \cdot (q-1)(q-2)$.

This series $H(w)$ reads

$$H(w) = q \cdot (q-1) \cdot w^2 + q \cdot (q-1)(q-2) \cdot \sum_{n=3}^{\infty} P_n(q) \cdot w^n, \qquad (6)$$

where the first terms of the sum reads:

$$\sum_{n=3}^{\infty} P_n(q) \cdot w^n = w^3 + (4q-9) \cdot w^4 + 3 \cdot (8q^2 - 37q + 43) \cdot w^5 + (176q^3 - 1245q^2 + 2951q - 2344) \cdot w^6 + (1456q^4 - 13\,935q^3 + 50\,273q^2 - 81\,036q + 49\,248) \cdot w^7 + \cdots. \qquad (7)$$

Of course there are many other solutions. For instance, with other initial conditions, namely $h_0 = 0$, but $h_1 \neq 0$, one deduces a *one-parameter family of solutions*:

$$H(w) \,=\, h_1 \cdot w + q \cdot \frac{U}{q + 4 \cdot h_1} \cdot w^2 + \frac{q^2 \cdot U\,V}{(q + 4 \cdot h_1)^3} \cdot w^3 \cdot h(z), \qquad (8)$$

where:

$$U = q \cdot (q-1) + (q-4) \cdot h_1, \qquad V = q \cdot (q-2) + 2 \cdot (q-4) \cdot h_1. \qquad (9)$$

and:

$$h(z) = 1 + (q \cdot (4q-9) + 9 \cdot (q-4) \cdot h_1) \cdot z + (129 \cdot (q-4)^2 \cdot h_1^2 + 3 \cdot q \cdot (37q - 86) \cdot (q-4) \cdot h_1 + 3 \cdot q^2 \cdot (8q^2 - 37q + 43)) \cdot z^2 + \cdots \quad \text{with:} \quad z = \frac{q\,w}{(q + 4\,h_1)^2}. \qquad (10)$$

When U or V in (8) are equal to zero, this yields two polynomial solutions of equation (4), valid *for any value* of q:

[7] As Bernardi and Bousquet-Mélou wrote it in [22], 'to date this recursion remains entirely mysterious and Tutte's tour de force has remained isolated'.

$$-\frac{q \cdot (q-1)}{q-4} \cdot w, \, -\frac{q \cdot (q-2)}{2\,(q-4)} \cdot w - \frac{q \cdot (q-4)}{2} \cdot w^2. \qquad (11)$$

Let us remark that, in the $h_1 = 0$ limit, the series (8) reduces, for any value of q, to the series (6).

2.1. The $q = 4$ subcase

In the $q = 4$ subcase the previous series (6) becomes:

$$H(w) = 12\, w^2 + 24\, w^3 + 168\, w^4 + 1656\, w^5 + 19\,296\, w^6 + 24\,8832\, w^7 + 3437\,424\, w^8 + 49\,923\,288\, w^9 + 753\,269\,856\, w^{10} + \cdots. \qquad (12)$$

If one considers the solutions of equation (4) with the initial conditions $h_0 = 0$ and $h_1 = 0$, but one does not impose $h_2 = q \cdot (q-1) = 12$, one finds a *one-parameter family* of solutions of equation (4), namely (here A denotes the parameter of this one-parameter family):

$$H_A(w) \; = \; -w + A^3 \cdot \left(\frac{w}{A^2} + H\left(\frac{w}{A^2}\right)\right), \qquad (13)$$

where the function H, in (13), is the previous series (12). This corresponds to a one-parameter group of symmetry of the nonlinear differential equation (4). Let us introduce the function $F(w) = H(w) + w$. It is solution of the (quite simple) nonlinear differential equation:

$$\left(3 \cdot w \cdot \frac{\mathrm{d}F(w)}{\mathrm{d}w} - 5 \cdot F(w)\right) \cdot \frac{\mathrm{d}^2F(w)}{\mathrm{d}w^2} + 48 \cdot w \; = \; 0, \qquad (14)$$

which has, clearly, the scaling symmetry $F(w) \; \rightarrow \; A^3 \cdot F(w/A^2))$. This suggests to define a function $G(w)$ such that $F(w) = w^{3/2} \cdot G(w)$. Introducing the homogeneous derivative

$$G_1(w) = w \cdot \frac{\mathrm{d}G(w)}{\mathrm{d}w}, \quad G_2(w) = w \cdot \frac{\mathrm{d}G_1(w)}{\mathrm{d}w}, \qquad (15)$$

one finds that the nonlinear differential equation (4), for $q = 4$, takes the very simple *autonomous*[8] form:

$$(G(w) - 6\, G_1(w)) \cdot (3\, G(w) + 8\, G_1(w) + 4\, G_2(w)) - 3 \cdot 2^7 = 0. \qquad (16)$$

As far as the singular points are concerned, this change of function suggests that the exponent $3/2$ should play a selected role.

In order to get very long series, we consider Tutte's recurrence (5) for $q = 4$. Using this recurrence we have been able to get 24 000 coefficients[9] of the series (12). This series has a finite radius of convergence $r \simeq 0.049\,65\ldots$, the coefficients growing like λ^N where $\lambda \simeq 20.1378\ldots$

We first tried to see if such very long series could actually correspond to a holonomic function using the same kind of tools we have already used in our (quite extreme) studies of n-fold integrals of the Ising type [3, 25, 26]. We seek linear differential operators, annihilating the series (12) given with N coefficients ($N = 10\,000, \cdots, 24\,000$), of order Q in the homogeneous derivative $\theta = w \cdot \mathrm{d}/\mathrm{d}w$ and of degree D for the polynomial coefficients in front of the θ^ns, where the order, degree, and number of coefficients are related by a simple relation[10]:

[8] As can be seen in equation (16) this equation has constant coefficients.
[9] This is a 376 Megaoctets file.
[10] This kind of relation corresponds to the so-called 'ODE formula' see, for instance, equation (26) in [26].

$$(Q + 1) \cdot (D + 1) = N - 1500. \tag{17}$$

For the series with $N = 24000$ coefficients we explored all the values of the order Q and degree D related [26] by the 'ODE formula' (17), and failed to find a linear differential operator annihilating (12). This seems to exclude the possibility that the series (12) could be a holonomic function.

A diff-Padé analysis[11] of this (probably non-holonomic) series gives a first set of singular points with their corresponding exponents. One gets the first set of singularities, namely one real singularity $w_1 = 0.049\,65\ldots$, and several complex singularities $0.202\,837\ldots \pm i \cdot 0.096\,4358\ldots$, $0.470\,420\ldots \pm i \cdot 0.377\,27\ldots$, $0.860\,28\ldots \pm i \cdot 0.925\,57\ldots$, $1.3784\ldots \pm i \cdot 1.822\,95\ldots$, $1.8007\ldots \pm i \cdot 0.487\,40\ldots$, $2.029\,904\ldots \pm i \cdot 3.150\,337\ldots$, all of them with the exponent $3/2$, the exponents at infinity being $-1/3$, $-2/3$, $-4/3$, $-5/3$,... It is possible that performing such kind of linear differential analysis of a (probably non-holonomic) series with longer series, one could, with higher order linear differential operators, see the emergence of more and more singularities: this could be a way to convince oneself that this series is non-holonomic. What is the validity of such a *linear approach for a typical non-linear function* is an open question, which certainly requires quite extensive studies[12] *per se*. Let us rather perform, in the following, some simpler clear-cut arithmetic calculations on this quite large series.

3. Reduction of the $q = 4$ series modulo primes

Recalling the results of a previous paper [1] where we have shown that the full susceptibility of the Ising model, which is a *non-holonomic* function [2, 3], actually reduces to *algebraic functions modulo any powers of the prime* 2, it is tempting to see if the series (12), for $q = 4$, actually reduces to *algebraic functions* modulo the first eight primes $2, 3, \ldots 19$.

Since we have developed some tools [25, 26] to find the (Fuchsian) linear differential operator annihilating a given series, let us first try (before seeking directly for algebraic relations on this series, see next section 4) to see if this series (12), modulo the first eight primes, is solution of a linear differential operator.

Since the coefficients of the series are all divisible by 12, and the series starts with w^2, we consider, instead of the series (12), this series divided by $12\, w^2$, modulo the first primes $2, 3, \ldots 17$, and search for *linear differential operators* annihilating these series modulo primes. It is only because we have a prejudice that this $q = 4$ series is 'very special' that we perform such calculations.

Caveat: Since we are going to use our tools [25, 26, 28–30] to find (Fuchsian) linear differential operators modulo rather small primes (the first eight primes), one may be facing a problem we do not encounter with our previous studies [25, 26] performed with rather large primes ($2^{15} - 19 = 32\,749, \ldots$). Modulo a prime p, *any* power series with *integer coefficients* is solution of the linear differential operators $\theta^p - \theta$, where θ denotes the homogeneous derivative $w \cdot d/dw$, or much more simply of the operator d^p/dw^p. Actually the linear differential operator, $\theta^p - \theta$ acting on w^n, gives (Fermat's little theorem):

$$(\theta^p - \theta)(w^n) = n^p - n = 0 \qquad \mod p. \tag{18}$$

[11] We do thank Hassani for providing this diff-Padé analysis.

[12] In the spirit of the calculations we performed in [27].

This is typically the reason why, when one is not in characteristic zero, the wording 'being holonomic' should be prohibited[13]. When one performs such linear differential operator guessing, modulo rather small prime p, it is important when one gets a result, to check, systematically, that the order of the linear differential operator one obtains is strictly smaller than p, in order to avoid being 'polluted' by such 'spurious' linear differential operators.

3.1. Reduction of the $q = 4$ series modulo the first eight primes: the results

To take into account the fact that all the integer coefficients of (12) are divisible by $q \cdot (q-1) = 12$ we will consider, instead of (12), the series (12) divided by $12\, w^2$:

$$S(w) = \frac{H(w)}{12\, w^2} = 1 + 2\, w + 14\, w^2 + 138\, w^3 + 1608\, w^4 + 20\,736\, w^5 + 286\,452\, w^6 + 4160\,274\, w^7 + 62\,772\,488\, w^8 + 976\,099\,152\, w^9 + \cdots. \tag{19}$$

From the previous recurrence relation (5) for $q = 4$ we obtained 24 001 coefficients of this series.

We *actually found linear differential operators* for the series (19), modulo the first primes $p = 2, 3, \ldots 17$. We denote L_p the linear differential operators annihilating, modulo the prime p, the series (12) divided by $12\, w^2$. In the spirit of previous linear differential operator guessing [3, 25, 26], we introduce the homogeneous derivative $\theta = w \cdot d/dw$. The linear differential operators L_p read respectively[14]:

$$L_3 = 2\, w + \theta + (w+1) \cdot \theta^2,$$
$$L_5 = 2\, w + (2 + 3\, w) \cdot \theta + (w+2) \cdot \theta^2,$$
$$L_7 = 3\, w^3 + (4 + w^3) \cdot \theta + (3\, w^3 + 3) \cdot \theta^3 + (5 + w^3) \cdot \theta^4,$$

$$\begin{aligned} L_{11} = \; & 9\, w^{15} + 5\, w^{10} + 5\, w^5 + (2\, w^{15} + 6\, w^{10} + 9\, w^5 + 6) \cdot \theta \\ & + (2\, w^{15} + 8\, w^{10} + 7\, w^5 + 1) \cdot \theta^2 + (5\, w^{15} + 7\, w^{10} + w^5) \cdot \theta^3 \\ & + (6 + 4\, w^5 + w^{10} + 2\, w^{15}) \cdot \theta^4 + (10\, w^{15} + 9\, w^{10} + 8\, w^5 + 10) \cdot \theta^5 \\ & + (8\, w^{15} + 8\, w^{10} + 5\, w^5 + 7) \cdot \theta^6 + (5\, w^{15} + 4\, w^5 + 6) \cdot \theta^7 \\ & + (w^{15} + w^5 + 8) \cdot \theta^8, \end{aligned} \tag{20}$$

and:

$$L_{13} = \sum_{n=0}^{8} p_n(w) \cdot \theta^n, \qquad L_{17} = \sum_{n=0}^{13} q_n(w) \cdot \theta^n, \tag{21}$$

where the polynomials p_n and q_n read respectively:

$$p_0(w) = 9\, w^{30} + 8\, w^{27} + 10\, w^{24} + 11\, w^{21} + 11\, w^{18} + 5\, w^{15} + 10\, w^{12} + 8\, w^9 + 2\, w^6,$$
$$p_1(w) = 11\, w^{30} + 4\, w^{27} + 7\, w^{24} + 4\, w^{21} + 7\, w^{18} + 12\, w^{15} + w^{12} + 2\, w^9 + 2\, w^6 + 9\, w^3 + 11,$$
$$p_2(w) = 3\, w^{30} + 7\, w^{27} + 12\, w^{24} + 2\, w^{21} + 9\, w^{15} + 7\, w^{12} + 5\, w^9 + 9\, w^6 + 2,$$

[13] Because of identity (18) every series is 'holonomic modulo a prime p': one must search for linear differential operators, getting rid of these spurious linear differential operators (18).

[14] Modulo 2 the series (12), divided by $12\, w^2$, is just the constant 1: the L_2 operator is trivially θ. Slight transformations of the series have to be performed to get a non-trivial result (see equation (23) in section 4 below).

$$p_3(w) = 6\,w^{30} + 10\,w^{27} + 7\,w^{24} + 12\,w^{21} + 9\,w^{18} + 10\,w^{15} + 4\,w^{12} + 2\,w^{9} + 2\,w^{3} + 6,$$
$$p_4(w) = w^{30} + w^{27} + 6\,w^{24} + 6\,w^{21} + 5\,w^{18} + 2\,w^{15} + 7\,w^{9} + 9\,w^{6} + 8\,w^{3} + 1,$$
$$p_5(w) = 12\,w^{30} + 9\,w^{27} + 4\,w^{24} + 5\,w^{21} + 10\,w^{15} + 3\,w^{12} + 3\,w^{9} + 9\,w^{6} + 6\,w^{3},$$
$$p_6(w) = 12\,w^{30} + w^{27} + 7\,w^{24} + 2\,w^{21} + 3\,w^{18} + 9\,w^{15} + 2\,w^{12} + 2\,w^{9} + 5\,w^{6} + 3\,w^{3} + 1,$$
$$p_7(w) = 10\,w^{30} + 6\,w^{24} + 5\,w^{18} + 3\,w^{15} + 9\,w^{12} + 9\,w^{9} + 3\,w^{6} + 9\,w^{3} + 9,$$
$$p_8(w) = w^{30} + 6\,w^{27} + 2\,w^{24} + 2\,w^{18} + 10\,w^{15} + 11\,w^{12} + 7\,w^{9} + 2\,w^{6} + 2\,w^{3} + 9,$$

and:

$$q_0(w) = 15\,w^{40} + 13\,w^{36} + 2\,w^{32} + 15\,w^{28} + 16\,w^{24} + 7\,w^{20} + w^{16} + 7\,w^{12},$$
$$q_1(w) = 15\,w^{40} + 5\,w^{36} + 5\,w^{32} + 4\,w^{28} + 12\,w^{24} + 15\,w^{20} + 11\,w^{16} + 2\,w^{12} + 15\,w^{8} + 16\,w^{4} + 5,$$
$$q_2(w) = 13\,w^{40} + 9\,w^{36} + 6\,w^{32} + w^{28} + 5\,w^{24} + 4\,w^{20} + 10\,w^{16} + 5\,w^{12} + 4\,w^{8} + 14\,w^{4} + 15,$$
$$q_3(w) = 15\,w^{40} + 10\,w^{36} + 12\,w^{32} + 2\,w^{28} + 14\,w^{24} + 10\,w^{20} + 15\,w^{16} + 5\,w^{12} + 13\,w^{8} + 10\,w^{4} + 6,$$
$$q_4(w) = 15\,w^{40} + w^{36} + 4\,w^{32} + 8\,w^{28} + 13\,w^{24} + 6\,w^{20} + 2\,w^{16} + 8\,w^{4} + 5,$$
$$q_5(w) = 4\,w^{40} + 5\,w^{36} + 11\,w^{32} + 16\,w^{28} + 13\,w^{24} + 6\,w^{20} + 16\,w^{16} + w^{12} + 14\,w^{8} + 4\,w^{4} + 16,$$
$$q_6(w) = 6\,w^{40} + 9\,w^{36} + 6\,w^{32} + 11\,w^{28} + w^{24} + 8\,w^{20} + 6\,w^{16} + 7\,w^{12} + 4\,w^{8} + 14\,w^{4} + 11,$$
$$q_7(w) = 14\,w^{40} + 5\,w^{36} + 11\,w^{32} + 7\,w^{24} + 8\,w^{20} + 11\,w^{16} + 8\,w^{12} + 2\,w^{8} + 11\,w^{4} + 10,$$
$$q_8(w) = 12\,w^{40} + 5\,w^{36} + 3\,w^{32} + 6\,w^{28} + 15\,w^{24} + 13\,w^{20} + 16\,w^{16} + 5\,w^{12} + 5\,w^{8} + 11\,w^{4} + 6,$$
$$q_9(w) = 14\,w^{40} + 15\,w^{36} + 11\,w^{32} + 4\,w^{28} + 14\,w^{24} + w^{20} + 14\,w^{16} + 12\,w^{12} + 13\,w^{8} + w^{4} + 2,$$
$$q_{10}(w) = 15\,w^{40} + 16\,w^{36} + 13\,w^{32} + 13\,w^{28} + 4\,w^{24} + 5\,w^{20} + 6\,w^{16} + 2\,w^{12} + 9\,w^{8} + 7\,w^{4} + 13,$$
$$q_{11}(w) = 5\,w^{40} + 2\,w^{36} + 9\,w^{32} + 13\,w^{28} + 2\,w^{24} + 16\,w^{20} + 11\,w^{16} + 9\,w^{12} + 2\,w^{8} + 4,$$
$$q_{12}(w) = 9\,w^{40} + 15\,w^{36} + 14\,w^{28} + 14\,w^{24} + 8\,w^{20} + 10\,w^{16} + 8\,w^{12} + 12\,w^{8} + 14\,w^{4} + 1,$$
$$q_{13}(w) = w^{40} + 9\,w^{36} + 9\,w^{32} + 12\,w^{28} + 6\,w^{24} + 12\,w^{20} + 7\,w^{16} + 14\,w^{12} + 9\,w^{8} + 9\,w^{4} + 8.$$

We tried to get the linear differential operator L_{19} for $p = 19$, but the calculations were too time consuming. We will come to this $p = 19$ case with another more direct approach (see section 4.1 below).

It is quite a surprise to find *linear* differential operators on such a *typically nonlinear, probably non-holonomic*, function. However, keeping in mind the results on the susceptibility of the Ising model [1], it is natural to ask if such results modulo various primes could correspond to reductions of the (probably non-holonomic) series (12) to *algebraic functions* modulo primes. This amounts to revisiting the previous series modulo primes, trying to see, directly, if they are algebraic functions modulo primes, seeking for a polynomial equation satisfied by these series modulo primes. Such calculations are performed in the next section. An alternative way amounts to calculating the *p-curvature* [46] of these linear differential operators known modulo the prime p: if these series are reductions of algebraic functions modulo primes, the p-curvature [46] *has to be equal to zero*.

Taking into account the fact that the primes, considered here, are small enough, one can actually calculate the p-curvature using some modular[15] algorithm [32, 33]. One actually finds that all these linear differential operators L_p, modulo the primes p, *have zero p-curvature*[16].

4. Algebraic functions modulo primes

Let us show that these series, modulo various primes, are actually *algebraic functions modulo primes*, by finding directly the polynomial equations they satisfy.

Let us introduce the following *lacunary functions* which will be used in the following:

$$\mathcal{L}_2(w) = \sum_{i=0}^{\infty} w^{2^i}, \qquad \mathcal{L}_3(w) = \sum_{i=0}^{\infty} w^{3^i}, \qquad \mathcal{L}_6(w) = \sum_{i=0}^{\infty} w^{2\cdot 3^i}. \tag{22}$$

Similarly to the calculations performed in [1] on the susceptibility of the Ising model, it is straightforward to see that, modulo the prime 2, a slight modification of the series (19) becomes the lacunary series $\mathcal{L}_2(w)$ which is well-known to satisfy a functional equation and an algebraic equation, namely $\mathcal{L}_2(w^2) = \mathcal{L}_2(w) - w = \mathcal{L}_2(w)^2 \mod 2$.

Modulo 2, we obtain:

$$\frac{w}{2} \cdot (S(w) - 1) + w \cdot (w^2 + 1) = \mathcal{L}_2(w). \tag{23}$$

Performing similar calculations, *modulo powers of the prime* 2, one gets similar results showing that the series *reduces to algebraic functions modulo powers of the prime* 2.

For instance, modulo 2^2, the following expression of $S(w)$ reduces, again, to the previous lacunary series:

$$\frac{w}{2} \cdot (S(w) - 1) + w \cdot (2\, w^6 + w^2 + 1) = \mathcal{L}_2(w). \tag{24}$$

Modulo 2^3, one has:

$$w \cdot (S(w) - 1) + w \cdot (4\, w^6 + 2\, w^2 + 2) = 2 \cdot \mathcal{L}_2(w). \tag{25}$$

[15] For larger prime numbers, one cannot, in practice, calculate the p-curvature that way, and one must use totally different algorithms [31].

[16] We thank Weil for providing this result using a modular algorithm.

Modulo 2^4, one verifies on the series of 24 001 coefficients, the following relation

$$w \cdot (S(w) - 1) = (2 + 8w) \cdot \mathcal{L}_2(w) + w \cdot (8w^{14} + 4w^6 + 8w^3 + 6w^2 + 8w + 14). \qquad (26)$$

Modulo 2^5, one can verify the more involved relation[17]:

$$w \cdot (S(w) - 1) = 24 \cdot \mathcal{L}_2(w)^2 + (16w^3 + 24w + 26) \cdot \mathcal{L}_2(w) + w \cdot (8w^{30} + 4w^{14} + 2w^6 + 8w^5 + 8w^4 + 4w^3 + 3w^2 + 12w + 3). \qquad (27)$$

Let us, now, consider the same series modulo the prime 3. One immediately sees the emergence of the lacunary series $\mathcal{L}_3(w)$:

$$\frac{w}{2} \cdot (S(w) - 1) + w \cdot (2w + 1) = \mathcal{L}_3(w) \mod 3. \qquad (28)$$

This new lacunary series $\mathcal{L}_3(w)$ satisfies, modulo 3, a simple functional equation, as well as a simple algebraic equation $\mathcal{L}_3(w^3) = \mathcal{L}_3(w) - w = \mathcal{L}_3(w)^3$. The series is thus an *algebraic function modulo* 3.

Modulo other primes (or power of primes) this guessing by lacunary series (along the line of [1]) is no longer well-suited.

4.1. Seeking algebraic relations modulo primes

A better approach to analyze these series is to seek, systematically, modulo a given prime p, for a *polynomial relation*: $P(w, S(w)) = 0 \mod p$.

As a first example, using equation (28), one can see that the series $S(w)$ satisfies, modulo $p = 3$, the polynomial relation:

$$w^2 \cdot S(w)^3 + 2S(w) + (1 + 2w + w^2 + w^5) = 0 \mod 3. \qquad (29)$$

Modulo powers of the prime $p = 3$, one also obtains reductions to algebraic functions, but the calculations are slightly more involved[18]. For instance, modulo $p = 3^2$ the series reads:

$$S(w) = 1 + 2w + 5w^2 + 3w^3 + 6w^4 + 6w^7 + 8w^8 + 3w^9 + 3w^{11} + 8w^{26} + 3w^{27} + 3w^{29} + 3w^{35} + 8w^{80} + 3w^{81} + 3w^{83} + 3w^{89} + 3w^{107} + 8w^{242} + \cdots. \qquad (30)$$

In fact the series (30) can actually be understood from the previously introduced lacunary series. The series (30) can in fact be seen to be equal, modulo 3^2, to:

$$\frac{1}{w}\left(\frac{3}{2}\mathcal{L}_3^2 + 8\mathcal{L}_3 + 3\mathcal{L}_6\right) + 2(3w^7 + 3w^4 + 3w^2 + w + 1). \qquad (31)$$

Note that these lacunary series satisfy (in characteristic zero) the functional equations

$$\mathcal{L}_3(w^3) - \mathcal{L}_3(w) + w = 0, \qquad \mathcal{L}_6(w^3) - \mathcal{L}_6(w) + w^2 = 0. \qquad (32)$$

[17] One may be surprised to see the occurrence of $\mathcal{L}_2^2$ in equation (27) if one has in mind the identity $\mathcal{L}_2 = w + \mathcal{L}_2^2$. Note that this identity holds modulo 2 and *not* modulo 2^5.
[18] As we are going to see below, see equation (37).

Therefore these lacunary series satisfy, modulo 3, the polynomial relations:

$$\mathcal{L}_3^3 - \mathcal{L}_3 + w = 0 \quad \text{mod } 3, \qquad \mathcal{L}_6^3 - \mathcal{L}_6 + w^2 = 0 \quad \text{mod } 3. \tag{33}$$

The lacunary function $\mathcal{L}_3$ satisfies, modulo 3^2, the slightly more involved polynomial relation:

$$w^2 + w \cdot \mathcal{L}_3 + 7 \cdot \mathcal{L}_3^2 + 2\, w \cdot \mathcal{L}_3^3 + \mathcal{L}_3^4 + \mathcal{L}_3^6 = 0 \quad \text{mod } 3^2. \tag{34}$$

Similarly, the lacunary function $\mathcal{L}_6$ satisfies, modulo 3^2, the polynomial relation:

$$w^4 + w^2 \cdot \mathcal{L}_6 + 7 \cdot \mathcal{L}_6^2 + 2\, w^2 \cdot \mathcal{L}_6^3 + \mathcal{L}_6^4 + \mathcal{L}_6^6 = 0 \quad \text{mod } 3^2. \tag{35}$$

The elimination of $\mathcal{L}_3$ and $\mathcal{L}_6$ in (31) gives a polynomial[19] relation of degree 36 in $S(w)$ and of degree 72 in w:

$$P(w, S(w)) = \sum_{n=0}^{36} P_n(w) \cdot S(w)^n = 0 \quad \text{mod } 3^2. \tag{36}$$

We will not give this polynomial here because it is a bit too large. What matters is that it exists. Now that we have these two degrees (36 in $S(w)$ and 72 in w) for a first example of polynomial relation, one can revisit this example trying to find, directly, simpler polynomial relations of lower degree (especially in $S(w)$). One actually finds the following polynomial relation of degree 6 in $S(w)$ and degree 17 in w:

$$\begin{aligned} w^3 \cdot\, & (8\, w^{17} + 6\, w^{14} + 3\, w^{13} + 6\, w^{12} + 6\, w^{11} + 6\, w^{10} + 5\, w^8 + 3\, w^6 + w^5 \\ & + 3\, w^4 + 3\, w^3 + 2\, w^2 + 6\, w + 3) \\ & + (5\, w^{15} + w^{12} + 5\, w^{11} + w^{10} + 5\, w^9 + 5\, w^8 + 5\, w^6 + 5\, w^5 + 6\, w^3) \cdot S(w) \\ & + 4\, w^5 \cdot (2\, w^5 + 2\, w^2 + w + 2) \cdot S(w)^2 \\ & + w^7 \cdot (w^{10} + 2\, w^7 + w^6 + 2\, w^5 + w^4 + w^3 + w + 1) \cdot S(w)^3 \\ & + w^7 \cdot (2\, w^5 + 2\, w^2 + w + 2) \cdot S(w)^4 \\ & + 3\, w^7 \cdot S(w)^5 + w^9 \cdot (2\, w^5 + 2\, w^2 + w + 2) \cdot S(w)^6 = 0 \quad \text{mod } 3^2. \end{aligned} \tag{37}$$

Because of the quite large size of these polynomial relations we will not, in the following, give these relations corresponding to the series modulo power of primes for the next primes.

Modulo $p = 5$, we obtained the polynomial relation:

$$w \cdot S(w)^2 + S(w) + 2\, w^2 + 2\, w + 4 = 0 \quad \text{mod } 5. \tag{38}$$

Modulo $p = 7$, we obtained the polynomial relation

$$\begin{aligned} w^4 \cdot S(w)^4 + w^2 \cdot (5\, w + 1) \cdot S(w)^3 + w \cdot (6\, w^2 + 5\, w + 2) \cdot S(w)^2 \\ + (w^2 + 2\, w + 6) \cdot S(w) + 2\, w^2 + 5\, w + 1 = 0 \quad \text{mod } 7. \end{aligned} \tag{39}$$

[19] This polynomial can easily be obtained performing resultants in Maple.

Modulo $p = 11$, we obtained the polynomial relation

$$\sum_{n=0}^{10} p_n(w) \cdot S(w)^n = 0, \qquad \text{where:}$$

$$p_0(w) = w^9 + 4w^8 + 2w^7 + 9w^6 + 2w^5 + w^4 + 8w^3 + 8w^2 + 3w + 3,$$
$$p_1(w) = 8w^9 + 8w^8 + 6w^7 + 7w^6 + 2w^4 + 10w^3 + 4w^2 + 9w + 8,$$
$$p_2(w) = w \cdot (4w^9 + w^8 + 2w^7 + 3w^5 + 7w^4 + 4w^3 + 3w^2 + 9w + 5),$$
$$p_3(w) = w^2 \cdot (8w^8 + 10w^7 + 2w^6 + w^5 + w^4 + 10w^3 + 4w^2 + 3w + 5),$$
$$p_4(w) = w^3 \cdot (2w^8 + 2w^7 + 3w^6 + 2w^5 + w^4 + 8w^3 + 3w^2 + 8),$$
$$p_5(w) = w^4 \cdot (3w^7 + 9w^6 + 8w^5 + 5w^4 + 10w^2 + 6),$$
$$p_6(w) = w^7 \cdot (6w^5 + 10w^4 + w^3 + 9w^2 + 9),$$
$$p_7(w) = 2w^{10} \cdot (3w^2 + 5w + 3),$$
$$p_8(w) = w^{12} \cdot (9w + 1), \quad p_9(w) = 10w^{13}, \quad p_{10}(w) = w^{14}. \tag{40}$$

One verifies that this polynomial equation is actually satisfied with our series of 24 001 coefficients modulo $p = 11$.

Modulo $p = 13$, we obtained the polynomial relation

$$\sum_{n=0}^{14} q_n(w) \cdot S(w)^n = 0, \qquad \text{where:}$$

$$q_0(w) = 11w^{14} + 6w^{13} + 9w^{12} + 2w^{11} + 6w^{10} + 9w^8 + 10w^7 + 4w^6 + 4w^5 + 11w^4 + 11w^3 + 5w^2 + 10w + 1,$$
$$q_1(w) = w^{14} + 3w^{13} + 7w^{12} + 11w^{11} + 3w^{10} + 4w^9 + 8w^8 + w^7 + 7w^6 + 5w^5 + 6w^4 + 5w^3 + 9w + 12,$$
$$q_2(w) = w \cdot (6w^{14} + 2w^{13} + 2w^{12} + 11w^8 + 11w^7 + 10w^6 + w^4 + 7w^3 + 11w^2 + 6w + 9),$$
$$q_3(w) = w^2 \cdot (3w^{13} + 6w^{12} + 11w^{11} + 6w^{10} + 11w^9 + 5w^8 + 5w^7 + 5w^6 + 5w^5 + 4w^4 + 8w^3 + 9w^2 + 9w + 1),$$
$$q_4(w) = w^3 \cdot (9w^{13} + 2w^{12} + 9w^{11} + 6w^{10} + 10w^8 + 12w^7 + 10w^6 + 10w^5 + 7w^4 + 7w^3 + 5w^2 + w + 9),$$
$$q_5(w) = w^4 \cdot (7w^{12} + 11w^{11} + 9w^{10} + 4w^9 + 5w^8 + 12w^7 + 7w^6 + 5w^5 + 7w^4 + 5w^3 + 9w^2 + 12),$$
$$q_6(w) = w^5 \cdot (w^{12} + w^{11} + 12w^{10} + 7w^9 + 4w^8 + 3w^7 + 8w^6 + 4w^5 + 5w^4 + 10w^3 + 2w^2 + 11),$$
$$q_7(w) = w^8 \cdot (9w^9 + 8w^8 + w^7 + 10w^6 + 2w^5 + 6w^4 + 10w^3 + 12w^2 + 5),$$
$$q_8(w) = w^9 \cdot (7w^9 + 10w^8 + w^7 + 2w^6 + 9w^5 + 6w^4 + 2w^3 + 2w^2 + 1),$$
$$q_9(w) = w^{12} \cdot (w + 1) \cdot (5w^5 - 4w^4 + 4w^3 + 7w^2 - w + 1),$$
$$q_{10}(w) = w^{13} \cdot (6w^6 + w^5 + 8w^3 + 6w^2 + 9), \quad q_{11}(w) = w^{16} \cdot (w^3 + 6),$$
$$q_{12}(w) = w^{17} \cdot (w^3 + 6), \quad q_{13}(w) = 12w^{20}, \; q_{14}(w) = w^{21}.$$

One verifies that this polynomial equation is actually satisfied with our series of 24 001 coefficients modulo $p = 13$.

Modulo $p = 17$, we obtained the polynomial relation

$$\sum_{n=0}^{24} r_n(w) \cdot S(w)^n \; = \; 0, \qquad (41)$$

where the polynomials $r_n(w)$ are given in appendix A.1. One verifies that this polynomial equation is actually satisfied with our 24 001 coefficients series modulo $p = 17$.

Modulo $p = 19$, we obtained the polynomial relation

$$\sum_{n=0}^{30} s_n(w) \cdot S(w)^n \; = \; 0, \qquad (42)$$

where the polynomials $s_n(w)$ are given in appendix A.2. One verifies that this polynomial equation is actually satisfied with 23 756 coefficients of our series.

After this accumulation of algebraic results, it seems reasonable to conjecture that the series (12), or equivalently (19), *reduces to algebraic functions modulo every prime* (and probably modulo power of primes, but it is much more difficult to confirm this statement modulo power of primes).

Remark: When one does not restrict to primes the results have to be taken 'cum grano salis'. For instance modulo 6, the series modulo 6 reads:

$$S(w) = \; 1 + 2\,w + 2\,w^2 + 2\,w^8 + 2\,w^{26} + 2\,w^{80} \; + 2\,w^{242} + 2\,w^{728} \\ + 2\,w^{2186} \; + 2\,w^{6560} \; + \; \cdots. \qquad (43)$$

If one considers the expression $w \cdot (1 + S(w))/2 - \; w^2$, one actually finds that it is nothing but the selected lacunary series $\sum w^{3^n} = \mathcal{L}_3(w)$:

$$\frac{w}{2} \cdot \; (1 + S(w)) - \; w^2 \; = \; \mathcal{L}_3(w) \\ = w \; + w^3 + w^9 \; + w^{27} \; + w^{81} \; + w^{243} \\ + w^{729} \; + w^{2187} \; + w^{6561} + w^{19\,683} \; + \; \cdots. \qquad (44)$$

Following the ideas displayed in [34], one can see that this series is not algebraic modulo 6. This series $S(w)$ is algebraic modulo 3 (because $S(w^3) = w + S(w)$ and $S(w)^3 = S(w^3) \mod 3$), but it is not algebraic modulo 2. If it were algebraic modulo 2, it would be[20] algebraic modulo 6.

5. Comparison with other reductions modulo primes

This first example of reduction to algebraic functions modulo primes, or power of primes, of (probably *non-holonomic*) functions, satisfying *nonlinear differentiable equation*s, is unexpected in a more general non-holonomic, nonlinear framework.

In order to have some perspective on such kind of results, let us consider series with *integer coefficients*, that are solutions of *linear* differential equations (holonomic). Let us consider *diagonals of rational functions* [36–41], and also holonomic *globally bounded* G-series which are *not known* to be diagonals of rational functions [42, 43].

[20] There is a theorem by Cobham [35] which says that if a series has only coefficients $+1$ it can be algebraic modulo two successive primes (here 2 and 3) only if it is rational. Furthermore if a series is algebraic modulo two relatively prime numbers, namely a prime p and also another prime q, it is algebraic modulo $p \cdot q$.

5.1. Reductions modulo primes of holonomic functions: diagonals of rational functions and beyond

Diagonals of rational functions are known to reduce to algebraic functions modulo any prime [42, 43] (or power of primes). Reductions modulo primes of diagonals of rational functions are, in general, quite easy and quick to perform. When the order of the linear differential operator is not too large one gets quite easily the algebraic functions corresponding to this reduction. One should note that diagonals of rational functions that are ${}_nF_{n-1}$ *hypergeometric series* are 'almost too simple' (see appendix B). The reduction of hypergeometric series are, most of the time, very simple algebraic functions of the form $P(x)^{-1/N}$ (where N is an integer and $P(x)$ is a polynomial), which correspond to the truncation of the series expansion of the hypergeometric series modulo the prime p. We sketch a few results of such reductions of hypergeometric series modulo primes in appendix B.

Along this hypergeometric line it is worth recalling the hypergeometric function ${}_3F_2([1/9,\, 4/9,\, 5/9],\, [1/3,\, 1],\, 3^6\, x)$ introduced by Christol [42–44], a few decades ago, to provide an example of holonomic G-series with *integer coefficients* that *may not be the diagonal of rational function*. After all these years, it is still an open question to see whether this function is, or is not, the diagonal of rational function. In such cases it is not guaranted[21] that the corresponding series modulo primes are algebraic functions (or that the series are 'automatic' [1]).

If one performs the same reductions modulo primes, one finds, in contrast with the previous studies of reductions modulo primes of diagonals of rational functions, that it becomes *quite hard to see whether a series like* ${}_3F_2([1/9,\, 4/9,\, 5/9],\, [1/3,\, 1],\, 3^6x)$*, modulo primes are algebraic functions* (they could be of the form $P(x)^{-1/N}$ where N is an extremely large integer, see appendix B.4, or they could satisfy polynomial relations of the 'Frobenius' type of large degree, see appendix B.4). Probably different strategies (p-automatic approaches) should be considered to find these polynomial relations (if any).

To sum up: As far as the reduction of *holonomic functions* modulo primes is concerned, we seem to have the following situation: either the holonomic function is actually the *diagonal of a rational function* [42, 43], the reduction to algebraic function modulo primes is thus guaranted, and one finds, very simply and quickly, these algebraic functions, or the holonomic function is *not* 'obviously' the diagonal of the rational function, and getting these algebraic functions can be very difficult (see appendix B.4).

This difficulty to find polynomial relations modulo rather small primes, for such a holonomic function (which is not obviously the diagonal of a rational function), has to be compared with the rather easy way we obtained, in section 4.1, polynomial relations for a (probably *non-holonomic*) series solution of the $q = 4$ *nonlinear differential equation* (4).

Remark: Modulo a prime p we have linear differential operators of two[22] different natures annihilating a given diagonal of rational function: one has linear differential operators of *nilpotent p-curvatures* [46] (which are the reduction, modulo p, of the globally nilpotent linear differential operators [46] annihilating the series in characteristic zero), and one also has linear differential operators of *zero p-curvatures*, corresponding to the fact that a diagonal of rational function reduces to algebraic functions modulo a prime p. For holonomic functions (in our case globally bounded [42] G-series), the order of the linear differential operator (of nilpotent p-curvature) 'saturates' with the order of the linear differential operator in

[21] The question to know if globally bounded D-finite formal power series (non-zero radius of convergence) are globally automatic (their reduction modulo all but finitely many primes p is p-automatic), *remains an open one*: see Question and remark, p 385 of [45].

[22] In fact three if one takes into account the 'spurious' linear differential operators (18).

characteristic zero. In contrast, for *selected non-holonomic functions, reducing to algebraic functions modulo primes*, one just has the second set of linear differential operators of zero p-curvature, their order having *no reason to have such an upper bound*. Increasing the value of the prime p in the modular guessing of the linear differential operator could, thus, be a way to *disentangle between holonomic functions and selected non-holonomic functions* reducing to algebraic functions modulo primes.

5.2. Reductions modulo primes of other selected non-holonomic functions

One would like to accumulate more examples of reductions modulo primes of other selected *non-holonomic* functions. In an integrable lattice model perspective where the theory of elliptic curves plays so often a crucial role (as well as mirror symmetries), a quite natural candidate amounts to considering the ratio of two selected holonomic functions, namely the *ratio of two periods of an elliptic curve* [47, 48]. Unfortunately, as can be seen in appendix C, one cannot perform such a reduction because one of the two holonomic functions, in such a ratio, is *not globally bounded* [42, 43], which means that the series cannot be recast into a series with integer coefficients: one cannot consider such series modulo primes[23].

Therefore let us rather consider non-holonomic functions that are, *not only the ratio of holonomic functions, but, in fact, the ratio of diagonals of rational functions*. Let us consider, for instance, the ratio of two simple hypergeometric functions *that are diagonals of rational functions* [42, 43]:

$$R(x) = \frac{{}_2F_1\left(\left[\frac{1}{3}, \frac{1}{3}\right], [1], \; 27\,x\right)}{{}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], \; 16\,x\right)}. \tag{45}$$

This ratio *satisfies a nonlinear differential equation* that can be obtained from the two order-two linear differential equations satisfied by these two simple hypergeometric functions. We give this nonlinear differential equation in appendix D.

The series expansion of this ratio (45) is a series with *integer coefficients*:

$$\begin{aligned} R(x) = 1 &- x + 4\,x^2 + 208\,x^3 + 5549\,x^4 + 133\,699\,x^5 + 3142\,224\,x^6 + 73\,623\,828\,x^7 \\ &+ 1733\,029\,548\,x^8 + 41\,095\,725\,700\,x^9 + 982\,470\,703\,424\,x^{10} + \cdots. \end{aligned} \tag{46}$$

These two hypergeometric functions are diagonals of a rational function: their reductions modulo primes must be algebraic functions. For instance, modulo $p = 7$, it reads:

$${}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], \; 16\,x\right) = (1 + 4\,x + x^2 + x^3)^{-1/6} \qquad \mod 7, \tag{47}$$

$${}_2F_1\left(\left[\frac{1}{3}, \frac{1}{3}\right], [1], \; 27\,x\right) = (1 + 3\,x + x^2)^{-1/6} \qquad \mod 7. \tag{48}$$

If one considers the *non-holonomic* series (46) corresponding to their ratio (45), it reduces modulo the prime 7, as it should, to the ratio of the (algebraic) reductions (47) and (48):

[23] An infinite number of primes occurs at the denominator of the successive coefficients of the series, preventing the consideration of such series modulo this infinite set of primes.

$$R(x) = \left(\frac{1 + 4x + x^2 + x^3}{1 + 3x + x^2}\right)^{1/6} \quad \mod 7. \tag{49}$$

The set of *non-holonomic series with integer coefficients, reducing to algebraic functions modulo every prime* (or power of prime), is clearly a very large set.

6. Conclusion

We have recalled that the full susceptibility series of the Ising model satisfies, modulo powers of the prime 2, exact algebraic equations [1] which is a consequence of the fact that, modulo 2^r, one cannot distinguish the full susceptibility from some simple diagonals of rational functions which reduce to algebraic functions modulo 2^r. We also recalled the nonlinear polynomial differential equation (4) obtained by Tutte for the generating function of the q-coloured rooted triangulations by vertices.

Along the line of a previous work [1] on the susceptibility model, we considered this series, solution of (4), modulo the first eight primes 2, 3, ... 19, and showed that this (probably non-holonomic) function actually reduces, modulo these primes, to algebraic functions. We conjecture that this *probably non-holonomic* function *reduces to algebraic functions modulo (almost) every primes*, or power of primes, numbers.

We believe that this result on the $q = 4$ solution of Tutte's nonlinear differential equation (4) for the generating function of the q-coloured rooted triangulations by vertices, is not an isolated curiosity, but corresponds to a first pedagogical example of a large class of remarkable non-holonomic functions in theoretical physics (lattice statistical physics, enumerative combinatorics ...) that reduce to algebraic functions modulo primes (and power of primes). It is important to understand these remarkable non-holonomic functions: Are they a ratio of holonomic functions (having in mind a *ratio of diagonals of rational functions*), or more generally algebraic functions of diagonals of rational functions [1], do the nonlinear differential equations they satisfy have the *Painlevé property*[24], etc ... ?

It is essential to build new tools and algorithms to see whether a given (large) series is solution of a *nonlinear differential equation*, and, in particular, of a *polynomial differential equation*. Too often Rubel's universal equation[25] is recalled to discourage any such '*nonlinear differential Padé* ' search. It must be clear that this kind of 'nonlinear differential Padé' analysis, should not be performed in the most general nonlinear framework: it must be performed with some assumptions, ansatz, corresponding to the problem of theoretical physics one considers (Painlevé property assumption [51], regular singularities assumptions, autonomous assumptions, see (16), nonlinear differential equations associated with *Schwarzian derivatives* [47, 52–54] or *modular forms* [55–60], ...).

It is crucial to build new tools and algorithms to see whether a given (large) series is a ratio of holonomic functions (having in mind the ratio of diagonals of rational functions), or more generally algebraic functions of diagonals of rational functions.

24 One can actually show that nonlinear equation (4) *does not have the Painlevé property*. We thank Ramani and Conte for two different proofs of this result.

25 Rubel's nonlinear differential equation [49] corresponds to a *homogeneous* polynomial differential equation such that any continuous function can be approximated, *on the real axis*, by a solution of this 'universal' equation. Other examples were obtained [50] which correspond to the idea of piecewise polynomial approximation on the real axis. This kind of *real analysis* theorem *do not mean* that any function of a complex variable is 'almost' a solution of a nonlinear differential equation in the complex plane, which would mean that any 'nonlinear differential Padé' would be pointless.

Such a result is clearly a strong incentive to obtain longer series (modulo some small primes $p = 3, \ldots$) for the full susceptibility of the Ising model to see if the susceptibility series reduces, for instance modulo 3, to an algebraic function.

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Appendix A. Polynomial relations modulo $p = 17$ and $p = 19$

Let us give the two polynomial relations satisfied by $S(w) = H(w)/(12\, w^2)$, namely the series (19), modulo $p = 17$ and $p = 19$.

A.1. Polynomial relation for $p = 17$

Modulo $p = 17$, we obtained the polynomial relation

$$\sum_{n=0}^{24} r_n(w) \cdot S(w)^n = 0, \qquad \text{(A.1)}$$

where:

$$\begin{aligned}
r_0(w) = {} & w^{27} + 10\, w^{26} + 9\, w^{25} + 8\, w^{24} + 14\, w^{23} + 12\, w^{22} + w^{21} + 7\, w^{20} + 8\, w^{19} \\
& + 3\, w^{18} + 6\, w^{17} + w^{16} + 16\, w^{15} + 3\, w^{14} + 4\, w^{13} \\
& + 5\, w^{12} + 6\, w^{11} + 2\, w^{10} + 9\, w^{9} \\
& + 12\, w^{8} + 4\, w^{7} + 11\, w^{6} + 11\, w^{5} + 4\, w^{4} \\
& + 4\, w^{3} + 5\, w^{2} + 9\, w + 10, \\
r_1(w) = {} & 2\, w^{27} + 7\, w^{26} + 15\, w^{25} + 12\, w^{24} \\
& + 15\, w^{23} + 5\, w^{22} + 7\, w^{21} + 14\, w^{20} + 6\, w^{19} \\
& + 7\, w^{18} + 11\, w^{17} + 3\, w^{16} + 3\, w^{15} + 4\, w^{14} + 8\, w^{13} + 16\, w^{12} \\
& + 8\, w^{11} + 15\, w^{10} + 15\, w^{9} + 2\, w^{8} + w^{7} \\
& + 16\, w^{6} + 16\, w^{5} + 7\, w^{4} + 6\, w^{3} + 7\, w^{2} + 6\, w + 7, \\
r_2(w) = {} & w \cdot (12\, w^{27} + 7\, w^{26} + 8\, w^{25} \\
& + 11\, w^{24} + 13\, w^{23} + 3\, w^{22} + 10\, w^{21} + 14\, w^{20} \\
& + 7\, w^{19} + 6\, w^{18} + 12\, w^{17} + 14\, w^{16} + 16\, w^{15} \\
& + 16\, w^{14} + 14\, w^{13} + 10\, w^{12} + 4\, w^{11} \\
& + 13\, w^{10} + 7\, w^{9} + 11\, w^{8} + 2\, w^{7} + 8\, w^{6} + 4\, w^{5} + 5w^{4} \\
& + 13\, w^{3} + 7\, w^{2} + 10\, w + 5),
\end{aligned}$$

$$
\begin{aligned}
r_3(w) = \;& w^2 \cdot \;(5\,w^{26} + 9\,w^{25} + 9\,w^{24}\\
& + 16\,w^{23} + 4\,w^{22} + 11\,w^{21} + 15\,w^{20} + 9\,w^{19}\\
& + 16\,w^{18} + 12\,w^{17} + 3\,w^{16} + 7\,w^{15} + 15\,w^{14}\\
& + 15\,w^{13} + 11\,w^{12} + 3\,w^{11} + 16\,w^{10}\\
& + 9\,w^{9} + 15\,w^{8} + 9\,w^{7} + w^{6} + 13\,w^{5}\\
& + w^{4} + 5\,w^{3} + 5\,w^{2} + 7\,w + 1),\\
r_4(w) = \;& w^3 \cdot \;(15\,w^{26} + 13\,w^{25} + 16\,w^{24} + 13\,w^{23} + 7\,w^{22} + 5\,w^{21} + 3\,w^{20} + 2\,w^{19}\\
& + 10\,w^{18} + 10\,w^{17} + 11\,w^{16} + 11\,w^{15}\\
& + 6\,w^{13} + 16\,w^{12} + 12\,w^{11} + 9\,w^{10}\\
& + 11\,w^{9} + w^{8} + 2\,w^{7} + 15\,w^{5} + 15\,w^{4} + 3\,w^{3} + 15\,w^{2} + 12\,w + 15),\\
r_5(w) = \;& w^4 \cdot \;(8\,w^{25} + 4\,w^{24} + 7\,w^{23} + 11\,w^{22} + 4\,w^{21} + w^{20} + 5\,w^{19}\\
& + 15\,w^{18} + 15\,w^{16} + 2\,w^{15} + w^{14} + 13\,w^{13}\\
& + 3\,w^{12} + 13\,w^{11} + 11\,w^{10} + w^{8}\\
& + 5\,w^{7} + 10\,w^{6} + 4\,w^{5} + 8\,w^{4} + 16\,w^{3} + 10\,w + 8),\\
r_6(w) = \;& w^5 \cdot \;(16\,w^{25} + 7\,w^{24} + 2\,w^{23} + w^{22} + 16\,w^{21} + 12\,w^{20} + 16\,w^{19} + 6\,w^{18}\\
& + 10\,w^{17} + 6\,w^{16} + 3\,w^{15} + 14\,w^{14} + 16\,w^{13}\\
& + 11\,w^{12} + 11\,w^{11} + w^{10} + 15\,w^{9}\\
& + 7\,w^{8} + 16\,w^{7} + 4\,w^{6} + 8\,w^{5} + 14\,w^{4} + 7\,w^{3} + 3\,w + 14),\\
r_7(w) = \;& w^6 \cdot \;(7\,w^{24} + 4\,w^{23} + 3\,w^{22} + 2\,w^{21} + 11\,w^{20} + 15\,w^{19} + w^{18}\\
& + 3\,w^{17} + 14\,w^{16} + 6\,w^{15} + 8\,w^{14} + 6\,w^{13}\\
& + 15\,w^{12} + 5\,w^{11} + 3\,w^{10} + 8\,w^{9}\\
& + 4\,w^{8} + 15\,w^{7} + 12\,w^{6} + 14\,w^{4} + 14\,w^{3} + 14),\\
r_8(w) = \;& w^7 \cdot \;(2\,w^{24} + w^{23} + 14\,w^{22} + 4\,w^{21} + 10\,w^{20} + 8\,w^{19} + 16\,w^{18}\\
& + 6\,w^{17} + 4\,w^{16} + 10\,w^{15} + 9\,w^{14}\\
& + 12\,w^{13} + 6\,w^{12} + 14\,w^{11} + 14\,w^{10}\\
& + 16\,w^{9} + 12\,w^{8} + 8\,w^{7} + 5\,w^{6} + 4\,w^{4} + 12\,w^{3} + 4),\\
r_9(w) = \;& w^{11} \cdot \;(14\,w^{20} + w^{19} + 15\,w^{18} + 7\,w^{17} + 10\,w^{16} + 6\,w^{15} + 14\,w^{14}\\
& + 10\,w^{13} + 4\,w^{12} + 14\,w^{11} + 11\,w^{10} + 6\,w^{9} + 9\,w^{8} + w^{7}\\
& + 9\,w^{6} + 4\,w^{5} + 10\,w^{4} + 4\,w^{3} + 15),\\
r_{10}(w) = \;& w^{12} \cdot \;(10\,w^{20} + 7\,w^{19} + 5\,w^{18} + w^{17} + 12\,w^{16} + 8\,w^{15} + 16\,w^{14}\\
& + 16\,w^{13} + 15\,w^{12} + 13\,w^{11} + 15\,w^{10} + 13\,w^{9} + 4\,w^{8}\\
& + 7\,w^{7} + 3\,w^{6} + 3\,w^{5} + 12\,w^{4} + 11\,w^{3} + 1),\\
r_{11}(w) = \;& w^{16} \cdot \;(6\,w^{16} + 11\,w^{15} + 5\,w^{14} + 2\,w^{12} + 8\,w^{11} + 12\,w^{10} + 16\,w^{8}\\
& + 16\,w^{7} + 14\,w^{6} + 6\,w^{4} + 10\,w^{3} + 15\,w^{2} + 7),
\end{aligned}
$$

$$
\begin{aligned}
r_{12}(w) = \;& w^{17} \cdot \;(6\,w^{16} + 16\,w^{15} + 8\,w^{14} + 2\,w^{12} + 7\,w^{11} + 9\,w^{10} + 16\,w^{8}\\
& + 14\,w^{7} + 2\,w^{6} + 6\,w^{4} + 13\,w^{3} + 7\,w^{2} + 7),\\
r_{13}(w) = \;& w^{20} \cdot \;(6\,w^{13} + 6\,w^{12} + 9\,w^{9} + 11\,w^{8} + w^{5} + 10\,w^{4} + 7\,w + 1),\\
r_{14}(w) = \;& w^{21} \cdot \;(10\,w^{13} + 13\,w^{12} + 15\,w^{9} + 4\,w^{8} + 13\,w^{5} + 16\,w^{4} + 6\,w + 5),\\
r_{15}(w) = \;& w^{22} \cdot \;(14\,w^{12} + 3\,w^{8} + 12\,w^{4} + 8),\\
r_{16}(w) = \;& w^{23} \cdot \;(2\,w^{12} + 15\,w^{8} + 9\,w^{4} + 6),
\end{aligned}
$$

$$r_{17}(w) = w^{27} \cdot (w^2 - w + 1) \cdot (7\,w^6 + 13\,w^5 + 10\,w^4 + 2\,w^3 + 9\,w + 9),$$
$$r_{18}(w) = w^{28} \cdot (16\,w^8 + 12\,w^7 + 2\,w^6 + 12\,w^5 + 11\,w^4 + 5\,w^3 + 6),$$
$$r_{19}(w) = w^{32} \cdot (8\,w^4 + 14\,w^3 + 7\,w^2 + 9),$$
$$r_{20}(w) = w^{33} \cdot (15\,w^4 + 15\,w^3 + 4\,w^2 + 2), \quad r_{21}(w) = w^{36} \cdot (5\,w + 14),$$
$$r_{22}(w) = 3\,w^{37} \cdot (4\,w + 3), \quad r_{23}(w) = 2\,w^{38}, \quad r_{24}(w) = w^{39}.$$

One verifies that this polynomial equation is actually satisfied with our series of 24 001 coefficients modulo $p = 17$.

A.2. Polynomial relation for $p = 19$

Modulo $p = 19$, we obtained the polynomial relation

$$\sum_{n=0}^{30} s_n(w) \cdot S(w)^n = 0, \tag{A.2}$$

where:

$$\begin{aligned}
s_0(w) = {} & 7\,w^{35} + 2\,w^{34} + 18\,w^{33} + 8\,w^{32} + w^{31} + 8\,w^{30} + 10\,w^{29} + 9\,w^{28} \\
& + 10\,w^{27} + 16\,w^{26} + 2\,w^{25} + 7\,w^{24} + 8\,w^{23} + 2\,w^{22} + 18\,w^{21} + 12\,w^{19} \\
& + 14\,w^{18} + 4\,w^{17} + 12\,w^{16} + 13\,w^{15} \\
& + 15\,w^{14} + 7\,w^{13} + 8\,w^{12} + 12\,w^{10} + 16\,w^9 \\
& + w^7 + 15\,w^6 + 17\,w^5 + 3\,w^4 + 7\,w^3 + 6\,w^2 + 14\,w + 13, \\
s_1(w) = {} & 10\,w^{35} + 9\,w^{34} + 6\,w^{33} + w^{32} + 9\,w^{31} + 8\,w^{30} + 10\,w^{28} + 17\,w^{27} \\
& + 5\,w^{26} + 7\,w^{25} + 4\,w^{24} + 16\,w^{22} + 15\,w^{21} \\
& + 9\,w^{20} + 16\,w^{19} + 16\,w^{18} + 11\,w^{17} \\
& + 5\,w^{16} + 5\,w^{15} + 14\,w^{14} + 5\,w^{13} + 13\,w^{12} + 3\,w^{11} + 6\,w^{10} + 16\,w^9 + 17\,w^8 \\
& + 17\,w^7 + 11\,w^6 + 3\,w^5 + 15\,w^4 + 10\,w^3 + 4\,w^2 + 9\,w + 6, \\
s_2(w) = {} & w \cdot (3\,w^{35} + 10\,w^{34} + 9\,w^{33} + 4\,w^{32} + 5\,w^{31} + 3\,w^{30} + 12\,w^{29} + 5\,w^{28} \\
& + w^{27} + 5\,w^{26} + 7\,w^{25} + 18\,w^{23} + 9\,w^{22} \\
& + 2\,w^{21} + 13\,w^{20} + 17\,w^{19} + 4\,w^{18} \\
& + 18\,w^{17} + 15\,w^{16} + w^{15} + 10\,w^{14} + 16\,w^{13} \\
& + 14\,w^{12} + 17\,w^{11} + 18\,w^{10} + w^8 \\
& + 2\,w^7 + 11\,w^6 + 12\,w^5 + 2\,w^4 + 13\,w^3 + w^2 + w + 3), \\
s_3(w) = {} & w^2 \cdot (4\,w^{34} + 11\,w^{33} + w^{32} + 8\,w^{31} + 5\,w^{30} + 8\,w^{29} + 4\,w^{28} + 12\,w^{27} \\
& + 11\,w^{26} + 14\,w^{25} + 9\,w^{24} + 10\,w^{23} + 10\,w^{22} + 3\,w^{21} + 11\,w^{20} + 15\,w^{19} \\
& + 7\,w^{18} + 13\,w^{17} + 7\,w^{16} + 18\,w^{15} + 13\,w^{14} \\
& + 18\,w^{13} + 4\,w^{12} + 4\,w^{11} + 13\,w^{10} \\
& + 7\,w^9 + 7\,w^8 + 16\,w^7 + w^6 + 3\,w^5 + 7\,w^4 + 18\,w^3 + 12\,w^2 + w + 8), \\
s_4(w) = {} & w^3 \cdot (12\,w^{34} + 4\,w^{33} + 15\,w^{32} + 16\,w^{31} + 18\,w^{30} + 16\,w^{29} + 6\,w^{28} \\
& + 18\,w^{27} + 17\,w^{26} + 15\,w^{25} + 12\,w^{24} + 5\,w^{23} + 15\,w^{22} + 15\,w^{21} + 5\,w^{20} \\
& + 8\,w^{19} + 18\,w^{18} + 8\,w^{17} + 15\,w^{15} \\
& + 12\,w^{14} + 17\,w^{13} + 4\,w^{12} + 6\,w^{11} + 3\,w^{10} \\
& + 6\,w^9 + 16\,w^8 + 10\,w^7 + 5\,w^6 + 7\,w^5 \\
& + 14\,w^4 + 6\,w^3 + 15\,w^2 + 12\,w + 11),
\end{aligned}$$

$$s_5(w) = w^4 \cdot \ (4\,w^{33} + 11\,w^{32} + 10\,w^{31} + 15\,w^{30} + 3\,w^{29} + 2\,w^{28} + 4\,w^{27} + 12\,w^{26} + 3\,w^{25} + 13\,w^{24} + 4\,w^{23} + 11\,w^{22} + 7\,w^{21} + 10\,w^{19} + 9\,w^{18} + 9\,w^{17} + w^{16} + 2\,w^{15} + 16\,w^{14} + 13\,w^{13} + 16\,w^{12} + 18\,w^{11} + 17\,w^{10} + 8\,w^9 + 18\,w^8 + 14\,w^5 + 4\,w^4 + 5\,w^3 + 17\,w^2 + 18\,w + 18),$$

$$s_6(w) = \ w^5 \cdot \ (8\,w^{33} + 9\,w^{32} + 5\,w^{31} + w^{30} + 14\,w^{29} + 11\,w^{28} + 5\,w^{27} + 11\,w^{26} + 10\,w^{25} + 17\,w^{24} + 17\,w^{23} + 17\,w^{22} + 2\,w^{21} + 8\,w^{20} + 2\,w^{18} + 13\,w^{17} + 16\,w^{16} + 2\,w^{15} + 2\,w^{14} + 16\,w^{13} + 15\,w^{12} + 15\,w^{11} + 7\,w^{10} + w^9 + 7\,w^8 + 4\,w^6 + 8\,w^5 + 2\,w^4 + 13\,w^3 + 15\,w + 11),$$

$$s_7(w) = w^6 \cdot \ (w^{32} + 18\,w^{31} + 13\,w^{30} + 2\,w^{29} + 13\,w^{28} + 13\,w^{27} + 2\,w^{26} + 6\,w^{25} + 7\,w^{24} + 2\,w^{23} + 13\,w^{21} + 12\,w^{20} + 6\,w^{19} + 2\,w^{18} + 11\,w^{17} + 2\,w^{16} + 2\,w^{15} + 3\,w^{14} + 12\,w^{13} + 5\,w^{12} + 13\,w^{11} + 18\,w^{10} + 18\,w^9 + 3\,w^8 + 17\,w^6 + 3\,w^5 + 11\,w^4 + 17\,w^3 + 4\,w + 5),$$

$$s_8(w) = w^7 \cdot \ (11\,w^{32} + 5\,w^{31} + 12\,w^{30} + 2\,w^{29} + 18\,w^{28} + 11\,w^{27} + 12\,w^{26} + 16\,w^{25} + 8\,w^{24} + 16\,w^{23} + 2\,w^{22} + 12\,w^{20} + 9\,w^{19} + 5\,w^{18} + 9\,w^{17} + 18\,w^{16} + 5\,w^{15} + 6\,w^{14} + 16\,w^{13} + 9\,w^{12} + 13\,w^{11} + 14\,w^{10} + 14\,w^9 + 18\,w^8 + 14\,w^6 + 14\,w^5 + 2\,w^4 + 4\,w^3 + 15),$$

$$s_9(w) = w^8 \cdot \ (7\,w^{31} + 7\,w^{30} + 16\,w^{29} + 2\,w^{28} + 3\,w^{27} + 10\,w^{26} + 18\,w^{25} + 5\,w^{24} + 13\,w^{23} + 17\,w^{22} + 12\,w^{21} + w^{20} + w^{19} + 17\,w^{18} + 17\,w^{17} + 6\,w^{16} + 15\,w^{15} + w^{14} + 11\,w^{13} + 10\,w^{12} + 9\,w^{11} + 10\,w^{10} + 2\,w^9 + 15\,w^8 + 4\,w^6 + 18\,w^4 + 15\,w^3 + 1),$$

$$s_{10}(w) = \ w^{12} \cdot \ (16\,w^{28} + 10\,w^{27} + w^{26} + 13\,w^{25} + 12\,w^{24} + 5\,w^{23} + 18\,w^{22} + 17\,w^{21} + 7\,w^{20} + 4\,w^{19} + 18\,w^{18} + 6\,w^{17} + 16\,w^{16} + 2\,w^{15} + 18\,w^{14} + 6\,w^{13} + 13\,w^{12} + 2\,w^{11} + 17\,w^{10} + 17\,w^9 + 16\,w^8 + 8\,w^7 + 9\,w^6 + 17\,w^5 + 6\,w^3 + 14\,w + 16),$$

$$s_{11}(w) = w^{13} \cdot \ (8\,w^{27} + 13\,w^{26} + 16\,w^{25} + 8\,w^{24} + 2\,w^{23} + 14\,w^{22} + 15\,w^{21} + 12\,w^{20} + 14\,w^{18} + 2\,w^{17} + 8\,w^{16} + 14\,w^{15} + 11\,w^{14} + 11\,w^{13} + 7\,w^{12} + 17\,w^{11} + 6\,w^9 + 18\,w^8 + 4\,w^7 + 6\,w^6 + 3\,w^5 + 12\,w^3 + 9),$$

$$s_{12}(w) = w^{14} \cdot \ (8\,w^{27} + w^{26} + 4\,w^{25} + 18\,w^{24} + 9\,w^{22} + 3\,w^{21} + 15\,w^{20} + 6\,w^{18} + 6\,w^{17} + 2\,w^{16} + 3\,w^{15} + 3\,w^{13} + 9\,w^{12} + 7\,w^{11} + 6\,w^9 + 16\,w^8 + w^7 + 4\,w^6 + 10\,w^3 + 9),$$

$$s_{13}(w) = w^{18} \cdot \ (15\,w^{23} + 6\,w^{22} + 12\,w^{21} + w^{19} + w^{18} + 2\,w^{17} + 14\,w^{14} + 3\,w^{13} + 2\,w^{12} + 13\,w^{10} + 3\,w^9 + 6\,w^8 + 12\,w^5 + 11\,w^4 + 9\,w^3 + 16),$$

$$s_{14}(w) = \ w^{19} \cdot \ (2\,w^{23} + 10\,w^{22} + 8\,w^{21} + 18\,w^{18} + 15\,w^{17} + 12\,w^{14} + 5\,w^{13} + 14\,w^{12} + 16\,w^9 + 7\,w^8 + 13\,w^5 + 12\,w^4 + 6\,w^3 + 3),$$

$$s_{15}(w) = w^{20} \cdot \, (16\, w^{22} + 18\, w^{21} + 3\, w^{18} + 17\, w^{17} + 8\, w^{13} + 3\, w^{12} + 9\, w^{9} + 13\, w^{8} + 4\, w^{4} + 4\, w^{3} + 10),$$

$$s_{16}(w) = w^{24} \cdot \, (12\, w^{19} + 12\, w^{18} + 16\, w^{14} + 6\, w^{10} + 2\, w^{9} + 10\, w^{5} + 3\, w + 9),$$
$$s_{17}(w) = 2\, w^{25} \cdot \, (4\, w^{18} + 9\, w^{14} + 7\, w^{9} + 8\, w^{5} + 3),$$
$$s_{18}(w) = w^{26} \cdot \, (18\, w^{18} + 3\, w^{9} + 4),$$
$$s_{19}(w) = w^{30} \cdot \, (13\, w^{14} + 15\, w^{13} + w^{12} + 17\, w^{11} + 18\, w^{10} + 17\, w^{8} + 4\, w^{7} + 15\, w^{6} + 17\, w^{5} + 14\, w^{4} + 8\, w^{3} + 14\, w + 8),$$

$$s_{20}(w) = w^{31} \cdot \, (4\, w^{14} + 8\, w^{13} + 3\, w^{12} + 3\, w^{11} + 15\, w^{10} + 4\, w^{9} + 18\, w^{8} + 3\, w^{7} + 15\, w^{6} + 14\, w^{5} + 10\, w^{4} + 5\, w^{3} + 2),$$
$$s_{21}(w) = w^{32} \cdot \, (w^{13} + 15\, w^{12} + 13\, w^{11} + 17\, w^{10} + 3\, w^{9} + w^{8} + 14\, w^{7} + 3\, w^{6} + 6\, w^{4} + 6\, w^{3} + 12),$$
$$s_{22}(w) = w^{36} \cdot \, (4\, w^{10} + 9\, w^{9} + 13\, w^{8} + w^{7} + 15\, w^{6} + 8\, w^{5} + 15\, w^{4} + 15\, w^{3} + 5\, w + 15),$$
$$s_{23}(w) = w^{37} \cdot \, (16\, w^{9} + 4\, w^{8} + 18\, w^{7} + 18\, w^{6} + 5\, w^{5} + 2\, w^{4} + 16\, w^{3} + 14),$$

$$s_{24}(w) = w^{38} \cdot \, (8\, w^{9} + 6\, w^{8} + 7\, w^{7} + 6\, w^{6} + 2\, w^{4} + 13\, w^{3} + 7),$$
$$s_{25}(w) = w^{42} \cdot \, (5\, w^{5} + 18\, w^{4} + 15\, w^{3} + 4\, w + 2),$$
$$s_{26}(w) = w^{43} \cdot \, (14\, w^{5} + 3\, w^{4} + w^{3} + 15),$$
$$s_{27}(w) = w^{44} \cdot \, (9\, w^{4} + 6\, w^{3} + 13),$$
$$s_{28}(w) = w^{48} \cdot \, (12\, w + 5), \; s_{29}(w) = 12\, w^{49}, \quad s_{30}(w) = w^{50}.$$

One verifies that this polynomial equation is actually satisfied with 23 756 coefficients of our series modulo $p = 19$.

Appendix B. Reduction of hypergeometric functions

B.1. Reduction of $_nF_{n-1}$ hypergeometric functions modulo primes

Let us consider the series expansions (with integer coefficients) of the hypergeometric function $_4F_3([1/2,\, 1/2,\, 1/2,\, 1/2],\, [1,\, 1,\, 1],\, 256\, x)$, which corresponds to a *Calabi–Yau operator* [47, 61]. It is the diagonal of a rational function [42, 43] since it is the *Hadamard product* [42] of four times the algebraic function $(1 \, - \, 4\, x)^{-1/2}$. This ensures that this series reduces to an algebraic function modulo any prime [42, 43] (or power of prime).

Let us perform the same calculations as in sections 3 and 4. The series reads:

$$_4F_3\left(\left[\frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2}\right],\, [1,\, 1,\, 1],\, 256\, x\right) = 1 + 16\, x + 1296\, x^2 + 160\,000\, x^3 + 24\,010\,000\, x^4 + 4032\,758\,016\, x^5 + 728\,933\,458\,176\, x^6 + 138\,735\,983\,333\,376\, x^7 + \cdots. \tag{B.1}$$

The reduction of this hypergeometric series is a very simple algebraic function of the form $P(x)^{-1/N}$ where N is an integer and where $P(x)$ is a polynomial, which corresponds to the truncation of the series expansion of the hypergeometric series modulo the prime p.

For instance, modulo 23, the hypergeometric function (B.1) becomes the algebraic function $1/P(x)^{1/22}$, where the polynomial $P(x)$ reads:

$$P(x) = \left({}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1,1,1],\, 256\,x\right)\right)^{-22} \qquad \mathrm{mod}\ 23$$
$$= 1 + 16\,x + 8\,x^2 + 12\,x^3 + x^4 + x^5 + 3\,x^6 + 4\,x^7 + 18\,x^8 + 16\,x^9 + 12\,x^{10} + x^{11} \qquad \text{(B.2)}$$

More generally one can conjecture that, modulo almost all prime p, the hypergeometric series to the power $-(p-1)$ is a polynomial:

$$P(x) = \left({}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1,1,1],\, 256\,x\right)\right)^{-(p-1)} \qquad \mathrm{mod}\ p. \qquad \text{(B.3)}$$

This polynomial is of degree 98 for the prime 197, of degree 411 for the prime 823, of degree 1121 for the prime 2243. One can conjecture, modulo almost all prime p, that the degree of this polynomial is $(p-1)/2$.

Remark: One remarks that the polynomial $P(x)$ corresponds to a truncation of the hypergeometric function we started from. For instance, modulo $p = 23$, the series expansion of the ${}_4F_3$ hypergeometric function reads:

$${}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1, 1, 1],\, 256\,x\right)$$
$$= 1 + 16\,x + 8\,x^2 + 12\,x^3 + x^4 + x^5 + 3\,x^6 + 4\,x^7 + 18\,x^8 + 16\,x^9 + 12\,x^{10} + x^{11}$$
$$+ 16\,x^{23} + 3\,x^{24} + 13\,x^{25} + 8\,x^{26} + 16\,x^{27} + \cdots \qquad \mathrm{mod}\ 23,$$
$$= \left({}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1,1,1],\, 256\,x\right)\right)^{-22}$$
$$+ 16\,x^{23} + 3\,x^{24} + 13\,x^{25} + 8\,x^{26} + 16\,x^{27} + \cdots \qquad \mathrm{mod}\ 23, \qquad \text{(B.4)}$$

which corresponds to the fact that:

$${}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1,1,1],\, 256\,x\right)^{23} - 1$$
$$= 16\,x^{23} + 8\,x^{46} + 12\,x^{69} + x^{92} + \cdots \qquad \mathrm{mod}\ 23. \qquad \text{(B.5)}$$

More generally, one has:

$${}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1,1,1],\, 256\,x\right)^{M} - 1 = 16\,M \cdot x + 16\,M \cdot (8\,M + 73) \cdot x^2$$
$$+ 256\text{-}3 \cdot M \cdot (8\,M^2 + 219\,M + 1648) \cdot x^3 + \cdots \qquad \text{(B.6)}$$

all the coefficients of this series (B.6) are of the form $M \cdot P(M)/d$ where $P(M)$ is a polynomial with integer coefficients, the denominator d is an integer. Modulo M the coefficients of this expansion are all equal to zero, except when the denominator of this coefficient is divisible by M.

B.2. Reduction of hypergeometric functions modulo power of primes

The algebraic expressions, corresponding to reductions of hypergeometric functions modulo *power of primes*, are much more complicated. Let us just consider the previous $_4F_3$ hypergeometric function, for instance, modulo 3^2. This series modulo 3^2 reads:

$$S = 1 + 7x + 7x^3 + 7x^4 + 7x^9 + 4x^{10} + 7x^{12} + 7x^{13} + 7x^{27} + 4x^{28} + 4x^{30} + 4x^{31} + 7x^{36} + 4x^{37} + 7x^{39} + 7x^{40} + \cdots \quad (B.7)$$

It is solution of the polynomial relation

$$(x^7 + 2x^6 + x^5 + x^2 + 2x + 1) \cdot S^4 + (x^6 + x^5 + x + 1) \cdot S^2 + 7 \cdot (1 + x^5) = 0 \mod 3^2. \quad (B.8)$$

B.3. More reduction of hypergeometric functions

Such result generalizes to other hypergeometric functions. For instance for the $_5F_4$ hypergeometric functions:

$$P(x) = 1 + 2x + x^2 = \left({}_5F_4\left([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}], [1{,}1,\ 1{,}1],\ 2^{10}x \right)\right)^{-4} \mod 5, \quad (B.9)$$

$$P(x) = 1 + 2x + 5x^2 = \left({}_5F_4\left([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}], [1{,}1,\ 1{,}1],\ 2^6\, 3^4\, x \right)\right)^{-6} \mod 7, \quad (B.10)$$

but

$$P(x) = 1 + 2x + 4x^2 + 3x^5 + x^6 + 2x^7 = \left({}_5F_4\left([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}], [1{,}1,\ 1{,}1],\ 2^6\, 3^4\, x \right)\right)^{-24} \mod 5. \quad (B.11)$$

In fact the hypergeometric series, modulo p, are of the form $P(x)^{-1/N}$ where N is an integer, not necessarily equal to $-(p-1)$, which is such that $-N = 1 \mod p$.

B.4. Reductions modulo primes of $_3F_2([1/9, 4/9, 5/9], [1/3, 1], 3^6\, x)$

Let us now consider the $_3F_2$ hypergeometric function $_3F_2([1/9, 4/9, 5/9], [1/3, 1], 3^6\, x)$. This hypergeometric function has a series expansion *with integer coefficients*:

$$ {}_3F_2\left([\frac{1}{9}, \frac{4}{9}, \frac{5}{9}], [\frac{1}{3}, 1], 3^6\, x \right) = 1 + 60\,x + 20\,475\,x^2 + 9373\,650\,x^3 + 4881\,796\,920\,x^4 + 2734\,407\,111\,744\,x^5 + 1605\,040\,007\,778\,900\,x^6 + 973\,419\,698\,810\,097\,000\,x^7 + \cdots. \quad (B.12)$$

This $_3F_2$ hypergeometric function has been introduced by Christol [42–44], a few decades ago, to provide an example of holonomic G-series with *integer coefficients* that may not be a diagonal of a rational function (it is still an open question to see whether this function is, or is not, the diagonal of rational function).

If this hypergeometric function were the diagonal of a rational function it would reduce to algebraic functions modulo every prime, in particular small primes like 2, 3, 5, 7.

Considering the series (B.12) modulo these primes, in order to see whether they reduce, or not, to algebraic functions modulo these primes, is certainely worth doing to have a better hint on the very nature of this hypergeometric function: diagonal of rational function, or not.

Considering the previous series expansion with integer coefficients (B.12), modulo the prime 2, we obtained a (quite lacunary) series of the first 533 000 coefficients:

$$S = 1 \; + x^2 + x^{128} + x^{130} \; + x^{8192} + x^{8194} + x^{8320} + x^{8322} + x^{524\,288} + x^{524\,290} + x^{524\,416} + x^{524\,418} + x^{532\,480} + x^{532\,482} + x^{532\,608} + \; O(x^{533\,000}). \tag{B.13}$$

In contrast with the calculations performed in sections 3 and 4, or in the previous section (B.1), it becomes hard to find the polynomial relation (if it exists !) this series (B.13) satisfies, even modulo 2. The reason is that the series (B.13) satisfies[26], modulo 2, an algebraic relation of slightly large degree $2^6 - 1 = 63$, namely $(1 + x^2) \cdot S^{63} - 1 = 0$. One can check directly that:

$$_3F_2\left(\left[\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\right], [\frac{1}{3}, 1], 3^6\, x\right) = (1 + x^2)^{-1/63} \qquad \text{mod } 2. \tag{B.14}$$

The series (B.12) becomes trivial modulo the prime 3, however, if one considers, instead, the series $S = 1 + ({}_3F_2([1/9, 4/9, 5/9], [1/3, 1], 3^6\, x) - 1)/15$, this series expansion, modulo 3, is the lacunary series $1 + \sum x^{3^n}$:

$$1 + x + x^3 + x^9 + x^{27} + x^{81} + x^{243} + x^{729} + x^{2187} + x^{6561} + x^{19\,683} + \cdots \tag{B.15}$$

which is algebraic since it satisfies, modulo 3, the polynomial relation $S^3 + x = S$.

Remark: Even in a holonomic framework, the property to reduce to an algebraic function modulo every prime (and power of prime) is probably more general than being the diagonal of a rational function. For holonomic G-series with integer coefficients that do not reduce to diagonal of a rational function, one must not search for polynomial relations $P(x, S) = 0$ where the degrees in x and S are not too drastically different, but one must rather seek polynomial relations of the 'Frobenius' type:

$$\sum a_i(x) \cdot S^{p^i} = 0 \qquad\qquad \text{mod } p \tag{B.16}$$

where the degree in S, namely p^N for some N integer, can be quite large.

Modulo 5 the series (B.12) becomes a function of the variable[27] x^5:

$$1 + 4\,x^5 + 2\,x^{10} + 3\,x^{25} + 2\,x^{30} + 2\,x^{35} + 2\,x^{50} + 3\,x^{55} + 4\,x^{250} + x^{255} + 3\,x^{260} + 2\,x^{275} + 3\,x^{280} + 3\,x^{285} + 3\,x^{300} + 2\,x^{305} + x^{375} + 4\,x^{380} + \cdots. \tag{B.17}$$

For this series (B.17), as well as the reduction of (B.12) modulo 7, it is extremely hard to see whether these series satisfy a polynomial relation, *even of the Frobenius type* (B.16).

Appendix C. Ratio of holonomic functions versus ratio of diagonal rational functions

Let us consider a quite pedagogical and important example related to the *theory of elliptic curves*, and the concept of *mirror maps* [47, 48].

[26] We thank Bostan for kindly providing this result.

[27] Sometimes called 'constant' by some authors because its derivative is $5 \cdot x^4$ which is zero mod 5.

Let us consider $\tau = -\pi\,\rho$ the ratio of the two periods of an elliptic function as a function of the lambda modulus $\lambda = k^2$:

$$\rho \, = \, \frac{{}_2F_1\Big(\Big[\frac{1}{2}, \frac{1}{2}\Big], [1], 1 - k^2\Big)}{{}_2F_1\Big(\Big[\frac{1}{2}, \frac{1}{2}\Big], [1], k^2\Big)}, \tag{C.1}$$

where the complete elliptic integral of the first kind and the complementary complete elliptic integral of the first kind have the series expansions

$${}_2F_1\Big(\Big[\frac{1}{2}, \frac{1}{2}\Big], [1], x^2\Big) = 1 + \frac{x^2}{4} + \frac{9}{64}x^4 + \frac{25}{256}x^6 + \frac{1225}{16\,384}x^8 + \cdots \tag{C.2}$$

and

$${}_2F_1\Big(\Big[\frac{1}{2}, \frac{1}{2}\Big], [1], 1 - x^2\Big) = \ln(x) \cdot {}_2F_1\Big(\Big[\frac{1}{2}, \frac{1}{2}\Big], [1], k^2\Big) + y_0(x) \qquad \text{where:}$$

$$y_0(x) = \frac{x^2}{4} + \frac{21\,x^4}{128} + \frac{185\,x^6}{1536} + \frac{18\,655\,x^8}{196\,608} + \frac{102\,501\,x^{10}}{1310\,720} + \frac{1394\,239\,x^{12}}{20\,971\,520} + \frac{33\,944\,053\,x^{14}}{587\,202\,560} + \frac{3074\,289\,075\,x^{16}}{60\,129\,542\,144} + \frac{99\,205\,524\,275\,x^{18}}{2164\,663\,517\,184} + \cdots. \tag{C.3}$$

Introducing the two second order linear differential operators (here $D_x = \mathrm{d}/\mathrm{d}x$)

$$L_2 = (x^2 - 1) \cdot x \cdot D_x^2 + (3\,x^2 - 1) \cdot D_x + x, \tag{C.4}$$

$$M_2 = (x^2 - 1) \cdot x^2 \cdot D_x^2 + (3\,x^2 - 1) \cdot x \cdot D_x + 1, \tag{C.5}$$

the complete elliptic integral of the first kind (C.2) is solution of L_2 when the series $y_0(x)$ in (C.3) is solution of the fourth-order linear differential operator $L_4 = M_2 \cdot L_2$. Therefore the ratio ρ in (C.1) reads

$$\rho = \ln(x) + r(x) \qquad \text{where:} \qquad r(x) = \frac{y_0}{{}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], x^2\Big)}. \tag{C.6}$$

It is well-known that the ratio τ (and thus the ratio ρ) satisfies a very simple *nonlinear 'Schwarzian differential equation'*:

$$\{\rho, \lambda\} = \frac{1}{2} \cdot \frac{(\lambda^2 - \lambda + 1)}{\lambda^2 \cdot (\lambda - 1)^2}, \tag{C.7}$$

where, if x is the modulus k of elliptic function, where λ denotes the 'lambda modulus' $\lambda = k^2 = x^2$, and where $\{\rho, \lambda\}$ denotes the *Schwarzian derivative*.

From (C.6) and (C.7) one immediately finds that $r(x)$, the ratio of two holonomic functions, satisfy a nonlinear differential equation, that we will not write here.

In order to have series with integer coefficients, let us scale x by a factor 4: $x \rightarrow 4\,x$. The elliptic integral (C.2), which is a diagonal of a rational function, has very simple reductions modulo primes. For instance, modulo $p = 7$, it reads:

$${}_2F_1\Big(\Big[\frac{1}{2}, \frac{1}{2}\Big], [1], 16\,x^2\Big) = (1 + 4\,x^2 + x^4 + x^6)^{-1/6} \qquad \mathrm{mod}\ 7. \tag{C.8}$$

Unfortunately one *cannot define the reduction of the holonomic series* y_0, the solution of a fourth-order linear differential operator. One sees that this series (even with a rescaling $x \rightarrow 4x$, or even any rescaling by an integer, cannot be recast into a series with integer coefficients: *it is not globally bounded* [42, 43]. In the denominators of the successive coefficients of this series almost every prime occurs, thus, one cannot look at this series modulo a prime[28].

Appendix D. Nonlinear differential equation for a ratio of diagonal rational functions

The series expansion (46) of the ratio of two ${}_2F_1$ hypergeometric series of section 5.2

$$R(x) = \frac{{}_2F_1\left(\left[\frac{1}{3}, \frac{1}{3}\right], [1], 27x\right)}{{}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], 16x\right)}, \tag{D.1}$$

is *solution of the nonlinear differential equation* (R denotes $R(x)$, and R_n denote $\mathrm{d}^nR/\mathrm{d}x^n$):

$$\begin{aligned}
&-2x^2 \cdot (27x-1)(16x-1) \cdot ((27x-1) \cdot (16x-1) \cdot R_1 - (72x+1) \cdot R) \cdot R_3 \\
&-2x \cdot (3x \cdot (16x-1)(72x+1)(27x-1) \cdot R_1 \\
&-(93\,312x^3 - 168x^2 - 297x + 4) \cdot R) \cdot R_2 \\
&+2 \cdot (29\,376x^3 + 5580x^2 - 221x + 1) \cdot R \cdot R_1 \\
&+3x^2 \cdot (27x-1)^2(16x-1)^2 \cdot R_2{}^2 \\
&+(16x-1)(1944x^3 - 1569x^2 + 58x - 1) \cdot R_1{}^2 \\
&+(144x^2 - 432x + 1) \cdot R^2 = 0.
\end{aligned} \tag{D.2}$$

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