Schwarzian conditions for linear differential operators with selected differential Galois groups (unabridged version)

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Abstract. We show that non-linear Schwarzian differential equations emerging from covariance symmetry conditions imposed on linear differential operators with hypergeometric function solutions can be generalized to arbitrary order linear differential operators with polynomial coefficients having selected differential Galois groups. For order three and order four linear differential operators we show that this pullback invariance up to conjugation eventually reduces to symmetric powers of an underlying order-two operator. We give, precisely, the conditions to have modular correspondences solutions for such Schwarzian differential equations, which was an open question in a previous paper. We analyze in detail a pullbacked hypergeometric example generalizing modular forms, that ushers a pullback invariance up to operator homomorphisms. We expect this new concept to be well-suited in physics and enumerative combinatorics. We finally consider the more general problem of the equivalence of two different order-four linear differential Calabi-Yau operators up to pullbacks and conjugation, and clarify the cases where they have the same Yukawa couplings.

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1. Introduction

In a previous paper [1] we focused on identities relating the *same* ${}_2F_1$ hypergeometric function with *two different‡ algebraic pullback transformations*. These identities correspond to *modular forms*, the algebraic transformations being solutions of a (non-linear) *differentially algebraic* [3, 4] *Schwarzian equation*, that also emerged in a paper by Casale on Galoisian envelopes [5, 6]. This covariance symmetry of ${}_2F_1$ hypergeometric functions could be regarded as one of the simplest illustrations of the concept of symmetries (of the renormalization group type [2, 7]) in physics or enumerative combinatorics, a *univariate function being covariant (automorphic) with respect to an infinite set of rational or algebraic transformations*. This paper [1] was essentially focused on ${}_nF_{n-1}$ *hypergeometric functions* and *modular forms* actually represented as ${}_2F_1$ hypergeometric function with two different algebraic pullback transformations (*modular correspondences* [1, 8]).

The applications of this Schwarzian equation [1] known to be associated to a quite large mathematical framework† (Malgrange's pseudogroup, Galois groupoid [9, 10, 11, 12, 13, 14, 15]), extend well beyond hypergeometric functions in physics. We have seen, for instance in [1], an example of identity relating the same Heun function with two different pullbacks††. This Heun example [1] could suggest that such Schwarzian differential equations emerge in physics with holonomic functions having a narrow set of singularities (three for hypergeometric functions, four for Heun functions, ...) like the Heun example in [1]. Going further we show, in this paper, that such *differentially algebraic* [3, 4] Schwarzian equations do emerge in a much more general holonomic framework.

We will show in section 2 that the covariance symmetry condition of *general* order-two linear differential operators with polynomial coefficients automatically yields this Schwarzian differential equation. We will then show in sections 3 and 4 that the covariance symmetry condition imposed on linear differential operators having order three and order four with respective orthogonal and symplectic differential Galois groups, yield Schwarzian differential equations like the one examined in [1]. When their respective symmetric and exterior powers are of order *five* (instead of six), one finds that these order-three and order-four operators reduce to symmetric square and symmetric cube of an underlying order-two operator. In section 5 we show that the Schwarzian condition can be derived for linear differential operators of arbitrary order N. The reduction of the solutions of this Schwarzian differential equation to only *modular correspondences* [8] was an open question in [1]: in section 6 a necessary condition to have such modular correspondences [8] is derived. In section 7 generalizations of modular forms provide examples of *pullback invariance of an operator, up to operator homomorphism*. This invariance should be important to describing the symmetries of linear differential operators and thus, is of relevance to physics. Finally in section 8, we consider the more general problem already addressed in [17] where Schwarzian differential equations also occurred, of the equivalence of *two*

‡ Beyond the $x \rightarrow 1-x, 1/x, \ldots$ known pullback symmetries of hypergeometric functions. The correspondence between the two pullbacks must be an *infinite order* rational or algebraic transformation [1, 2].

† In Casale's paper [5, 6] the Schwarzian equation is associated with meromorphic functions instead of the rational functions of our paper [1]. See also [9, 10, 11].

†† This Heun function being not, in general, reducible to a ${}_2F_1$ pullbacked hypergeometric function [16].

different order-four linear differential *Calabi-Yau operators* [18] up to pullbacks and conjugation, possibly yielding the *same Yukawa couplings* [17], and we will generalize it to linear differential operators of arbitrary orders.

2. Beyond hypergeometric and Heun functions: order-two linear differential operators

We will show here that non-linear ODEs involving Schwarzian derivatives (cf. equation (9) below), that we will call "Schwarzian ODEs"‡, obtained in [1] for hypergeometric and Heun functions [22, 23], can be generalized to arbitrary globally nilpotent [24] linear differential operators having an arbitrary numbers of singularities (as opposed to three and four singularities for hypergeometric and Heun functions).

Let us consider a linear differential operator of order two

$$L_2 \;=\; D_x^2 \;+ p(x) \cdot D_x \;+\; q(x), \qquad \text{where:} \qquad D_x \;=\; \frac{d}{dx}, \tag{1}$$

and let us also introduce two other linear differential operators of order two: the operator $L_2^{(c)} \;=\; 1/v(x) \cdot L_2 \cdot v(x)$ being the conjugate of (1) by a function $v(x)$, and the pullbacked operator $L_2^{(p)}$ which amounts to changing $x \, \rightarrow \, y(x)$ in (1), the head coefficient being normalized† to 1. These two linear differential operators read respectively:

$$L_2^{(c)} \;=\; D_x^2 \;+ \Big(p(x) \,+\, 2 \cdot \frac{v'(x)}{v(x)}\Big) \cdot D_x \;+\; q(x) \;+ p(x) \cdot \frac{v'(x)}{v(x)} \;+\; \frac{v''(x)}{v(x)}, \tag{2}$$

where

$$v'(x) \;=\; \frac{dv(x)}{dx}, \qquad\qquad v''(x) \;=\; \frac{d^2 v(x)}{dx^2}, \tag{3}$$

and

$$L_2^{(p)} \;=\; D_x^2 \;+ \Big(p(y(x)) \,\cdot\, y'(x) \,-\, \frac{y''(x)}{y'(x)}\Big) \cdot D_x \;+\; q(y(x)) \cdot y'(x)^2, \tag{4}$$

where:

$$y'(x) \;=\; \frac{dy(x)}{dx}, \qquad\qquad y''(x) \;=\; \frac{d^2 y(x)}{dx^2}. \tag{5}$$

The identification of these two linear differential operators $L_2^{(c)} \;=\; L_2^{(p)}$ gives two conditions:

$$p(x) \;+ 2 \cdot \frac{v'(x)}{v(x)} \;=\; p(y(x)) \,\cdot\, y'(x) \;-\, \frac{y''(x)}{y'(x)}, \tag{6}$$

$$q(x) \;+ p(x) \cdot \frac{v'(x)}{v(x)} \;+\; \frac{v''(x)}{v(x)} \;=\; q(y(x)) \cdot y'(x)^2. \tag{7}$$

Since

$$\frac{v''(x)}{v(x)} \;=\; \frac{d}{dx}\Big(\frac{v'(x)}{v(x)}\Big) \;+\; \Big(\frac{v'(x)}{v(x)}\Big)^2, \tag{8}$$

‡ See [1, 19] for a definition. See also [20, 21].
† Throughout the paper we consider, for clarity and simplicity, this normalized form for the linear differential operators. The "true" pullbacked operator which amounts to changing $x \, \rightarrow \, y(x)$ (see the command "dchange" in PDEtools in Maple) is in fact $1/y'(x)^2 \cdot L_2^{(p)}$ where $L_2^{(p)}$ is given by (4).

one can eliminate the log-derivative $v'(x)/v(x)$ between (6) and (7), and obtain the *Schwarzian condition* previously given in [1]

$$W(x) \quad - W(y(x)) \cdot y'(x)^2 \quad + \{y(x),\, x\} \quad = \quad 0, \tag{9}$$

where

$$W(x) \; = \; \frac{dp(x)}{dx} \; + \; \frac{p(x)^2}{2} \; - 2 \cdot q(x), \tag{10}$$

and where $\{y(x),\, x\}$ denotes the *Schwarzian derivative* [19]:

$$\{y(x),\, x\} \; = \; \frac{y'''(x)}{y'(x)} \; - \; \frac{3}{2} \cdot \Big(\frac{y''(x)}{y'(x)}\Big)^2 \; = \; \frac{d}{dx}\Big(\frac{y''(x)}{y'(x)}\Big) \; - \frac{1}{2} \cdot \Big(\frac{y''(x)}{y'(x)}\Big)^2,$$

$$\text{where:} \qquad y'''(x) \; = \; \frac{d^3y(x)}{dx^3}, \qquad y''(x) \; = \; \frac{d^2y(x)}{dx^2}, \qquad y'(x) \; = \; \frac{dy(x)}{dx}.$$

Unlike in [1], the *number of singularities of the second order operator* (1) *is arbitrary*: it does not need to be three or four like in the hypergeometric or Heun examples in [1]. The second order linear differential operator L_2 is a *general* order-two linear differential operator with polynomial coefficients. Introducing $w(x)$ the wronskian of L_2

$$p(x) \; = \; - \frac{w'(x)}{w(x)} \qquad \text{where:} \qquad w'(x) \; = \; \frac{dw(x)}{dx}, \tag{11}$$

we see that the LHS and RHS of the first condition (6) are both log-derivatives. Thus one can immediately integrate the first condition (6) and get (up to a multiplicative factor μ) the conjugation function $v(x)$ in terms of the wronskian $w(x)$ and the pullback function $y(x)$:

$$v(x) \; = \; \mu \cdot \Big(\frac{w(x)}{w(y(x)) \cdot y'(x)}\Big)^{1/2}. \tag{12}$$

Remark 1: When the wronskian $w(x)$ is an N-th root of a rational function, the exact expression (12) for the conjugation function $v(x)$, becomes an algebraic function when $y(x)$ *is an algebraic function*. This is actually the case when the order-two linear differential operator L_2 is *globally nilpotent* [24]. In this case the linear differential operator is simply conjugated to its adjoint through its wronskian $w(x)$ which is a N-th root of a rational function:

$$L_2 \cdot w(x) \quad = \quad w(x) \cdot adjoint(L_2). \tag{13}$$

Remark 2: If the linear differential operator is *not* globally nilpotent [24] the wronskian is *not* necessarily an algebraic function. Introducing $L_v(x)$, the log-derivative of the conjugation function $v(x)$, one can rewrite the two conditions (6) and (7) as:

$$p(x) \; + 2 \cdot L_v(x) \; = \; p(y(x)) \cdot y'(x) \; - \; \frac{y"(x)}{y'(x)}, \tag{14}$$

$$q(x) \; + p(x) \cdot L_v(x) \; + \; \frac{dL_v(x)}{dx} + L_v(x)^2 \; = \; q(y(x)) \cdot y'(x)^2. \tag{15}$$

The elimination of $L_v(x)$ in (14) and (15) gives the Schwarzian condition (9) with (10), however the conjugation function $v(x)$ *is no longer an algebraic function when* $y(x)$ *is an algebraic function* (see (12)): it is a transcendental function, and we certainly move away from a modular correspondence [1, 8] framework†.

† For modular correspondences see also the concept of modular equations [25, 26, 27, 28].

3. Order-three linear differential operators

3.1. General order-three linear differential operators.

Considering an *irreducible* order-three linear differential operator

$$L_3 \; = \; D_x^3 \; + p(x) \cdot D_x^2 \; + \; q(x) \cdot D_x \; + \; r(x), \qquad (16)$$

we introduce two other linear differential operators of order three defined as previously in section 2: the operator $L_3^{(c)}$ conjugated of (16) by a function $v(x)$, namely $L_3^{(c)} \; = \; 1/v(x) \cdot L_3 \cdot v(x)$, and the pullbacked† operator $L_3^{(p)}$ which amounts to changing $x \, \rightarrow \, y(x)$ in L_3. These two linear differential operators read respectively

$$\begin{aligned} L_3^{(c)} \; = \; & D_x^3 \; + \Big(p(x) \, + 3 \cdot {{v'(x)} \over {v(x)}}\Big) \cdot D_x^2 \\ & + \; \Big(q(x) \, + 2 \cdot p(x) \cdot {{v'(x)} \over {v(x)}} \, + 3 \cdot {{v"(x)} \over {v(x)}}\Big) \cdot D_x \qquad (17) \\ & \quad + \; r(x) \; + q(x) \cdot {{v'(x)} \over {v(x)}} \; + p(x) \cdot {{v"(x)} \over {v(x)}} \; + \; {{v^{(3)}(x)} \over {v(x)}}, \end{aligned}$$

and:

$$\begin{aligned} L_3^{(p)} \; = \; & D_x^3 \; + \Big(p(y(x)) \cdot y'(x) \; - \; 3 \, {{y"(x)} \over {y'(x)}}\Big) \cdot D_x^2 \\ & + \; \Big(q(y(x)) \cdot y'(x)^2 \; - p(y(x)) \cdot y"(x) \; - \; {{y^{(3)}(x)} \over {y'(x)}} \; + 3 \cdot \Big({{y"(x)} \over {y'(x)}}\Big)^2\Big) \cdot D_x \\ & \quad + r(y(x)) \cdot y'(x)^3. \qquad (18) \end{aligned}$$

The equality of these two order-three linear differential operators gives three conditions $\mathcal{C}_n$, with $n \, = \, 0, \, 1, \, 2$, corresponding, respectively, to the identification of the D_x^n coefficients of $L_3^{(p)}$ and $L_3^{(c)}$. Introducing the wronskian $w(x)$ of L_3, the LHS and RHS of condition $\mathcal{C}_2$ being, again, log-derivatives, one can easily integrate condition $\mathcal{C}_2$ and get the exact expression of the conjugation function $v(x)$ in terms of the wronskian of L_3 and of the pullback $y(x)$:

$$v(x) \; = \; \mu \cdot \Big({{w(x)} \over {w(y(x)) \cdot y'(x)^3}}\Big)^{1/3}. \qquad (19)$$

Similarly the elimination of the log-derivative $v'(x)/v(x)$ between condition $\mathcal{C}_2$ and condition $\mathcal{C}_1$ yields the Schwarzian condition

$$W(x) \; - W(y(x)) \cdot y'(x)^2 \; + \{y(x), \, x\} \; = \; 0, \qquad (20)$$

where this time $W(x)$ reads:

$$W(x) \; = \; {{1} \over {2}} \cdot {{dp(x)} \over {dx}} \; + {{p(x)^2} \over {6}} \; - {{q(x)} \over {2}}. \qquad (21)$$

3.2. Symmetric Calabi-Yau condition.

Let us consider the condition corresponding to imposing the *symmetric square* of L_3 to be of order *five* instead of the generic order six. This ("symmetric" Calabi-Yau [35])

† The D_x^3 coefficient is normalized to 1.

condition reads:

$$r(x) \;=\; -\frac{2}{27}\cdot p(x)^3 \;+\frac{1}{3}\cdot p(x)\cdot q(x) \;-\frac{1}{3}\cdot p(x)\cdot \frac{dp(x)}{dx} \;+\frac{1}{2}\cdot \frac{dq(x)}{dx} \;-\frac{1}{6}\cdot \frac{d^2p(x)}{dx^2}. \qquad (22)$$

For a globally nilpotent [24] linear differential operator, this (symmetric Calabi-Yau) condition (22) together with (11) yields an order-three linear differential operator (16) simply conjugated to its adjoint:

$$L_3\cdot w(x)^{2/3} \;=\; w(x)^{2/3}\cdot adjoint(L_3), \qquad (23)$$

where the wronskian $w(x)$ is a N-th root of a rational function.

Again for a globally nilpotent [24] linear differential operator, the exact expression (19) for the conjugation function $v(x)$, becomes an algebraic function *when* $y(x)$ *is an algebraic function*.

The symmetric square of an order-two linear differential operator $L_2 = D_x^2 + A(x)\cdot D_x + B(x)$ is an order-three linear differential operator (16) with the following coefficients:

$$p(x) \;=\; 3\cdot A(x), \qquad q(x) \;=\; 2\cdot A(x)^2 \;+4\cdot B(x) \;+\frac{dA(x)}{dx}, \qquad (24)$$

$$r(x) \;=\; 4\cdot B(x)\cdot A(x) \;+2\cdot \frac{dB(x)}{dx}. \qquad (25)$$

These coefficients (24), (25) *automatically verify the (symmetric Calabi-Yau) condition* (22): the symmetric square of a symmetric square of an order-two linear differential operator is of order *five* instead of the generic order six. Conversely, the (symmetric Calabi-Yau) condition (22) can be parametrized† by (24) and (25) and amounts to imposing the order-three linear differential operator (16) to be exactly the symmetric square of an order-two operator.

Thus our calculations show that the pullback-compatibility of an order-three linear differential operator is equivalent to saying that this order-three operator *reduces to* (the symmetric square of) an underlying *order-two linear differential operator.* The Schwarzian condition (20) with $W(x)$ given by (21), is *thus inherited from the Schwarzian condition* (9) *of the underlying order-two linear differential operator.*

4. Order-four linear differential operators

Consider the *irreducible* order-four linear differential operator

$$L_4 \;=\; D_x^4 \;+p(x)\cdot D_x^3 \;+\; q(x)\cdot D_x^2 \;+\; r(x)\cdot D_x \;+\; s(x), \qquad (26)$$

and introduce two other linear differential operators of order four defined as previously in sections 2 and 3.1: the linear differential operator $L_4^{(c)}$ conjugated of (26) by a function $v(x)$ and the (normalized) pullbacked operator $L_4^{(p)}$. These two linear differential operators read respectively

$$L_4^{(c)} \;=\; D_x^4 \;+\Big(p(x)\;+\;4\cdot\frac{v'(x)}{v(x)}\Big)\cdot D_x^3 \qquad (27)$$

† Note that rewriting the exact expression of $W(x)$ given by (21) in terms of $A(x)$ and $B(x)$ using (24) one recovers (10), $p(x)$ and $q(x)$ in (10) being now $A(x)$ and $B(x)$.

$$+ \Big(q(x) \, + 3 \cdot p(x) \cdot \frac{v'(x)}{v(x)} \, + 6 \cdot \frac{v''(x)}{v(x)}\Big) \cdot D_x^2$$
$$+ \Big(r(x) \, + 2 \cdot q(x) \cdot \frac{v'(x)}{v(x)} \, + 3 \cdot p(x) \cdot \frac{v''(x)}{v(x)} \, + 4 \cdot \frac{v^{(3)}(x)}{v(x)}\Big) \cdot D_x$$
$$+ \, s(x) \, + \, r(x) \cdot \frac{v'(x)}{v(x)} \, + q(x) \cdot \frac{v''(x)}{v(x)} \, + p(x) \cdot \frac{v^{(3)}(x)}{v(x)} \, + \frac{v^{(4)}(x)}{v(x)},$$

and:

$$L_4^{(p)} \, = \, D_x^4 \, + \Big(p(y(x)) \cdot y'(x) \, - 6 \cdot \frac{y''(x)}{y'(x)}\Big) \cdot D_x^3$$
$$+ \Big(q(y(x)) \cdot y'(x)^2 \, - 3 \cdot p(y(x)) \cdot y''(x) \, - 4 \cdot \frac{y^{(3)}(x)}{y'(x)} + 15 \cdot \Big(\frac{y''(x)}{y'(x)}\Big)^2\Big) \cdot D_x^2$$
$$+ \Big(r(y(x)) \cdot y'(x)^3 \, - q(y(x)) \cdot y'(x) \cdot y''(x) \, - p(y(x)) \cdot y^{(3)}(x)$$
$$+ 3 \cdot p(y(x)) \cdot \frac{y''(x)^2}{y'(x)} \, - \frac{y^{(4)}}{y'(x)} \, + 10 \cdot \frac{y''(x) \cdot y^{(3)}}{y'(x)^2} \, - 15 \cdot \Big(\frac{y''(x)}{y'(x)}\Big)^3\Big) \cdot D_x$$
$$+ \, s(y(x)) \cdot y'(x)^4. \qquad (28)$$

The identification of these two order-four linear differential operators $L_4^{(p)}$ and $L_4^{(c)}$ gives this time four conditions $\mathcal{C}_n$, $n = 0, 1, 2, 3$, corresponding, respectively, to the identification of the D_x^n coefficients of $L_4^{(p)}$ and $L_4^{(c)}$.

Eliminating once again the log-derivative $v'(x)/v(x)$ between $\mathcal{C}_3$ and $\mathcal{C}_2$ one deduces a Schwarzian condition

$$W(x) \, - W(y(x)) \cdot y'(x)^2 \, + \{y(x), x\} \, = \, 0, \qquad (29)$$

where this time:

$$W(x) \, = \, \frac{3}{10} \cdot \frac{dp(x)}{dx} \, + \frac{3}{40} \cdot p(x)^2 \, - \frac{q(x)}{5}. \qquad (30)$$

Introducing the wronskian $w(x)$ of the order-four linear differential operator L_4 with (11), the condition $\mathcal{C}_3$ just corresponds to log-derivatives and can be easily integrated giving the exact expression of the conjugation function $v(x)$ as:

$$v(x) \, = \, \Big(\frac{w(x)}{w(y(x)) \cdot y'(x)^6}\Big)^{1/4}. \qquad (31)$$

The next conditions $\mathcal{C}_1$ and $\mathcal{C}_0$ yield extremely involved non-linear differential conditions on the miscellaneous derivatives of the various coefficients. It turned out to be very difficult to proceed with such huge expressions. Yet when the linear differential operator L_4 has a selected (symplectic) differential Galois group one can go much further in the calculations, as we will see in the coming subsection.

4.1. Calabi-Yau condition (exterior square).

Imposing the *Calabi-Yau condition* [29, 30] in the case of an order-four linear differential operator gives:

$$r(x) \, = \, \frac{p(x) \cdot q(x)}{2} \, - \frac{p(x)^3}{8} \, + \frac{dq(x)}{dx} \, - \frac{3}{4} \cdot p(x) \cdot \frac{dp(x)}{dx} \, - \frac{1}{2} \cdot \frac{d^2p(x)}{dx^2}. \qquad (32)$$

In this case the exterior square of the order-four operator L_4 has order *five* instead of order six.

When condition (32) is verified, the order-four linear differential operator L_4 has a *symplectic* differential Galois group $Sp(4, \mathbb{C})$. Note that if condition (32) is verified, the Calabi-Yau conditions for the pullbacked and conjugated operators $L_4^{(p)}$ and $L_4^{(c)}$ *are automatically verified*: this is a consequence of the fact that the Calabi-Yau condition (32) is left invariant by conjugation and pullback¶. In other words the *following identification of the* D_x *coefficients of* $L_4^{(p)}$ and $L_4^{(c)}$ *is automatically verified when the Calabi-Yau condition* (32) *is verified.*

Recall that the Calabi-Yau condition (32) together with the globally nilpotent condition [24] automatically yields L_4 to be conjugated to its adjoint

$$L_4 \cdot w(x)^{1/2} \quad = \quad w(x)^{1/2} \cdot adjoint(L_4), \qquad (33)$$

where $w(x)$ is a N-root of a rational function.

At the last step we consider the identification of the constant terms in D_x in $L_4^{(p)}$ and $L_4^{(c)}$. After injecting in this "large" non-linear differential equation, equation (11), the Schwarzian condition (29) with $W(x)$ given by (30), and the Calabi-Yau condition (32), we eventually find that this last "large" equation *becomes independent of the pullback* $y(x)$ and reduces to a quite simple condition *giving* $s(x)$ *as a polynomial expression in the two coefficients* $p(x)$ *and* $q(x)$ *and their derivatives*:

$$s(x) \; = \; {{9} \over {100}} \cdot q(x)^2 \; - {{1} \over {200}} \cdot q(x) \cdot p(x)^2 \; + {{1} \over {4}} \cdot p(x) \cdot {{dq(x)} \over {dx}} \; - {{1} \over {50}} \cdot q(x) \cdot {{dp(x)} \over {dx}} \\ + {{3} \over {10}} \cdot {{d^2q(x)} \over {dx^2}} \; - {{11} \over {1600}} \cdot p(x)^4 \; - {{9} \over {50}} \cdot p(x)^2 \cdot {{dp(x)} \over {dx}} \; - {{21} \over {100}} \cdot \Bigl({{dp(x)} \over {dx}}\Bigr)^2 \\ - {{1} \over {5}} \cdot {{d^3p(x)} \over {dx^3}} \; - {{7} \over {20}} \cdot p(x) \cdot {{d^2p(x)} \over {dx^2}}. \qquad (34)$$

In order to understand what this new condition (34) coming on top of the Calabi-Yau condition (32) really means, we calculated, for various MUM† order-four linear differential operators L_4 verifying (32) and (34), the corresponding nome and *Yukawa couplings* [31]. The corresponding Yukawa couplings *were actually found to be trivial*: $K_q \, = \, 1$!!

This amounts to saying that combining the two conditions (32) and (34) corresponds to a drastic reduction: the (irreducible) order-four linear differential operator L_4 is not a "true" order-four operator. Typically one can imagine that L_4 reduces to an order-two operator, being homomorphic to the *symmetric cube* of an underlying order-two linear differential operator. In fact it is exactly the symmetric cube of an order-two operator.

Let us consider the *symmetric cube* of an *order-two* linear differential operator $L_2 \, = \, D_x^2 \, + \, A(x) \cdot D_x \, + \, B(x)$ which is an order-four linear differential (26) with the following coefficients:

$$p(x) \; = \; 6 \cdot A(x), \qquad\qquad q(x) \; = \; 11 \cdot A(x)^2 \; + 4 \cdot {{dA(x)} \over {dx}} \; + 10 \cdot B(x),$$

$$r(x) \; = \; 6 \cdot A(x)^3 \; + 7 \cdot A(x) \cdot {{dA(x)} \over {dx}} \; + 30 \cdot B(x) \cdot A(x) \; + {{d^2A(x)} \over {dx^2}} \; + 10 \cdot {{dB(x)} \over {dx}},$$

¶ To see that the Calabi-Yau condition is preserved by conjugation is straightforward. However, as remarked in [17], to see that the Calabi-Yau condition is preserved by pullback transformations is very hard to see by direct computation, since one gets an enormous fourth-order nonlinear differential equation.

† Maximal unipotent monodromy (MUM) linear operators [24, 31].

$$s(x) \; = \; 18 \cdot A(x)^2 \cdot B(x) \; + 6 \cdot B(x) \cdot {{dA(x)} \over {dx}} \; + 15 \cdot {{dB(x)} \over {dx}} \cdot A(x) \\ \qquad + 9 \cdot B(x)^2 \; + 3 \cdot {{d^2B(x)} \over {dx^2}}. \qquad (35)$$

One finds straightforwardly that the coefficients given by (35) verify the Calabi-Yau condition (32), *as well as the new condition* (34). In this case the differential Galois group is no longer the symplectic differential Galois group $Sp(4, \mathbb{C})$, but actually reduces‡ to the differential Galois group of the underlying order-two linear differential operator, namely $SL(2, \mathbb{C})$. The fact that the Calabi-Yau condition (32) is verified is not a surprise: the exterior square of a symmetric cube is naturally of order less than six. The fact that being the symmetric cube of an underlying order-two operator verifies automatically the new condition (34) emerging from a compatibility condition of an order-four linear differential operator by pullback is far less obvious. The "parametrization" (35) necessarily yields the Calabi-Yau condition (32) and the new condition (34), and, conversely, (32) and (34) can be parametrized by (35).

Our large calculations thus show that the pullback-compatibility of an order-four linear differential operator which verifies the Calabi-Yau condition (32), amounts to saying that this order-four linear differential operator *reduces to* (the symmetric cube of) an underlying *order-two linear differential operator*. The Schwarzian condition (29) with $W(x)$ given by (30), is *thus inherited from the Schwarzian condition* (9) *of the underlying order-two linear differential operator*.

4.2. Reducible operators

Throughout the paper we make the assumption that the linear differential operators are *irreducible*. The reduciblility of the linear differential operators is not an academic scenario: it is the situation *we encounter* (almost) *systematically with the linear differential operators emerging in physics*, typically in the case of the n-fold integral $\chi^{(n)}$ of the two-dimensional Ising model [32, 33, 34]. When the linear differential operators are *reducible*, it is clear that all the calculations of this paper must be revisited, taking into account the miscellaneous factorization scenarios.

Sketching the kind of situation we may encounter, let us consider an order-four linear differential operator $L_4 \, = \, D_x^4 + p_r(x) \cdot D_x^3 \, + q_r(x) \cdot D_x^2 + \cdots$ which factorizes into the product of two order-two linear differential operators:

$$L_4 \, = \, M_2 \cdot L_2, \qquad \qquad \text{where:}$$

$$L_2 \, = \, D_x^2 \, + p(x) \cdot D_x \, + q(x), \qquad M_2 \, = \, D_x^2 \, + \tilde{p}(x) \cdot D_x \, + \tilde{q}(x), \qquad (36)$$

$$p_r(x) \, = \, p(x) \, + \tilde{p}(x), \quad q_r(x) \, = \, \tilde{p}(x) \cdot p(x) \, + \tilde{q}(x) \, + 2 \cdot {{dp(x)} \over {dx}} \, + q(x), \quad \cdots$$

The simple case where the two operators M_2 and L_2 are identical is sketched in Appendix A. In general the exterior square of L_4 is an order-six linear differential operator which is the product of an order-one operator, of the symmetric product of L_2 and M_2, and of the order-one linear differential operator $D_x + p(x)$. Therefore, this reducible order-four linear differential operator L_4 *does not verify in general* the Calabi-Yau condition (32).

Imposing the (normalized) pullback by $y(x)$ of this reducible order-four linear differential operator $L_4 \, = \, M_2 \cdot L_2$ to be equal to a conjugation by a function $v(x)$

‡ When an order-four linear differential operator is the symmetric cube of an underling order-two operator its symmetric square is no longer of order 10 but reduces to order 7.

of that operator, it is important to remember that a change of variable $x \rightarrow y(x)$ on a linear differential operator which is the product of two operators, is the product of these two linear differential operators on which this change of variable has been performed. One gets a set of equations where one can disentangle two Schwarzian equations

$$W(x) \quad - W(y(x)) \cdot y'(x)^2 \quad + \{y(x), x\} \quad = \quad 0, \tag{37}$$

$$\tilde{W}(x) \quad - \tilde{W}(y(x)) \cdot y'(x)^2 \quad + \{y(x), x\} \quad = \quad 0, \tag{38}$$

where $W(x)$ and $\tilde{W}(x)$ are the functions (10) already encountered in the analysis of order-two linear differential operators

$$W(x) \;=\; \frac{dp(x)}{dx} \;+\; \frac{p(x)^2}{2} \;-\; 2 \cdot q(x), \tag{39}$$

$$\tilde{W}(x) \;=\; \frac{d\tilde{p}(x)}{dx} \;+\; \frac{\tilde{p}(x)^2}{2} \;-\; 2 \cdot \tilde{q}(x), \tag{40}$$

corresponding to the Schwarzian conditions written separately on L_2 and M_2, together with another relation which couples L_2 and M_2:

$$4 \cdot \frac{y''(x)}{y'(x)} \;+ \tilde{p}(x) \;- p(x) \;=\; \Big(\tilde{p}(y(x)) \;- p(y(x))\Big) \cdot y'(x). \tag{41}$$

Among the four solutions of the order-four operators $L_4 = M_2 \cdot L_2$, the two solutions of the order-two linear differential operator L_2 transform nicely under the pullback $x \rightarrow y(x)$, provided the Schwarzian condition (37) is satisfied, but this just corresponds to a *partial symmetry*. In general the set of equations (37), (38), (41) seems to be too rigid to allow solutions other than trivial symmetries or partial symmetries.

It is however worth mentioning a quite curious result. If one imposes the reducible order-four linear differential operator $L_4 = M_2 \cdot L_2$ to verify the Calabi-Yau condition (32) (i.e. to be such that the exterior square of that operator is order five instead of order six), one gets a condition that becomes remarkably simple when written in terms of the functions $W(x)$ and $\tilde{W}(x)$ given by (39) and (40). Introducing the difference $\Delta W(x) = W(x) - \tilde{W}(x)$, the Calabi-Yau condition (32) simply reads:

$$2 \cdot \frac{d\Delta W(x)}{dx} \;=\; (p(x) \;-\; \tilde{p}(x)) \cdot \Delta W(x). \tag{42}$$

Therefore, if one restricts oneself to $W(x) = \tilde{W}(x)$ which identifies the two Schwarzian conditions (37) and (38), one sees that condition (42) is automatically verified: condition $W(x) = \tilde{W}(x)$ *is thus a sufficient condition for the Calabi-Yau condition* (32).

The analysis of pullback symmetry on *reducible* linear differential operators is clearly an interesting and challenging problem in physics. It will require many more calculations to explore the arborescence of these various factorization scenarios.

4.3. Symmetric Calabi-Yau condition

The condition, we called in [35, 36] *symmetric Calabi-Yau condition* for the order-four linear differential operator L_4 (which correspond to impose that its symmetric square is of order less than 10), is a huge† polynomial condition on the coefficients of L_4 and

† This polynomial is the sum of 3548 monomials in the coefficients of L_4 and their derivatives.

their derivatives. This condition is invariant by pullback and conjugation. Provided the Schwarzian condition (29) with $W(x)$ given by (30) is satisfied, this symmetric Calabi-Yau condition alone is not sufficient to have $L_4^p \,=\, L_4^c$. Similarly to what we saw with the Calabi-Yau condition (32), would a supplementary condition to the symmetric Calabi-Yau condition be sufficient to have $L_4^p \,=\, L_4^c$? Could one also have, in this selected subcase, a reduction of L_4 to an underlying order-two operator? This scenario remains open.

Working with such huge polynomials will not get us far. In order to advance, let us introduce a parametrization based on the ideas explained in [36], namely that an order-four linear differential operator L_4, with an orthogonal differential Galois group $SO(4, \mathbb{C})$ and such that its symmetric square is of order less than 10, is necessarily of the form‡

$$L_4 \,\,=\,\, (U_1 \cdot U_3 \,+\, 1) \cdot d(x), \tag{43}$$

where U_1 and U_3 are order-one and order-three *self-adjoint* linear differential operators:

$$U_3 \,=\, a(x) \cdot D_x^3 \,+\, \frac{3}{2} \cdot \frac{da(x)}{dx} \cdot D_x^2 \,+\, b(x) \cdot D_x \,+\, \frac{1}{2} \cdot \frac{db(x)}{dx} \,-\, \frac{1}{4} \cdot \frac{d^3a(x)}{dx^3}, \tag{44}$$

$$U_1 \,\,=\,\, c(x) \cdot D_x \,\,+\,\, \frac{1}{2} \cdot \frac{dc(x)}{dx}. \tag{45}$$

This yields a parametrization of this huge polynomial differential (symmetric Calabi-Yau) condition:

$$p(x) \,\,=\,\, \frac{5}{2} \cdot \frac{a'(x)}{a(x)} \,+\, \frac{1}{2} \cdot \frac{c'(x)}{c(x)} \,+\, 4 \cdot \frac{d'(x)}{d(x)}, \tag{46}$$

$$\begin{aligned} q(x) \,\,=\,\, & \frac{b(x)}{a(x)} \,+\, \frac{3}{2} \cdot \frac{a''(x)}{a(x)} \,+\, \frac{3}{4} \cdot \frac{a'(x)}{a(x)} \cdot \frac{c'(x)}{c(x)} \,+\, 6 \cdot \frac{d''(x)}{d(x)} \\ & + \frac{15}{2} \cdot \frac{a'(x)}{a(x)} \cdot \frac{d'(x)}{d(x)} \,+\, \frac{3}{2} \cdot \frac{c'(x)}{c(x)} \cdot \frac{d'(x)}{d(x)}, \end{aligned} \tag{47}$$

$$\begin{aligned} r(x) \,\,=\,\, & \frac{1}{2} \cdot \frac{c'(x)}{c(x)} \cdot \frac{b(x)}{a(x)} \,+\, 4 \cdot \frac{d'''(x)}{d(x)} \,+\, 4 \cdot \frac{a'(x)}{a(x)} \cdot \frac{c'(x)}{c(x)} \cdot \frac{d'(x)}{d(x)} \\ & + \frac{3}{2} \cdot \frac{d''(x)}{d(x)} \cdot \frac{c'(x)}{c(x)} \,-\, \frac{1}{4} \cdot \frac{a'''(x)}{a(x)} \,+\, \frac{3}{2} \cdot \frac{b'(x)}{a(x)} \,+\, \frac{15}{2} \cdot \frac{d''(x)}{d(x)} \cdot \frac{a'(x)}{a(x)} \\ & + 2 \cdot \frac{d'(x)}{d(x)} \cdot \frac{b(x)}{a(x)} \,+ 3 \cdot \frac{d'(x)}{d(x)} \cdot \frac{a''(x)}{a(x)}, \end{aligned} \tag{48}$$

$$\begin{aligned} s(x) \,\,=\,\, & \frac{d^{(4)}}{d(x)} \,+\, \frac{1}{2} \cdot \frac{c'(x)}{c(x)} \cdot \frac{d'''(x)}{d(x)} \,+\, \frac{1}{2} \cdot \frac{b"(x)}{a(x)} \,-\, \frac{1}{4} \cdot \frac{a^{(4)}(x)}{a(x)} \\ & - \frac{1}{8} \cdot \frac{a'''(x)}{a(x)} \cdot \frac{c'(x)}{c(x)} \,+\, \frac{1}{4} \cdot \frac{b'(x)}{a(x)} \cdot \frac{c'(x)}{c(x)} \,+\, \frac{1}{a(x)\, c(x)} \\ & - \frac{1}{4} \cdot \frac{a'''(x)}{a(x)} \cdot \frac{d'(x)}{d(x)} + \frac{3}{2} \cdot \frac{b'(x)}{a(x)} \cdot \frac{d'(x)}{d(x)} \,+\, \frac{b(x)}{a(x)} \cdot \frac{d"(x)}{d(x)} \\ & + \frac{3}{2} \cdot \frac{a''(x)}{a(x)} \cdot \frac{d''(x)}{d(x)} \,+\, \frac{5}{2} \cdot \frac{a'(x)}{a(x)} \cdot \frac{d'''(x)}{d(x)} \end{aligned} \tag{49}$$

‡ The differential Galois group $SO(4, \mathbb{C})$ with an order-10 symmetric square situation corresponds to a decomposition $L_4 \,=\, (U_3 \cdot U_1 \,+\, 1) \cdot d(x)$, see [36].

$$+ \ \frac{1}{2} \cdot \frac{c'(x)}{c(x)} \cdot \frac{d'(x)}{d(x)} \cdot \frac{b(x)}{a(x)} \ + \frac{3}{4} \cdot \frac{a'(x)}{a(x)} \cdot \frac{c'(x)}{c(x)} \cdot \frac{d''(x)}{d(x)}.$$

One easily verifies that this parametrization (46) ... (49) is such that the polynomial encoding the symmetric Calabi-Yau condition, *is identically equal to zero.* Moreover one verifies that the order-four linear differential operator (43), with parametrization (46), (47), (48), (49), is, generically, such that its symmetric square has order 9 (instead of 10), its exterior square being of order 6.

Imposing $L_4^{(p)} \, = \, L_4^{(c)}$ for an order-four linear differential operator, corresponding to this parametrization (such that it verifies the symmetric Calabi-Yau condition, and such that its symmetric square is of order nine), one naturally finds the Schwarzian condition (29) with (30), as well as the exact expression (31) for the conjugation function $v(x)$. Taking into account the Schwarzian condition (29), the identification of the coefficients of D_x for $L_4^{(p)}$ and $L_4^{(c)}$ yields a relation of the form $\Phi(x) \, = \, \Phi(y(x)) \cdot y'(x)^3$, where $\Phi(x)$ is a rational function. Together with the last condition, this gives an invariance of the form $\Psi(x) \, = \, \Psi(y(x))$ yielding only trivial cases‡ for $L_4^{(p)} \, = \, L_4^{(c)}$.

This symmetric Calabi-Yau condition, *even if it is invariant by pullback and conjugation*, is thus *not sufficient to get* $L_4^{(p)} \, = \, L_4^{(c)}$. We have here a situation similar to the one described in the previous section 4.1, with the emergence of the additional condition (34) on top of the Calabi-Yau condition (32). However here the calculations are way too large: finding the additional condition(s) together with the symmetric Calabi-Yau condition yielding $L_4^{(p)} \, = \, L_4^{(c)}$, is beyond our reach for now. The case, described in the previous section 4.1, where the order-four operator (43) is the symmetric cube of an underlying order-two operator is also such that the symmetric square of L_4 is not of the generic order 10, but, in fact, of order 7: in this case the coefficients of L_4 *verify*† *the symmetric Calabi-Yau condition.* Since the calculations are way too large, it is not possible for now to tell if the additional condition(s) to the symmetric Calabi-Yau condition, also gives eventually a linear differential operator that is the symmetric cube of an order-two operator, as described in the previous section 4.1, or whether it would give something else. This would mean the emergence of the "classic" Calabi-Yau condition (32) combined with the condition (34). This remains an open question.

5. Order-N linear differential operators

The analysis of *irreducible* order-five operators is sketched in Appendix B. Let us now consider an *irreducible* order-N linear differential operator

$$L_N \; = \; D_x^N \; + p(x) \cdot D_x^{N-1} \; + \; q(x) \cdot D_x^{N-2} \; + \; \cdots \qquad (50)$$

and let us also introduce two other linear differential operators of order N: the operator $L_N^{(c)}$ conjugated of (50) by a function $v(x)$, namely $L_N^{(c)} \, = \, 1/v(x) \cdot L_N \cdot v(x)$, and the (normalized) pullbacked operator $L_N^{(p)}$ which amounts to changing $x \, \rightarrow \, y(x)$ in L_N. The pullbacked operator $L_N^{(p)}$ reads

$$L_N^{(p)} \;\; = \;\; D_x^N \;\; + \Big(p(y(x)) \cdot y'(x) \, - \, \frac{N \cdot (N-1)}{2} \cdot \frac{y''(x)}{y'(x)} \Big) \cdot D_x^{N-1}$$

‡ See [1] for similar calculations.
† This can be verified straightforwardly substituting (35) in the 3548 monomials symmetric Calabi-Yau condition.

$$+ \Big(q(y(x)) \cdot y'(x)^2 \, - \frac{(N-2) \cdot (N-1)}{2} \cdot p(y(x)) \cdot y''(x)$$
$$- \frac{N \cdot (N-1) \cdot (N-2)}{6} \cdot \frac{y^{(3)}}{y'(x)} \qquad (51)$$
$$- \frac{(N+1) \cdot N \cdot (N-1) \cdot (N-2)}{8} \cdot \Big(\frac{y^{(2)}}{y'(x)}\Big)^2 \Big) \cdot D_x^{N-2} \quad + \, \cdots$$

and the conjugate of (50) reads:

$$L_N^{(c)} \,\, = \,\, D_x^N \,\,\, + \Big(p(x) \, + N \cdot \frac{v'(x)}{v(x)} \Big) \cdot D_x^{N-1} \qquad (52)$$
$$+ \Big(q(x) \,\, + (N-1) \cdot \frac{v'(x)}{v(x)} \cdot p(x) \,\, + \frac{N \cdot (N-1)}{2} \cdot \frac{v''(x)}{v(x)} \Big) \cdot D_x^{N-2} \quad + \, \cdots$$

We impose the identification of these two order-N linear differential operators:

$$\frac{1}{v(x)} \cdot L_N \cdot v(x) \,\, = \,\, pullback\Big(L_N, \, y(x)\Big). \qquad (53)$$

The identification of the D_x^{N-1} coefficients gives the exact expression of $v(x)$ in terms of the wronskian $w(x)$ and of the pullback $y(x)$:

$$v(x) \,\, = \,\, y'(x)^{-(N-1)/2} \cdot \Big(\frac{w(x)}{w(y(x))}\Big)^{1/N} \qquad \text{where:} \qquad p(x) \,\, = \,\, - \frac{w'(x)}{w(x)}. \qquad (54)$$

Injecting this exact expression in (52), or eliminating the log-derivative $v'(x)/v(x)$, the identification of the D_x^{N-2} coefficients gives the following Schwarzian equation

$$W(x) \quad - W(y(x)) \cdot y'(x)^2 \quad + \{y(x), \, x\} \,\, = \,\, 0, \qquad (55)$$

where

$$W(x) \,\, = \,\, \frac{6}{(N+1) \cdot N} \cdot \frac{dp(x)}{dx} \,\, + \, \frac{6 \cdot p(x)^2}{(N+1) \cdot N^2} \,\, - \, \frac{12 \cdot q(x)}{(N+1) \cdot N \cdot (N-1)}, \qquad (56)$$

i.e.

$$W(x) \,\, = \,\, \frac{6}{(N+1) \cdot N} \cdot \mathcal{W}(x) \qquad\qquad \text{where:} \qquad (57)$$
$$\mathcal{W}(x) \,\, = \,\, \frac{dp(x)}{dx} \, + \, \frac{p(x)^2}{N} \,\, - \, \frac{2 \cdot q(x)}{N-1} \,\, = \,\, N \cdot \frac{z''(x)}{z(x)} \,\, - \, \frac{2 \cdot q(x)}{N-1}, \qquad (58)$$

where:

$$z(x) \,\, = \,\, w(x)^{-1/N}, \qquad\qquad p(x) \,\, = \,\, -\frac{w'(x)}{w(x)}. \qquad (59)$$

This is in agreement with the fact that the symmetric $(N-1)$-th power of an order-two linear differential operator $L_2 \, = \, D_x^2 \, + A(x) \cdot D_x \, + \, B(x)$ gives an order-N linear differential operator $L_N \, = \, D_x^N \, + p(x) \cdot D_x^{N-1} \, + \, q(x) \cdot D_x^{N-2} \, + \, \cdots$ such that

$$p(x) \,\, = \,\, \frac{N \cdot (N-1)}{2} \cdot A(x),$$
$$q(x) \,\, = \,\, \frac{(3\,N-1) \cdot N \cdot (N-1) \cdot (N-2)}{24} \cdot A(x)^2 \, + \, \frac{N \cdot (N-1) \cdot (N+1)}{6} \cdot B(x)$$
$$+ \, \frac{N \cdot (N-1) \cdot (N-2)}{6} \cdot \frac{dA(x)}{dx}, \qquad (60)$$

and thus conversely:

$$A(x) \;=\; \frac{2}{N \cdot (N-1)} \cdot p(x),$$

$$B(x) \;=\; \frac{6 \cdot q(x)}{(N+1) \cdot N \cdot (N-1)} \;-\; \frac{(3\,N\,-1) \cdot (N-2) \cdot p(x)^2}{(N+1) \cdot N^2 \cdot (N-1)^2} \;-\; \frac{2 \cdot (N-2)}{(N+1) \cdot N \cdot (N-1)} \cdot \frac{dp(x)}{dx}. \qquad (61)$$

Injecting (61) in the expression of $W(x)$ for an order-two linear differential operator L_2 (see (10))

$$W(x) \;=\; \frac{dA(x)}{dx} \;+\; \frac{A(x)^2}{2} \;-\; 2 \cdot B(x), \qquad (62)$$

one gets again the expression (56) for $W(x)$ for an order-N linear differential operator $L_N \;=\; D_x^N \;+p(x) \cdot D_x^{N-1} \;+q(x) \cdot D_x^{N-2} \;+\; \cdots$

Remark: the Schwarzian condition (55) and the associated function $W(x)$ given by (56), correspond to an elimination of the conjugation function $v(x)$ in (53). If one changes the order-N linear differential operator L_N by conjugation, $L_N \;\rightarrow\; \tilde{L}_N \;=\; 1/\rho(x) \cdot L_N \cdot \rho(x)$, one gets again (53), L_N being replaced by $\tilde{L}_N$ and $v(x)$ being replaced by $\tilde{v}(x)$:

$$v(x) \qquad \longrightarrow \qquad \tilde{v}(x) \;=\; \frac{v(x) \cdot \rho(y(x))}{\rho(x)}. \qquad (63)$$

Consequently one gets again the same Schwarzian condition (55) with the function $W(x)$ given by (56), since they are obtained by elimination of the conjugation functions $v(x)$ or $\tilde{v}(x)$. Therefore $W(L_N, \, x)$ given by (56), which is invariant by the conjugation $L_N \;\rightarrow\; 1/\rho(x) \cdot L_N \cdot \rho(x)$, is left invariant by:

$$p(L_N, \, x) \quad \longrightarrow \qquad p(L_N, \, x) \quad + \; N \cdot \frac{\rho'(x)}{\rho(x)}, \qquad (64)$$

$$q(L_N, \, x) \quad \longrightarrow$$
$$q(L_N, \, x) \quad + \; (N-1) \cdot \frac{\rho'(x)}{\rho(x)} \cdot p(L_N, \, x) \quad + \; \frac{N \cdot (N-1)}{2} \cdot \frac{\rho''(x)}{\rho(x)}. \qquad (65)$$

Conversely imposing this invariance by conjugation (64), (65), on a function of the form $W(x) = \alpha_N \cdot p'(x) + \beta_N \cdot p(x)^2 + \gamma \cdot q(x)$ gives (56) *up to an overall constant factor*.

6. Solutions of the Schwarzian conditions

Let us study the solutions $y(x)$ of the Schwarzian equation (55) that emerge for any pullback-symmetry condition of linear differential operators of arbitrary order N. This should provide valuable information on the pullbacks that are symmetries of linear differential operators.

6.1. Solutions of the Schwarzian equation that are diffeomorphisms of the identity: a condition on $W(x)$

The Schwarzian condition (9) has been shown in [1] to be compatible under the *composition of the pullback-functions* $y(x)$ verifying (9). The fact that the composition of two solutions $y(x)$ of the Schwarzian condition (9) is also a solution‡ of the Schwarzian condition (9), is crucial to describe the set of solutions $y(x)$ of (9). Once a solution $y(x)$ of the Schwarzian condition (9) is known, the n-th composition $y^{(n)}(x) \, = \, y(y(\cdots y(x) \cdots)))$, yields automatically a commuting set of solutions¶ of (9). By obtaining the series expansions of these solutions, one can extend to non integer complex values of n, and in order to build a one-parameter family of *commuting* solution series, consider the infinitesimal composition [2]:

$$y_\epsilon(x) \quad = \quad x \quad + \epsilon \cdot F(x) \quad + \quad \cdots \qquad (66)$$

The one-parameter family of commuting solution series $y^{(n)}(x)$ commutes with (66) yielding the functional equations [2]:

$$F(x) \cdot \frac{dy^{(n)}(x)}{dx} \quad = \quad F(y^{(n)}(x)), \qquad F(x) \cdot \frac{dy_\epsilon(x)}{dx} \quad = \quad F(y_\epsilon(x)). \qquad (67)$$

Inserting (66) in the Schwarzian condition (9), one sees that $F(x)$ is actually *holonomic* being solution of the linear differential equation of *order-three*:

$$\frac{d^3F(x)}{dx^3} \quad - \, 2 \cdot W(x) \cdot \frac{dF(x)}{dx} \quad - \frac{dW(x)}{dx} \cdot F(x) \quad = \quad 0, \qquad (68)$$

whose corresponding order-three linear differential operator $\mathcal{L}_3$ is *exactly the symmetric square of an underlying order-two linear differential operator*§ $\mathcal{L}_2$:

$$\mathcal{L}_3 \;=\; D_x^3 \;-\; 2 \cdot W(x) \cdot D_x \;-\; \frac{dW(x)}{dx} \quad = \quad Sym^2\Big(D_x^2 \;-\; \frac{W(x)}{2}\Big). \qquad (69)$$

Conversely $W(x)$ can be expressed in terms of $F(x)$ as follows:

$$W(x) \quad = \quad \frac{F"(x)}{F(x)} \; - \frac{1}{2} \cdot \Big(\frac{F'(x)}{F(x)}\Big)^2 \;\; + \;\; \frac{\lambda}{F(x)^2} \qquad (70)$$

$$= \quad \frac{d}{dx}\Big(\frac{F'(x)}{F(x)}\Big) \; + \; \frac{1}{2} \cdot \Big(\frac{F'(x)}{F(x)}\Big)^2 \;\; + \;\; \frac{\lambda}{F(x)^2}. \qquad (71)$$

This last result (70) is easily obtained by multiplying the LHS of (68) by $F(x)$ and integrating the result. One gets this way†:

$$F(x) \cdot \frac{d^2F(x)}{dx^2} \quad - \frac{1}{2} \cdot \Big(\frac{dF(x)}{dx}\Big)^2 \; + \lambda \;\; - F(x)^2 \cdot W(x) \quad = \quad 0, \qquad (72)$$

which is (70). Thus, for a given pullback $y(x)$, or for a given *one-parameter* family of commuting solution series (66), or for a given $F(x)$, one has a one-parameter family (70) of $W(x)$ in the Schwarzian equation (9). Conversely, for a given $W(x)$, one has *at least* a one-parameter family of commuting solution series (66).

‡ See Appendix D in [1].

¶ Cum grano salis: when the pullbacks $y(x)$ are algebraic functions, they are *multivalued functions*. The composition of multivalued functions is limited to their analytic series expansions (setting aside Puiseux series).

§ The reduction of $\mathcal{L}_3$ to a symmetric square (69) *does not mean that* $F(x)$ is solution of a second order linear differential (Liouvillian) equation $F"(x)/F(x) \; = \; W(x)/2$.

† This "gauge" $W(x) \, \rightarrow \, W(x) \, + \, \lambda/F(x)^2$ in (70) corresponds to the fact that because of (67) one has $\lambda/F(x)^2 \, - \lambda/F(y(x))^2 \cdot y'(x)^2 \, = 0$ which allows to change $W(x) \, \rightarrow \, W(x) \, + \, \lambda/F(x)^2$ in the Schwarzian equation (9), as well as in the third order linear differential ODE (68). One easily verifies that inserting (70) in (68) *gives an identity*.

6.1.1. Selected subcase of the Schwarzian equation.

Let us consider an order-two linear differential operator $L_2 \, = \, D_x^2 \, + \, A(x) \cdot D_x \, + \, B(x)$ (where $A(x)$ and $B(x)$ are rational functions), such that its corresponding function $W(x) \, = \, A'(x) \, + A(x)^2/2 \, - 2\, B(x)$ (see (10)) in the Schwarzian equation (9), is of the form (see subsection 6.2 of [1])

$$W(x) \, = \, \frac{dA_R(x)}{dx} \, + \, \frac{A_R(x)^2}{2}, \qquad (73)$$

where $A_R(x)$ is a *rational function*. Introducing the rational function $C(x) \, = \, (A(x) \, - \, A_R(x))/2$, the identification of the expression of $W(x)$, namely $W(x) \, = \, A'(x) \, + A(x)^2/2 \, - 2\, B(x)$ with (73), gives $B(x)$ in terms of $A_R(x)$ and $C(x)$

$$B(x) \, = \, \frac{dC(x)}{dx} \, + C(x) \cdot \, (C(x) \, + \, A_R(x)), \qquad (74)$$

which is the condition for the order-two linear differential operator L_2 *to factorize into two order-one linear differential operators*:

$$L_2 \, = \, \Big(D_x \, + \, A_R(x) \, + \, C(x)\Big) \cdot \Big(D_x \, + \, C(x)\Big). \qquad (75)$$

In other words, condition (73) with $A_R(x)$ a rational function, is the condition of factorization of the order-two linear differential operator L_2. In this case, the Schwarzian equation (9) reduces to a simpler second order *non-linear* differential equation (that was studied extensively in [1, 2]):

$$\frac{d^2y(x)}{dx^2} \, = \, A_R(y(x)) \cdot \Big(\frac{dy(x)}{dx}\Big)^2 \, - \, A_R(x) \cdot \frac{dy(x)}{dx}. \qquad (76)$$

Seeking the following one-parameter solutions (66), $y_\epsilon(x) \, = \, x \, + \, \epsilon \cdot \, F(x) \, + \, \cdots$, one finds that $F(x)$ verifies a linear differential equation of order two [2]

$$\frac{d^2F(x)}{dx^2} \, - \, A_R(x) \cdot \frac{dF(x)}{dx} \, - \, \frac{dA_R(x)}{dx} \cdot F(x) \, = \, 0, \qquad (77)$$

corresponding to the linear differential operator of order two†:

$$\mathcal{L}_F \, = \, D_x^2 \, - \, A_R(x) \cdot D_x \, - \, \frac{dA_R(x)}{dx} \, = \, D_x \cdot \Big(D_x \, - A_R(x)\Big). \qquad (78)$$

Introducing the wronskian $w(x)$, $A_R(x)$ reads $A_R(x) \, = \, -w'(x)/w(x)$. Thus the linear differential operator (78) has two solutions: $1/w(x)$ which is the solution of the right factor $D_x \, - A_R(x)$, and another (transcendental) solution that we denote $\mathcal{S}_F$. The function $F(x)$ corresponds to this last (transcendental) solution, and *not the* $1/w(x)$ *solution*. Conversely $A_R(x)$ can be expressed‡ in terms of $F(x)$ as follows:

$$A_R(x) \, = \, \frac{F'(x)}{F(x)} \, + \, \frac{\mu}{F(x)}. \qquad (79)$$

One easily verifies that by inserting (79) in (77) ones gets an identity, and that by inserting (79) in (73) one recovers (71) with $\lambda \, = \, \mu^2/2$. Here the $\mu/F(x)$ *term is crucial*, because when $\mu \, = \, 0$ condition (79) with $A_R(x) \, = \, -w'(x)/w(x)$ yield the trivial result, $F(x) \, = \, 1/w(x)$ which is different from the transcendental (holonomic)

† In fact the order-two operator $\mathcal{L}_F$ is the adjoint of the operator $\Omega = (D_x + A_R(x)) \cdot D_x$ (see [2]). When $A_R(x) \, = \, -w'(x)/w(x)$ the linear differential operator $\mathcal{L}_F$ is conjugated by the wronskian $w(x)$ to the linear differential operator Ω, namely $\Omega \cdot w(x) = w(x) \cdot \mathcal{L}_F$.

‡ Just integrate the LHS of (77).

function we are looking for. For instance in the example detailed in [2], we had the condition (79) verified with $\mu \neq 0$, namely $\mu = 1/4$:

$$F(x) \, = \, x \cdot (1-x)^{1/2} \cdot {}_2F_1\Big([\frac{1}{2}, \, \frac{1}{4}], [\frac{5}{4}], \, x\Big), \qquad A_R(x) \, = \, \frac{3 \, - \, 5\,x}{4\,x\,(1\,-\,x)}. \tag{80}$$

At first sight one expects the order-two linear differential equation (77) on $F(x)$ to be a simple limit of the order-three linear differential equation (68) when the condition (73) is imposed. This reduction is not obvious however and the interested reader can find it explained in Appendix C.

Remark: the global nilpotence of the linear differential operators gives an $A_R(x)$ of the form $A_R(x) \, = \, -w'(x)/w(x)$, where the wronskian $w(x)$ is an N-th root of a rational function [24]. Using $A_R(x) \, = \, -w'(x)/w(x)$, condition (76) can be easily integrated into

$$\frac{dy(x)}{dx} \, = \, c_1 \cdot \frac{w(x)}{w(y(x))} \qquad \text{or:} \tag{81}$$

$$\Theta(y(x)) \, = \, c_1 \cdot \Theta(x) \, + \, c_2 \qquad \text{with:} \qquad \Theta(x) \, = \, \int^x w(x)\,dx \tag{82}$$

where c_1 and c_2 are constants of integration.

Now let us describe this one-parameter family of commuting solution series (66) of the Schwarzian equation (9).

6.2. Solutions of the Schwarzian equation that are diffeomorphisms of the identity: the general formal solution

Let us consider (66) as a series in ϵ:

$$y_\epsilon(x) \, = \, x \, + \epsilon \cdot F(x) \, + \sum_{n=2}^{\infty} \frac{\epsilon^n}{n!} \cdot F(x) \cdot Q_n(x), \tag{83}$$

solution of the functional equation (67). This is sufficient to find, order by order in ϵ, the solution (83) of (67) where the $Q_n(x)$ are given by

$$Q_1(x) \, = \, F(x), \qquad Q_2(x) \, = \, F(x) \cdot \frac{dQ_1(x)}{dx} \, = \, F(x) \cdot \frac{dF(x)}{dx},$$

$$Q_3(x) \, = \, F(x) \cdot \frac{d}{dx} Q_2(x) \, = \, F(x) \cdot \Big(F(x) \cdot F''(x) \, + \, F'(x)^2\Big),$$

$$Q_4(x) \, = \, F(x) \cdot \frac{d}{dx} Q_3(x), \qquad Q_5(x) \, = \, F(x) \cdot \frac{d}{dx} Q_4(x),$$

$$\cdots \qquad Q_{n+1}(x) \, = \, F(x) \cdot \frac{d}{dx} Q_n(x), \tag{84}$$

the most general solution (83) of (67) corresponding to linear combinations of the Q_n's which amounts to changing ϵ in (83) into:

$$\epsilon \quad \longrightarrow \quad \epsilon \cdot (1 \, + \, \lambda_1 \cdot \epsilon \, + \, \lambda_2 \cdot \epsilon^2 \, + \, \lambda_3 \cdot \epsilon^3 \, + \, \cdots). \tag{85}$$

Note that all the Q_n's are *polynomial expressions of* $F(x)$ *and its derivatives.*

The functional equation (67) corresponds to the one-form $d\Theta \, = \, dx/F(x) \, = \, dy/F(y)$ giving:

$$\Theta(x) \, = \, \int^x \frac{dx}{F(x)}, \qquad \frac{d}{d\Theta} \, = \, F(x) \cdot \frac{d}{dx}. \tag{86}$$

Seeing x as a function of Θ, one finds that the series (83) together with the recursion (84), gives the well-known Taylor expansion

$$y_\epsilon(x(\Theta)) \;=\; x(\Theta) \;+\sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \cdot \frac{d^n\, x(\Theta)}{d\Theta^n} \;=\; x(\Theta\,+\,\epsilon), \qquad (87)$$

meaning that $x \,\rightarrow\, y_\epsilon(x)$ *is just a shift in* Θ

$$\Theta_x \quad \longrightarrow \quad \Theta_y \;=\; \Theta_x \,+\, \epsilon, \qquad (88)$$

corresponding to the integration of the one-form $d\Theta \,=\, dx/F(x) \,=\, dy/F(y)$. The two transformations $y_{\epsilon_1}(x)$ and $y_{\epsilon_2}(x)$ of the one-parameter family clearly commute‡:

$$y_{\epsilon_1}(y_{\epsilon_2}(x(\Theta))) \;=\; y_{\epsilon_1}(x(\Theta\,+\,\epsilon_2)) \;=\; x(\Theta\,+\,\epsilon_1\,+\,\epsilon_2). \qquad (89)$$

One verifies order by order in ϵ, that the one-parameter family of commuting series (83) with (84) *is solution of the Schwarzian equation*

$$W(x) \quad - W(y_\epsilon(x)) \cdot y'_\epsilon(x)^2 \quad + \{y_\epsilon(x),\, x\} \;=\; 0, \qquad (90)$$

where $W(x)$ is given by (70). In terms of Θ, the expression (70) for $W(x)$ can be written using the Schwarzian derivative:

$$W(x) \;+\; \{\Theta(x),\, x\} \;-\; \lambda \cdot \Bigl(\frac{d\Theta(x)}{dx}\Bigr)^2 \;=\; 0. \qquad (91)$$

Recalling the chain rule for the Schwarzian derivative of a composition of functions†† and the fact that $d\Theta(y(x))/dx \,=\, d\Theta(x)/dx$, one finds that the Schwarzian condition (90) corresponds to the equality of the two Schwarzian derivatives:

$$\{\Theta(y(x)),\, x\} \;=\; \{\Theta(x),\, x\},$$

which is verified since $d\Theta(y(x))/dx \,=\, d\Theta(x)/dx$. This is another way to see that the one-parameter family of commuting series (83) (with the Q_n's given by (84)) is solution of the Schwarzian equation.

6.3. A simple modular form example.

We have considered in [1, 29, 30, 31, 37] many examples of *modular forms* represented as pullbacked ${}_2F_1$ hypergeometric functions. Each time the one-parameter commuting series combined with the modular correspondences [8] series yields one-parameter series of the form $y_n(x) \,=\, a_n \cdot x^n \,+\, \cdots, \;\; n \,=\, 2, 3, 4, \cdots$ that are solutions of the Schwarzian equation (90).

In [1] the pullback symmetry of the order-two linear differential operator was given as a covariance of its solution, namely a hypergeometric function with *two different*¶ *pullbacks* related by modular equations§

$${}_2F_1\Bigl([\frac{1}{12},\, \frac{5}{12}],\, [1],\, y(x)\Bigr) \;=\; \mathcal{A}(x) \cdot {}_2F_1\Bigl([\frac{1}{12},\, \frac{5}{12}],\, [1],\, x\Bigr), \qquad (92)$$

the pullback $y(x)$ being solution of the Schwarzian condition (90).

‡ This can also be checked directly using (83) with (84) for any rational function $F(x)$.

†† Namely $\{\Theta(y(x)),\, x\} \;=\; \{\Theta(y(x)),\, y(x)\} \cdot y'(x)^2 \;+ \{y(x),\, x\}$.

¶ We exclude the trivial well-known changes of variables on hypergeometric functions $x \,\rightarrow\, 1-x,\, 1/x, ...$ The transformation $x \,\rightarrow\, y(x)$ must be an *infinite order* transformation symmetry.

§ The emergence of a *modular form* [29, 30, 38] corresponds to the emergence of a selected hypergeometric function having an exact covariance property [39, 40] with respect to an *infinite order algebraic transformation* (the modular correspondences).

In this example, the pullback $y_\epsilon(x)$ is solution of the Schwarzian solution (90) with $w(x)$ and $F(x)$ given by†:

$$W(x) = \, -\frac{32\,x^2-41\,x+36}{72\,x^2\cdot\,(x-1)^2}, \quad F(x) = \, x\cdot\,(1-x)^{1/2}\cdot\,{}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1],\,x\Big)^2. \quad (93)$$

One can also check that these expressions (93) verify (70) with‡ $\lambda \, = \, 0$, *thus providing a quite non-trivial (non-linear differential) identity between the rational function* $W(x)$ *and the holonomic function* $F(x)$.

The one-parameter commuting family (66) solution of the Schwarzian equation (90) can be expressed using the two (mirror maps) *differentially algebraic* [3, 4] functions $P(x)$ and $Q(x)$ described in [1] and in Appendix D, as $y_1(a_1,\, x) \, = \, P(a_1\cdot\, Q(x))$:

$$y_1(a_1,\, x) \, = \, a_1\cdot\, x \, -\frac{31\,a_1\cdot\,(a_1\,-1)}{72}\cdot\, x^2\, + \frac{a_1\cdot\,(9907\,a_1{}^2-30752\,a_1+20845)}{82944}\cdot\, x^3$$

$$-\frac{a_1\cdot\,(a_1-1)\cdot\,(4386286\,a_1{}^2-20490191\,a_1+27274051)}{161243136}\cdot\, x^4 \quad + \cdots \quad (94)$$

where $a_1 \, = \, \exp(\epsilon)$.

Besides this one-parameter commuting family (66), the Schwarzian equation (90) has a remarkable (infinite) set of algebraic functions solutions [1] $y(x)$ defined by the corresponding *modular equations* [25, 41, 42, 43, 44, 45]. Their series expansions near $x \, = \, 0$ read:

$$y_n(x) \quad = \quad P(Q^n(x)) \quad = \quad 1728\cdot\Big(\frac{x}{1728}\Big)^n \quad + \; \cdots \quad (95)$$

where n is an integer $n \, = \, 2,\, 3,\, 4, \cdots$ These series expansions *commute for different values of the integer* n. This is a consequence of the fact that, up to the previous change of variables $P(x)$, $Q(x)$, these *modular correspondences* (95) correspond to taking the n-th power of the nome: $q \, \rightarrow \, q^n$ (see [1] for more details).

6.3.1. A pre-modular concept.

The composition of the one-parameter series (66) (which corresponds to $q \, \rightarrow \, a_1\cdot q$) and of the modular correspondences (95), yields an *infinite set of one-parameter series* $y_n(x) \, = \, a_n\cdot\, x^n \, + \, \cdots, \; n \, = \, 2,\, 3,\, 4,\, \cdots$ for instance [1]:

$$y_3 \, = \, a_3\cdot\, x^3\, + \frac{31\,a_3}{24}\cdot\, x^4\, + \frac{36221\,a_3}{27648}\cdot\, x^5\, - \frac{a_3\cdot\,(23141376\,a_3-66458485)}{53747712}\cdot\, x^6 + \; \cdots$$

These one-parameter series do not commute but verify [1] the simple composition formulae¶:

$$y_n(a_n,\; y_m(a_m,\; x)) \;\; = \;\; y_{nm}(a_n a_m^n,\; x), \qquad n,\, m \; = \; 1,\, 2,\, 3,\; \cdots \quad (96)$$

When the a_n are arbitrary rational numbers the corresponding series $y_n(a_n,\, x)$ are *not globally bounded series* [31] in general. Therefore, they *cannot be the series expansion of an algebraic function*: they are *differentially algebraic* [3, 4] since they are solutions of the Schwarzian equation (90).

In general, finding the Schwarzian equation (90) is easy, and getting solutions order by order as series expansions is also easy. However finding the selected values of

† One can easily check that these expressions (93) for $W(x)$ and $F(x)$ verify (68).
‡ This selected value of λ has to be compared with the value $\mu = 1/4$ in (80).
¶ Consequence of the fact, in the nome, they correspond to the composition of transformations like $q \, \rightarrow \, a_n\cdot\, q^n$.

the rational numbers a_n such that the *differentially algebraic* [3, 4] series $y_n(a_n, x)$ are globally bounded and *thus can be algebraic functions*, and, possibly, modular correspondences, is a quite difficult task‡.

We will call "pre-modular♯" the existence of an infinite set of one-parameter differentially algebraic series (solution of the Schwarzian equation) of the form $y_n(x) = a_n \cdot x^n + \cdots$ which verify (96), but for which *one does not know if there exist some selected values of the parameter* a_n such that these *differentially algebraic* series [3, 4] become *algebraic functions*.

In the next section, we will characterize the Schwarzian equations corresponding to these "pre-modular" structure, thus finding *conditions that are necessary* for the emergence of modular forms.

6.4. Schwarzian equation: conditions for modular correspondence

In the previous sections it was shown that the pullback symmetry condition of *arbitrary* order-two linear differential operators yields Schwarzian equation (90). The solutions of these order-two linear differential operators *are much more general than hypergeometric functions and Heun functions* [1]: they can have an *arbitrary number of singularities*. Let us see which Schwarzian equation (90), or equivalently, which function $W(x)$ gives relations (96) corresponding to *rigid constraints necessary to have modular correspondences* [1].

Series calculations give the conditions on $W(x)$ such that series solutions of the form $y_n(x) = a_n \cdot x^n + \cdots$ are solutions of the Schwarzian equation with these $y_n(x)$'s verifying relations (96). These constraints are conditions on the *Laurent series* of $W(x)$. For the solution series of the Schwarzian equation to have the pre-modular structure (96), i.e. the same structure as modular correspondences, the Laurent series of $W(x)$ must be of the form:

$$W(x) \quad = \quad -\frac{1}{2\,x^2} \; + \frac{b_1}{x} \; + \sum_{m=0}^{\infty} a_m \cdot x^m. \qquad (97)$$

One easily verifies that this is the case for the previous modular form example where $W(x)$ reads (93), as well as for all the other modular forms emerging in physics or enumerative combinatorics we mentioned in previous papers [29, 30, 31, 35, 37].

Condition (97) is a result whose scope transcends the hypergeometric functions framework. In order to show this, let us apply this result on the open problem of finding Heun functions† that could be modular forms [38], or pullbacked ${}_2F_1$ functions [16, 50]. The Heun function $HeunG\,(a, q, \alpha, \beta, \gamma, \delta, x)$ is solution of a linear differential operator of order two $L_2 = D_x^2 + A(x) \cdot D_x + B(x)$ where $A(x)$

‡ Similar to finding the selected values of the parameters so that a quantum Hamiltonian becomes integrable, or finding modular forms among Beukers' second order differential equations depending on three parameters [46] (36 cases emerging of a numerical exploration of 10 millions triples).

♯ Of course, this "pre-modular" term should not be confused with the term premodular in premodular categories, (ribbon fusion categories). Here we mean prerequisites for the emergence of modular forms.

† Finding the selected values of the parameters of a Heun function [47] (in particular the *accessory parameter* [48]) such that its series expansion is a series with *integer coefficients* (or more generally is globally bounded [31]), or such that the corresponding order-two linear differential operator is *globally nilpotent* [24] is a difficult problem. These classification problems are closely related to finding the Heun functions reducible to pullbacked hypergeometric functions [49], and to modular forms [46].

and $B(x)$ read:

$$A(x) \; = \; \frac{(\alpha+\beta+1)\cdot x^2 \, - \, ((\delta+\gamma)\cdot a \, + \alpha \, - \delta \, + \beta \, + 1)\cdot x \; + \gamma \cdot a}{x \cdot (x-1)\cdot (x\, - a)}, \qquad (98)$$

$$B(x) \; = \; \frac{\alpha \, \beta \cdot x \; - q}{x \cdot (x-1)\cdot (x\, - a)}. \qquad (99)$$

The corresponding function $W(x)$ is easily deduced from the formula (10) given by $W(x) \, = \, A'(x)\, A^2(x)/2 \, - 2\, B(x)$. It has the following *Laurent series* expansion:

$$W(x) \; = \; \frac{\gamma \cdot (\gamma-2)}{2\, x^2} \; - \frac{a\,\delta\,\gamma \, + \alpha\,\gamma + \beta\,\gamma \, - \delta\,\gamma - \gamma^2 + \gamma - 2\,q}{a\; x} \; + \; \cdots, \quad (100)$$

and has the form (97) given by $-1/2/x^2 \, + \, \cdots$ *only when* $\gamma \, = \, 1$. Thus a general analytical constraint like (97) yields a simple exact constraint on the intriguing problem of the classification of the Heun functions that can be modular forms, and more specifically on the necessary conditions for the Heun functions to have a “pre-modular” structure.

6.4.1. Schwarzian equation for $W(x) \, = \, -1/2/x^2$.

In order to understand the Laurent series condition (97), let us try to see what is so “special” in the case where $W(x) \, = \, -1/2/x^2$. For

$$W(x) \; = \; -\frac{1}{2\,x^2} \; = \; -\{\ln(x), \, x\}, \qquad (101)$$

the most general solutions of corresponding Schwarzian equation read:

$$y(x) \; = \; \exp\Bigl(\frac{a \cdot \ln(x) \, + \, b}{c \cdot \ln(x) \, + \, d}\Bigr), \qquad (102)$$

which just amounts to a simple transformation on $\ln(x)$:

$$\ln(x) \quad \longrightarrow \quad \ln(y(x)) \; = \; \frac{a \cdot \ln(x) \, + \, b}{c \cdot \ln(x) \, + \, d}. \qquad (103)$$

The solutions of the form $y_n(x) \, = \, a_n \cdot x^n \, + \, \cdots$ are given by $y_n(x) \, = \, a_n \cdot x^n$ and are thus “trivial”: this is the case because the nome‡ q *is nothing but the* x *variable!* Similarly, the ratio of periods τ is just $\ln(x)$, and thus the condition $W(x) \, = \, -1/2/x^2$ is a “trivialization” of the mirror map.

6.4.2. Rank-two condition (76) and pre-modular structures.

The factorization of the order-two linear differential operator which corresponds to $W(x)$ of the form (73), yields the rank-two *non-linear* differential equation (76) (see section 6.1.1). We would like to know when the modular correspondences structures (existence of solutions series $y_n(x) \, = \, a_n \cdot x^n \, + \, \cdots, \; n \, = \, 2, \, 3, \, 4, \cdots$ such that (96), thus requiring $W(x) \, = \, -1/2/x^2 \, + \, \cdots$) are compatible with a factorization of the order-two linear differential operator and thus with condition (73). Imposing

$$W(x) \; = \; \frac{dA_R(x)}{dx} \; + \; \frac{A_R(x)^2}{2} \; = \; -\frac{1}{2\,x^2} \; + \; \cdots \qquad (104)$$

‡ Such that the transformations $x \, \rightarrow \, y_n(x) \, = \, a_n \cdot x^n \, + \, \cdots$ simply reduce to $q \, \rightarrow \, a_n \cdot q^n$, see the concept of mirror maps [1].

where $A_R(x)$ is a rational function, one finds that $A_R(x)$ must have the following Laurent series expansion:

$$A_R(x) \;\; = \;\; \frac{1}{x} \; + \sum_{m=0}^{\infty} r_m \cdot x^m. \qquad (105)$$

This result (105) can be directly obtained by looking for the *Laurent series* for $A_R(x)$ with a pre-modular structure, i.e. such that the series $y_n(x) \, = \, a_n \cdot x^n \, + \, \cdots$, $n \, = \, 2, \, 3, \, 4, \cdots$ are solutions of condition (76). As a byproduct, one finds that in the case (105) the solutions $y_n(x) \, = \, a_n \cdot x^n \, + \, \cdots$ are such that (96). In particular the solution $y_1(x) \, = \, a_1 \cdot x \, + \, \cdots$ is a one-parameter family of commuting series. The case $W(x) \, = \, -1/2/x^2$, or $A_R(x) \, = \, 1/x$, corresponds to the simple order-two linear differential operator θ^2 where θ is the homogeneous derivative $\theta \, = \, x \cdot D_x$.

More specifically, if one revisits our Heun classification problems, imposing the *factorization condition* (see the analysis sketched in Appendix E) *together* with the condition (97) required for the *emergence of modular correspondence structure* (96), one gets the following Laurent series expansion (see (E.4) for the definition of the u, v, w parameters):

$$W(x) \;\; = \;\; \frac{v \cdot (v \, - 2)}{2 \cdot x^2} \;\; - \frac{v \cdot (a \, w \, + u)}{a \, \cdot \, x} \;\; + \; \cdots \qquad (106)$$

This gives the condition $v \, = \, 1$ (in agreement with condition (105)) and four other conditions. Excluding the case $a \, = \, 0$ corresponding to the reduction from the four singularities of the Heun function to three singularities, one gets $\gamma \, = \, v \, = \, 1$. The Heun function $HeunG\,(a, \, 0, \, 0, \, \beta, \, 1, \, \delta, \, x)$ is a (Liouvillian) solution of a reducible linear differential operator of order two $L_2 \, = \, (D_x \, + A_R(x)) \cdot D_x$, where $A_R(x)$ then reads:

$$A_R(x) \;\; = \;\; \frac{1}{x} \; + \frac{\delta}{x \, - 1} \; + \frac{\beta \, - \delta}{x \, - a}. \qquad (107)$$

The pullbacks $y(x)$ are solutions of the rank-two non-linear differential equation (76) which can easily be integrated into (see (81), (82)):

$$x \cdot \frac{y'(x)}{y(x)} \;\; = \;\; c_1 \cdot \frac{(y(x) - 1)^{\delta} \cdot (y(x) - a)^{\beta \, - \delta}}{(x - 1)^{\delta} \cdot (x - a)^{\beta \, - \delta}}, \qquad (108)$$

giving a functional equation on the pullbacks $y(x)$ with an Abel integral $\Theta(x)$:

$$\Theta(y(x) \; = \; c_1 \cdot \Theta(x) \, + \, c_2 \quad \text{where:} \quad \Theta(x) \;\; = \;\; \int^x \frac{dx}{x \cdot (x \, - 1)^{\delta} \cdot (x - a)^{\beta \, - \delta}}. \qquad (109)$$

One has for instance the following one-parameter series solutions for the pullback $y(x)$, which verify (96):

$$y_1 \;\; = \;\; a_1 \cdot x \;\; - a_1 \cdot (a_1 \, - 1) \cdot \frac{a \, \delta \, + \beta \, - \delta}{a} \cdot x^2 \;\; + \;\; \cdots \qquad (110)$$

$$y_2 \;\; = \;\; a_2 \cdot x^2 \;\; + \; 2 \cdot \frac{a \, \delta \, + \beta \, - \delta}{a} \cdot a_2 \cdot x^3 \;\; + \; \cdots \qquad (111)$$

The fact that solutions of the form $y(x) \, = \, a_n \cdot x^n \, + \, \cdots$ occur can be clearly seen on equation (108). Even if the “pre-modular” conditions (96) are verified for this example, this Heun function $HeunG\,(a, \, 0, \, 0, \, \beta, \, 1, \, \delta, \, x)$ will not be necessarily a modular form represented as a pullbacked ${}_2F_1$ hypergeometric function with more than one pullback for generic parameters†.

† The exponent-differences at the four singularities are: $0, 1 - \delta, \, 1 + \delta - \beta, \beta$. Introducing e_1, e_2, e_3 the exponents difference of the three singular points of the ${}_2F_1$ hypergeometric function each the previous exponent-differences must be a multiple of the e_i's.

7. Pullback symmetry of an operator up to equivalence of operators

With the aim of generalizing covariance (92), we introduce the derivative of ${}_2F_1([1/12, 5/12], [1], x)$

$$\Phi(x) \; = \; {{d} \over {dx}} \Big({}_2F_1\Big([{{1} \over {12}}, \, {{5} \over {12}}], \, [1], \, x\Big)\Big) \; = \; {{5} \over {144}} \cdot {}_2F_1\Big([{{13} \over {12}}, \, {{17} \over {12}}], \, [2], \, x\Big), \tag{112}$$

which *does not* correspond to a modular form, *since the derivative of a modular form is not a modular form.* A derivative of the simple covariance identity (92) gives

$$\Phi(y(x)) \cdot y'(x) \; = \; \mathcal{A}(x) \cdot \Phi(x) \; + \mathcal{A}'(x) \cdot {}_2F_1\Big([{{1} \over {12}}, \, {{5} \over {12}}], \, [1], \, x\Big). \tag{113}$$

Using the order-two linear differential equation verified by ${}_2F_1([1/12, 5/12], [1], x)$, one can rewrite the ${}_2F_1([1/12, 5/12], [1], x)$ in the RHS of (113), as a linear combination of $\Phi(x)$ and its derivative $\Phi'(x)$. One then deduces from relation (113) a slightly more general relation than the initial simple covariance (92)

$$\Phi(y(x)) \; = \; \Big(\mathcal{A}_\Phi(x) \cdot {{d} \over {dx}} \; + \mathcal{B}_\Phi(x)\Big) \cdot \Phi(x), \tag{114}$$

where $\mathcal{A}_\Phi(x)$ and $\mathcal{B}_\Phi(x)$ read in this particular example‡:

$$\mathcal{A}_\Phi(x) \; = \; {{144 \cdot \, x \cdot \, (x-1) \cdot \, \mathcal{A}(x)} \over {5 \cdot \, y'(x)}}, \qquad \mathcal{B}_\Phi(x) \; = \; {{5 \cdot \, \mathcal{A}(x) \, + \, 72 \cdot \, (2 \, - 3\, x) \cdot \, \mathcal{A}'(x)} \over {5 \cdot \, y'(x)}}.$$

Recalling two Hauptmoduls $p_1(x)$ and $p_2(x)$

$$p_1(x) \; = \; {{1728 \cdot \, x} \over {(x \, + 16)^3}}, \qquad \qquad p_2(x) \; = \; {{1728 \cdot \, x^2} \over {(x \, + 256)^3}}, \tag{115}$$

one can also write relation (114) in a more "balanced" form (see equation (7) in [2]). Introducing the two algebraic functions $A_1(x)$ and $A_2(x)$

$$A_1(x) \; = \; \Big(1 \, + \, {{x} \over {16}}\Big)^{-1/4}, \qquad \qquad A_2(x) \; = \; \Big(1 \, + \, {{x} \over {256}}\Big)^{-1/4}, \tag{116}$$

one has the (modular form) hypergeometric identity:

$$A_1(x) \cdot {}_2F_1\Big([{{1} \over {12}}, \, {{5} \over {12}}], \, [1], \, p_1(x)\Big) \; = \; A_1(x) \cdot {}_2F_1\Big([{{1} \over {12}}, \, {{5} \over {12}}], \, [1], \, p_2(x)\Big). \tag{117}$$

After performing calculations of a similar nature of the ones previously seen, one deduces the $1 \leftrightarrow 2$ balanced relation on $\Phi(x)$:

$$\begin{aligned} & 144 \cdot \, p_1(x) \cdot \, (p_1(x) \, - 1) \cdot \, {{dA_1(x)} \over {dx}} \cdot \, \Phi'(p_1(x)) \\ & \quad + \Big(72 \cdot \, (3\, p_1(x) \, - 2) \cdot \, {{dA_1(x)} \over {dx}} \, - 5 \cdot \, A_1(x) \cdot \, {{dp_1(x)} \over {dx}}\Big) \cdot \, \Phi(p_1(x)) \\ = \; & 144 \cdot \, p_2(x) \cdot \, (p_2(x) \, - 1) \cdot \, {{dA_2(x)} \over {dx}} \cdot \, \Phi'(p_2(x)) \\ & \quad + \Big(72 \cdot \, (3\, p_2(x) \, - 2) \cdot \, {{dA_2(x)} \over {dx}} \, - 5 \cdot \, A_1(x) \cdot \, {{dp_2(x)} \over {dx}}\Big) \cdot \, \Phi(p_2(x)), \end{aligned} \tag{118}$$

which should be viewed as a (rational) parametrization of the relation having the form (114).

‡ If instead of the simple derivative (112) we had introduced $\Phi(x) \, = L_1({}_2F_1([1/12, 5/12], [1], x))$ where L_1 is an arbitrary order-one linear differential operator, we would have also obtained a relation of the form (114) but where $\mathcal{A}_\Phi(x)$ and $\mathcal{B}_\Phi(x)$ are much more involved expressions.

The interested reader shall find in Appendix F a detailed (and we hope pedagogical) analysis of the more general relation (114) given for a selected hypergeometric function¶ solution ${}_2F_1([-1/4, 3/4], [1], x)$.

Let us provide an example of the relevance of the relation (114) in the context of integrable models in physics. In the case of the two-dimensional Ising model, the covariance (114) is instantiated on $\tilde{\chi}^{(2)}$, the simplest of the low-temperature n-fold integrals $\tilde{\chi}^{(n)}$ occurring in the decomposition of the susceptibility of the square Ising model [32, 33, 34] (see subsection 5.1 in [54]). When applied to $\tilde{\chi}^{(2)}$, the *Landen transformation* $k \rightarrow k_L = \frac{2\sqrt{k}}{1+k}$, which provides an exact representation of a generator of the renormalization group [2, 7, 53], gives the following covariance relation (see equation‡ (64) in [54]):

$$\tilde{\chi}^{(2)}\Big(\frac{2\sqrt{k}}{1\,+\,k}\Big) \;=\; 4 \cdot \frac{1\,+\,k}{k} \cdot \frac{d\,\tilde{\chi}^{(2)}(k)}{dk}, \qquad (119)$$

$$\text{where:} \qquad \tilde{\chi}^{(2)}(k) \;=\; \frac{k^4}{4^3} \cdot {}_2F_1\Big([\frac{3}{2},\, \frac{5}{2}],\; [3],\; k^2\Big). \qquad (120)$$

This relation (119) can also be written as

$$\tilde{\chi}^{(2)}(k) \;=\; \frac{1}{4} \cdot \Big(k \cdot (k-1) \cdot \frac{d}{dk} \,+\, \frac{k^2\,+k\,+2}{k\,+1}\Big)\, \tilde{\chi}^{(2)}\Big(\frac{2\sqrt{k}}{1\,+\,k}\Big), \qquad (121)$$

or introducing the *inverse Landen transformation* (descending Landen transformation):

$$\frac{1\,-\,(1\,-k^2)^{1/2}}{1+(1\,-k^2)^{1/2}} \;=\; \frac{k^2}{4} \,+\frac{k^4}{8} \,+\frac{5}{64}k^6 \,+\frac{7}{128}k^8 \,+\frac{21}{512}k^{10} \;+\; \cdots, \qquad (122)$$

$$\tilde{\chi}^{(2)}\Big(\frac{1\,-\,(1\,-k^2)^{1/2}}{1\,+(1\,-k^2)^{1/2}}\Big) \;=\; \Big(\frac{(k^2\,-2)\cdot\,(1\,-k^2)^{1/2}\,+\,2}{4\,k^2}\Big) \cdot \tilde{\chi}^{(2)}(k)$$
$$+\; \frac{k^2\,-1}{4\,k} \cdot \Big(1\,-\,(1\,-k^2)^{1/2}\Big) \cdot \frac{d\tilde{\chi}^{(2)}(k)}{dk}. \qquad (123)$$

Remark: Note that the premodular condition (97), $W(x) \,=\, -1/2/x^2 \,+\, \cdots$, has no reason to be verified for such generalizations of modular forms (112), (114). For instance for $\tilde{\chi}^{(2)}$ given by (121), the function $W(x) \,=\, p'(x) \,+ p(x)^2/2 \,-\, 2\,q(x)$ (see (10)) has the following Laurent series expansion (here $x \,=\, k$):

$$W(x) \;=\; \frac{3}{2} \cdot \frac{x^2\,-5}{x^2\,\cdot\,(x^2\,-1)} \;=\; \frac{15}{2} \cdot \frac{1}{x^2} \;+6\;+6\,x^2\;+\;6\,x^4\;+\;\cdots \qquad (124)$$

More generally these (hypergeometric) examples provide simple illustrations of a more general pullback symmetry, where one imposes the pullback of an order N linear differential operator to be *homomorphic to that operator.* In this case there exists two intertwiners (of order $N-1$ in general) L_{N-1} and M_{N-1}, such that:

$$M_{N-1} \cdot L_N \;=\; pullback\Big(L_N,\, y(x)\Big) \cdot L_{N-1}. \qquad (125)$$

¶ We thank A.J. Guttmann for showing us this remarkable hypergeometric function emerging in a dual context of combinatorics and random-matrix theory, counting the number of avoiding permutations [51, 52].

‡ Note a misprint in the expression of the Landen transformation in the unlabelled equation above equation (62) in [54].

The pullback symmetry up to conjugation studied in sections 2, 3, 4, 5, 6 is appropriate for modular forms [29, 30, 31, 37], but *not for derivatives of modular forms* that *also occur in physics* (see for instance the previous relation (119) on the square Ising model). The emergence of such generalized covariance (125) for the representation of the Landen transformation (and more generally the modular correspondences providing exact representations of the generators of the renormalization group) on the other n-fold integrals $\tilde{\chi}^{(n)}$'s of the susceptibility of the Ising model [32, 33, 34] is a *challenging open problem*, that will require one to consider *reducible operators* (see subsection 4.2).

Analyzing these more general constraints (125) will require many additional assumptions (beyond the one of having selected differential Galois group) on the linear differential operator L_N to be able to perform more calculations.

8. Schwarzian conditions for different Calabi-Yau operators with the same Yukawa couplings

In the previous sections we have analyzed the question of the covariance under algebraic pullbacks of a linear differential operator of arbitrary order N, i.e. the question of linear differential operators with algebraic pullback symmetries. Let us consider here the more general problem of the equivalence under pullbacks up to conjugations of *two different linear differential operators*, which is an enlightening sieve when one tries to classify selected linear differential operators in theoretical physics (Calabi-Yau linear differential operators [17, 18]). The interested reader will find in Appendix G an illustration of this important question where we revisit in detail some calculations of a paper by Almkvist, van Straten and Zudilin [17]. This calculation reexamines the question of pullback equivalence up to conjugation, of two selected order-four operators L_4 and $\mathcal{L}_4$ verifying the Calabi-Yau condition:

$$v(x) \cdot \mathcal{L}_4 \cdot \frac{1}{v(x)} \; = \; pullback\Big(L_4, \, \frac{-4\,x}{(1\,-\,x)^2}\Big), \qquad (126)$$

$$\text{with:} \qquad v(x) \; = \; \Big(\frac{x \cdot (1\,+x)}{1\,-x}\Big)^{1/2}. \qquad (127)$$

One finds that a Schwarzian equation verified by these two order-four linear differential operators L_4 and $\mathcal{L}_4$ reads:

$$\hat{U}_R(x) \;\; - U_M(y(x)) \cdot y'(x)^2 \;\; + \{y(x), \, x\} \;\; = \;\; 0, \qquad (128)$$

where $U_M(x)$ and $\hat{U}_R(x)$ are given by (30), and where $p(x)$ and $q(x)$ are the coefficients of D_x^3 and D_x^2 for respectively L_4 and $\mathcal{L}_4$, (see (G.12) and (G.13) in Appendix G).

One sees on this example that the nome and Yukawa couplings, expressed in terms of the x variable, are related (see (G.16), (G.18)) by the pullback transformation. Yet, the Yukawa couplings of the two linear differential operators *expressed in term of the nome*, are related in an even simpler and "universal" way: $K_q(\mathcal{L}_4) \; = \; K_q(L_4)(-4 \cdot q)$, as shown in Appendix E of [31]. For a pullback $y(x)$ with a series expansion of the form

$$y(x) \;\; = \;\; \lambda \cdot x^n \;\; + \; \cdots \qquad (129)$$

the nome and Yukawa couplings expressed in terms of the x variable, of two order-four operators such that

$$v(x) \cdot \mathcal{L}_4 \cdot \frac{1}{v(x)} \;\; = \;\; pullback\Big(L_4, \, y(x)\Big), \qquad (130)$$

are simply related through the relations

$$q_x(\mathcal{L}_4)^n \;=\; \frac{1}{\lambda} \cdot q_x(L_4)\Big(y(x)\Big), \qquad K_x(\mathcal{L}_4) \;=\; K_x(L_4)\Big(y(x)\Big). \qquad (131)$$

The Yukawa couplings *expressed in terms of the nome‡, are related in an even simpler "universal" way* as so:

$$K_q(\mathcal{L}_4) \;=\; K_q(L_4)(\lambda \cdot q^n). \qquad (132)$$

The previous example (126) corresponds to $n = 1$ and $\lambda = -4$. In the case $n = 1$ and $\lambda = 1$, the pullback is a deformation of the identity $y(x) = x + \cdots$ and the Yukawa couplings expressed in terms of the nome, of the two linear differential operators are equal. Thus one recovers Proposition (6.2) of Almkvist et al. paper [17] where the Yukawa couplings coincide.

Since the Schwarzian equation (128) corresponds to the equivalence of two linear differential operators by pullback with remarkably simple relations (132) on their Yukawa couplings expressed in terms of the nome, the Schwarzian equation (128) can be seen as a condition to have simply related Yukawa couplings. In the case of *deformation of the identity* $y(x) = x + \cdots$ pullbacks, it can be seen as a condition of preservation of the Yukawa couplings (*seen as functions of the nome*). These results *are not restricted to order-four linear differential operators* (see Appendix E of [31] and Appendix G). For instance, one can impose that *two different pullbacks of the same order-N linear differential operator* L_N *are homomorphic*, i.e. there exist two intertwiners (of order $N-1$ in general) L_{N-1} and M_{N-1} such that:

$$pullback\Big(L_N,\, p_1(x)\Big) \cdot L_{N-1} \;=\; M_{N-1} \cdot pullback\Big(L_N,\, p_2(x)\Big). \qquad (133)$$

This last generalization turns out to be instructive for physics and enumerative combinatorics.

9. Conclusion

In a previous paper [1] we focused on identities relating the *same* ${}_2F_1$ hypergeometric function with *two different algebraic pullback transformations*

$$\mathcal{A}(x) \cdot {}_2F_1\Big([a,b],[c],x\Big) \;=\; {}_2F_1\Big([a,b],[c],y(x)\Big), \qquad (134)$$

along with the existence of ${}_nF_{n-1}$ analogues of the previous relation. Such remarkable identities correspond to *modular forms* that emerged in the analysis of multiple integrals related to the square Ising model [29, 30, 31, 35] or in other enumerative combinatorics context [37]. They can be seen as a simple occurence of *infinite order*†† covariance symmetries in physics [2] or enumerative combinatorics.

The current paper generalizes these previous results beyond hypergeometric functions¶, analyzing the conditions for order-N linear differential operators with an arbitrary number of singularities† to be *pullback invariant up to conjugations*:

$$\frac{1}{v(x)} \cdot L_N \cdot v(x) \;=\; pullback\Big(L_N,\, y(x)\Big). \qquad (135)$$

‡ This function is often viewed as a function of the nome $q = e^\tau$, since its q-expansion in the case of degenerating family of Calabi-Yau 3-folds is supposed to encode the counting of rational curves of various degrees on a mirror manifold.

†† We have for instance in mind to provide exact representations of the renormalization group [2, 7, 53].

¶ Or even Heun functions, see [1].

† Far beyond operators with hypergeometric solutions, or pullbacked hypergeometric solutions.

One finds that the pullbacks $y(x)$ are *differentially algebraic* [3, 4], being *necessarily solutions of the same Schwarzian equations* as in [1]

$$W(x) \quad - W(y(x)) \cdot y'(x)^2 \quad + \{y(x), \, x\} \quad = \quad 0, \qquad (136)$$

where the function $W(x)$ encoding the Schwarzian equation (136) is a simple expression of the first two coefficients of the linear differential operator (see (56)). For order-two linear differential operators this Schwarzian condition turns out to be sufficient. In the case of linear differential operators with selected differential Galois groups however, we showed, for orders three and four, that the "Calabi-Yau" conditions (see sections 4.1) are rigid enough to force the pullbacked-invariant (up to conjugation) operators (see (135)) to reduce to symmetric powers of an order-two linear differential operator.

The reduction of the solutions of this Schwarzian differential equation to *modular correspondences* was an open question in [1]. Modular correspondences require the existence, for any integer n, of solutions of the Schwarzian equation (136) of the form $y_n(x) = a_n \cdot x^n + \cdots$ such that, for any integer m and n, the following "pre-modular" condition is satisfied:

$$y_n(a_n, \; y_m(a_m, \; x)) \;\; = \;\; y_{nm}(a_n a_m^n, \; x). \qquad (137)$$

We derived in this paper a necessary and sufficient condition to obtain such "pre-modular" solutions for the "Schwarzian condition" (136). This condition turns out to be a simple condition on the Laurent series of $W(x)$ encoding the Schwarzian condition:

$$W(x) \;\; = \;\; -\frac{1}{2 \cdot x^2} \;\; + \frac{b}{x} \;\; + \sum_{m=0}^{\infty} a_m \cdot x^m. \qquad (138)$$

In light of what we have discussed so far, the current paper generates more questions than answers that give directions for further research. We have seen for example that (138) is a necessary and sufficient condition for obtaining "pre-modular" solutions for the "Schwarzian condition", corresponding, in general, to a *transcendental*‡ *declination of modular correspondences*. To have modular correspondences one needs the existence of *selected values* of the parameters such that the solution series $y_n(x) = a_n \cdot x^n + \cdots$ (see (96)) actually *reduce to algebraic functions*. Is it only in the case of modular correspondences that such algebraic reductions for selected values take place ?

Then we showed that an order-two linear differential operator emerging in the context of avoiding permutations counting [51, 52], provides a good illustration of a generalization of the pullback-covariance (134) or of the pullback invariance up to conjugation (135): the ${}_2F_1\Big([-1/4, 3/4], [1], x\Big)$ that comes up in the context of avoiding permutations counting [51, 52], verify a relation (see (F.9), (F.11)), whose general form is given by

$$\Phi(y(x)) \quad = \quad \Big(\mathcal{A}(x) \cdot \frac{d}{dx} \;\; + \mathcal{B}(x)\Big) \cdot \Phi(x), \qquad (139)$$

giving a non-trivial explicit example of a *pullback invariance of an operator up to operator homomorphisms* (see (125))

$$M_{N-1} \cdot L_N \quad = \quad pullback\Big(L_N, \, y(x)\Big) \cdot L_{N-1}. \qquad (140)$$

‡ The series $y_n(x)$ (see (137)) are *differentially algebraic*, but, *not necessarily algebraic functions*.

Equation (119) providing an exact representation of the *Landen* transformation (generator of the renormalization group) on $\tilde{\chi}^{(2)}$, together with the explicit calculations of section 7, make quite clear that conditions like (139) provide a natural and interesting generalization of *modular forms*, going beyond the Schwarzian equation (136).

At last, we examined the equivalence of two different linear differential operators, under pullback and conjugation, yielding again some Schwarzian condition relating these two linear differential operators (see relation (G.26)), and we discussed the consequence of such equivalence on the corresponding Yukawa couplings. These results revisiting and complementing the results of [17], provide powerful tools to analyze various symmetry and classification problems of selected linear differential operators, in particular linear differential operators of the Calabi-Yau type [18] (*not necessarily of order four* [31]).

When dealing with *linear* differential operators, we have seen the emergence of Schwarzian derivatives, consequence of the fact that *the Schwarzian derivative is appropriate for the composition of functions* [19] (see the chain rule of the Schwarzian derivative of the composition of function). Do *higher order Schwarzian derivatives* [55, 56, 57, 58] occur for pullback-symmetries of *non-linear* ODE's, or, more generally, for *functional* equations?

Restraining oneself to the univariate *linear* differential operators case, let us remark that if condition (134), or (135), describe effectively all the modular forms that often occur in physics [29, 30, 35], or enumerative combinatorics [37], a pullback symmetry up to conjugation constraint like (135) could be restrictive in some sense since it seems to yield systematic reduction¶ to order-two linear differential operators. In contrast the simple hypergeometric example of section 7 seems to provide a natural generalization of *modular forms*: the *pullback invariance of an operator up to operator homomorphisms* condition (140) promises to cover a larger ensemble of exact representations of symmetries in physics or enumerative combinatorics. In particular the emergence of conditions like (139) of higher order, namely generalized covariance (140) for the representation of the Landen transformation† on the other n-fold $\tilde{\chi}^{(n)}$'s of the Ising susceptibility (see [32, 33, 34]), together with their corresponding large order *reducible* linear differential operators, is a challenging open problem.

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Appendix A. A simple reducible linear differential operator of order four

Let us consider an order-four linear differential operator which is the square of an order-two linear differential operator: $L_4 = L_2 \cdot L_2$, where $L_2 = D_x^2 + p(x) \cdot D_x + q(x)$. This reducible order-four linear differential operator L_4 is of the form

¶ At least in the case where the operators verify Calabi-Yau conditions and thus have selected differential Galois groups.

† And more generally the modular correspondences providing exact representations of the generators of the renormalization group [2, 53].

$D_x^4 \, + \, p_r(x) \cdot \, D_x^3 \, + \, q_r(x) \cdot \, D_x^2 \, + \, \cdots$ where the two coefficients $p_r(x)$ and $q_r(x)$ read respectively:

$$p_r(x) \; = \; 2 \cdot \, p(x), \qquad\qquad q_r(x) \; = \; p(x)^2 \; + 2 \cdot \, q(x) \; + 2 \cdot \, {{dp(x)} \over {dx}}. \qquad\qquad (A.1)$$

The coefficients of the order-four operator L_4 *verify the Calabi-Yau condition*‡ (32). We even have the identity†† that the exterior square of $L_4 \, = \, L_2^2$ is the product of an order-one operator (having the wronskian of L_4 as solution), an order-three operator which is the symmetric square of L_2 and again the same order-one operator:

$$Ext^2(L_2 \cdot \, L_2) \; = \; (D_x \, + p(x)) \cdot \, Sym^2(L_2) \cdot \, (D_x \, + p(x)). \qquad\qquad (A.2)$$

For this reducible order-four linear differential operator $L_4 \, = \, L_2^2$ the first steps of the $L_4^{(p)} \, = \, L_4^{(c)}$ calculations give a function $W(x)$ given by (30), namely $W_r(x) \, = \, 3/10 \cdot \, p_r'(x) \, + 3/40 \cdot \, p_r(x)^2 \, - \, q_r(x)/5$. Using (A.1) one can rewrite $W_r(x)$ in terms of $p(x)$ and $q(x)$. One gets an expression similar to (10) but *different*, namely $W_r(x) \, = \, (p'(x) + p(x)^2/2 \, - 2 \, q(x))/5$, which is exactly (10) *but divided by* 5. Therefore the pullback condition on this square operator $L_4 \, = \, L_2^2$ *does not reduce* to the pullback condition on the (underlying) L_2.

The change of variable $x \, \rightarrow \, y(x)$ on a linear differential operator which is the product of two operators, is the product of these two linear differential operators on which this change of variable has been performed. More precisely *with our normalization of the pullback* of a linear differential operator a condition $L_4^{(p)} \, = \, L_4^{(c)}$ would give the relation

$$\begin{aligned} & {{1} \over {y'(x)^4}} \cdot \, pullback(L_2^2, y(x)) \; = \\ & \quad = \; \Bigl({{1} \over {y'(x)^2}} \cdot \, pullback(L_2, y(x))\Bigr) \cdot \, \Bigl({{1} \over {y'(x)^2}} \cdot \, pullback(L_2, y(x))\Bigr) \\ & \quad = \; \Bigl({{1} \over {y'(x)^2}} \cdot \, {{1} \over {v(x)}} \cdot \, L_2 \cdot \, v(x)\Bigr) \cdot \, \Bigl({{1} \over {y'(x)^2}} \cdot \, {{1} \over {v(x)}} \cdot \, L_2 \cdot \, v(x)\Bigr) \qquad\qquad (A.3) \\ & \quad = \; {{1} \over {y'(x)^4}} \cdot \, \Bigl({{1} \over {v(x)}} \cdot \, M_2 \cdot \, L_2 \cdot \, v(x)\Bigr) \qquad \hbox{where:} \quad M_2 \, = \, y'(x)^2 \cdot \, L_2 \cdot \, {{1} \over {y'(x)^2}}. \end{aligned}$$

In other words the pullback of $L_4 \, = \, L_2^2$ corresponds to a conjugate of *another* order-four linear differential operator $M_4 \, = \, M_2 \cdot \, L_2$, which is not L_4 but is also reducible into two different order-two linear differential operators. Note that the order-two linear differential operator M_2 depends on the change of variable $x \, \rightarrow \, y(x)$.

Appendix B. Order-five linear differential operators

Let us consider an *irreducible* order-five linear differential operator

$$L_5 \; = \; D_x^5 \, + \, p(x) \cdot \, D_x^4 \, + \, q(x) \cdot \, D_x^3 \, + \, r(x) \cdot \, D_x^2 \, + \, s(x) \cdot \, D_x \, + \, t(x), \qquad\qquad (B.1)$$

and let us also introduce two other linear differential operator of order five, the operator $L_5^{(c)}$ conjugated of (B.1) by a function $v(x)$, namely $L_5^{(c)} \, = \, 1/v(x) \cdot L_5 \cdot v(x)$,

‡ The *exterior square* of that an order-four operator $L_4 \, = \, L_2^2$ is of order *five* instead of order six. This is a general result: the order of the symmetric squares of operators $L_{2\,n} \, = \, L_2^n$ is less than $2\,n\,(2\,n \, - 1)/2$. Such n-th powers verify higher order Calabi-Yau conditions.

†† More generally, the order-one linear differential operator $D_x + p(x)$ rightdivises the exterior square of the n-th power of L_2, for any integer n.

and the pullbacked operator $L_5^{(p)}$ which amounts to changing $x \, \rightarrow \, y(x)$ in L_5. Imposing a generalized (symmetric) Calabi-Yau condition amounts to imposing that the symetric square of (B.1) is of order less than (the generic order) 15. Using this (symmetric) Calabi-Yau condition to perform any calculation is a very difficult task since this condition corresponds to a huge polynomial in the coefficients and their derivatives. However, similarly to what we did in section 4.3 we can introduce a parametrization, similar to (46) of this huge (symmetric) Calabi-Yau condition. We saw in [36] that the (symmetric) Calabi-Yau condition for an order-five linear differential operator L_5 (which amounts to saying that the symmetric square of L_5 is of order less than 15), amounts to saying that L_5 has the following decomposition

$$L_5 \,\,\, = \,\,\, (U_1 \cdot V_1 \cdot U_3 \, + \, U_1 \, + U_3) \cdot e(x), \qquad \text{(B.2)}$$

where U_1 and U_3 are order-one, order-one, and order-three *self-adjoint* linear differential operators of the form previously given with (44) and (45), and V_1 is another order-one self-adjoint operator:

$$V_1 \,\,\, = \,\,\, e(x) \cdot D_x \, + \, {{1} \over {2}} \cdot {{d\, e(x)} \over {dx}}, \qquad \text{(B.3)}$$

It is straightforward to get the coefficients of the order-five operator (B.1):

$$p(x) \,\, = \,\, {{7} \over {2}} \cdot {{a'(x)} \over {a(x)}} \, + {{1} \over {2}} \cdot {{c'(x)} \over {c(x)}} \, + 4 \cdot {{d'(x)} \over {d(x)}} \, + {{3} \over {2}} \cdot {{e'(x)} \over {e(x)}}, \qquad \cdots \qquad \text{(B.4)}$$

This gives a parametrization of the (symmetric) Calabi-Yau condition and thus a way to perform calculations for an order-five operator that verifies this huge (symmetric) Calabi-Yau condition. Again, one finds that just imposing this (symmetric) Calabi-Yau condition is not sufficient to have $L_5^{(p)} \, = \, L_5^{(c)}$.

There is one subcase of that huge polynomial condition that can be written explicitly (in a similar manner we wrote the Calabi-Yau (32) and symmetric Calabi-Yau (22) conditions, see (B.6), (B.7), (B.8) below).

Let us consider an order-two linear differential operator $L_2 = \, D_x^2 \, + A(x) \cdot D_x$ $+B(x)$, and the symmetric fourth power of L_2, the coefficients of that order-five operator read:

$$p(x) \, = \, 10 \cdot A(x), \qquad q(x) \,\, = \,\, 35 \cdot A(x)^2 \, + 20 \cdot B(x) \, + 10 \cdot {{dA(x)} \over {dx}},$$

$$r(x) \,\, = \,\, 50 \cdot A(x)^3 \, + 120 \cdot B(x) \cdot A(x) \, + 45 \cdot A(x) \cdot {{dA(x)} \over {dx}}$$
$$\qquad + 30 \cdot {{dB(x)} \over {dx}} \, + 5 \cdot {{d^2A(x)} \over {dx^2}}, \qquad \text{(B.5)}$$

$$s(x) \,\, = \,\, 24 \cdot A(x)^4 \, + 208 \cdot A(x)^2 \cdot B(x) \, + 46 \cdot A(x)^2 \cdot {{dA(x)} \over {dx}}$$
$$+ 120 \cdot {{dB(x)} \over {dx}} \cdot A(x) \, + 11 \cdot A(x) \cdot {{d^2A(x)} \over {dx^2}} \, + 64 \cdot B(x)^2$$
$$+ 56 \cdot B(x) \cdot {{dA(x)} \over {dx}} + 7 \cdot \Bigl({{dA(x)} \over {dx}}\Bigr)^2 \, + 18 \cdot {{d^2B(x)} \over {dx^2}} \, + {{d^3A(x)} \over {dx^3}},$$

$$t(x) \,\, = \,\, 96 \cdot A(x)^3 \cdot B(x) \, + 104 \cdot A(x)^2 \cdot {{dB(x)} \over {dx}} \, + 128 \cdot A(x) \cdot B(x)^2$$
$$+ 80 \cdot A(x) \cdot B(x) \cdot {{dA(x)} \over {dx}} \, + 36 \cdot {{d^2B(x)} \over {dx^2}} \cdot A(x) \, + 64 \cdot B(x) \cdot {{dB(x)} \over {dx}}$$
$$+ 8 \cdot B(x) \cdot {{d^2A(x)} \over {dx^2}} \, + 28 \cdot {{dB(x)} \over {dx}} \cdot {{dA(x)} \over {dx}} \, + 4 \cdot {{d^3B(x)} \over {dx^3}}.$$

Conversely this means $A(x) = p(x)/10$ and

$$r(x) = -\frac{4}{25}\cdot p(x)^3 - \frac{6}{5}\cdot p(x)\cdot \frac{dp(x)}{dx} + \frac{3}{5}\cdot p(x)\cdot q(x) - \frac{d^2p(x)}{dx^2} + \frac{3}{2}\cdot \frac{dq(x)}{dx}, \qquad (B.6)$$

$$s(x) = -\frac{9}{625}\cdot p(x)^4 - \frac{58}{125}\cdot p(x)^2\cdot \frac{dp(x)}{dx} - \frac{1}{125}\cdot p(x)^2\cdot q(x) - \frac{28}{25}\cdot p(x)\cdot \frac{d^2p(x)}{dx^2} + \frac{3}{5}\cdot p(x)\cdot \frac{dq(x)}{dx} - \frac{17}{25}\cdot \Big(\frac{dp(x)}{dx}\Big)^2 - \frac{1}{25}\cdot \frac{dp(x)}{dx}\cdot q(x) + \frac{4}{25}\cdot q(x)^2 - \frac{4}{5}\cdot \frac{d^3p(x)}{dx^3} + \frac{9}{10}\cdot \frac{d^2q(x)}{dx^2}, \qquad (B.7)$$

$$t(x) = -\frac{11}{25}\cdot \frac{dp(x)}{dx}\cdot \frac{d^2p(x)}{dx^2} - \frac{8}{25}\cdot p(x)\cdot \frac{d^3p(x)}{dx^3} + \frac{4}{625}\cdot p(x)^3\cdot \frac{dp(x)}{dx} - \frac{11}{625}\cdot p(x)^3\cdot q(x) - \frac{17}{125}\cdot p(x)^2\cdot \frac{d^2p(x)}{dx^2} - \frac{1}{250}\cdot p(x)^2\cdot \frac{dq(x)}{dx} - \frac{3}{25}\cdot p(x)\cdot \Big(\frac{dp(x)}{dx}\Big)^2 + \frac{4}{125}\cdot p(x)\cdot q(x)^2 + \frac{9}{50}\cdot p(x)\cdot \frac{d^2q(x)}{dx^2} - \frac{1}{50}\cdot \frac{dp(x)}{dx}\cdot \frac{dq(x)}{dx} - \frac{3}{25}\cdot q(x)\cdot \frac{d^2p(x)}{dx^2} + \frac{4}{25}\cdot q(x)\cdot \frac{dq(x)}{dx} - \frac{17}{125}\cdot p(x)\cdot q(x)\cdot \frac{dp(x)}{dx} - \frac{1}{5}\cdot \frac{d^4p(x)}{dx^4} + \frac{1}{5}\cdot \frac{d^3q(x)}{dx^3} + \frac{7}{3125}\cdot p(x)^5. \qquad (B.8)$$

When the three conditions (B.6), (B.7), (B.8) are verified, the symmetric square of the order-five linear differential operator L_5 is of order 9 instead of 15 (and thus its differential Galois group is $SO(5, \mathbb{C})$). The three conditions (B.6), (B.7), (B.8) are necessary for L_5 to be reducible to the symmetric cube of an underlying order-two linear differential operator. If one imposes the three conditions (B.6), (B.7), (B.8), the order-five linear differential operator is simply conjugated to its adjoint:

$$L_5 \cdot w(x)^{2/5} = w(x)^{2/5}\cdot adjoint(L_5), \qquad (B.9)$$

where $w(x)$ denotes the wronskian of L_5. Recalling that an order-two linear differential operator $L_2 = D_x^2 + A(x)\cdot D_x + B(x)$, having a wronskian $w_2(x)$ is such that $L_2 \cdot w_2(x) = w_2(x)\cdot adjoint(L_2)$, the identity (B.9) is a simple consequence of the fact that the order-five operator reduces to the symmetric fourth power of an order-two linear differential operator.

Note that by imposing the two conditions† (B.6), (B.7), the symmetric square of the order-five operator L_5 *becomes of the generic order* 15, yet the symmetric square of L_5 does not have a rational solution (the operator L_5 and its adjoint are not homomorphic: the differential Galois group of L_5 is not equal, or included, in the orthogonal group $SO(5, \mathbb{C})$).

The identification of these two order-four linear differential operators $L_5^{(p)}$ and $L_5^{(c)}$ gives four conditions $\mathcal{C}_n$, $n = 4, 3, 2, 1, 0$, corresponding respectively to identification of the D_x^n coefficients of $L_5^{(p)}$ and $L_5^{(c)}$.

Performing the same pullback-compatibility calculations we did for order-three, and order-four operators for L_5 is a tremendously difficult task in a general framework.

† We have the same result imposing the two conditions (B.6) and (B.8), or (B.7) and (B.8)

The first calculation steps can be performed, giving the exact expression of the conjugation function $v(x)$ from $\mathcal{C}_4$ as:

$$v(x) \quad = \quad y'(x)^{-2} \cdot \Bigl(\frac{w(x)}{w(y(x))} \Bigr)^{1/5}. \tag{B.10}$$

and, eliminating the log-derivative $v'(x)/v(x)$ between $\mathcal{C}_4$ and $\mathcal{C}_5$, giving the Schwarzian equation

$$W(x) \quad - W(y(x)) \cdot y'(x)^2 \quad + \{y(x), \, x\} \quad = \quad 0, \tag{B.11}$$

where, this time:

$$W(x) \quad = \quad \frac{1}{5} \cdot \frac{dp(x)}{dx} \quad + \frac{1}{25} \cdot p(x)^2 \quad - \frac{q(x)}{10}. \tag{B.12}$$

Again one finds that the expression of $W(x)$ given by (B.12) gives back the expression (10) when $p(x)$ and $q(x)$ are deduced from (B.5) ($p(x)$ and $q(x)$ becoming $A(x)$ and $B(x)$).

The condition that we called in [35] *symmetric Calabi-Yau condition* for the operator L_5 (corresponding to impose that its symmetric square is of order less than 15) is a huge polynomial condition on the coefficients of L_5 and its derivative. Seeing if these pullback-compatibility calculations yield necessarily the huge (symmetric Calabi-Yau‡) condition and the *three* conditions (B.6), (B.7), (B.8), or, in other words, that the order-five linear differential operator necessarily reduces again to (a symmetric fourth power of) an underlying order-two linear differential operator, remains an open question.

Appendix C. Reduction of the order-three ODE (68) to the order-two ODE (77) in the rank-two case (73).

The order-three linear differential equation (68) on $F(x)$ should reduce to the order-two linear ODE (77) in the rank-two subcase (73). When $A_R(x) \, = \, -w'(x)/w(x)$, the order-three linear differential operator $\mathcal{L}_3$ (see (69)) has three solutions:

$$\frac{1}{w(x)}, \qquad \mathcal{S}_F, \qquad \text{and:} \qquad w(x) \cdot \mathcal{S}_F^2. \tag{C.1}$$

This can be seen as a consequence of the fact that the order-two linear differential operator $\mathcal{L}_F$ rightdivides the order-three operator $\mathcal{L}_3$:

$$\mathcal{L}_3 \quad = \quad \Bigl(D_x \, + A_R(x)\Bigr) \cdot D_x \cdot \Bigl(D_x \, - A_R(x)\Bigr) \, = \, \Bigl(D_x \, + A_R(x)\Bigr) \cdot \mathcal{L}_F. \tag{C.2}$$

In this rank-two subcase (73), the function $F(x)$ is $\mathcal{S}_F$ *and not the third solution* $w(x) \cdot \mathcal{S}_F^2$ *which prevails in the general Schwarzian case* (see (93)). The form of the last solution $w(x) \cdot \mathcal{S}_F^2$ can be deduced from the fact that order-three linear differential operator $\mathcal{L}_3$ is the symmetric square of an order-two *self-adjoint* operator $\mathcal{L}_2$ (see (69)) which is simply conjugated to the order-two operator $\mathcal{L}_F$ given by (78):

$$\begin{aligned} \mathcal{L}_2 \quad &= \quad D_x^2 - \frac{W(x)}{2} \quad = \quad \Bigl(D_x \, + \frac{A_R(x)}{2}\Bigr) \cdot \Bigl(D_x \, - \frac{A_R(x)}{2}\Bigr) \\ &= \quad w(x)^{1/2} \cdot \mathcal{L}_F \cdot w(x)^{-1/2} \end{aligned} \tag{C.3}$$

which has clearly the solution $w(x)^{1/2} \cdot \mathcal{S}_F$ as well as the solution $w(x)^{1/2} \cdot w(x)^{-1}$ $= \, w(x)^{-1/2}$, deduced from the solutions of $\mathcal{L}_F$ (see (78)). The three solutions (C.1)

‡ Meaning that the order-five operator has a $SO(5, \, \mathbb{C})$ differential Galois group.

correspond to all the products of these two solutions namely the square of $w(x)^{-1/2}$ and $w(x)^{1/2} \cdot \mathcal{S}_F$, and their product. Note that the factorization (C.3) *requires condition* (73) *to be satisfied*.

Remark : Recalling (69), (78), (C.3), one can see that the rightdivision (C.2) can be seen as a consequence of the identity

$$Sym^2\Big(\mathcal{L}_2\Big) \, = \, Sym^2\Big(\Big(D_x \, + \frac{A_R(x)}{2}\Big) \cdot \Big(D_x \, - \frac{A_R(x)}{2}\Big)\Big)$$
$$= \, \Big(D_x \, + A_R(x)\Big) \cdot D_x \cdot \Big(D_x \, - A_R(x)\Big). \qquad \text{(C.4)}$$

Appendix D. Mirror maps for ${}_2F_1([1/12, 5/12], [1], x)$.

The modular correspondences $x \, \rightarrow \, y(x)$ are *infinite order* algebraic transformations such that

$${}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], y(x)\Big) \, = \, \mathcal{A}(x) \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], x\Big), \qquad \text{(D.1)}$$

where $\mathcal{A}(x)$ is an algebraic function. The modular correspondences $y(x)$ are solutions of the Schwarzian condition (90), where $W(x)$ simply related to the function $F(x)$ (see (68)) are given by equations (93). These modular correspondences have series expansion at $x \, = \, 0$ of the form

$$y_n(x) \, = \, P(Q^n(x)) \, = \, 1728 \cdot \Big(\frac{x}{1728}\Big)^n \, + \, \cdots \qquad n \, = \, 2, \, 3, \, 4, \, \cdots \qquad \text{(D.2)}$$

where $P(x)$ and $Q(x)$ are such that $P(Q(x)) \, = \, Q(P(x)) \, = \, x$, corresponding to the "simplest" examples of *mirror maps* [1]. More precisely, the well-known "mirror maps" [61, 62, 63, 64] are often described as series with *integer coefficients* [65, 66]. These series correspond to a rescaling of $P(x)$ and $Q(x)$ by 1728, namely [1]:

$$\frac{Q(1728 \cdot x)}{1728} \, = \, x \, + 744\, x^2 \, + 750420\, x^3 \, + 872769632\, x^4 \, + 1102652742882\, x^5 \, + \, \cdots$$

and:

$$\frac{P(1728 \cdot x)}{1728} \, = \, x \, - 744\, x^2 \, + 356652\, x^3 \, - 140361152\, x^4 \, + 49336682190\, x^5 \, + \, \cdots$$

The two functions $P(x)$ and $Q(x)$ are *differentially algebraic* [3, 4], but *not holonomic* functions. Introducing the function $Q(x) \, = \, \exp(\Theta(x))$, equation (70) with $\lambda \, = \, 0$ yields the following Schwarzian relations on $Q(x)$

$$W(x) \, + \, \{Q(x), \, x\} \, + \, \frac{1}{2} \cdot \Big(\frac{Q'(x)}{Q(x)}\Big)^2 \, = \, 0, \qquad \text{or:} \qquad \text{(D.3)}$$

$$W(x) \, + \, \{\ln(Q(x)), \, x\} \, = \, 0 \qquad \text{where:} \qquad \frac{Q'(x)}{Q(x)} \, = \, \frac{1}{F(x)}, \qquad \text{(D.4)}$$

when $P(x)$ the (composition) inverse of $Q(x)$ verifies the functional equation and Schwarzian equation:

$$x \cdot \frac{dP(x)}{dx} \, = \, F(P(x)), \qquad \{P(x), \, x\} \, - \frac{1}{2 \cdot x^2} \, - W(P(x)) \, = \, 0. \qquad \text{(D.5)}$$

Note that the one-parameter commuting family (66) solution of the Schwarzian equation (90), can be expressed using these two functions $P(x)$ and $Q(x)$ as $y_1(a_1, \, x) = P(a_1 \cdot Q(x))$ where $a_1 \, = \, \exp(\epsilon)$.

Appendix E. Selected subcase: Heun function examples.

Since the classification of Heun function is an interesting non trivial problem, let us use the condition (73), $W(x) \,=\, A'_R(x) \,+\, A_R(x)^2/2$, to find the Heun functions corresponding to such factorizations (like the example analysed in detail in [1]). The Heun function $HeunG\,(a,\, q,\, \alpha,\, \beta,\, \gamma,\, \delta,\, x)$ is solution of a linear differential operator of order two $L_2 \,=\, D_x^2 \,+ A(x)\cdot D_x \,+ B(x)$ where $A(x)$ and $B(x)$ read:

$$A(x) \;=\; \frac{(\alpha+\beta+1)\cdot x^2 \,-\, ((\delta+\gamma)\cdot a +\alpha -\delta +\beta +1)\cdot x \,+\gamma\cdot a}{x\cdot (x-1)\cdot (x\,-a)}, \tag{E.1}$$

$$B(x) \;=\; \frac{\alpha\,\beta\cdot x \,-q}{x\cdot (x-1)\cdot (x\,-a)}. \tag{E.2}$$

One thus simply deduces the corresponding function $W(x)$ function from the formula (10), namely $W(x) \,=\, A'(x) + A^2(x)/2 - 2\,B(x)$. At first sight we exclude the values $a \,=\, 0$ and $a \,=\, 1$ in order to have Heun functions with four singularities $0,\, 1,\, a,\, \infty$ to avoid trivial subcases where the Heun functions could reduce to ${}_2F_1$ hypergeometric functions. If one imposes that the function $W(x)$ is of the form (73), the rational function $A_R(x)$ must be of the form:

$$A_R(x) \;=\; \frac{u}{x\,-a} \;+\; \frac{v}{x} \;+\; \frac{w}{x\,-1}. \tag{E.3}$$

The identification of $W(x)$ given by (73) with A_R of the form (E.4), with $W(x) \,=\, A'(x) + A^2(x)/2 - 2\,B(x)$ where $A(x)$ and $B(x)$ are given by (E.1) and (E.2), gives a set of five equations in the parameters of the Heun function and in the three coefficients $u,\, v,\, w$ in (E.4), the simplest one being

$$a^2\,\cdot(\gamma\,-v)\cdot(\gamma\,-2+v) \;=\; 0. \tag{E.4}$$

The example analysed in [1] corresponding to the factorization condition (73) corresponds to the following values of these parameters:

$$a \,=\, M, \quad q \,=\, (M+1)/4, \quad \alpha = 1/2, \quad \beta = 1, \quad \gamma = 3/2, \quad \delta = 1/2, \tag{E.5}$$

$$\text{with:} \qquad u \,=\, 1/2, \quad v \,=\, 1/2, \quad w \,=\, 1/2, \tag{E.6}$$

which corresponds to the $\gamma\,-2+v \,=\, 0$ branch of (E.4). The analysis of these five equations gives four solutions that we have excluded at first sight because they corresponds to $a \,=\, 1$ and yield reduction to ${}_2F_1$ hypergeometric functions†, except when a bunch of conditions occur

$$\alpha -\gamma +1 \,=\, 0, \quad \beta -\delta -1 \,=\, 0, \quad \alpha -\delta -\gamma +2 \,=\, 0, \tag{E.7}$$

$$\text{and} \quad \alpha -\gamma -1 \,=\, 0, \quad \alpha -\delta -\gamma \,=\, 0, \quad \beta -\delta +1 \,=\, 0, \quad \cdots \tag{E.8}$$

The example analyzed in [1] corresponding to (E.5) is equivalent to

$$\alpha \,-\gamma \,+1 \;=\; 0. \tag{E.9}$$

Appendix F. Pullback invariance up to operator homomorphisms: a simple hypergeometric example.

Let us consider the order-two linear differential operator

$$\mathcal{L}_2 \;=\; D_x^2 \;+\; \frac{3\,x\,-2}{2\cdot x\cdot (x-1)}\cdot D_x \;-\; \frac{3}{16\cdot x\cdot (x-1)}, \tag{F.1}$$

† Like for instance $a \,=\, 1$, $q \,=\, \beta\cdot\gamma$, with $u \,=\, \alpha-\beta-\delta-\gamma+1$, $v \,=\, \gamma$, $w \,=\, \delta$.

which has the hypergeometric function solution ${}_2F_1([-1/4, 3/4], [1], x)$. We have the following homomorphism of the type (133) between $\mathcal{L}_2$ pullbacked by two simple different rational functions $p_1(x)$ and $p_2(x)$:

$$pullback\Big(\mathcal{L}_2,\; p_1(x)\Big) \cdot L_1 \cdot \alpha(x) \;\; = \;\; \alpha(x) \cdot M_1 \cdot pullback\Big(\mathcal{L}_2,\; p_2(x)\Big), \qquad \text{(F.2)}$$

$$\text{where:} \qquad p_1(x) \; = \; \frac{-\,64\,x}{(1-x)\cdot(1-9\,x)^3}, \qquad p_2(x) \; = \; \frac{-\,64\,x^3}{(1-x)^3\cdot(1-9\,x)}, \qquad \text{(F.3)}$$

$$\alpha(x) \; = \; x^3 \cdot \Big(\frac{1-x}{1-9\,x}\Big)^{1/2}, \qquad M_1 \; = \; 8 \cdot \frac{(1-9\,x)}{(1-x)\cdot x^2} \cdot D_x \; + \frac{171\,x^2 - 142\,x + 19}{(1-x)^2\cdot x^3},$$

$$\text{and:} \qquad L_1 \;\; = \;\; 8 \cdot \frac{(1-9\,x)}{(1-x)\cdot x^2} \cdot D_x \;\; - \frac{189\,x^2 - 226\,x + 21}{(1-x)^2\cdot x^3}. \qquad \text{(F.4)}$$

Denoting A and B the two rational pullbacks $p_1(x)$ and $p_2(x)$ in (F.2) one finds that they are related by the following rational algebraic curve:

$$\Gamma_3(A,\,B) \; = \; 4096 \cdot A\,B \cdot (A^2B^2 + 1) \; - 4608 \cdot A\,B \cdot (AB + 1) \cdot (A + B)$$
$$- \,(A^4 - 900\,A^3B + 28422\,A^2B^2 - 900\,AB^3 + B^4) \;\; = \;\; 0. \qquad \text{(F.5)}$$

The two *Hauptmoduls* parametrizing the *modular equation*‡ corresponding to the representation of $\tau \rightarrow 3\,\tau$, are given as follows:

$$P_1(x) \;\; = \;\; \frac{1728\;x}{(x+27)\cdot(x+3)^3}, \qquad P_2(x) \;\; = \;\; \frac{1728\;x^3}{(x+27)\cdot(x+243)^3}. \qquad \text{(F.6)}$$

Note that we have the following relations between $p_1(x)$ and $p_2(x)$, and the two Hauptmoduls $P_1(x)$ and $P_2(x)$:

$$p_1(x) \;\; = \;\; P_1(-27\,x), \qquad\qquad p_2(x) \;\; = \;\; P_2(-243\,x), \qquad \text{(F.7)}$$

which explain the compatibility between the two relations:

$$p_2(x) \;\; = \;\; p_1\Big(\frac{1}{9\,x}\Big), \qquad\qquad P_2(x) \;\; = \;\; P_1\Big(\frac{729}{x}\Big). \qquad \text{(F.8)}$$

Relation (F.2) yields the following identity on the ${}_2F_1$ hypergeometric function

$${}_2F_1\Big([-\frac{1}{4},\,\frac{3}{4}],\,[1],\,p_1(x)\Big) \;\;\; = \;\;\; \mathcal{L}_1\Big(\,{}_2F_1\Big([-\frac{1}{4},\,\frac{3}{4}],\,[1],\,p_2(x)\Big)\Big), \qquad \text{(F.9)}$$

$$\text{where:} \qquad \mathcal{L}_1 \;\; = \;\; \frac{8\cdot(1-9\,x)^{1/2}}{3\cdot(1-\,x)^{1/2}} \cdot x \cdot \frac{d}{dx} \;\; + \frac{1 - 3\,x - 45\,x^2 - 81\,x^3}{(1-x)^{3/2}\cdot(1-9\,x)^{3/2}}, \qquad \text{(F.10)}$$

$${}_2F_1\Big([-\frac{1}{4},\,\frac{3}{4}],\,[1],\,p_2(x)\Big) \;\;\; = \;\;\; \mathcal{L}_2\Big(\,{}_2F_1\Big([-\frac{1}{4},\,\frac{3}{4}],\,[1],\,p_1(x)\Big)\Big), \qquad \text{(F.11)}$$

$$\text{where:} \qquad \mathcal{L}_2 \;\; = \;\; -\,\frac{8\cdot(1-\,x)^{1/2}}{3\cdot(1-\,9\,x)^{1/2}} \cdot x \cdot \frac{d}{dx} \;\; + \frac{1 + 5\,x + 3\,x^2 - 9\,x^3}{(1-x)^{3/2}\cdot(1-9\,x)^{3/2}}. \qquad \text{(F.12)}$$

Introducing the order-two linear differential operator H_1 annihilating the pullbacked hypergeometric function ${}_2F_1([-1/4,\,3/4],\,[1],\,p_1(x))$:

$$H_1 \;\; = \;\; D_x^2 \;\; + \frac{(1-3\,x)^2}{x\cdot(1-x)\cdot(1-9\,x)} \cdot D_x \;\; + \frac{12}{x\cdot(1-x)^2\cdot(1-9\,x)^2}, \qquad \text{(F.13)}$$

the compatibility between relation (F.9) and (F.11) is a consequence of the identity

$$\mathcal{L}_1 \cdot \mathcal{L}_2 \;\;\; = \;\;\; 1 \;\; - \;\frac{64\,x^2}{9} \;\cdot\; H_1, \qquad \text{(F.14)}$$

‡ See equation (108) in subsection 5.1 of [1].

namely that the product $\mathcal{L}_1 \cdot \mathcal{L}_2$ is equal to 1 modulo H_1. Of course introducing the order-two linear differential operator H_2 annihilating the pullbacked hypergeometric function ${}_2F_1([-1/4,\, 3/4],\, [1],\, p_2(x))$ one also has a very similar identity:

$$\mathcal{L}_2 \cdot \mathcal{L}_1 \quad = \quad 1 \;\; - \; \frac{64\, x^2}{9} \;\cdot\; H_2, \tag{F.15}$$

which means that the product $\mathcal{L}_2 \cdot \mathcal{L}_1$ is equal to 1 modulo H_2.

One can get rid of the unpleasant square roots in (F.10), (F.12) introducing instead of the pullbacked hypergeometric functions ${}_2F_1([-1/4,\, 3/4],\, [1],\, p_2(x))$ and ${}_2F_1([-1/4,\, 3/4],\, [1],\, p_1(x))$, the functions

$$\Xi_2(x) \quad = \quad x \cdot (1\,-x)^{3/4} \cdot (1\,-9\,x)^{1/4} \cdot {}_2F_1\Big([-\frac{1}{4},\, \frac{3}{4}],\, [1],\, p_2(x)\Big), \tag{F.16}$$

$$\Xi_1(x) \quad = \quad 3^{7/2} \cdot \Xi_2\Big(\frac{1}{9\,x}\Big) \quad = \quad \frac{(1\,-x)^{1/4} \cdot (1\,-9\,x)^{3/4}}{x^2} \cdot {}_2F_1\Big([-\frac{1}{4},\, \frac{3}{4}],\, [1],\, p_1(x)\Big).$$

$\Xi_2(x)$ is a series with integer coefficients

$$\Xi_2(x) \quad = \quad x \;\; -3\,x^2 \;\; -6\,x^3 \;\; -22\,x^4 \;\; -108\,x^5 \;\; -612\,x^6 \;\; -3786\,x^7 \;\; -24858\,x^8$$
$$-\,170406\,x^9 \;\; -1207014\,x^{10} \;\; -8771850\,x^{11} \;\; + \;\; \cdots$$

when $\Xi_1(x)$ is a Laurent series with integer coefficients. These two functions are simply related as follows:

$$\Xi_1(x) \quad = \quad \mathcal{M}_1\Big(\Xi_2(x)\Big) \qquad\qquad \text{where:} \tag{F.17}$$

$$\mathcal{M}_1 \quad = \quad \frac{8}{3} \cdot \frac{(1\,-9\,x)}{x^2 \cdot (1\,-x)} \cdot D_x \;\; + \frac{117\,x\,-5}{3 \cdot x^3 \cdot (1\,-x)}. \tag{F.18}$$

In fact the function $\Xi_2(x)$ is solution of the order-two linear differential operator Ω_2

$$\Omega_2 \quad = \quad D_x^2 \;\; - \frac{(1\,-3\,x)}{x \cdot (1\,-x)} \cdot D_x \;\; + \frac{1\,-9\,x\,+36\,x^2}{x^2 \cdot (1\,-9\,x) \cdot (1\,-x)}, \tag{F.19}$$

with a remarkable duality property. *It is homomorphic to its pullback by* $x \,\rightarrow\, 1/9/x$:

$$pullback\Big(\Omega_2,\, \frac{1}{9\,x}\Big) \cdot \mathcal{M}_1 \quad = \quad \mathcal{N}_1 \cdot \Omega_2 \tag{F.20}$$

$$\text{where:} \qquad \mathcal{N}_1 \quad = \quad \frac{8 \cdot (1\,-9\,x)}{3 \cdot (1\,-x) \cdot x^2} \cdot D_x \;\; - \frac{27\,x\,-11}{3 \cdot (1\,-x) \cdot x^3}. \tag{F.21}$$

The simple relation (F.17), which is a rewriting of (F.9) with the order-one operator $\mathcal{L}_1$ being replaced by the order-one operator $\mathcal{M}_1$, is an obvious consequence of the homomorphism (F.20). Of course we also have the (mirror) relation‡, compatible with (F.17), which is a rewriting of (F.11) with the order-one operator $\mathcal{L}_2$ being replaced by the order-one operator $\mathcal{M}_2$

$$\Xi_2(x) \quad = \quad \mathcal{M}_2\Big(\Xi_1(x)\Big) \qquad\qquad \text{where:} \tag{F.22}$$

$$\mathcal{M}_2 \quad = \quad -\,\frac{8 \cdot (1\,-\,x) \cdot x^4}{3 \cdot (1\,-9\,x)} \cdot D_x \;\; + \frac{(5\,x\,-13) \cdot x^3}{3 \cdot (1\,-9\,x)}. \tag{F.23}$$

Note that $\mathcal{M}_1$ and $\mathcal{M}_2$ given by (F.18) and (F.23) are related by the involutive change of variable $x \,\rightarrow\, 1/9/x$:

$$\mathcal{M}_1 \quad = \quad 6561 \cdot pullback\Big(\mathcal{M}_2,\, \frac{1}{9\,x}\Big), \qquad 6561 \cdot \mathcal{M}_2 \quad = \quad pullback\Big(\mathcal{M}_1,\, \frac{1}{9\,x}\Big). \tag{F.24}$$

‡ Consequence of the (mirror) homomorphism relation: $\mathcal{N}_2 \cdot pullback((\Omega_2,\, 1/9/x) \;\; = \;\; \Omega_2 \cdot \mathcal{M}_2$.

Denoting Ω_1 the order-two operator annihilating Ξ_1, the compatibility between the relations (F.17) and (F.22) corresponds to the relations:

$$\mathcal{M}_1 \cdot \mathcal{M}_2 \, = \, 1 \, - \, \frac{64\, x^2}{9} \cdot \Omega_1, \qquad \mathcal{M}_2 \cdot \mathcal{M}_1 \, = \, 1 \, - \, \frac{64\, x^2}{9} \cdot \Omega_2, \qquad \text{(F.25)}$$

which should be compared with (F.15) and (F.41).

Relations (F.11), or† (F.22), can be seen as a particular case of a generalized pullback symmetry condition of the form

$$_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, y(x)\Big) \,\, = \,\, \Big(\mathcal{A}(x) \cdot \frac{d}{dx} \, + \mathcal{B}(x)\Big) \cdot \, _2F_1\Big([\alpha,\, \beta],\, [\gamma],\, x\Big), \qquad \text{(F.26)}$$

where $\mathcal{A}(x)$ and $\mathcal{B}(x)$ are algebraic functions. Identities like (F.9) can be seen as generalizations of the identities $_2F_1([\alpha,\, \beta],\, [\gamma],\, y(x)) \, = \, \mathcal{A}(x) \cdot \, _2F_1([\alpha,\, \beta],\, [\gamma],\, x)$ analysed in [1].

Appendix F.1. Representation of the composition of the algebraic transformations $x \, \rightarrow \, y(x)$.

We want to see the algebraic transformations $x \, \rightarrow \, y(x)$ as symmetries. In particular we want to have a representation of the composition of these algebraic transformations, like:

$$_2F_1\Big([\alpha,\, \beta],\, [\gamma],\, y(y(x))\Big) \,\, = \,\, \Big(\mathcal{A}_2(x) \cdot \frac{d}{dx} \, + \mathcal{B}_2(x)\Big) \cdot \, _2F_1\Big([\alpha,\, \beta],\, [\gamma],\, x\Big). \qquad \text{(F.27)}$$

Let us show here that by building on the previous example we can actually provide identities of the type (F.27). Introducing

$$q_1(x) \,\, = \,\, \frac{-1728 \cdot \, x \cdot \, (1 \, - 81\, x \, + 2187\, x^2)}{(1 \, - 81\, x)^9 \cdot \, (1 \, - 27\, x) \cdot \, (1 \, + 2187\, x^2)}, \qquad \text{(F.28)}$$

$$q_2(x) \,\, = \,\, q_1\Big(\frac{1}{2187\, x}\Big) \,\, = \,\, \frac{-1728 \cdot \, 3^{24} \cdot \, x^9 \cdot \, (1 \, - 81\, x \, + 2187\, x^2)}{(1 \, + 2187\, x^2) \cdot \, (1 \, - 27\, x)^9 \cdot \, (1 \, - 81\, x)}. \qquad \text{(F.29)}$$

one has the new pullback symmetry relation similar to (F.9):

$$_2F_1\Big([-\frac{1}{4},\, \frac{3}{4}],\, [1],\, q_1(x)\Big) \,\, = \,\, \hat{L}_1\Big(\, _2F_1\Big([-\frac{1}{4},\, \frac{3}{4}],\, [1],\, q_2(x)\Big)\Big), \qquad \text{(F.30)}$$

where:

$$\hat{L}_1 \,\, = \,\, \frac{32}{9} \cdot \frac{x \cdot \, (1 \, - 81\, x \, + 2187\, x^2) \cdot \, U_1(x)}{(1 \, - 81\, x) \cdot (1 \, - 27\, x)^5} \cdot \, D_x \, + \, \frac{V_1(x)}{(1 \, - 108\, x \, + 2187\, x^2) \cdot \, (1 \, - 81\, x) \cdot \, (1 \, - 27\, x)^5}, \qquad \text{(F.31)}$$

$$U_1(x) \,\, = \,\, 1 \, - 81\, x \, + 4374\, x^2 \, - 177147\, x^3 \, + 4782969\, x^4, \qquad \text{(F.32)}$$

$$V_1(x) \,\, = \,\, 1 \, - 26244\, x^2 \, + 3779136\, x^3 \, - 277412202\, x^4 \, + 12397455648\, x^5 \, - 311486073156\, x^6 \, + 3012581722464\, x^7 \, + 22876792454961\, x^8. \qquad \text{(F.33)}$$

One also has the new pullback symmetry relation similar to (F.11)

$$_2F_1\Big([-\frac{1}{4},\, \frac{3}{4}],\, [1],\, q_2(x)\Big) \,\, = \,\, \hat{L}_2\Big(\, _2F_1\Big([-\frac{1}{4},\, \frac{3}{4}],\, [1],\, q_1(x)\Big)\Big), \qquad \text{(F.34)}$$

† Or relations (F.9) or (F.17), but in that case the series corresponding to $y(x)$ are Puiseux series : $y(x) \, = \, x^{1/3} \, + \cdots$

$$\hat{L}_2 \; = \; -\,\frac{32}{9}\cdot \frac{x\cdot\,(1\,-81\,x\,+2187\,x^2)\cdot\, U_2(x)}{(1\,-81\,x)^5\cdot\,(1\,-27\,x)}\cdot\, D_x$$
$$+\,\frac{V_2(x)}{(1\,-108\,x\,+2187\,x^2)\cdot\,(1\,-81\,x)^5\cdot\,(1\,-27\,x)}, \qquad (F.35)$$

$$U_2(x) \; = \; 1\;\; -81\,x\;\; +4374\,x^2\;\; -177147\,x^3\;\; +4782969\,x^4, \qquad (F.36)$$

$$V_2(x) \; = \; 1\;\; +288\,x\;\; -65124\,x^2\;\; +5668704\,x^3\;\; -277412202\,x^4\;\; +8264970432\,x^5$$
$$-\,125524238436\,x^6\,+22876792454961\,x^8. \qquad (F.37)$$

Let us introduce the order-two linear differential operator $\hat{H}_1$ annihilating the pullbacked hypergeometric function ${}_2F_1([-1/4,\, 3/4],\, [1],\, q_1(x))$:

$$\hat{H}_1 \; = \; D_x^2\;\; +\,\frac{\alpha_1(x)}{(1\,-81\,x)\cdot\,(1\,-27\,x)\cdot\,(1\,+2187\,x^2)\cdot\,(1\,-81\,x+2187\,x^2)\cdot\, x}\cdot\, D_x$$
$$-\,\frac{324}{x\cdot\,(1\,-81\,x+2187\,x^2)\cdot\,(1\,+2187\,x^2)^2\cdot\,(1-81\,x)^2\cdot\,(1\,-27\,x)^2}, \qquad (F.38)$$

where

$$\alpha_1(x) \; = \; 1\;\; +2187\,x^2\;\; -354294\,x^3\;\; +23914845\,x^4\;\; -774840978\,x^5\;\; +10460353203\,x^6.$$

The compatibility between relation (F.9) and (F.11) is a consequence of the identity:

$$\hat{L}_1\cdot\,\hat{L}_2 \; = \; 1\;\; +R_{1,2}(x)\,\cdot\,\hat{H}_1, \qquad \text{where:} \qquad (F.39)$$

$$R_{1,2}(x) \; = \; -\,\frac{1024}{81}\cdot\,\frac{x^2\cdot\,(1\,-81\,x\,+2187\,x^2)^4\cdot\,(1\,+2187\,x^2)^2}{(1\,-81\,x)^6\cdot\,(1\,-27\,x)^6}. \qquad (F.40)$$

Of course introducing the order-two linear differential operator $\hat{H}_2$ annihilating the pullbacked hypergeometric function ${}_2F_1([-1/4,\, 3/4],\, [1],\, q_2(x))$, one also has a similar identity *with the same rational function* $R_{1,2}(x)$:

$$\hat{L}_2\cdot\,\hat{L}_1 \; = \; 1\;\; +R_{1,2}(x)\,\cdot\,\hat{H}_2. \qquad (F.41)$$

Again we have that $\hat{L}_1$ and $\hat{L}_2$ are obtained from each other by the (involutive) change of variable $x \longleftrightarrow 1/2187/x$:

$$-\,9\,\cdot\,\hat{L}_1 \; = \; pullback\Big(\hat{L}_2,\, \frac{1}{2187\,x}\Big), \qquad \hat{L}_2 \; = \; -9\,\cdot\, pullback\Big(\hat{L}_1,\, \frac{1}{2187\,x}\Big). \qquad (F.42)$$

Note that the two pullbacks $q_1(x)$ and $q_2(x)$ (see (F.28), (F.29)) are related to the two previous pullbacks $p_1(x)$ and $p_2(x)$ (see (F.3)):

$$q_1(x) \; = \; p_1\Big(27\cdot\, x\cdot\,(1\,-81\,x\,+2187\,x^2)\Big), \qquad (F.43)$$

$$q_2(x) \; = \; p_2\Big(\frac{19683\cdot\, x^3}{1\,-81\,x\,+2187\,x^2}\Big) \; = \; p_1\Big(\frac{1\,-81\,x\,+2187\,x^2}{177147\cdot\, x^3}\Big). \qquad (F.44)$$

Recalling $\Phi(x) \, = \, {}_2F_1([-1/4,\, 3/4],\, [1],\, p_1(x))$ the new identities (F.30) and (F.34) read

$$\Phi\Big(27\cdot\, x\cdot\,(1\,-81\,x\,+2187\,x^2)\Big) \; = \; \hat{L}_1\Big(\Phi\Big(\frac{1\,-81\,x\,+2187\,x^2}{177147\cdot\, x^3}\Big)\Big), \qquad (F.45)$$

$$\Phi\Big(\frac{1\,-81\,x\,+2187\,x^2}{177147\cdot\, x^3}\Big) \; = \; \hat{L}_2\Big(\Phi\Big(27\cdot\, x\cdot\,(1\,-81\,x\,+2187\,x^2)\Big)\Big), \qquad (F.46)$$

or, introducing $\Psi(x) \, = \, {}_2F_1([-1/4,\, 3/4],\, [1],\, q_1(x))$:

$$\Psi(x) \; = \; \hat{L}_1\Big(\Psi\Big(\frac{1}{2187\cdot\, x}\Big)\Big), \qquad \Psi\Big(\frac{1}{2187\cdot\, x}\Big) \; = \; \hat{L}_2\Big(\Psi(x)\Big). \qquad (F.47)$$

Denoting A and B the two pullbacks in (F.45), (F.46),

$$A \,=\, 27 \cdot x \cdot (1\,-81\,x\,+2187\,x^2), \qquad B \,=\, \frac{1\,-81\,x\,+2187\,x^2}{177147 \cdot x^3}, \qquad \text{(F.48)}$$

one sees that they are related by the simple A, B symmetric algebraic curve:

$$9\,A^3B^3\,\,-30\,A^2B^2\,\,+12\,AB \cdot (A+B)\,\,-A^2\,-AB\,-B^2 \,=\, 0. \qquad \text{(F.49)}$$

Let us consider the algebraic equation (F.5), that we denote $\Gamma_3(A,\, B) \,=\, 0$ because it is so closely related to the modular equation representing $\tau \,\rightarrow\, 3\,\tau$ (see their close relation with the Hauptmoduls (F.6) and (F.8)). Performing the resultant in B of the polynomial $\Gamma_3(A,\, B)$ with the same one $\Gamma_3(B,\, C)$ one gets a new algebraic equation $\Gamma_9(A,\, C) \,=\, 0$. The two pullbacks $q_1(x)$ and $q_2(x)$ *are actually a rational parametrization of that new algebraic equation* $\Gamma_9(A,\, C) \,=\, 0$. In other words, if we think identity (F.11) as a symmetry transformation identity of the type (F.26), the new identity (F.30) must be seen as the identity for the iteration of that transformation:

$${}_2F_1\Big([\alpha,\,\beta],\,[\gamma],\,y(y(x))\Big) \,\,=\,\, \Big(\mathcal{A}_2(x) \cdot \frac{d}{dx}\,\,+\mathcal{B}_2(x)\Big) \cdot {}_2F_1\Big([\alpha,\,\beta],\,[\gamma],\,x\Big). \qquad \text{(F.50)}$$

We are very close to a modular form, the previous algebraic curve (F.5) playing the role of the *modular equation*‡ (see (F.8)), and the algebraic curve $\Gamma_9(A,\, C) \,=\, 0$ playing the role of the modular equation corresponding to $\tau \rightarrow 9 \cdot \tau$.

Note that if one calculates the function $W(x) \,=\, A'(x) + A(x)^2/2\,-2\,B(x)$ corresponding to the order-two operator $\mathcal{L}_2$, one gets

$$W(x) \,\,=\,\, \frac{x\,-4}{8 \cdot (x-1) \cdot x} \,\,=\,\, -\frac{1}{2\,x^2}\,\,-\frac{7}{8\,x}\,\,-\frac{5}{4}\,\,-\frac{13}{8}\,x\,\,-2\,x^2\,\,+\,\cdots \qquad \text{(F.51)}$$

which is of the form $W(x) \,=\, -1/2/x^2\,+\,\cdots$ (in contrast with the result for $\tilde{\chi}^{(2)}$, see (124)).

Appendix G. Schwarzian conditions for different Calabi-Yau operators with related Yukawa couplings

Appendix G.1. Revisiting a Calabi-Yau operator in [17]

Following Almkvist, van Straten and Zudilin [17], let us consider the order-four linear differential operator L_4 such that its exterior square annihilates¶

$${}_5F_4\Big([\frac{1}{2},\,a,\,1-a,\,b,\,1-b],[1,\,1,\,1,\,1],\,x\Big). \qquad \text{(G.1)}$$

This order-four linear differential operator such that its exterior square is order-five (it verifies the Calabi-Yau condition (32)) reads

$$L_4 \,\,=\,\, D_x^4\,\,+P(x) \cdot D_x^3\,\,+Q(x) \cdot D_x^2\,\,+R(x) \cdot D_x\,\,+S(x), \qquad \text{(G.2)}$$

where $P(x)$ and $Q(x)$ read:

$$P(x) \,\,=\,\, \frac{4\,-5\,x}{x \cdot (1\,-x)},$$

$$Q(x) \,\,=\,\, \frac{(3\,x\,-2) \cdot (11\,x\,-10)}{8 \cdot x^2 \cdot (x-1)^2}\,\,+\,\frac{a \cdot (1-a)\,+b \cdot (1-b)}{2 \cdot x \cdot (x-1)}. \qquad \text{(G.3)}$$

‡ Given by equation (108) in subsection 5.1.1 in [1].
¶ See also [59].

The other rational functions $R(x)$ and $S(x)$ are more involved rational functions that will not be given here. The operator L_4 can be seen as the "exterior (or antisymmetric) square root†" of the order-five linear differential operator that annihilates the ${}_5F_4$ hypergeometric function (G.1).

Remark: In [17] the authors introduce a proxy of the exact "exterior square root" L_4 namely the so-called Yifan Yang pullback, given in general by the equations in the section "Definition" page 10 of [60]‡ and, in this example, by equations (3.11), page 278 in [17], which reads

$$M_4 \, = \, D_x^4 \, + P_{YY}(x) \cdot D_x^3 \, + Q_{YY}(x) \cdot D_x^2 \, + R_{YY}(x) \cdot D_x \, + S_{YY}(x), \qquad \text{(G.4)}$$

where $P_{YY}(x)$ and $Q_{YY}(x)$ read:

$$P_{YY}(x) \, = \, \frac{2 \cdot (3 - 5x)}{x \cdot (1-x)},$$
$$Q_{YY}(x) \, = \, \frac{99x^2 - 122x + 28}{4 \cdot x^2 \cdot (x-1)^2} \, + \frac{a \cdot (1-a) + b \cdot (1-b)}{2 \cdot x \cdot (x-1)}, \qquad \text{(G.5)}$$

the other rational functions $R_{YY}(x)$ and $S_{YY}(x)$ being more involved rational functions that will not be given here. The "Yifan Yang pullback" M_4 is related to the exact "exterior square root" L_4 by a simple conjugation $M_4 \cdot u(x) \, = \, u(x) \cdot L_4$, with $u(x) \, = \, x^{-1/2} \cdot (1-x)^{-3/4}$. In general one may prefer to introduce the Yifan Yang pullback defined page 10 and 11 of [60] instead of the exact "exterior square root", because the corresponding formulae are simpler. It does not make any difference however since the two operators are simply conjugated.

Let us consider the order-four linear differential operator $\mathcal{L}_4$ given on page 284 of [17] which annihilates the Hadamard product of two simple ${}_2F_1$ hypergeometric functions:

$$\Bigl(\frac{1}{1-x} \cdot {}_2F_1([a, 1-a], [1], x)\Bigr) \star \Bigl(\frac{1}{1-x} \cdot {}_2F_1([b, 1-b], [1], x)\Bigr). \qquad \text{(G.6)}$$

This order-four operator $\mathcal{L}_2$ reads

$$\mathcal{L}_4 \, = \, D_x^4 \, + \hat{P}(x) \cdot D_x^3 \, + \hat{Q}(x) \cdot D_x^2 \, + \hat{R}(x) \cdot D_x \, + \hat{S}(x), \qquad \text{(G.7)}$$

where:

$$\hat{P}(x) \, = \, 2 \, \frac{5x^2 + 4x - 3}{x \cdot (x+1)(x-1)},$$
$$\hat{Q}(x) \, = \, 2 \cdot \frac{a \cdot (1-a) + b \cdot (1-b)}{x \cdot (x-1)^2} \, + \frac{25x^4 + 40x^3 - 16x^2 - 32x + 7}{x^2 \cdot (x+1)^2 (x-1)^2}. \qquad \text{(G.8)}$$

Introducing the pullback $y(x)$ and the function $v(x)$

$$y(x) \, = \, \frac{-4 \cdot x}{(1-x)^2}, \qquad v(x) \, = \, \Bigl(\frac{x \cdot (1+x)}{1-x}\Bigr)^{1/2}, \qquad \text{(G.9)}$$

one has the relation

$$v(x) \cdot \mathcal{L}_4 \cdot \frac{1}{v(x)} \, = \, pullback\Bigl(L_4, \, \frac{-4x}{(1-x)^2}\Bigr). \qquad \text{(G.10)}$$

† See the concept of Yifan Yang pullback introduced in [60].
‡ The author of [60] has benefited from an unpublished result by Yifan Yang. Note that there is a misprint in [60] in the "Definition" of Yifan Yang pullback: on top of page 11, the term $b_3 \, b_4/25$ *should be replaced by* $b_3 \, b_4'/25$. With this correction the exact 'exterior square root" L_4 and the Yifan Yang pullback M_4 are related by a simple conjugation $M_4 \cdot u(x) \, = \, u(x) \cdot L_4$, where $3/10 \cdot b_4 \, = \, -u'(x)/u(x)$.

and one verifies that a Schwarzian equation (G.11) is actually verified for (G.5) and (G.8)

$$\hat{U}_R(x) \;\; - \, U_M(y(x)) \cdot \, y'(x)^2 \;\; + \, \{y(x), \, x\} \;\; = \;\; 0, \qquad \text{(G.11)}$$

with:

$$U_M(x) \;\; = \;\; -\frac{Q(x)}{5} \; + \; \frac{3}{40} \cdot \, P(x)^2 \; + \; \frac{3}{10} \cdot \, \frac{dP(x)}{dx}, \qquad \text{(G.12)}$$

$$\hat{U}_R(x) \;\; = \;\; -\frac{\hat{Q}(x)}{5} \; + \; \frac{3}{40} \cdot \, \hat{P}(x)^2 \; + \; \frac{3}{10} \cdot \, \frac{d\hat{P}(x)}{dx}. \qquad \text{(G.13)}$$

This Schwarzian equation (G.11), together with the definitions (G.12) and (G.13), are exactly the Schwarzian equation (6.5) together with definition (6.4), page 290 of [17].

Appendix G.1.1. Schwarzian conditions for Calabi-Yau operators and Yukawa couplings.

Let us calculate the series expansion of the nome and *Yukawa couplings* [31] of L_4 and $\mathcal{L}_2$. In order to perform the calculations for arbitrary values of a and b, let us introduce the same variables s and p as the one introduced by [17]:

$$s \; = \; a \cdot \, (1-a) \, + b \cdot \, (1-b), \qquad\qquad p \; = \; a \cdot \, b \cdot \, (1-a) \cdot \, (1-b). \qquad \text{(G.14)}$$

Considering the subcase $a = \, 3$ and $b = \, 5$, the nome of L_4 reads

$$\begin{aligned} q_x(L_4) \;\; = \;\; & x \; + \, (2\,p \, - s \, +1) \cdot \, \frac{x^2}{2} \\ & + \, (93\,p^2 - 98\,ps + 26\,s^2 + 112\,p - 60\,s + 40) \cdot \, \frac{x^3}{128} \\ & + \, (27748\,p^3 - 45289\,p^2 s + 24798\,ps^2 - 4554\,s^3 + 55759\,p^2 \\ & \qquad - 61734\,ps + 17190\,s^2 + 43848\,p - 24516\,s + 13608) \cdot \, \frac{x^4}{62208} \;\; + \; \cdots, \end{aligned} \qquad \text{(G.15)}$$

while the nome of $\mathcal{L}_4$ reads:

$$\begin{aligned} q_x(\mathcal{L}_4) \;\; = \;\; & -\frac{1}{4} \cdot \, q_x(L_4)\Big(\frac{-4 \cdot \, x}{(1-x)^2}\Big) \;\; = \;\; x \;\; - 2 \cdot \, (2\,p \, - s) \cdot \, x^2 \\ & + \, \Big((93\,p^2 \, - 98\,ps \, + 26\,s^2 \, - 16\,p \, + 4\,s\Big) \cdot \, \frac{x^3}{8} \\ & - \, \Big(27748\,p^3 \, - 45289\,p^2 s \, + 24798\,ps^2 \, - 4554\,s^3 \, + 9708\,ps \, - 12038\,p^2 \\ & \qquad - 1764\,s^2 \, + 1080\,p \, - 216\,s\Big) \cdot \, \frac{x^4}{972} \;\; + \; \cdots \end{aligned} \qquad \text{(G.16)}$$

The respective Yukawa couplings of L_4 and $\mathcal{L}_4$ read:

$$\begin{aligned} K_x(L_4) \;\; = \;\; & 1 \;\; - (5\,p + 1 - 2\,s) \cdot \, x \;\; + \Big(825\,p^2 \, - 638\,ps \, + 120\,s^2 \, + 244\,p \, - 80\,s\Big) \cdot \, \frac{x^2}{64} \\ & - \, \Big(119240\,p^3 \, - 133883\,p^2 s \, + 48642\,ps^2 \, - 5688\,s^3 \, - 20346\,ps \, + 35609\,p^2 \\ & \qquad + 2448\,s^2 \, - 3420\,p \, + 1728\,s\Big) \cdot \, \frac{x^3}{5184} \;\; + \; \cdots \end{aligned} \qquad \text{(G.17)}$$

$$K_x(\mathcal{L}_4) \;\; = \;\; K_x(L_4)\Big(\frac{-4 \cdot \, x}{(1-x)^2}\Big) \;\; = \;\; 1 \;\; + 4 \cdot \, (5\,p \, - 2\,s \, + 1) \cdot \, x$$

$$+ \Big(825\,p^2\,-638\,ps\,+120\,s^2\,+404\,p\,-144\,s\,+32\Big)\cdot\,\frac{x^2}{4}$$
$$+ \Big(119240\,p^3\,-133883\,p^2s\,+48642\,ps^2\,-5688\,s^3\,-72024\,ps\,+102434\,p^2$$
$$+12168\,s^2\,+21204\,p\,-6696\,s\,+972\Big)\cdot\,\frac{x^3}{81}\quad+\,\cdots \qquad (G.18)$$

In terms of the nome the Yukawa couplings read:

$$K_q(L_4) \;=\; 1\;-\,(5\,p\,-2\,s\,+1)\cdot\,q$$
$$+\,\Big(1145\,p^2\,-926\,ps\,+184\,s^2\,+468\,p\,-176\,s\,+32\Big)\cdot\,\frac{q^2}{64} \qquad (G.19)$$
$$-\,\Big(571795\,p^3\,-698524\,p^2s\,+280506\,ps^2\,-36972\,s^3\,+355447\,p^2$$
$$-\,273162\,ps\,+51390\,s^2\,+54072\,p\,-18900\,s\,+1944\Big)\cdot\,\frac{q^3}{10368}\quad+\,\cdots$$

and

$$K_q(\mathcal{L}_4) \;=\; K_q(L_4)(-4\cdot\,q) \;=\; 1\;+4\cdot\,(5\,p\,-2\,s\,+1)\cdot\,q$$
$$+\,\Big(1145\,p^2\,-926\,ps\,+184\,s^2\,+468\,p\,-176\,s\,+32\Big)\cdot\,\frac{q^2}{4} \qquad (G.20)$$
$$+\,\Big(571795\,p^3\,-698524\,p^2s\,+280506\,ps^2\,-36972\,s^3\,+355447\,p^2$$
$$-\,273162\,ps\,+51390\,s^2\,+54072\,p\,-18900\,s\,+1944\Big)\cdot\,\frac{q^3}{162}\quad+\,\cdots$$

On this example we see that the nome and Yukawa couplings expressed in terms of the x variable, are simply related (see (G.16), (G.18)) by the pullback transformation. The Yukawa couplings expressed in term of the nome of the two linear differential operators are related in an even more simple and "universal" way: $K_q(\mathcal{L}_4) \;=\; K_q(L_4)(-4\cdot\,q)$. This is a general result (see Appendix E of [31]). For a pullback $y(x)$ with a series expansion of the form

$$y(x) \;=\; \lambda\cdot x^n \;+\;\cdots, \qquad (G.21)$$

the nome and Yukawa couplings expressed in terms of the x variable of two order-four linear differential operators such that

$$v(x)\cdot\,\mathcal{L}_4\cdot\,\frac{1}{v(x)} \;=\; pullback\Big(L_4,\,y(x)\Big), \qquad (G.22)$$

are simply related as follows:

$$q_x(\mathcal{L}_4)^n \;=\; \frac{1}{\lambda}\cdot\,q_x(L_4)\Big(y(x)\Big), \qquad K_x(\mathcal{L}_4) \;=\; K_x(L_4)\Big(y(x)\Big). \qquad (G.23)$$

Their Yukawa couplings, expressed in terms of the nome, *are related in an even simpler "universal" way*:

$$K_q(\mathcal{L}_4) \;=\; K_q(L_4)(\lambda\cdot\,q^n). \qquad (G.24)$$

The previous example corresponded to the case $n\,=\,1$ and $\lambda\,=\,-4$. In the case $n\,=\,1$ and $\lambda\,=\,1$, the pullback is a deformation of the identity $y(x)\,=\,x+\,\cdots$ and the Yukawa couplings expressed in terms of the nome of the two operators are equal. One thus recovers Proposition (6.2) of [17] where the Yukawa couplings coincide.

Appendix G.2. Schwarzian conditions for Calabi-Yau operators related by pullback and conjugation.

In fact the Schwarzian condition (G.11) can be obtained in a *totally general framework* where two order-four linear differential operators are equal up to pullback and conjugation. Let us consider two order-four operators L_4 and M_4 such that

$$v(x) \cdot M_4 \cdot \frac{1}{v(x)} \quad = \quad pullback\Big(L_4, \, y(x)\Big). \qquad \text{(G.25)}$$

A straightforward calculation similar to the one performed in section 4 yields the Schwarzian relation‡

$$W(M_4, \, x) \; - \; W(L_4, \, y(x)) \cdot y'(x)^2 \; + \{y(x), \, x\} \; = \; 0, \qquad \text{(G.26)}$$

where the $W(M_4, \, x)$ and $W(L_4, \, x)$ are given by (30), the $p(x)$ and $q(x)$ being the ones of the corresponding operators M_4 and L_4:

$$W(M_4, \, x) \; = \; \frac{3}{10} \cdot \frac{dp(M_4, \, x)}{dx} \; + \; \frac{3}{40} \cdot p(M_4, \, x)^2 \; - \frac{q(M_4, \, x)}{5}, \qquad \text{(G.27)}$$

$$W(L_4, \, x) \; = \; \frac{3}{10} \cdot \frac{dp(L_4, \, x)}{dx} \; + \; \frac{3}{40} \cdot p(L_4, \, x)^2 \; - \frac{q(L_4, \, x)}{5}. \qquad \text{(G.28)}$$

Remark 1: There is nothing specific with order-four linear differential operators, one has the same result for two operators of *arbitrary orders* N equal up to pullback and conjugation (see (G.25)): the expressions of $W(M_N, \, x)$ and $W(L_N, \, x)$ being the ones given in (56), (57). One also has

$$W(M_N, \, x) \; - \; W(L_N, \, y(x)) \cdot y'(x)^2 \; + \{y(x), \, x\} \; = \; 0. \qquad \text{(G.29)}$$

Remark 2: The expressions of $W(M_N, \, x)$ and $W(L_N, \, x)$ are related by (G.29). Let us assume that $W(L_N, \, x)$ is compatible with the modular correspondences structures (existence of solutions of the Schwarzian equations of the form $y(x) = a_n \cdot x^n + \, \cdots$ with (96)). One thus has $W(L_N, \, x) \, = \, -1/2/x^2 + \, \cdots$ Is this condition automatically satisfied for $W(M_N, \, x)$ as a consequence of (G.29) ? For pullbacks of the form $y(x) \, = \, a_n \cdot x^n + \, \cdots$, the function $W(M_N, \, x)$ deduced from (G.29), reads:

$$W(M_N, \, x) \; = \; W(L_N, \, y(x)) \cdot y'(x)^2 \; - \, \{y(x), \, x\}$$
$$= \; \Big(-\frac{n^2}{2\,x^2} \, + \cdots\Big) \; + \; \Big(\frac{n^2 \, - 1}{2\,x^2} \, + \cdots\Big) \; = \; -\frac{1}{2\,x^2} \; + \cdots \qquad \text{(G.30)}$$

The condition (97) *for the modular correspondences structures is thus preserved by pullbacks.*

Appendix G.3. More general framework

For arbitrary orders we observed that the functions $W(x)$ that occur in the Schwarzian conditions are left invariant under conjugations of the operators (64) and (65). More generally, one can consider operators that are not conjugated by a function $\rho(x)$, yet homomorphic, in the sense of the equivalence of operators†. For a given operator L_N of order-N, one can easily obtain operators $\tilde{L}_N$ homomorphic to L_N. For instance, for an order-two linear differential operator $L_2 = \, D_x^2 + A(x)\, D_x + B(x)$, introducing

‡ This result is the same as the one in [17].

† Two linear differential operators L_N and $\tilde{L}_N$ of order N are homomorphic [35, 36] when there exists operators (intertwiners) of order at most $N-1$, such that $M_{N-1}\, L_N - \tilde{L}_N \, \tilde{M}_{N-1} = 0$.

the order-one operator $L_1 = \eta(x)\, D_x + \rho(x)$, an order-two operator $\tilde{L}_2$ homomorphic to L_2 is easily obtained performing†† the rightdivision by L_1 of the LCLM of L_2 and L_1. If one now compares the functions $W(x)$ corresponding respectively to L_2 and $\tilde{L}_2$, one sees that they are *quite different*, except when $\eta(x) = 0$, in which case one reduces the operator equivalence to a conjugation by a function $\rho(x)$. The analysis of the conditions for two order-N operators L_N and M_N to be homorphic up to pullback

$$M_{N-1} \cdot M_N \;\; = \;\; pullback\Big(L_N,\, y(x)\Big) \cdot L_{N-1}, \qquad \text{(G.31)}$$

is a much more general problem corresponding to massive calculations even if one restricts to operators that are homomorphic to their adjoint (thus corresponding to selected, orthogonal or symplectic, differential Galois groups¶). Performing such calculations will require new tools and ideas. This cannot be performed in general (like we did in the first section of this paper) but could be considered on particular problems emerging from physics or enumerative combinatorics, where the operators will be of some "selected" form.

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†† In Maple just to rightdivision(LCLM(L_2, L_1), L_1).

¶ In that general framework (G.31), we do not have the Calabi-Yau, or symmetric Calabi-Yau, equations that help us to perform our calculations.

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