Modular correspondences and replicable functions

J.-M. Maillard[£]

[£] LPTMC, UMR 7600 CNRS, Sorbonne Université, Tour 23, 5ème étage, case 121, 4 Place Jussieu, 75252 Paris Cedex 05, France

E-mail: `maillard@lptmc.jussieu.fr`

Abstract.
Landen transformation, and more generally modular correspondences, can be seen to be exact symmetries of some Yang-Baxter integrable lattice models, like the square Ising model, or the Baxter model. They are solutions of remarkable Schwarzian equations and have some compositional properties. Most of the known examples correspond, in an elliptic curves framework, to an automorphy property of pullbacked $_2F_1$ hypergeometric functions, associated with modular forms. It is, however, important to underline that these Schwarzian equations go beyond an elliptic curves, and hypergeometric functions framework. The question of a modular correspondence interpretation of the solutions of these "Schwarzian" equations was clearly an open question. This paper tries to shed some light on this open question. We first shed some light on the very nature of a one-parameter series solution of the Schwarzian equation. This one-parameter series is not generically a modular correspondence series, but it actually reduces to an infinite set of modular correspondence series for an infinite set of (N-th root of unity) values of the parameter. We also provide an example of two-parameter series, with a compositional property, solution of a Schwarzian equation. We finally provide simple pedagogical examples that are very similar to modular correspondence series, but are far beyond the elliptic curves framework. These last examples show that the modular correspondence-like series, or the nome-like series, are not necessarily globally bounded. The results of this paper can be seen as an incentive to study differentially algebraic series with integer coefficients, in physics or enumeratice combinatorics.

PACS: 05.50.+q, 05.90.+m, 05.10.-a, 02.30.Hq, 02.30.lk,02.30.Gp, 02.40.Xx
AMS Classification scheme numbers: 34M55, 47Exx, 32Hxx, 32Nxx, 34Lxx, 34Mxx, 14Kxx, 14H52

Key-words: Replicable functions, modular equations, modular correspondences, modular curves, modular forms, mirror maps, Schwarzian derivative, Fuchsian linear differential equations, hypergeometric functions, Square Ising model, globally nilpotent linear differential operators, Hauptmoduls, elliptic functions, Shimura curves, automorphic forms, renormalization group, Malgrange pseudo-group.

21 May 2025

1. Introduction: infinite order symmetries.

There is no need to underline the crucial role played by the concept of symmetry in physics, theoretical physics, mathematical physics. We will not consider here continuous symmetry groups (Lie groups) but rather *discrete* symmetries. Examples of such *discrete symmetries* are, for instance, *birational transformations* [1, 2], which are known to be (infinite order) discrete symmetries of (Yang-Baxter) integrable models [3, 4]. Such discrete symmetries can be studied, *per se*, in a discrete dynamical perspective‡.

The simplest example of such discrete symmetries corresponds to a (univariate) transformation $x \rightarrow y(x)$ preserving some structures†. These structures must be *invariant*, or *covariant*, under the previous transformations $x \rightarrow y(x)$. The simplest example of "structure" is certainely just a function. Let us consider a function $\Phi(x)$, let us discard the (too simple) invariance situation, where we have a functional equation $\Phi(y(x)) = \Phi(x)$, and let us consider the following "covariance" property for a function $\Phi(x)$

$$\Phi\Big(y(x)\Big) \quad = \quad \mathcal{A}(x) \cdot \, \Phi(x), \qquad (1)$$

where the "automorphy" cofactor $\mathcal{A}(x)$ can be described in terms of the symmetry transformation $y(x)$. From a mathematical view-point such an "automorphy property" (1) is reminiscent of the theory of *automorphic forms* [10, 11, 12, 13] (which can be generalized to Hilbert modular forms for two, or more, variables), which generalizes the theory of *modular forms* [14, 15, 16, 17, 18, 19]. In the case where $y(x)$ is not only a rational function, but a linear fractional transformation, the "covariance" property (1) can be illustrated by the Poincaré series [20, 21, 22], and other Theta-Fuchsian functions or series [13, 23, 24, 25]. From a physics view-point such an "automorphy property" (1) is reminiscent of the renormalization group theory, revisited by Wilson [26, 27, 28], seen as a fundamental symmetry in lattice statistical mechanics or field theory.

In the following we will not restrict the transformation symmetry $y(x)$ to be a linear fractional transformation: the function $y(x)$ is a series, analytic at $x = 0$, it can be a rational function, an algebraic function, a D-finite function, a D-D-finite function¶, a *differentially algebraic* function††, ...

To be more specific, let us give a simple, but highly pedagogical, illustration of a "covariance" property (1), which corresponds to $\Phi(x)$ being a selected $_2F_1$ hypergeometric function [30, 31]

$${}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, y(x)\Big) \quad = \quad \mathcal{A}(x) \cdot \, {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, x\Big), \qquad (2)$$

where the "automorphic prefactor" $\mathcal{A}(x)$ reads

$$\mathcal{A}(x) \; = \; \lambda \cdot \, \Big(\frac{u(x)}{u(y(x))} \cdot \, y'(x)\Big)^{1/2}, \qquad (3)$$

and where $u(x)$ is related [30, 31] to the wronskian of the order-two linear differential operator annihilating $\Phi(x)$, namely the $_2F_1$ hypergeometric function $_2F_1([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, x)$.

‡ One can recall that the theory of iteration of rational functions was seen, in the pioneering work of Julia, Fatou and Ritt, as a method for investigating functional equations [5, 6, 7, 8]

† These structures can be linear (or non-linear) differential equations, systems of partial differential equations [9], functional equations, etc ...

¶ A D-D-finite function is a function solution of a linear differentiable operator with D-finite function coefficients [29].

†† A differentially algebraic function is a function solution of a non-linear differential equation of the form $P(x, \, y, \, y', \, y'', \, \cdots \, y^{(n)}) \, = \, 0$, where P is a polynomial.

1.1. Modular forms, correspondences and physics.

The simplest example of a transformation $x \rightarrow y = y(x)$ occurring in the "automorphy" relation (2), or occurring as an *exact generator* of the *renormalization group* of the square Ising model, or even of the Baxter model [32], corresponds to the *Landen transformation* [32, 33]

$$k \quad \longrightarrow \quad k_L \, = \, \frac{2\sqrt{k}}{1+k}, \tag{4}$$

or to its compositional inverse, the inverse Landen transformation. As it should, the *critical point* of the square Ising model (resp. Baxter model) is a *fixed point* [32] of the Landen transformation: $k = 1$.

Let us introduce the j-invariant♯ of an elliptic curve of modulus k, and its transform by the Landen transformation (4)

$$j(k) \, = \, 256 \cdot \frac{(1-k^2+k^4)^3}{k^4 \cdot (1-k^2)^2}, \qquad j(k_L) \, = \, 16 \cdot \frac{(1+14\,k^2+k^4)^3}{(1-k^2)^4 \cdot k^2}, \tag{5}$$

and let us also introduce the two corresponding *Hauptmoduls* [32]:

$$x \, = \, \frac{1728}{j(k)}, \qquad y \, = \, \frac{1728}{j(k_L)}. \tag{6}$$

These two Hauptmodul variables (6) are related by the *modular equation* [35, 36, 37, 38, 39, 40]:

$$\begin{aligned} &1953125\, x^3y^3 \, - 187500\, x^2y^2 \cdot (x+y) \, + 375\, xy \cdot (16\,x^2 - 4027\,xy + 16\,y^2) \\ &\quad - 64\, (x+y) \cdot (x^2 + 1487\,xy + y^2) \, + 110592\,xy \, = \, 0. \end{aligned} \tag{7}$$

The algebraic function $y = y(x)$, defined from the modular curve (7), is a *multivalued function*, but we can, for instance, single out the (algebraic) series expansion††:

$$y \, = \, \frac{1}{1728} \cdot x^2 \, + \frac{31}{62208} \cdot x^3 \, + \frac{1337}{3359232} \cdot x^4 \, + \frac{349115}{1088391168} \cdot x^5 \, + \, \cdots \tag{8}$$

1.2. Schwarzian condition

More generally, the Gauss hypergeometric function ${}_2F_1([\alpha, \beta], [\gamma], x)$ is solution of the second order linear differential operator:

$$\Omega \, = \, D_x^2 \, + A(x) \cdot D_x \, + B(x), \qquad \text{where:} \tag{9}$$

$$A(x) \, = \, \frac{(\alpha+\beta+1) \cdot x \, - \gamma}{x \cdot (x-1)} \, = \, \frac{u'(x)}{u(x)}, \qquad B(x) \, = \, \frac{\alpha\,\beta}{x \cdot (x-1)}.$$

An automorphy relation, like (2) but on ${}_2F_1([\alpha, \beta], [\gamma], x)$, amounts to saying that the second order linear differential operator (9), pullbacked by $x \rightarrow y(x)$, reduces to the conjugate of the linear differential operator (9). Let us assume that the pullback $y(x)$ is an algebraic series like in (8). A straightforward calculation† allows to find relation (3) giving the cofactor $\mathcal{A}(x)$ in terms of the pullback $y(x)$. Eliminating the cofactor $\mathcal{A}(x)$, the identification of these two linear differential operators thus corresponds to

♯ The j-invariant [32, 34] (see also Klein's modular invariant) regarded as a function of a complex variable τ (the ratio of periods), is a modular function of weight zero for $SL(2, \mathbb{Z})$.

†† This series (8) has a radius of convergence 1.

† Relation (3) for $\mathcal{A}(x)$ amounts to imposing [31] that the two order-two linear differential operators have the same D_x coefficient. If the pullback $y(x)$ is an algebraic series, like in (8), the cofactor $\mathcal{A}(x)$ will be an algebraic function.

just one (non-linear) condition that can be rewritten (after some algebra ...) in the following *Schwarzian* form:

$$W(x) \quad - W(y(x)) \cdot y'(x)^2 \quad + \{y(x),\, x\} \quad = \quad 0, \qquad (10)$$

where

$$W(x) \;=\; A'(x) \;+\; \frac{A(x)^2}{2} \;\; - 2 \cdot B(x), \qquad (11)$$

and where $\{y(x),\, x\}$ denotes the *Schwarzian derivative* [41]:

$$\{y(x),\, x\} \;=\; \frac{y'''(x)}{y'(x)} \;-\; \frac{3}{2} \cdot \Big(\frac{y''(x)}{y'(x)}\Big)^2 \;=\; \frac{d}{dx}\Big(\frac{y''(x)}{y'(x)}\Big) \;-\; \frac{1}{2} \cdot \Big(\frac{y''(x)}{y'(x)}\Big)^2. \qquad (12)$$

For ${}_2F_1\Big([\frac{1}{12},\, \frac{5}{12}],\, [1],\, x\Big)$, the "automorphy" condition (2) yields the Schwarzian condition (10) with:

$$W(x) \;=\; -\frac{32\,x^2 - 41\,x + 36}{72 \cdot x^2 \cdot (x\, - 1)^2}. \qquad (13)$$

In previous papers [30, 31] we have seen that all the algebraic series (like (8)), associated with different modular equations (like (7)), are solutions of the *same* Schwarzian condition (10) with $W(x)$ given by (13). These *modular correspondences* series (8), associated with *modular curves*, are thus *algebraic series*. We are going to revisit these results, with a different normalization of the pullbacks, in a forthcoming section (2).

1.3. One-parameter solution series of the Schwarzian condition (10)

Trying to generalize the modular equation (7), and its associated algebraic series (8), let us try to find the series of the form $a \cdot x^2 + \cdots$, solutions of the Schwarzian equation (10) with $W(x)$ given by (13). It is straightforward to find that such series is, in fact, the following *one-parameter* series:

$$y_2 \;=\; a \cdot x^2 \;+ \frac{31 \cdot a x^3}{36} \;-\; \frac{a \cdot (5952\,a - 9511)}{13824} \cdot x^4 \;\; + \; \cdots \qquad (14)$$

which actually reduces to (8) for $a \,=\, 1/1728$.

Remark 1.1: Generically the series (14) is differentially *algebraic series* (being solution of a Schwarzian condition (10), with $W(x)$ given by (13)). For *selected* values of the parameter, like $a \,=\, 1/1728$, the series becomes an *algebraic series* (actually a *correspondence* associated with a *modular curve*). Are there other selected values of the parameters for which the series becomes an algebraic series ? Are there selected values of the parameter for which the series (14) becomes a (non algebraic) D-finite series ? Are there selected values of the parameter for which the series become D-D-finite† series [42, 43] ?

1.4. The nome and mirror maps

† D-finite functions are solutions of linear differential operators with polynomial coefficients, D-D-finite series are solutions of linear differential operators with D-finite function coefficients, etc ...

Let us recall the concept of *mirror map* [46, 47, 48, 49, 50, 51, 52] relating the reciprocal of the j-function and the nome, with the well-known series with *integer* coefficients† :

$$\tilde{X}(q) \, = \, q \, - 744\, q^2 \, + 356652\, q^3 \, - 140361152\, q^4 \, + 49336682190\, q^5 \, - 16114625669088\, q^6 \, + 4999042477430456\, q^7 \, + \, \cdots \tag{15}$$

and the *nome* which is its *compositional inverse*:

$$\tilde{Q}(x) \, = \, x \, + 744\, x^2 \, + 750420\, x^3 \, + 872769632\, x^4 \, + 1102652742882\, x^5 \, + 1470561136292880\, x^6 \, + 2037518752496883080\, x^7 \, + \, \cdots \tag{16}$$

The series (15) corresponds to x being the reciprocal of the j-function: $1/j$. As a consequence of the (modular form) hypergeometric identities (2) (see (6)), we need x to be identified with the *Hauptmodul* $1728/j$.

The nome series (16) and the mirror map series (15), are, respectively, solutions of the following Schwarzian equations

$$\{\tilde{Q}(x), \, x\} \, + {{1} \over {2 \cdot \, \tilde{Q}(x)^2}} \cdot \Bigl({{d\tilde{Q}(x)} \over {dx}}\Bigr)^2 \, + W(x) \, = \, 0, \tag{17}$$

and

$$\{\tilde{X}(x), \, x\} \, - {{1} \over {2 \cdot \, x^2}} \, - W\Bigl(\tilde{X}(x)\Bigr) \cdot \Bigl({{d\tilde{X}(x)} \over {dx}}\Bigr)^2 \, = \, 0, \tag{18}$$

where‡:

$$W(x) \, = \, -{{1} \over {2}} \cdot {{1 \, - 1968\, x \, + 2654208\, x^2} \over {x^2 \cdot \, (1 \, - 1728\, x)^2}}. \tag{19}$$

Let us introduce

$$X(q) \, = \, 1728 \cdot \, \tilde{X}(q), \qquad Q(x) \, = \, \tilde{Q}\Bigl({{x} \over {1728}}\Bigr). \tag{20}$$

The one-parameter series (14) is actually of the form $X\Bigl(a \cdot Q(x)^2\Bigr)$. More generally, all the series

$$y_n(a, \, x) \, = \, X\Bigl(a \cdot Q(x)^n\Bigr), \tag{21}$$

are solutions of the Schwarzian condition (10) with (13). For the selected values $a \, = \, 1/1728^{n-1}$ these series (21) turn out to be *algebraic series*: they are series actually associated with *correspondences*, *modular curves*. The composition of two such series is also solution of the Schwarzian condition (10). One easily finds that

$$y_n\Bigl(a, \, y_m(b, \, x)\Bigr) \, = \, y_{m\,n}(a \cdot \, b^n, \, x) \, = \, a \cdot \, b^n \cdot \, x^{m\,n} \, + \, \cdots$$

$$y_m\Bigl(b, \, y_n(a, \, x)\Bigr) \, = \, y_{m\,n}(b \cdot \, a^m, \, x) \, = \, b \cdot \, a^m \cdot \, x^{m\,n} \, + \, \cdots \tag{22}$$

Generically the two series y_n and y_m *do not commute.*

Remark 1.2: Note that if one assumes that the parameters a (resp. b) are of the form ρ^{n-1} (resp. ρ^{m-1}) with ρ different from $1/1728$ or 1, the series $y_n(a, \, x)$ and $y_m(b, \, x)$ *commute*¶, even if they are *not* algebraic series but *only differentially*

† In Maple the series (15) can be obtained substituting $L \, = \, EllipticModulus(q^{1/2})^2$, in $1/j \, = \, L^2 \cdot \, (L \, - 1)^2/(L^2 \, - L \, + 1)^3/256$. See https://oeis.org/A066395 for the series (15) and https://oeis.org/A091406 for the series (16).

‡ Note that (19) is nothing but (13) where x has been changed into $1728\, x$.

¶ In terms of the nome, this amounts to noticing that transformations $q \, \rightarrow \, \alpha^{n-1} \cdot \, q^n$ and $q \rightarrow \alpha^{m-1} \cdot q^m$ commute.

algebraic series. The compositional identities (22) are inherited from the fact that the composition of two algebraic series is an algebraic series, and that the *composition of two solutions of the Schwarzian condition* (10) *must*† *also be a solution of the Schwarzian condition* (10). Such properties are reminiscent of the concept of *replicable* functions [53, 54, 55, 56, 57, 58, 59, 60].

Remark 1.3: It is straightforward to see that the series $X\Big(a \cdot Q(x)\Big)$ is an order-N transformation when the parameter a is a N-th root of unity: $a^N = 1$. These N-th root of unity are, thus, clearly *selected values* of the parameter. Are all these N-th root of unity series algebraic series, or just D-finite series, or simply differentially algebraic series ?

1.5. Multivaled functions and reversibility

The Landen algebraic transformation (4) amounts to multiplying (*or dividing* because of the modular group symmetry $\tau \leftrightarrow 1/\tau$) the ratio τ of the two periods of the elliptic curves: $\tau \longleftrightarrow 2\,\tau$. The other (isogeny) transformations†† correspond to $\tau \leftrightarrow N \cdot \tau$, for various integers N.

We, thus, see that a modular equation, like (7), yields *multivalued* functions corresponding to the different series solutions of the modular equation (for instance (8) and its compositional inverse). More generally, for $\tau \leftrightarrow N \cdot \tau$, we will have series like $1/1728^{N-1} \cdot x^N + \cdots$ and also (their compositional inverse Puiseux series) $1728^{(N-1)/N} \cdot x^{1/N} + \cdots$

In the textbooks the renormalization group is often presented as a semi-direct group♯. In fact the renormalization group generators have no reason to be such irreversible transformations. They are, at first sight, *reversible transformations*. The modular equation (7) has a $x \leftrightarrow y$ symmetric polynomial, corresponding to the Landen transformation, *as well as its compositional inverse*, the inverse Landen transformation. These two transformations are both *exact generators* of the renormalization group of the square Ising model, or of the Baxter model [32]. With this exact renormalization group representation we see that the modular equation restores, *as a consequence of its* $x \leftrightarrow y$ *symmetry*, the *reversible character of the renormalization group*, the price to pay being that the function $y(x)$ is actually *multivalued*.

The Schwarzian condition (10) encapsulates [30, 31] an *infinite number of modular correspondences* associated with their modular curves and modular forms [14, 15, 16]. In these cases the automorphy relation (2) corresponds to *algebraic function* prefactors $\mathcal{A}(x)$. However, for series with one-parameter, like (14), which are *generically* differentially algebraic, we still have an "automorphy" relation (2), but with *differentially algebraic* "automorphy" prefactors $\mathcal{A}(x)$ (see (3)). We cannot expect a modular equation, but is there a way to still see a transformation like (14), as a "correspondence" with some "appropriate" generalization of the concept of correspondences ?

† This is also a clear consequence of the automorphy property (1).
††See for instance (2.18) in [34].
♯ In most of the graduate text book on renormalization group, the critical fixed point is an attractive fixed point. There is an "arrow of time". The renormalization group is seen as an irreversible process.

1.6. Correspondences, Schwarzian conditions and replicable functions

The Schwarzian condition (10) coincides exactly with one of the conditions G. Casale obtained [61, 62, 63, 64, 65, 66, 67] in a classification of Malgrange's $\mathcal{D}$-envelopes and $\mathcal{D}$-groupoids [68] on $\mathbb{P}_1$. Denoting $y'(x)$, $y''(x)$ and $y'''(x)$ the first, second and third derivative of $y(x)$ with respect to x, these conditions‡ read respectively

$$\mu(y) \cdot y'(x) \; - \mu(x) \; + \; \frac{y''(x)}{y'(x)} \; = \; 0, \qquad (23)$$

$$\nu(y) \cdot y''(x)^2 \; - \nu(x) \; + \; \frac{y'''(x)}{y'(x)} \; - \frac{3}{2} \cdot \Big(\frac{y''(x)}{y'(x)}\Big)^2 \; = \; 0, \qquad (24)$$

together with $\gamma(y) \cdot y'(x)^n - \gamma(x) = 0$ and $h(y) = h(x)$, corresponding respectively to rank two, rank three, together with rank one and rank nul groupoïds, where $\nu(x)$, $\mu(x)$, $\gamma(x)$ are *meromorphic* functions ($h(x)$ is holomorph).

The previous examples of Schwarzian condition (10) correspond to *elliptic curves* (modular curves, modular forms and modular correspondences [70]), through pullbacked ${}_2F_1$ hypergeometric functions [19]. In subsection 3.2 of [30] we have seen that the Schwarzian condition (10) can actually occur with *Heun* functions *which cannot be reduced to pullbacked ${}_2F_1$ hypergeometric functions*††, and which *do not* correspond to globally bounded [72, 73] series. Similarly, we have seen Schwarzian conditions (10) corresponding to (non globally bounded) pullbacked ${}_2F_1$ hypergeometric functions, associated with *Shimura curves* [74, 71, 70]. The Malgrange-Casale approach for Schwarzian conditions (24) suggests that one should be able to find examples of such Schwarzian conditions *far beyond modular curves, or even Shimura curves* (and their associated modular forms [14, 15, 16] and automorphic forms [10]). If such generalizations exist, are they also associated with *one-parameter* series ? How to describe them ? Can they necessarily be seen, eventually, as generalization of correspondences ?

In the next section we will first revisit the previous "classical" modular correspondence results *with a different normalization* of the pullback (see (25) below) which makes the occurrence of series with *integer* coefficients crystal clear. Revisiting these calculations, with a key role played by a function $F(x)$ defined below by (104), we will be able to find some new partial differential equations (see (129), or (130) below), in the parameter of the series†. These new equations will help us to find many examples of replicable-like [54, 55, 56, 57, 58] functions *far beyond modular curves or Shimura curves* [74, 70].

2. Revisiting the modular equations with a different normalization of the pullbacks.

Some part of this section will be reminiscent of the results explained in [30], with the difference that we have *another normalization* of the pullback, corresponding to change $x \rightarrow 1728\,x$, the "automorphy" relation (2) thus becoming

$${}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot y\Big) \; = \; \mathcal{A}(x) \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot x\Big), \qquad (25)$$

‡ Casale's condition (23) is *exactly the same condition as* the one we already found in [69], and this is not a coincidence.

††See for instance the two Heun functions given by (164) in [71].

† See also (162) below for more parameters.

where¶:

$$\mathcal{A}(x) \;=\; \lambda \cdot \Bigl(\frac{u(x)}{u(y(x))} \cdot y'(x)\Bigr)^{1/2}. \qquad (26)$$

As a consequence the (pullback) algebraic series $y = y(x)$, corresponding to isogenies like (8) ... are normalized as $x \longrightarrow x^N + \cdots$, and are series with *integer* coefficients.

In our case, taking into account the exact expression of the wronskian, one has $u(x) = x \cdot (1 - 1728\, x)^{1/2}$, and, thus, we get:

$$\mathcal{A}(x) \;=\; \lambda \cdot \Bigl(\frac{x \cdot (1 - 1728\, x)^{1/2}}{y \cdot (1 - 1728\, y)^{1/2}} \cdot y'(x)\Bigr)^{1/2}. \qquad (27)$$

Taking the square of (25) we can, thus, rewrite the "automorphic" relation (25) as

$$\lambda \cdot y \cdot (1 - 1728 \cdot y)^{1/2} \cdot {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot y\Bigr)^2$$
$$= \; x \cdot (1 - 1728 \cdot x)^{1/2} \cdot {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot x\Bigr)^2 \cdot \frac{dy}{dx}, \qquad (28)$$

which is, in fact, nothing but

$$\lambda \cdot \frac{dx}{F(x)} \;=\; \frac{dy}{F(y)}. \qquad (29)$$

where $F(x)$ reads:

$$F(x) \;=\; x \cdot (1 - 1728 \cdot x)^{1/2} \cdot {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot x\Bigr)^2 \qquad (30)$$
$$= \; x - 744\, x^2 - 393768\, x^3 - 357444672\, x^4 - 394896727080\, x^5 + \cdots$$

The elimination of the "automorphic" cofactor $\mathcal{A}(x)$ gives the Schwarzian equation on $y(x)$

$$W(x) \; - W(y(x)) \cdot y'(x)^2 \; + \{y(x), x\} \;=\; 0, \qquad (31)$$

where now

$$W(x) \;=\; -\frac{1}{2} \cdot \frac{1 - 1968\, x + 2654208\, x^2}{x^2 \cdot (1 - 1728\, x)^2}, \qquad (32)$$

namely:

$$-\frac{1}{2} \cdot \frac{1 - 1968\, x + 2654208\, x^2}{x^2 \cdot (1 - 1728\, x)^2} \qquad (33)$$
$$+ \frac{1}{2} \cdot \frac{1 - 1968\, y(x) + 2654208\, y(x)^2}{y(x)^2 \cdot (1 - 1728\, y(x))^2} \cdot y'(x)^2 \quad + \{y(x), x\} \;=\; 0.$$

3. Modular equation, modular correspondence

3.1. $q \longrightarrow q^2$

Let us consider the *modular equation*†:

$$\Gamma_2(x, y) \;=\; x\, y \; - (x+y) \cdot (x^2 + 1487\, x\, y + y^2)$$
$$+ 10125 \cdot x\, y \cdot (16\, x^2 - 4027\, x\, y + 16\, y^2) \qquad (34)$$
$$- 8748000000 \cdot x^2\, y^2 \cdot (x+y) + 157464000000000 \cdot x^3\, y^3 \;=\; 0,$$

¶ Note a typo in (92) in [30]. the exponent $-1/2$ in (92) must be changed into $1/2$.
† Which is nothing but (7) with the change of variables $x \rightarrow x/1728$, $y \rightarrow y/1728$.

which has the following rational parametrization [19]:

$$x \;=\; \frac{t}{(t+16)^3} \qquad \text{and:} \qquad y \;=\; \frac{t^2}{(t+256)^3}. \tag{35}$$

It has the following *algebraic series* solutions with *integer* coefficients

$$y_2 \;=\; x^2 \; + 1488\,x^3 \; + 2053632\,x^4 \; + 2859950080\,x^5 \; + 4062412996608\,x^6 \\ + 5882951135920128\,x^7 \; + 8664340079503736832\,x^8 \; + \; \cdots \tag{36}$$

and

$$y_{1/2} \;=\; \omega\cdot\, x^{1/2} \; - 744\cdot\, x^{2/2} \; + 357024\cdot\,\omega\cdot\, x^{3/2} \; - 140914688\cdot\, x^{4/2} \\ + 49735011840\cdot\,\omega\cdot\, x^{5/2} \; - 16324041375744\cdot\, x^{6/2} \; + \; \cdots \tag{37}$$

where $\omega^2 \,=\, 1$ (i.e. $\omega \,=\, \pm 1$). These two algebraic series can be written respectively:

$$\tilde{X}\Big(\tilde{Q}(x)^2\Big) \qquad \text{and:} \qquad \tilde{X}\Big(\omega\cdot\,\tilde{Q}(x)^{1/2}\Big). \tag{38}$$

They amount, respectively, to changing the nome as follows: $q \longrightarrow q^2$, *together with its compositional inverse* $q \longrightarrow \omega\cdot q^{1/2}$, where $\omega^2 \,=\, 1$. These two series, (36) and (37), are actually solutions of the Schwarzian equation (31) with $W(x)$, now, given by (32). Note that we have the following relation:

$$2\cdot\, y_2\cdot\, (1\,-1728\cdot\, y_2)^{1/2}\cdot\, {}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1],\,1728\cdot\, y_2\Big)^2 \\ =\; x\cdot\, (1\,-1728\cdot\, x)^{1/2}\cdot\, {}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1],\,1728\cdot\, x\Big)^2\cdot\,\frac{dy_2}{dx}. \tag{39}$$

We have a similar relation for $y_{1/2}$. Relation (39), and the corresponding one for $y_{1/2}$, are nothing but:

$$2\cdot\,\frac{dx}{F(x)} \;=\; \frac{dy_2}{F(y_2)} \qquad \text{and:} \qquad \frac{1}{2}\cdot\,\frac{dx}{F(x)} \;=\; \frac{dy_{1/2}}{F(y_{1/2})}. \tag{40}$$

3.2. Linear ODE for $q \longrightarrow q^2$

The previous *algebraic series* (36), (37) are solutions of an order-three linear differential operator $M_3 \,=\, M_1 \oplus M_2$ which is the *direct sum* (LCLM) of an order-two linear differential operator M_2, and an order-one linear differential operator M_1 with a rational function solution

3.3. $q \longrightarrow q^3$

Let us consider the *modular equation*:

$$1855425871872000000000\cdot\, x^3\, y^3\cdot\, (y+x) \\ + 16777216000000\cdot\, y^2\, x^2\cdot\, (27\,x^2 - 45946\,x\,y + 27\,y^2) \\ + 36864000\cdot\, x\,y\cdot\, (y+x)\cdot\, (x^2 + 241433\,x\,y + y^2) \\ + (x^4 - 1069956\,x^3\,y + 2587918086\,x^2\,y^2 - 1069956\,x\,y^3 + y^4) \\ + 2232\cdot\, x\,y\cdot\, (y+x) \; - x\,y \;=\; 0, \tag{41}$$

which has the following rational parametrization [19]:

$$x \;=\; \frac{t}{(t+27)\cdot\,(t+3)^3} \qquad \text{and:} \qquad y \;=\; \frac{t^3}{(t+27)\cdot\,(t+243)^3}. \tag{42}$$

This *modular equation* (41) has the following *algebraic* series solutions

$$y_3 \; = \; x^3 \; + 2232\,x^4 \; + 3911868\,x^5 \; + 6380013816\,x^6 \; + 10139542529238\,x^7 \\ + 15969813236020944\,x^8 \; + 25104342383076998772\,x^9 \; + \; \cdots \qquad (43)$$

and its compositional inverse

$$y_{1/3}(\omega,\, x) \; = \; \omega \cdot x^{1/3} \; - 744 \cdot \omega^2 \cdot x^{2/3} \; + 356652 \cdot x^{3/3} \; - 140360904 \cdot \omega \cdot x^{4/3} \\ + 49336313166 \cdot \omega^2\, x^{5/3} \; - 16114360320000 \cdot x^{6/3} \; + \; \cdots \qquad (44)$$

where $\omega^3 \, = \, 1$. The radius of convergence of the series (43) is $R \, = \, 1/1728$, corresponding to the vanishing of the discriminant of the modular equation (41). These two series can be written respectively

$$\tilde{X}\Big(\tilde{Q}(x)^3\Big) \qquad \text{and:} \qquad \tilde{X}\Big(\omega \cdot \tilde{Q}(x)^{1/3}\Big), \qquad (45)$$

where $\omega^3 \, = \, 1$. They amount, respectively, to changing the nome as follows: $q \, \longrightarrow \, q^3$, together with its compositional inverse $q \, \longrightarrow \, \omega \cdot q^{1/3}$ where $\omega^3 \, = \, 1$. These two algebraic series, (43) and (44), are actually solutions of the Schwarzian equation (31), with $W(x)$ given by (32). Note that we have the following relation:

$$3 \cdot y_3 \cdot (1 - 1728 \cdot y_3)^{1/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot y_3\Big)^2 \\ = \; x \cdot (1 - 1728 \cdot x)^{1/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot x\Big)^2 \cdot \frac{dy_3}{dx}. \qquad (46)$$

We have a similar relation for $y_{1/3}$. Relation (46), and the corresponding one for $y_{1/3}$, are nothing but:

$$3 \cdot \frac{dx}{F(x)} \; = \; \frac{dy_3}{F(y_3)} \qquad \text{and:} \qquad \frac{1}{3} \cdot \frac{dx}{F(x)} \; = \; \frac{dy_{1/3}}{F(y_{1/3})}. \qquad (47)$$

3.4. Linear ODE for $q \longrightarrow q^3$

The previous algebraic series (43), (44) are solutions of an order-four linear differential operator $M_3 \, = \, M_1 \oplus M_3$, which is the direct sum (LCLM) of an order-three linear differential operator M_3, and an order-one linear differential operator M_1 with a rational function solution.

3.5. $q \longrightarrow q^5$

We are not going to give explicitely the modular equation corresponding to $q \, \longrightarrow \, q^5$ because it starts becoming a bit too large. Let us just say that it can (easily) be obtained by the elimination of t in its rational parameterization [19]:

$$x \; = \; \frac{t}{(t^2 + 10\,t + 5)^3} \qquad \text{and:} \qquad y \; = \; \frac{t^5}{(t^2 + 250\,t + 3125)^3}. \qquad (48)$$

This modular curve $\Gamma_5(x,\, y) \; = \; \Gamma_5(y,\, x) \; = \; 0$, has the following *algebraic series* solutions

$$y_5 \; = \; x^5 \; + 3720\,x^6 \; + 9287460\,x^7 \; + 19648405600\,x^8 \; + 38124922672650\,x^9 \\ + 70330386411705000\,x^{10} \; + 125698841122545005000\,x^{11} \; + \; \cdots \qquad (49)$$

and

$$y_{1/5} \;=\; \omega \cdot x^{1/5} \;-744 \cdot \omega^2 \cdot x^{2/5} \;+356652\,\omega^3 \cdot x^{3/5} \;-140361152 \cdot \omega^4 \cdot x^{4/5} \;+49336682190 \cdot x^{5/5} \;-\frac{80573128344696}{5} \cdot \omega \cdot x^{6/5} \;+\; \cdots \qquad (50)$$

where $\omega^5 \,=\, 1$. The series (49) and (50) are (algebraic) solutions of an order-six linear differential operator $L_6 \,=\, L_1 \oplus L_5$, which is the direct sum of an order-one linear differential operator with a rational function solution, and an irreducible order-five linear differential operator operator L_5. The series (49) and (50) can be written respectively

$$y_5 \,=\, \tilde{X}\Big(\tilde{Q}(x)^5\Big) \qquad \text{and:} \qquad y_{1/5} \,=\, \tilde{X}\Big(\omega \cdot \tilde{Q}(x)^{1/5}\Big), \qquad (51)$$

where $\omega^5 \,=\, 1$. They amount, respectively, to changing the nome as follows: $q \,\longrightarrow\, q^5$, and its compositional inverse $q \,\longrightarrow\, \omega \cdot q^{1/5}$ where $\omega^5 \,=\, 1$. These two series, (49) and (50), are actually solutions of the Schwarzian equation (31), with $W(x)$ given by (32). Note that we have the following relation:

$$5 \cdot y_5 \cdot (1 \,-1728 \cdot y_5)^{1/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot y_5\Big)^2 \;=\; x \cdot (1 \,-1728 \cdot x)^{1/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot x\Big)^2 \cdot \frac{dy_5}{dx}, \qquad (52)$$

i.e.

$$5 \cdot F(y_5) \;=\; F(x) \cdot \frac{dy_5}{dx}, \qquad (53)$$

and:

$$5 \cdot \frac{dx}{F(x)} \;=\; \frac{dy_5}{F(y_5)} \qquad \text{and:} \qquad \frac{1}{5} \cdot \frac{dx}{F(x)} \;=\; \frac{dy_{1/5}}{F(y_{1/5})}. \qquad (54)$$

Remark 3.1: The series (37), (44), (50) (and also (57) below) *can be seen to be functions of* $\omega \cdot x^{1/N}$ with $\omega^N \,=\, 1$.

3.6. $q \longrightarrow q^4$

We are not going to give explicitely the modular equation corresponding to $q \,\longrightarrow\, q^4$ because it becomes a bit too large. Let us just say that it can (easily) be obtained by the elimination of t in its rational parameterization [19]:

$$x \;=\; \frac{t \cdot (t+16)}{(t^2\,+16\,t\,+16)^3} \qquad \text{and:} \qquad y \;=\; \frac{t^4 \cdot\, (t+16)}{(t^2\,+256\,t\,+4096)^3}. \qquad (55)$$

This *modular curve* $\Gamma_4(x, y) \;=\; \Gamma_4(y, x) \;=\; 0$ can also be obtained from the elimination of the variable z between the (fundamental) modular equation $\Gamma_2(x, z) \,=\, 0$, given by (34), and the same modular equation $\Gamma_2(z, y) \,=\, 0$. The calculation of the resultant, in z, between $\Gamma_2(x, z)$ and $\Gamma_2(z, y)$ factorizes, and gives $(x-y)^2 \cdot \Gamma_4(x, y)$. This modular curve $\Gamma_4(x, y) \;=\; \Gamma_4(y, x) \;=\; 0$, has the following algebraic series solutions

$$y_4 \;=\; x^4 \;+2976\,x^5 \;+6322896\,x^6 \;+11838151424\,x^7 \;+20872495228416\,x^8 \;+35647177050980352\,x^9 \;+59796357134115627008\,x^{10} \;+99264875397039869263872\,x^{11} \;+\; \cdots \qquad (56)$$

$$y_{1/4}(\omega,\, x) \;=\; \omega \cdot x^{1/4} \;-744 \cdot \omega^2 \cdot x^{1/2} \;+356652 \cdot \omega^3 \cdot x^{3/4} \;-140361152 \cdot x^{4/4} \;+49336682376 \cdot \omega \cdot x^{5/4} \;-16114625945856 \cdot \omega^2 \cdot x^{6/4} \;+4999042676442272 \cdot \omega^3 \cdot x^{7/4} \;-1492669488513712128 \cdot x^{8/4} \;+432762805367932714848 \cdot \omega \cdot x^{9/4} \;+\; \cdots \qquad (57)$$

where $\omega^4 = 1$, *together* with the (*involutive*) series:

$$y_1 \;=\; -x \;-1488\, x^2 \;-2214144\, x^3 \;-3337633792\, x^4 \;-5094329942016\, x^5 \;-7859077093785600\, x^6 \;-12234039128005541888\, x^7 \;-19190712499154486034432\, x^8 \;-30301349938167862039412736\, x^9 \;+\; \cdots \qquad (58)$$

The radius of convergence of the series (56), or (58), is $R = 1/1728$, corresponding to the vanishing of the discriminant of the modular equation $\Gamma_4(x,\, y) = \Gamma_4(y,\, x) = 0$. These three series (56), (57) and (58), can be written respectively

$$\tilde{X}\Big(\tilde{Q}(x)^4\Big) \qquad \text{and:} \qquad \tilde{X}\Big(\omega \cdot \tilde{Q}(x)^{1/4}\Big) \qquad \text{and:} \qquad \tilde{X}\Big(-\tilde{Q}(x)\Big), \qquad (59)$$

where $\omega^4 = 1$. These series can be obtain from the series (36) and (37) of subsection (3.1). It is straightforward to see† that $y_4(x) = y_2(y_2(x))$, and that $y_{1/4}(x) = y_{1/2}(y_{1/2}(x))$, which amounts, *on the nome*, to performing $q \longrightarrow q^2 \longrightarrow (q^2)^2 = q^4$ and similarly $q \longrightarrow \pm\, q^{1/2} \longrightarrow \pm\,(\pm\, q^{1/2})^{1/2} = \omega \cdot q^{1/4}$, where $\omega^4 = 1$. However, the composition of y_2 and $y_{1/2}$ also corresponds, on the nome, to

$$q \,\rightarrow\, \pm\, q^{1/2} \,\rightarrow\, (\pm\, q^{1/2})^2 \,=\, q \qquad \text{or:} \qquad q \,\rightarrow\, q^2 \,\rightarrow\, \pm\,(q^2)^{1/2} \,=\, \pm\, q. \qquad (60)$$

Getting rid of the identity transformation, we get $q \rightarrow -q$, which, precisely, corresponds to the *involutive* series (58). These three series (56), (57) and (58) are actually solutions of the Schwarzian equation (31), with $W(x)$ given by (32). Note that we have the following relation:

$$4 \cdot y_4 \cdot (1 - 1728 \cdot y_4)^{1/2} \cdot {}_2F_1\Big([\frac{1}{12},\, \frac{5}{12}],\, [1],\, 1728 \cdot y_4\Big)^2 \;=\; x \cdot (1 - 1728 \cdot x)^{1/2} \cdot {}_2F_1\Big([\frac{1}{12},\, \frac{5}{12}],\, [1],\, 1728 \cdot x\Big)^2 \cdot \frac{dy_4}{dx}, \qquad (61)$$

$$4 \cdot \frac{dx}{F(x)} \;=\; \frac{dy_4}{F(y_4)} \qquad \text{and:} \qquad \frac{1}{4} \cdot \frac{dx}{F(x)} \;=\; \frac{dy_{1/4}}{F(y_{1/4})} \qquad \text{and:} \qquad \frac{dx}{F(x)} \;=\; \frac{dy_1}{F(y_1)}. \qquad (62)$$

3.7. Linear differential operators for $q \rightarrow q^4$

The previous algebraic series (56), (57) and (58) are solutions of an order-six linear differential operator $M_6 = M_1 \oplus M_2 \oplus M_3$, which is the *direct sum* (LCLM) of an order-three linear differential operator M_3, an order-three linear differential operator M_2, and an order-one linear differential operator M_1 with a rational function solution.

Remark 3.2: The order of the linear operator M_6, corresponds to the four series (57) of the form $y_{1/4}$, together with the series y_4, and the series y_1, namely $6 = 4 + 1 + 1$ series. The series y_4 (given by (56)) can be seen to be an (algebraic) *analytic continuation of the involutive series* y_1 (given by (58)).

† The composition/iteration of *multivalued* functions, like algebraic functions, is a bit tricky, we have, however, no problem to compose *algebraic series*, for instance $x \rightarrow y_2(x) \rightarrow y_4(x) = y_2(y_2(x))$.

Remark 3.3: The algebraic series $y_1,\ y_4,\ y_{1/4}(\omega^n,\, x)$, solutions of the modular equation $\Gamma_4(x,\, y) \, = \, 0$, can be expressed as linear combinations of the solutions of the three linear differential operators $M_n, \ \ n = 1,\, 2,\, 3$. If one introduces the (finite) Galois group of the polynomial associated with the modular equation $\Gamma_4(x,\, y) = \, 0$, and the differential Galois groups of the three linear differential operators M_n, one sees that the relation between these different Galois groups is far from being straightforward.

3.8. More correspondence series

Let us display¶ more correspondence series.

- The (algebraic) series

$$\tilde{X}\Big(\tilde{Q}(x)^6\Big) \; = \; x^6 \; + 4464\, x^7 \; + 12805560\, x^8 \; + 30222607872\, x^9 \; + 64062187946172\, x^{10} \; + \; \cdots \tag{63}$$

is solution of a modular equation $\Gamma_6(x,\, y) \, = \, \Gamma_6(y,\, x) \, = \, 0$, that will not be written here, but can easily be obtained from its rational parametrization [19]:

$$x \; = \; \frac{t \cdot \, (t+8)^3 \cdot \, (t+9)^2}{(t+6)^3 \cdot \, (t^3 \, + 18\, t^2 \, + 84\, t \, + 24)^3},$$

$$y \; = \; \frac{t^6 \cdot \, (t+8)^2 \cdot \, (t+9)^3}{(t+12)^3 \cdot \, (t^3 \, + 252\, t^2 \, + 3888\, t \, + 15552)^3}. \tag{64}$$

This series (63) is solution of an order-twelve linear differential operator $L_{12} \, = \, L_1 \oplus \, L_{11}$, which is the *direct sum* of an order-one linear differential operator L_1 with a rational function solution, and an order-eleven linear differential operator L_{11}.

- We can also consider

$$\tilde{X}\Big(\tilde{Q}(x)^{13}\Big) \; = \; x^{13} \; + 9672\, x^{14} \; + 52931268\, x^{15} \; + 216226356320\, x^{16} \; + \; \cdots \tag{65}$$

which is solution of a modular equation† $\Gamma_{13}(x,\, y) \, = \, \Gamma_{13}(y,\, x) \, = \, 0$, that we will not write here, but can easily be obtained from its rational parametrization [19]:

$$x \; = \; \frac{t}{(t^2 \, + 5\, t \, + 13) \cdot \, (t^4 \, + 7\, t^3 \, + 20\, t^2 \, + 19\, t \, + 1)^3},$$

$$y \; = \; \frac{t^{13}}{(t^2 \, + 5\, t \, + 13) \cdot \, (t^4 \, + 247\, t^3 \, + 3380\, t^2 \, + 15379\, t \, + 28561)^3}. \tag{66}$$

This series (65) is solution of an order-fourteen linear differential operator $L_{14} \, = \, L_1 \oplus \, L_{13}$, which is the direct sum of an order-one linear differential operator L_1 with a rational function solution, and an irreducible order-thirteen linear differential operator L_{13}.

- Let us consider

$$\tilde{X}\Big(\tilde{Q}(x)^9\Big) \; = \; x^9 \; + 6696\, x^{10} \; + 26681076\, x^{11} \; + 82647211104\, x^{12} \; + \; \cdots \tag{67}$$

which is solution of a modular equation $\Gamma_9(x,\, y) \, = \, \Gamma_9(y,\, x) \, = \, 0$, that we will not write here, but can easily be obtained from its rational parametrization [19]:

$$x = \frac{t \cdot \, (t^2 + 9\, t \, + 27))}{(t+3)^3 \cdot \, (t^3 \, + 9\, t^2 \, + 27\, t \, + 3)^3}, \quad y = \frac{t^9 \cdot \, (t^2 + 9\, t \, + 27))}{(t+9)^3 \cdot \, (t^3 \, + 243\, t^2 \, + 2187\, t \, + 6561)^3}.$$

¶ For all these examples we used gfun of Bruno Salvy. We used the following commands: algeqtodiffeq, diffeqtohomdiffeq, de2diffop, algeqtoseries, formal_sols.

† The polynomial $\Gamma_{13}(x,\, y)$ is of degree 14 in y (or x).

The polynomial $\Gamma_9(x, y)$ is of degree 12 in y (resp. in x). We thus have twelve algebraic solutions-series of the *modular equation* $\Gamma_9(x, y) = 0$. This series (67) is solution of an order-twelve linear differential operator $L_{12} = L_1 \oplus L_{11}$, which is the direct sum of an order-one operator L_1 with a rational function solution, and an order-eleven linear differential operator L_{11}. The (nine) series which are compositional inverse of the series (67), are also solutions of the modular equation $\Gamma_9(x, y) = 0$, read:

$$\begin{aligned} \tilde{X}\Big(\tilde{Q}(x)^{1/9}\Big) \quad = \quad & \omega \cdot x^{1/9} \quad - 744 \cdot \omega^2 \cdot x^{2/9} \quad + 356652 \cdot \omega^3 \cdot x^{1/3} \\ & - 140361152 \cdot \omega^4 \cdot x^{4/9} \, + 49336682190 \cdot \omega^5 \cdot x^{5/9} \, - 16114625669088 \cdot \omega^6 \cdot x^{2/3} \\ & + 4999042477430456 \cdot \omega^7 \cdot x^{7/9} \quad + \quad \cdots \end{aligned} \tag{68}$$

where $\omega^9 = 1$. These (nine) series (68) are solutions of the order-twelve linear differential operator L_{12}. Note that the (two) *order-three* series

$$\begin{aligned} y_\omega(x) \, = \, y_{1/3}\Big(y_3(x)\Big) \, = \, & \omega \cdot x \quad - 744 \cdot \omega \cdot (\omega - 1) \cdot x^2 \\ & + 36 \cdot \omega \cdot (\omega - 1) \cdot (9907\,\omega - 20845) \cdot x^3 \\ & - 32 \cdot \omega \cdot (\omega - 1) \cdot (-24876477\,\omega + 22887765) \cdot x^4 \quad + \quad \cdots \end{aligned} \tag{69}$$

where $\omega^2 + \omega + 1 = 0$, are also solutions of the modular equation $\Gamma_9(x, y) = 0$, and are also of the order-twelve operator L_{12}. We thus have $1 + 2 + 9 = 12$ *algebraic* solutions of the modular equation $\Gamma_9(x, y) = 0$, and solutions of L_{12}.

4. The one-parameter series solutions of the Schwarzian equation.

The Schwarzian equation (31) has more solutions than the infinite discrete set of algebraic series (see (36), (43), (49), (56), (63), (65), ...) corresponding to *modular correspondences*. One actually has a series *depending on one parameter*, namely:

$$\begin{aligned} y(a, x) \, = \, & a \cdot x \quad - 744 \cdot a \cdot (a-1) \cdot x^2 \, + 36 \cdot a \cdot (a-1) \cdot (9907\,a - 20845) \cdot x^3 \\ & - 32 \cdot a \cdot (a-1) \cdot (4386286\,a^2 - 20490191\,a + 27274051) \cdot x^4 \\ & + 6 \cdot a \cdot (a-1) \cdot (8222780365\,a^3 - 61396351027\,a^2 \\ & \quad + 171132906629\,a - 183775457147) \cdot x^5 \\ & - 144 \cdot a \cdot (a-1) \cdot (111907122702\,a^4 - 1162623833873\,a^3 + 5000493989295\,a^2 \\ & \quad - 10801207072185\,a + 10212230113145) \cdot x^6 \\ & + 8 \cdot a \cdot (a-1) \cdot (624880309678807\,a^5 - 8367080813672297\,a^4 \\ & \quad + 48909476982869878\,a^3 - 158792594445015178\,a^2 \\ & \quad + 293243568886999823\,a - 254689844062110385) \cdot x^7 \\ & - 192 \cdot a \cdot (a-1) \cdot (7774319708776120\,a^6 - 127824707491524999\,a^5 \\ & \quad + 946950323149342341\,a^4 - 4101941044701784034\,a^3 \\ & \quad + 11156847890086765926\,a^2 - 18508096006772656203\,a \\ & \quad + 15126379507970624425) \cdot x^8 \quad + \quad \cdots \end{aligned} \tag{70}$$

Note that all the algebraic series (58), (69), (see also (92) below), ... associated with modular equations, are of the form (70), where the parameter is a N-th root of unity: $a^N = 1$.

Note that this one-parameter series (70) is a series of the form

$$y(a,\, x) \;=\; a \cdot x \;+ a \cdot (a-1) \cdot \sum_{n=2}^{\infty} P_n(a) \cdot x^n, \qquad (71)$$

where the polynomials $P_n(a)$ are polynomials of degree $n-2$ in the parameter a, *with integer coefficients*‡.

This one-parameter series (70), (71) verifies the following composition rule:

$$y\Big(a,\, y(a',\, x)\Big) \;=\; y\Big(a',\, y(a,\, x)\Big) \;=\; y(a\, a',\; x). \qquad (72)$$

These series *commute*. One can verify that this one-parameter series (70) can, in fact, be written

$$y(a,\; x) \;=\; \tilde{X}\Big(a \cdot \tilde{Q}(x)\Big), \qquad (73)$$

where

$$\tilde{X}(q) \;=\; q \;-744\; q^2 \;+356652\; q^3 \;-140361152\; q^4 \;+49336682190\; q^5 \\ -16114625669088\, q^6 \;+4999042477430456\, q^7 \;+\; \cdots \qquad (74)$$

and† its composition inverse:

$$\tilde{Q}(x) \;=\; x \;+744\, x^2 \;+750420\, x^3 \;+872769632\, x^4 \;+1102652742882\, x^5 \\ +1470561136292880\, x^6 +2037518752496883080\, x^7 \;+\; \cdots \qquad (75)$$

The nome series (75) has a radius of convergence $R \,=\, 1/1728 \,=\, 0.00057870370 \cdots$

In the $a \rightarrow 0$ limit one has

$$\lim_{a \rightarrow 0} \frac{y(a,\; x)}{a} \;=\; x \;+744\, x^2 \;+750420\, x^3 \;+872769632\, x^4 \;+1102652742882\, x^5 \\ +1470561136292880\, x^6 \;+2037518752496883080\, x^7 \;+2904264865530359889600\, x^8 \\ +4231393254051181981976079\, x^9 \;+\; \cdots \qquad (76)$$

which is nothing but the nome series $\tilde{Q}(x)$ given by (75). In the $a \rightarrow \infty$ limit one has

$$\lim_{a \rightarrow \infty} y\Big(a,\; \frac{x}{a}\Big) \;=\; x \;-744\, x^2 \;+356652\, x^3 \;-140361152\, x^4 \;+49336682190\, x^5 \\ -16114625669088\, x^6 \;+4999042477430456\, x^7 \;-1492669384085015040\, x^8 \\ +432762759484818142437\, x^9 \;+\; \cdots \qquad (77)$$

which is nothing but $\tilde{X}$, the (elliptic modulus) series (74).

Let us introduce the ratio of the polynomials in expansion (71):

$$R_n(a) \;=\; \frac{P_n(a)}{P_{n+1}(a)}. \qquad (78)$$

One sees, in the $n \rightarrow \infty$ and $a \rightarrow 0$ limit, that the ratio (78) becomes (as it should) $1/1728 \,=\, 0.00057870 \cdots$ For miscellaneous small values of the parameter a, one can see, that this ratio (78) also becomes $1/1728$ in the $n \rightarrow \infty$ limit.

‡ This can be seen as a consequence of the fact that $y(a,\, x) \,=\, \tilde{X}\Big(a \cdot \tilde{Q}(x)\Big)$, where $\tilde{X}(x)$ and $\tilde{Q}(x)$ are actually series *with integer coefficients* (see (15) and (16)).

† In Maple the $\tilde{X}(q)$ series (15), (74) can be obtained substituting $L \,=\, EllipticModulus(q^{1/2})^2$, in $1/j \,=\, L^2 \cdot (L-1)^2/(L^2-L+1)^3/256$. See https://oeis.org/A066395 for the series (15) and https://oeis.org/A091406 for the series (16).

In the last $n \rightarrow \infty$ and $a \rightarrow \infty$ limit (77), the ratio (78) becomes†† $-0.004316810242 \cdots$ which corresponds to the radius of convergence of the series (15), (74). This radius of convergence is according to Vaclav Kotesovec¶

$$\exp\Big(-\sqrt{3}\cdot\pi\Big) \, = \, 0.004333420501 \cdots \tag{79}$$

which is reminiscent of the selected values (see equation (55) in [32]):

$$t \, = \, \exp\Big(i\,\pi\,\frac{1+i\sqrt{3}}{2}\Big) \, = \, i\cdot\exp\Big(-\frac{\sqrt{3}}{2}\cdot\pi\Big) \quad \text{or:} \quad j\Big(\frac{1+i\sqrt{3}}{2}\Big) \, = \, 0. \tag{80}$$

The nearest to $x = 0$ singularity of $\tilde{X}$ is thus $x_c = t^2 = -\exp(-\sqrt{3}\cdot\pi)$. We have seen that the radius of convergence of the (involutive) series (58) (i.e. $a = -1$) is $R = 1/1728$, corresponding to the vanishing of the discriminant of the modular equation $\Gamma_4(x, y) = \Gamma_4(y, x) = 0$, and more generally, for $|a| = 1$, one can see that the radius of convergence of the series (70), (71), for N-th root of unity, $a^N = 1$, is also‡ $R = 1/1728$.

More generally, the radius of convergence of series (70), (71) corresponds to the singularities of (73), namely the $x = 1/1728$ singularity of $\tilde{Q}(x)$, and to the values of x such that $a\cdot\tilde{Q}(x) = -\exp\Big(-\sqrt{3}\cdot\pi\Big)$, associated with the singularity of $\tilde{X}(x)$, namely:

$$x \, = \, \tilde{X}\Big(-\frac{1}{a}\cdot\,\exp\Big(-\sqrt{3}\cdot\pi\Big)\Big). \tag{81}$$

When the parameter a is large enough ($|a| > \simeq 7.5$), the radius of convergence no longer corresponds to $R = 1/1728$, but to the singularity (81).

This transcendental value (79), for the radius of convergence of the series $\tilde{X}(q)$, is a strong incentive to understand the "very nature" of the one-parameter series (70), (71), especially since it can be written in the simple form (73). Generically the one-parameter series (70), being solution of a Schwarzian equation, is a *differentially algebraic series*, but is it possible that this series could be, only *for selected values* of the parameter, an algebraic series, or just a D-finite series, or possibly a D-D-finite series ?

5. Trying to understand the one-parameter series solutions.

5.1. When the one-parameter series becomes an algebraic series

For $a = -1$ the (*involutive*) series $y(a, x)$ (see series (58))

$$-x \,\, - 1488\,x^2 \,\, - 2214144\,x^3 \,\, - 3337633792\,x^4 \,\, - 5094329942016\,x^5$$
$$- 7859077093785600\,x^6 \,\, - 12234039128005541888\,x^7 \,\, + \,\cdots \tag{82}$$

has a radius of convergence $1/1728$. Let us generalize what we have seen in subsection (3.6) with series (58). Let us first recall the algebraic series y_3 (corresponding to $q \rightarrow q^3$), given by (43), and $y_{1/3}$, given by (44), where $\omega^3 = 1$, and let us compose y_3 and $y_{1/3}$. We first get:

$$y_3\Big(y_{1/3}(x)\Big) \, = \, x. \tag{83}$$

†† Obtained with 421 coefficients.
¶ See https://oeis.org/A066395 and https://oeis.org/A066395/b066395.txt for the reciprocal of j-function. See also in [75], $Q(\exp(-\sqrt{3}\cdot\pi) = 0$ or $J(\exp(-\sqrt{3}\cdot\pi) = 0$, where Q is the Eisenstein series E_4 and J is the Klein modular invariant.
‡ This also corresponds to vanishing of the discriminant of the corresponding modular equations.

More interestingly, we also get the following algebraic series

$$y_\omega(x) \;=\; y_{1/3}\Big(y_3(x)\Big) \;=\; \omega \cdot x \;-744 \cdot \omega \cdot (\omega-1) \cdot x^2$$
$$+\,36 \cdot \omega \cdot (\omega-1) \cdot (9907\,\omega\,-20845) \cdot x^3 \qquad (84)$$
$$-\,32 \cdot \omega \cdot (\omega-1) \cdot (22887765\,-24876477\,\omega) \cdot x^4 \;+\;\cdots$$

where $\omega^3 = 1$. One can verify that series (84) is actually series (70) when $a^3 = 1$. One can verify that this series is (for $\omega \neq 1$) a series of order 3:

$$y_\omega(y_\omega(y_\omega(x))) \;=\; x. \qquad (85)$$

Let us also recall the algebraic series (corresponding to $q \rightarrow q^5$) y_5, given by (49), and its compositional inverse $y_{1/5}$, given by (50), where $\omega^5 = 1$, and let us compose y_5 and $y_{1/5}$. We first get:

$$y_5\Big(y_{1/5}(x)\Big) \;=\; x. \qquad (86)$$

More interestingly, we also get the following series:

$$y_\omega(x) \;=\; y_{1/5}\Big(y_5(x)\Big) \;=\; \omega \cdot x \;-744 \cdot \omega \cdot (\omega-1) \cdot x^2$$
$$+\,36 \cdot \omega \cdot (\omega-1) \cdot (9907\,\omega-20845) \cdot x^3$$
$$-\,32 \cdot \omega \cdot (\omega-1) \cdot (4386286\,\omega^2-20490191\,\omega+27274051) \cdot x^4$$
$$+\,6 \cdot \omega \cdot (\omega-1) \cdot (8222780365\,\omega^3-61396351027\,\omega^2$$
$$+\,171132906629\,\omega-183775457147) \cdot x^5$$
$$-\,144 \cdot \omega \cdot (\omega-1) \cdot (-1274530956575\,\omega^3+4888586866593\,\omega^2$$
$$-\,10913114194887\,\omega+10100322990443) \cdot x^6 \;+\;\cdots \qquad (87)$$

where $\omega^5 = 1$.

One can verify that (87) is actually (70) when $a^5 = 1$. One can verify that the series (87) is (for $\omega \neq 1$) a series of order 5:

$$y_\omega(y_\omega(y_\omega(y_\omega(y_\omega(x))))) \;=\; x. \qquad (88)$$

This is a straight consequence of (73) with $a^5 = 1$. Similarly, let us now consider

$$y_{13} \;=\; \tilde{X}\Big(\tilde{Q}(x)^{13}\Big) \;=\; x^{13} \;+9672\,x^{14} \;+52931268\,x^{15} \;+216226356320\,x^{16}$$
$$+\,735033166074714\,x^{17} \;+2200510278533887632\,x^{18} \;+\cdots \qquad (89)$$

Its compositional inverse (Puiseux) series reads

$$y_{1/13} \;=\; \tilde{X}\Big(\tilde{Q}(x)^{1/13}\Big) \;=\; \omega \cdot x^{1/13} \;-744 \cdot \omega^2 \cdot x^{2/13} \;+356652 \cdot \omega^3 \cdot x^{3/13}$$
$$-\,140361152 \cdot \omega^4 \cdot x^{4/13} \;+49336682190 \cdot \omega^5 \cdot x^{5/13}$$
$$-\,16114625669088 \cdot \omega^6 \cdot x^{6/13} \;+\;\cdots \qquad (90)$$

where $\omega^{13} = 1$. Let us compose y_{13} and $y_{1/13}$. We first get

$$y_{13}\Big(y_{1/13}(x)\Big) \;=\; x, \qquad (91)$$

which corresponds, on the nome, to: $\quad q \;\longrightarrow\; \omega\, q^{1/13} \;\longrightarrow\; \Big(\omega\, q^{1/13}\Big)^{13} \;=\; q.$

More interestingly, we also get the following algebraic series:

$$y_\omega(x) \; = \; y_{1/13}\Big(y_{13}(x)\Big) \; = \; \omega \cdot x \; - 744 \cdot \omega \cdot (\omega - 1) \cdot x^2$$
$$+ 36 \cdot \omega \cdot (\omega - 1) \cdot (9907\,\omega - 20845) \cdot x^3$$
$$- 32 \cdot \omega \cdot (\omega - 1) \cdot (4386286\,\omega^2 - 20490191\,\omega + 27274051) \cdot x^4$$
$$+ 6 \cdot \omega \cdot (\omega - 1) \cdot (8222780365\,\omega^3 - 61396351027\,\omega^2$$
$$+ 171132906629\,\omega - 183775457147) \cdot x^5$$
$$- 144 \cdot \omega \cdot (\omega - 1) \cdot (111907122702\,\omega^4 - 1162623833873\,\omega^3 + 5000493989295\,\omega^2$$
$$- 10801207072185\,\omega + 10212230113145) \cdot x^6 \; + \; \cdots \qquad (92)$$

where $\omega^{13} \, = \, 1$. This series corresponds to $q \, \rightarrow \, q^{13} \, \rightarrow \, \omega \cdot \, (q^{13})^{1/13} \, = \, \omega \cdot \, q$. One can verify that the series (92) is (for $\omega \, \neq \, 1$) a series of order 13:

$$y_\omega(y_\omega(y_\omega(y_\omega(y_\omega(y_\omega(y_\omega(y_\omega(y_\omega(y_\omega(y_\omega(y_\omega(y_\omega(x))))))))))))) \; = \; x. \qquad (93)$$

If we compare the series (92) with the one-parameter series (70), we see that:

$$y_\omega(x) \; = \; y(\omega, \, x) \qquad \text{where:} \qquad \omega^{13} \, = \, 1. \qquad (94)$$

Conversely, denoting $y(a, \, x) \; = \; y_a(x)$ we see that the N-times composition of $y_a(x)$ becomes the identity transformation when $a^N \, = \, 1$:

$$y_a(y_a(\cdots \, (y_a(x)) \, \cdots)) \; = \; x \qquad \Longleftrightarrow \qquad a^N \, = \, 1. \qquad (95)$$

The series (92) is the solution series of the *modular equation* associated with the nome-transformation $q \, \rightarrow \, q^{13^2}$, corresponding to the composition of the modular equation associated with $q \, \longrightarrow \, q^{13}$ with itself.

More generally the modular equation $\Gamma_{N^2}(x, \, y) \; = \; 0$, corresponding to $q \, \rightarrow \, q^{N^2}$, will have $1 + (N-1) + N^2 \, = \, N \cdot (N+1)$ algebraic solution-series, corresponding respectively to the series

$$y_{N^2} \; = \; x^{N^2} \; + 744 \cdot \, N^2 \cdot \, x^{N^2+1} \; + \; \cdots \qquad (96)$$

together with the $N-1$ (order- N) series

$$y_{1/N}(y_N(x)) \; = \; \omega \cdot x \; - 744 \cdot \omega \cdot (\omega - 1) \cdot x^2 \; + \; \cdots \qquad (97)$$

with $\omega^N \, = \, 1$ but $\omega \, \neq \, 1$, and the N^2 compositional inverse (Puiseux) series of the series (96). Series (58), (83) and (87) are such examples.

5.1.1. The one-parameter series (70) is not generically a D-finite series

The one-parameter series (70) becomes an *algebraic series* when the parameter is a N-th root of unity. All the previous *algebraic series* associated with *modular equations*, can also be seen as D-finite series as displayed in the previous section (3.7). Along this line it is crucial to note that these series are solutions of a linear differential operator (like M_3 in the previous section (3.7)) *of order increasing with* N. Therefore, we see that *one cannot expect the one-parameter series* (70) *to be generically* D*-finite*, being solution a finite order linear differential operator with polynomial coefficients in x and in the parameter a, since the order of this linear differential operator *grows with* N when the parameter is a N-th root of unity.

5.2. When the one-parameter series becomes a globally bounded series

Note that, for *integer* values of the parameter a, the series $y(a,\ x)$ are actually series with *integer* coefficients. More generally, one can see easily that *such series are globally bounded* [72, 73] *for any rational number* $a\ =\ P/Q$: the series (70) can be recast into a series with *integer* coefficients if one rescales x as follows: $x\ \longrightarrow\ Q\cdot x$.

If one of these series is D-finite, the series should be, according to Christol's conjecture [76], a diagonal of a rational (or algebraic) function [72]. In particular this series should *reduce to algebraic function modulo any prime number* [72, 73]. Let us focus, for instance, on the particular value $a=3$. For $a\ =\ 3$ the series $y(a,\ x)$ is a series with *integer* coefficients

$$S\ =\ \ 3\,x\ \ -4464\,x^2\ \ +1917216\,x^3\ \ -1013769984\,x^4\ \ -33437759328\,x^5 \\ -420498625999104\,x^6\ \ -452363497164804864\,x^7\ \ +\ \cdots \tag{98}$$

which has a radius of convergence $1/1728\ =\ 0.00057870\cdots$ If one considers the series (98) modulo different primes p, it is very difficult to see (for p large enough) if this series (98) is an *algebraic series* modulo p, or, even, is D-finite modulo p. We have, however, found the following result. Introducing

$$\sigma\ \ =\ \ \frac{S\,-3\,x}{3\cdot 2^5\cdot x}\ \ +\frac{99}{2}\cdot x\ \ +1\ \ =\ \ 1\ +\ 3\,x\ +\ 19971\,x^2\ \ -10560104\,x^3 \\ -348309993\,x^4\ \ -4380194020824\,x^5\ \ -4712119762133384\,x^6\ \ +\ \cdots \tag{99}$$

this series reduces, modulo $p=2$, to the algebraic series

$$\sigma(x)\ \ =\ \ 1\ \ +x\ \ +x^2\ \ +x^4\ +x^8\ +x^{16}\ +x^{32}\ +x^{64}\ +x^{128}\ +x^{256}\ \ +\ \ \cdots \tag{100}$$

solution, modulo $p=2$, of the algebraic polynomial:

$$\sigma(x^2)\ \ -\sigma(x)\ \ +x\ \ \ =\ \ \ \sigma(x)^2\ \ -\sigma(x)\ \ +x\ \ =\ \ \ 0. \tag{101}$$

The nature of the series (98), or more generally of (70) for integer, or rational values of the parameter a, remains an open question. It seems that such globally bounded series are not D-finite. At least, one has an *infinite number of differentially algebraic series.* Are these globally bounded series D-D-finite series [42, 43] ?

5.3. Miscellaneous calculations.

Let us introduce the hypergeometric function:

$$F(x)\ \ =\ \ x\cdot\,(1\,-1728\cdot\,x)^{1/2}\cdot\,{}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1],\,1728\cdot\,x\Big)^2. \tag{102}$$

Note that the Schwarzian equation (17), on $\tilde{Q}(x)$, can be seen to be a consequence of (see (109) below):

$$F(x)\ \ =\ \ \frac{\tilde{Q}(x)}{\tilde{Q}(x)'}\qquad \hbox{together with:}\qquad W(x)\ \ =\ \ \frac{F''(x)}{F(x)}\ \ -\frac{1}{2}\cdot\Big(\frac{F'(x)}{F(x)}\Big)^2. \tag{103}$$

Therefore the nome $\tilde{Q}(x)$ is *also* solution of the order-one linear differential operator:

$$\mathcal{L}_1\ \ =\ \ F(x)\cdot\,D_x\ \ -1\qquad\qquad \hbox{where:} \tag{104}$$

$$F(x)\ \ =\ \ x\cdot\,(1\,-1728\cdot\,x)^{1/2}\cdot\,{}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1],\,1728\cdot\,x\Big)^2.$$

It is thus DD-finite†:

$$\frac{\tilde{Q}(x)'}{\tilde{Q}(x)} \, = \, \frac{1}{F(x)} \qquad \hbox{or:} \qquad \tilde{Q}(x) \, = \, \exp\Bigl(\int^x \frac{dx}{F(x)}\Bigr). \tag{105}$$

The one-parameter series $y(x) \, = \, y(a, \, x)$, given by (70), is solution of the rank-two equation (see (23))

$$A_R(x) \, \, \, - A_R(y(x)) \cdot y'(x) \, \, \, + \frac{y''(x)}{y'(x)} \, \, = \, \, 0, \tag{106}$$

with

$$A_R(x) \, \, = \, \, \frac{F'(x)}{F(x)}, \tag{107}$$

and also solution of the Schwarzian condition

$$W(x) \, \, \, - W(y(x)) \cdot y'(x)^2 \, \, \, + \{y(x), \, x\} \, \, \, = \, \, \, 0, \tag{108}$$

where:

$$W(x) \, = \, \frac{F''(x)}{F(x)} \, - \frac{1}{2} \cdot \Bigl(\frac{F'(x)}{F(x)}\Bigr)^2 \, = \, A_R'(x) \, + \, \frac{A_R(x)^2}{2}$$
$$= \, -\frac{1}{2} \cdot \, \frac{1 \, - 1968 \, x \, + 2654208 \, x^2}{x^2 \cdot \, (1 \, - 1728 \, x)^2}. \tag{109}$$

Note that $W(x)$ is a *rational function*, but this is far from being the case for $A_R(x)$. We will see, in the following, that the one-parameter series $y(x) \, = \, y(a, \, x)$, given by (70), *is also solution of*:

$$a \cdot \frac{\partial y(a, \, x)}{\partial a} \, = \, F(y(a, \, x)) \, = \, F(x) \cdot \frac{\partial y(a, \, x)}{\partial x}. \tag{110}$$

5.4. More one-parameter series solutions.

If one combines y_2, the "correspondence" series (36) solution of the modular equation (34), with the one-parameter series (70), one gets a one-parameter series

$$y_2^{(a)} \, = \, y(a, \, y_2) \, = \, \tilde{X}\Bigl(a \cdot \tilde{Q}(x)^2\Bigr) \, = \, a \cdot x^2 \, + 1488 \cdot a \cdot x^3$$
$$- 24 \cdot a \cdot (31 \, a - 85599) \cdot x^4 \, - 256 \cdot a \cdot (8649 \, a - 11180329) \cdot x^5$$
$$+ 12 \cdot a \cdot (29721 \, a^2 - 392019552 \, a + 338926406215) \cdot x^6 \tag{111}$$
$$+ 192 \cdot a \cdot (8292159 \, a^2 - 45872836768 \, a + 30686235044193) \cdot x^7 \, + \, \cdots$$

This series (111) is also solution of the Schwarzian equation (33). Furthermore we have:

$$2 \cdot y_2^{(a)} \cdot (1 \, - 1728 \cdot y_2^{(a)})^{1/2} \cdot {}_2F_1\Bigl([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, 1728 \cdot y_2^{(a)}\Bigr)^2$$
$$= \, x \cdot (1 \, - 1728 \cdot x)^{1/2} \cdot {}_2F_1\Bigl([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, 1728 \cdot x\Bigr)^2 \cdot \frac{dy_2^{(a)}}{dx}. \tag{112}$$

When $a \, = \, 1$, the radius of convergence of (111) is $1/1728 \, = \, 0.000578703703 \cdots$, and this is also the case for any a, N-th root of unity $a^N \, = \, 1$.

† See [42, 43].

More generally, all the series

$$\tilde{X}\Big(a \cdot \tilde{Q}(x)^n\Big) \;=\; a \cdot x^n \;+\; \cdots \tag{113}$$

have a radius of convergence corresponding, for a small enough, to the occurrence of the singularity of the nome-like series $\tilde{Q}(x)$, namely $x \,=\, 1/1728$.

Similarly to (111), if one combines y_3, the "correspondence" series (43) solution of the modular equation (41), with the one-parameter series (70), one gets a one-parameter series

$$\begin{aligned} y_3^{(a)} \;=\; y(a,\, y_3) \;=\; \tilde{X}\Big(a \cdot \tilde{Q}(x)^3\Big) \;=\; & \; a \cdot x^3 \;+ 2232 \cdot a \cdot x^4 \;+ 3911868 \cdot a \cdot x^5 \\ & - 24 \cdot a \cdot (31\,a - 265833940) \cdot x^6 \;- 54 \cdot a \cdot (61504\,a - 187769367601) \cdot x^7 \\ & - 1296 \cdot a \cdot (7351340\,a - 12322394107529) \cdot x^8 \;+\; \cdots \end{aligned} \tag{114}$$

This series (114) is also solution of the Schwarzian equation (33). Furthermore we have:

$$\begin{aligned} & 3 \cdot y_3^{(a)} \cdot (1 - 1728 \cdot y_3^{(a)})^{1/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot y_3^{(a)}\Big)^2 \\ & \quad = \; x \cdot (1 - 1728 \cdot x)^{1/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot x\Big)^2 \cdot \frac{dy_3^{(a)}}{dx}. \end{aligned} \tag{115}$$

Similarly:

$$\begin{aligned} y_5^{(a)}(x) \;=\; y\Big(a,\, y_5(x)\Big) \;=\; \tilde{X}\Big(a \cdot \tilde{Q}(x)^5\Big) \;=\; & \; a \cdot x^5 \;+ 3720 \cdot a \cdot x^6 \\ & + 9287460 \cdot a \cdot x^7 \;+ 19648405600 \cdot a \cdot x^8 \;+ 38124922672650 \cdot a \cdot x^9 \\ & - 24 \cdot a \cdot (31\,a - 2930432767154406) \cdot x^{10} \\ & - 40 \cdot a \cdot (138384\,a - 3142471028063763509) \cdot x^{11} \\ & - 960 \cdot a \cdot (25120323\,a - 229208433006295134073) \cdot x^{12} \;+\; \cdots \end{aligned} \tag{116}$$

This series (116) is also solution of the Schwarzian equation (33). Furthermore we have:

$$\begin{aligned} & 5 \cdot y_5^{(a)} \cdot (1 - 1728 \cdot y_5^{(a)})^{1/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot y_5^{(a)}\Big)^2 \\ & \quad = \; x \cdot (1 - 1728 \cdot x)^{1/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot x\Big)^2 \cdot \frac{dy_5^{(a)}}{dx}. \end{aligned} \tag{117}$$

One also easily gets:

$$5 \cdot F(y_5^{(a)}(x)) \;=\; F(x) \cdot \frac{dy_5^{(a)}(x)}{dx} \;=\; 5 \cdot a \cdot \frac{\partial y_5^{(a)}(x)}{\partial a}. \tag{118}$$

More generally, let us introduce the modular correspondence series $y_n(x) \,=\, x^n + 744 \cdot n \cdot x^{n+1} + \cdots$ (for $n \ge 2$), one can verify *that these series commute*. These modular correspondences $y_n(x)$ can easily be generalized to *one-parameter* series $y\Big(a,\, y_n(x)\Big)$ which are also solutions of the Schwarzian equations:

$$y\Big(a,\, y_n(x)\Big) \;=\; a \cdot x^n \;+ 744 \cdot n \cdot a \cdot x^{n+1} \;+\; \cdots \tag{119}$$

5.5. Composition in general

The one-parameter series (119) can be written

$$y_n^{(a)}(x) \; = \; \tilde{X}\Big(a \cdot \tilde{Q}(x)^n\Big). \tag{120}$$

We have the following composition:

$$y_n^{(a)}\Big(y_m^{(b)}(x)\Big) \; = \; \tilde{X}\Big(a \cdot \tilde{Q}\Big(\tilde{X}\Big(b \cdot \tilde{Q}(x)^m\Big)\Big)^n\Big) \; = \; \tilde{X}\Big(a \cdot \Big(b \cdot \tilde{Q}(x)^m\Big)^n\Big)$$
$$= \; \tilde{X}\Big(a \cdot b^n \cdot \tilde{Q}(x)^{m\,n}\Big) \; = \; y_{m\,n}^{(a\,b^n)}(x). \tag{121}$$

Note that the condition to have series solutions of the Schwarzian equation of the form $y_n^{(a)}(x) \; = \; a \cdot x^n \; + \; \cdots$, with $n \, \geq \, 2$, amounts to having [30, 31] $W(x)$ of the form $W(x) \; = \; -1/2/x^2 \; + \; \cdots$ which is satisfied when $F(x) \; = \; \alpha \cdot x \; + \; \cdots$, or $\tilde{Q}(x) \; = \; \rho \cdot x^{1/\alpha} \; + \; \cdots$

6. The one-parameter series (4) seen as a ϵ-expansion.

In the $a \; \rightarrow \; 1$ limit, let us denote $\epsilon \; = \; a \, - \, 1$. The one-parameter series $y(x) \; = \; y(a, \, x)$, given by (70), can, thus, be seen as an ϵ-expansion:

$$y(a, \, x) \quad = \quad x \quad + \sum_{n=1}^{\infty} \epsilon^n \cdot B_n(x), \tag{122}$$

where $B_1(x) \; = \; F(x)$, with $F(x)$ given by (102), and where $B_2(x)$ reads (see also equation (115) in [30]):

$$B_2(x) \; = \; \frac{1}{2} \cdot F(x) \cdot \Big(\frac{dB_1(x)}{dx} \; - 1\Big). \tag{123}$$

Assuming that (122) is solution of the Schwarzian condition (108) (with $W(x)$ given by (104)), we actually obtained the next $B_n(x)$'s:

$$B_3(x) \; = \; \frac{1}{3} \cdot F(x) \cdot \Big(\frac{dB_2(x)}{dx} \; - \; \frac{dB_1(x)}{dx} \; + 1\Big),$$
$$B_4(x) \; = \; \frac{1}{4} \cdot F(x) \cdot \Big(\frac{dB_3(x)}{dx} \; - \; \frac{dB_2(x)}{dx} + \frac{dB_1(x)}{dx} \; - 1\Big), \tag{124}$$
$$B_5(x) \; = \; \frac{1}{5} \cdot F(x) \cdot \Big(\frac{dB_4(x)}{dx} \; - \; \frac{dB_3(x)}{dx} + \frac{dB_2(x)}{dx} - \frac{dB_1(x)}{dx} \; + 1\Big),$$
$$B_6(x) \; = \; \frac{1}{6} \cdot F(x) \cdot \Big(\frac{dB_5(x)}{dx} \; - \; \frac{dB_4(x)}{dx} + \frac{dB_3(x)}{dx} - \frac{dB_2(x)}{dx} + \frac{dB_1(x)}{dx} \; - 1\Big), \quad \cdots$$

More generally, one easily discovers the recursion

$$(n+1) \cdot B_{n+1} \; + n \cdot B_n \; = \; F(x) \cdot \frac{dB_n(x)}{dx}, \tag{125}$$

which yields on the series (122)

$$\sum_n (n+1) \cdot B_{n+1} \cdot \epsilon^n \; + \sum_n n \cdot B_n \cdot \epsilon^n \; = \; F(x) \cdot \Big(\sum_n \frac{dB_n(x)}{dx} \cdot \epsilon^n\Big), \tag{126}$$

or

$$\frac{\partial \sum_n B_{n+1} \cdot \epsilon^{n+1}}{\partial \epsilon} \; + \epsilon \cdot \frac{\partial \sum_n B_n \cdot \epsilon^n}{\partial \epsilon} \; = \; F(x) \cdot \Big(\frac{\partial \sum_n B_n(x) \cdot \epsilon^n}{\partial x}\Big), \tag{127}$$

yielding finally

$$(1 + \epsilon) \cdot \frac{\partial y(a, x)}{\partial \epsilon} \;=\; F(x) \cdot \frac{\partial y(a,\, x)}{\partial x}, \tag{128}$$

namely:

$$a \cdot \frac{\partial y(a, x)}{\partial a} \;=\; F(x) \cdot \frac{\partial y(a,\, x)}{\partial x}. \tag{129}$$

Note that $y(a,\, x)$ is also solution of:

$$F\Bigl(y(a,\, x)\Bigr) \;=\; F(x) \cdot \frac{\partial y(a,\, x)}{\partial x}. \tag{130}$$

Recalling some relation on the nome q (see equation (33) in [30]):

$$\frac{q'}{q} \;=\; \frac{1}{F(x)} \qquad \hbox{or:} \qquad q \cdot \frac{d}{dq} \;=\; F(x) \cdot \frac{d}{dx}, \tag{131}$$

we see that relation (129) also reads more simply:

$$a \cdot \frac{\partial y(a, x)}{\partial a} \;=\; q \cdot \frac{\partial y(a, x)}{\partial q}. \tag{132}$$

which is reminiscent of the fact that changing $x \, \rightarrow \, y(a, x)$ just amounts, on the nome, to changing $q \, \rightarrow \, a \cdot q$. Equation (129) means that $y(a,\, x)$ is a function of

$$\int \Bigl(\frac{da}{a} \; + \frac{dx}{F(x)}\Bigr) \;=\; \ln(a) \; + \int \Bigl(\frac{dx}{F(x)}\Bigr), \tag{133}$$

or, recalling (105), a function of:

$$\exp\Bigl(\int \Bigl(\frac{da}{a} \; + \frac{dx}{F(x)}\Bigr)\Bigr) \;=\; a \cdot \tilde{Q}(x). \tag{134}$$

This is actually the case since $y(a,\, x)$ is nothing but $\tilde{X}\Bigl(a \cdot \tilde{Q}(x)\Bigr)$ (see (73)).

Remark 6.1 : Do note that the previous calculations *are still valid* when $F(x)$ *is not given by* (102). One can verify, for *any function* $F(x)$, that the ϵ-expansion (122) with coefficients B_n given by (123), (124), (125), *is actually solution of the Schwarzian relation* (108), with $W(x)$ given by (see (103), (109)):

$$W(x) \;=\; \frac{F''(x)}{F(x)} \; - \frac{1}{2} \cdot \Bigl(\frac{F'(x)}{F(x)}\Bigr)^2. \tag{135}$$

7. Generalization of $W(x)$ in the Schwarzian equation: adding an extra parameter α.

For a given function $F(x)$ let us consider the relation

$$F(y(x)) \;=\; F(x) \cdot \frac{dy(x)}{dx}, \tag{136}$$

which corresponds to:

$$\frac{dy}{F(y)} \;=\; \frac{dx}{F(x)} \;=\; \frac{dq}{q}. \tag{137}$$

From (136), namely $F(y) \, = \, F(x) \cdot y'$, one gets

$$F'(y) \cdot y' \;=\; F'(x) \cdot y' \; + F(x) \cdot y'', \tag{138}$$

or

$$\frac{F'(y)}{F(y)} \cdot y' \; = \; \frac{F'(x)}{F(x)} \; + \frac{y''}{y'}. \tag{139}$$

or, more generally, using (136) in order to introduce an extra parameter α:

$$\Big(\frac{F'(y)}{F(y)} \; + \frac{\alpha}{F(y)} \Big) \cdot y' \; = \; \Big(\frac{F'(x)}{F(x)} \; + \frac{\alpha}{F(x)} \Big) \; + \frac{y''}{y'}. \tag{140}$$

Let us introduce

$$A_R(x) \; = \; \frac{F'(x)}{F(x)} \; + \frac{\alpha}{F(x)}, \tag{141}$$

we see that (140) can be written

$$A_R(x) \; - A_R(y) \cdot y' \; + \frac{y''}{y'} \; = \; 0. \tag{142}$$

which is (23) of section (1.6). From (138), that we rewrite

$$F'(y) \; = \; F'(x) \; + F(x) \cdot \frac{y''}{y'}, \tag{143}$$

one gets

$$F''(y) \cdot y' \; = \; F''(x) \; + F'(x) \cdot \frac{y''}{y'} \; + F(x) \cdot \Big(\frac{y''}{y'} \Big)', \tag{144}$$

or, using (136), written $F(x) \; = \; F(y)/y'$:

$$\frac{F''(y)}{F(y)} \cdot y'^2 \; = \; \frac{F''(x)}{F(x)} \; + \frac{F'(x)}{F(x)} \cdot \frac{y''}{y'} \; + \Big(\frac{y''}{y'} \Big)'. \tag{145}$$

Taking the square of (139) one gets (up to a factor 2):

$$\frac{1}{2} \cdot \Big(\frac{F'(y)}{F(y)} \Big)^2 \cdot y'^2 \; = \; \frac{1}{2} \cdot \Big(\frac{F'(x)}{F(x)} \Big)^2 + \frac{1}{2} \cdot \Big(\frac{y''}{y'} \Big)^2 + \frac{F'(x)}{F(x)} \cdot \frac{y''}{y'}. \tag{146}$$

From (145) and (146) we deduce:

$$\Big(\frac{F''(y)}{F(y)} - \frac{1}{2} \cdot \Big(\frac{F'(y)}{F(y)} \Big)^2 \Big) \cdot y'^2 \; = \; \frac{F''(x)}{F(x)} \; - \frac{1}{2} \cdot \Big(\frac{F'(x)}{F(x)} \Big)^2 \; + \Big(\frac{y''}{y'} \Big)' - \frac{1}{2} \cdot \Big(\frac{y''}{y'} \Big)^2,$$

or, recalling the Schwarzian derivative,

$$\frac{F''(x)}{F(x)} \; - \frac{1}{2} \cdot \Big(\frac{F'(x)}{F(x)} \Big)^2 \; - \Big(\frac{F''(y)}{F(y)} \; - \frac{1}{2} \cdot \Big(\frac{F'(y)}{F(y)} \Big)^2 \Big) \cdot y'^2 \; + \{y(x), \, x\} \; = \; 0,$$

or, more generally, using (136), which allows to introduce an extra parameter α

$$\frac{F''(x)}{F(x)} \; - \frac{1}{2} \cdot \Big(\frac{F'(x)}{F(x)} \Big)^2 \; + \frac{1}{2} \cdot \frac{\alpha^2}{F(x)^2} \tag{147}$$

$$- \Big(\frac{F''(y)}{F(y)} - \frac{1}{2} \cdot \Big(\frac{F'(y)}{F(y)} \Big)^2 \; + \frac{1}{2} \cdot \frac{\alpha^2}{F(y)^2} \Big) \cdot y'^2 \; + \{y(x), \, x\} \; = \; 0.$$

Note that (147) is actually of the form

$$W(x) \; - W(y(x)) \cdot y'(x)^2 \; + \{y(x), \, x\} \; = \; 0, \tag{148}$$

where (A_R given by (141)):

$$W(x) \; = \; \frac{F''(x)}{F(x)} \; - \frac{1}{2} \cdot \Big(\frac{F'(x)}{F(x)} \Big)^2 \; + \frac{1}{2} \cdot \frac{\alpha^2}{F(x)^2} \; = \; A'_R(x) \; + \; \frac{A_R(x)^2}{2}. \tag{149}$$

Remark 7.1: Note that these calculations *also work* with

$$\mu \cdot F(y(x)) \; = \; F(x) \cdot \frac{dy(x)}{dx}, \tag{150}$$

which corresponds to (40), (47), (54), (62).

8. An "academical" Schwarzian equation: $W(x)$ is no longer a rational function

Recalling

$$F(x) \;=\; x \cdot (1 - 1728 \cdot x)^{1/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot x\Big)^2, \qquad (151)$$

the one-parameter series $y(x) = y(a, x)$, given by (70), is, for any value of α, solution of the rank-two equation

$$A_R(x) \;-A_R(y(x)) \cdot y'(x) \;+\frac{y''(x)}{y'(x)} \;=\; 0, \qquad (152)$$

with

$$A_R(x) \;=\; \frac{F'(x)}{F(x)} \;+\frac{\alpha}{F(x)}, \qquad (153)$$

but is also solution of the Schwarzian condition

$$W(x) \;-W(y(x)) \cdot y'(x)^2 \;+\{y(x), x\} \;=\; 0, \qquad (154)$$

where:

$$W(x) \;=\; \frac{F''(x)}{F(x)} \;-\frac{1}{2} \cdot \Big(\frac{F'(x)}{F(x)}\Big)^2 \;+\frac{1}{2} \cdot \frac{\alpha^2}{F(x)} \;=\; A'_R(x) \;+\; \frac{A_R(x)^2}{2}$$
$$=\; -\frac{1}{2} \cdot \frac{1 - 1968\, x + 2654208\, x^2}{x^2 \cdot (1 - 1728\, x)^2} \;+\frac{1}{2} \cdot \frac{\alpha^2}{F(x)}. \qquad (155)$$

For *generic* values of α, the solution-series of the form $a \cdot x + \cdots$, of the rank-two equation (152), as well as the Schwarzian equation (154), with $W(x)$ given by (155), is just the one-parameter series $y(x) = y(a, x)$, given by (70). However, for a selected set of values of α, namely (non-zero) *integer values*, the solution-series of the form $a \cdot x + \cdots$, becomes a *two-parameters series*. For instance, for $\alpha = \pm 1$, the extra parameter occurs with the coefficient of x^2, for $\alpha = \pm 2$ the extra parameter occurs with the coefficient of x^3, ... and, more generally, for $\alpha = \pm N$ the extra parameter occurs with the coefficient of x^{N+1}. Let us display the $\alpha = 1$ case in detail.

8.1. The $\alpha = 1$ case: two-parameters series

Let us consider the case $\alpha = 1$ in the Schwarzian equation (154) with (155), or in the rank-two relation (152) with (153).

The *two-parameter* series

$$y(a, b, x) \;=\; a \cdot x \;+\Big(1728 \cdot b - 744 \cdot a \cdot (a-1)\Big) \cdot x^2$$
$$+\Big(2985984 \cdot a \cdot b^2 \;- 2571264 \cdot a \cdot (a-1) \cdot b \qquad (156)$$
$$+ 36 \cdot a \cdot (a-1) \cdot (9907\, a - 20845)\Big) \cdot x^3$$
$$+\Big(5159780352 \cdot a \cdot b^3 \;- 6664716288 \cdot a \cdot (a-1) \cdot b^2$$
$$+ 186624 \cdot (9907\, a^2 - 30752\, a + 19022) \cdot a \cdot b$$
$$- 32 \cdot a \cdot (a-1) \cdot (4386286\, a^2 - 20490191\, a + 27274051)\Big) \cdot x^4 \;+\; \cdots$$

is *actually solution of the Schwarzian equation* (154) with (155), or the rank-two relation (152) with (153), for $\alpha \, = \, 1$. Note that the *two-parameter* series (156) is also solution† of

$$a \cdot \, {{\partial y(a, \, b, \, x) } \over {\partial a}} \, \, \, + b \cdot \, {{\partial y(a, \, b, \, x) } \over {\partial b}} \,\, \, = \, \, \, F\Big(y(a, \, b, \, x)\Big), \tag{157}$$

with $F(x)$ given by (151). We have the following composition rules for the *two-parameter* series (156):

$$y\Big(a', \, b', \, y(a, \, b, \, x)\Big) \, \, \, = \, \, \, y\Big(a\, a', \, a^2 \, b' \, + a'\, b, \, x\Big). \tag{158}$$

Let us introduce an *alternative parametrization* of the two-parameter series (156), changing b into $a\,b$, in (156):

$$Y(a, \, b, \, x) \, \, = \, \, \, a \cdot \, x \, \, \, + \Big(1728 \cdot \, a\,b \, - 744 \cdot \, a \cdot \, (a-1) \Big) \cdot \, x^2$$
$$+ \Big(2985984 \cdot \, a^3 \cdot \, b^2 \, \, - 2571264 \cdot \, a^2 \cdot \, (a-1) \cdot \, b$$
$$+ 36 \cdot \, a \cdot \, (a-1) \cdot \, (9907\,a -20845) \Big) \cdot \, x^3 \, \, \, + \, \, \cdots \tag{159}$$

We have the following composition rules for the *two-parameter* series (159)

$$Y\Big(a', \, b', \, Y(a, \, b, \, x)\Big) \, \, \, = \, \, \, Y\Big(a\, a', \, a\, b' \, + b, \, x\Big), \tag{160}$$

which corresponds to:

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix} \, \, = \, \, \begin{bmatrix} a\,a' & a\,b' \, + b \\ 0 & 1 \end{bmatrix}. \tag{161}$$

The series (159) is, now, solution of:

$$a \cdot \, {{\partial Y(a, \, b, \, x) } \over {\partial a}} \, \, \, = \, \, \, F\Big(Y(a, \, b, \, x)\Big), \tag{162}$$

with $F(x)$ given by (151). Let us introduce the $a \, \rightarrow \, 0$ limit:

$$Q_b(x) \, \, = \, \, \, \lim_{a \rightarrow \, 0} \, {{ Y(a, \, b, \, x)} \over {a}}$$
$$= \, \, \, x \, \, \, + (744 + 1728\,b) \cdot \, x^2 \, \, \, + (750420 + 2571264\,b + 2985984\,b^2) \cdot \, x^3$$
$$+ (872769632 + 3549961728\,b \, + 6664716288\,b^2 + 5159780352\,b^3) \cdot \, x^4$$
$$+ (1102652742882 + 4945819779072\,b + 11680775258112\,b^2$$
$$+ 15355506327552\,b^3 + 8916100448256\,b^4) \cdot \, x^5$$
$$+ (1470561136292880 + 7027977959274240\,b + 19050621395927040\,b^2$$
$$+ 32624754548539392\,b^3 + 33167893667512320\,b^4 + 15407021574586368\,b^5) \cdot \, x^6$$
$$+ \, \, \cdots \tag{163}$$

In the $b \, \rightarrow \, 0$ limit, this series (163) reduces to the nome series (75) or (76).

In the $a \, \rightarrow \, \infty$ limit one gets:

$$X_b(x) \, \, = \, \, \, \lim_{a \rightarrow \, \infty} \, Y\Big(a, \, b, \, {{x} \over {a}}\Big) \, \, \, = \, \, \, x \, \, \, - 744\,x^2 \, \, + 356652\,x^3 \, \, - 140361152\,x^4$$
$$+ 49336682190\,x^5 \, \, - 16114625669088\,x^6 \, \, + 4999042477430456\,x^7$$
$$- 1492669384085015040\,x^8 \, + 432762759484818142437\,x^9 \, \, \, + \, \, \cdots \tag{164}$$

† However it is *not* solution of $F(x) \cdot \, y' \, = \, F(y)$ or $F(x) \cdot \, y' \, = \, a \, {{\partial y} \over {\partial a}}$.

This series (164) is nothing but (74) or (77), and, thus, *does not depend* on the second parameter b.

One actually finds that the two parameter series (159) is *nothing but*:

$$Y(a,\, b,\, x) \;=\; X_b\Big(a \cdot Q_b(x)\Big). \tag{165}$$

From (165) we can also deduce that (162), is, in fact, nothing but equation:

$$a \cdot \frac{\partial X_b(a \cdot x)}{\partial a} \;=\; F\Big(X_b(a \cdot x)\Big). \tag{166}$$

Furthermore, since $a \cdot \frac{\partial X_b(a \cdot x)}{\partial a} \;=\; x \cdot \frac{\partial X_b(a \cdot x)}{\partial x}$, relation (166) also gives:

$$x \cdot \frac{\partial X_b(a \cdot x)}{\partial x} \;=\; F\Big(X_b(a \cdot x)\Big). \tag{167}$$

In contrast with the $b = 0$ case, the two functions, Q_b and X_b, given by the two limits (163), (164), are *not compositional inverse*. In the $a \to 1$ limit, the decomposition (165) becomes:

$$\begin{aligned} Y(1,\, b,\, x) \;&=\; X_b\Big(Q_b(x)\Big) \\ &=\; x \;+ 1728 \cdot b \cdot x^2 \,+ 2985984 \cdot b^2 \cdot x^3 \,+ 186624 \cdot (27648\, b^2 \,- 1823) \cdot b \cdot x^4 \\ &\quad + 110592 \cdot (80621568\, b^3 \,- 15947604\, b \,- 5249233) \cdot b \cdot x^5 \;+\; \cdots \end{aligned} \tag{168}$$

The series (168) is a one-parameter family of commuting series:

$$Y\Big(1,\, b,\, Y(1,\, b',\, x)\Big) \;=\; Y\Big(1,\, b',\, Y(1,\, b,\, x)\Big) \;=\; Y(1,\, b + b',\, x). \tag{169}$$

In particular the compositional inverse of $Y(1,\, b,\, x)$ is $Y(1,\, -b,\, x)$:

$$Y\Big(1,\, b,\, Y(1,\, -b,\, x)\Big) \;=\; Y\Big(1,\, -b,\, Y(1,\, b,\, x)\Big) \;=\; x. \tag{170}$$

Note that:

$$Q_b\Big(X_b(x)\Big) \;=\; \frac{x}{1 \,- 1728 \cdot b \cdot x} \;=\; x \;+ 1728 \cdot b \cdot x^2 \;+ \cdots \tag{171}$$

From (171) we deduce an alternative expression for $Q_b(x)$ in terms of the nome (16) (i.e. the compositional inverse of (164), or, equivalently $Q_b(x)$ for $b = 0$):

$$Q_b(x) \;=\; \frac{Q_0(x)}{1 \,- 1728 \cdot b \cdot Q_0(x)}. \tag{172}$$

Note that the composition rule relation (160) can, now, be seen as a straightforward consequence of relation (172). From relation (172) one can see that the radius of convergence of the series (163) corresponds, for small enough values of the additional parameter b, to the singularity of $Q_0(x)$, (i.e. $R = 1/1728$), and for large enough values of the parameter b, to the singularity $Q_0(x) = 1/1728/b$, namely:

$$x \;=\; X_b\Big(\frac{1}{1728\, b}\Big) \;=\; \tilde{X}\Big(\frac{1}{1728\, b}\Big). \tag{173}$$

Remark 8.1: Do note that, in contrast with the $\alpha = 0$ case, *there is no* solution-series of the form $a \cdot x^2 + \cdots$ or, more generally, of the form $a \cdot x^N + \cdots$ with $N \neq 1$, of the Schwarzian equation (154), when $W(x)$ is given by (155). This corresponds to the fact that, when $\alpha \neq 0$, $W(x)$ is no longer of the form $W(x) = -1/2/x^2 + \cdots$ (see [30, 31]).

9. Polynomial examples for $F(x)$.

Modular correspondences, modular curves, correspond to a (transcendental) function $F(x)$ associated to elliptic functions like (102), (151).

Let us now recall the general results of section (6), which describes the *one-parameter* solution-series (122) of the Schwarzian equation (108), and also the partial differential equations (129), (130), and the fact that these equations *are actually valid for any function* $F(x)$.

Let us consider, here, the one-parameter functions $y(a, x)$, corresponding to miscellaneous *polynomial* examples of functions $F(x)$, that are, thus, far from being associated with the previous "classical" modular forms [14, 15, 19] and hypergeometric/elliptic functions [44, 45].

From the general results of the previous section (6) we will thus get a set of miscellaneous examples. All the corresponding one-parameter series, below, will verify the composition rule:

$$y\Bigl(a, y(a', x)\Bigr) \;=\; y(a\,a',\; x). \qquad (174)$$

All these one-parameter series will also verify:

$$F\Bigl(y(a, x)\Bigr) \;=\; a \cdot {{\partial y(a, x)} \over {\partial a}} \;=\; F(x) \cdot {{\partial y(a, x)} \over {\partial x}}. \qquad (175)$$

One will also consider a polynomial that will be the truncation of the hypergeometric function (102). One will, then, get a one-parameter solution-series, very similar‡ to (70), which also verifies the composition rule (72), but does not correspond to globally bounded series [72].

9.1. A first simple polynomial example for $F(x)$

Let us now consider the polynomial

$$F(x) \;=\; x \cdot (1\,-373 \cdot x) \cdot (1\,-371 \cdot x) \;=\; x \;-744\,x^2 \;+138383\,x^3, \qquad (176)$$

which has the *same first two terms* as the series expansion of the hypergeometric function (102). The function $W(x)$ in the Schwarzian equation, given by (135), reads:

$$W(x) \;=\; -{{1} \over {2}} \cdot {{1\,-830298\,x^2 \,+411827808\,x^3 \,-57449564067\,x^4} \over {x^2 \cdot (1\,-373\,x)^2 \cdot (1\,-371\,x)^2}}. \qquad (177)$$

A solution of the Schwarzian equation, with $W(x)$ given by (177), reads:

$$y(a, x) \;=\; a \cdot x \,-744 \cdot a \cdot (a-1) \cdot x^2 \,+{{1} \over {2}} \cdot a \cdot (1245455\,a \,-968689) \cdot (a-1) \cdot x^3$$
$$-\,620 \cdot a \cdot (885656\,a \,-470507) \cdot (a-1)^2 \cdot x^4 \;+\; \cdots \qquad (178)$$

Let us introduce the two limits

$$\tilde{Q}(x) \;=\; \lim_{a \rightarrow 0} {{y(a,\; x)} \over {a}} \;=\; x \;+744\,x^2 \;+{{968689} \over {2}}\,x^3 \;+291714340\,x^4 \;+\; \cdots \qquad (179)$$

and:

$$\tilde{X}(x) \;=\; \lim_{a \rightarrow \infty} y\Bigl(a,\; {{x} \over {a}}\Bigr) \;=\; x \;-744\,x^2 \;+{{1245455} \over {2}} \cdot x^3 \;-549106720\,x^4$$
$$+{{3989599188003} \over {8}} \cdot x^5 \;-461623555588416\,x^6 \;+{{6928370820171415659} \over {16}} \cdot x^7$$
$$-\,410201463628637176320\,x^8 \;+\; \cdots \qquad (180)$$

‡ The three first terms are the same.

The one-parameter series (178) can actually be written explicitly as:

$$y(a,\, x) \;\; = \;\; \tilde{X}\Bigl(a \cdot \tilde{Q}(x)\Bigr). \qquad (181)$$

The series (178) clearly reduces, at $a = 1$, to $y(1,\, x) = x$. Using this remark, together with the decomposition (181), we see that the series $\tilde{X}(x)$ *must be* the *compositional inverse* of the "nome-like" series $\tilde{Q}(x)$.

These two series, (179) and (180), are solutions of the two Schwarzian equations†

$$\{\tilde{Q}(x),\, x\} \;\; + \frac{1}{2 \cdot \tilde{Q}(x)^2} \cdot \Bigl(\frac{d\tilde{Q}(x)}{dx}\Bigr)^2 \;\; + W(x) \;\; = \;\; 0, \qquad (182)$$

and

$$\{\tilde{X}(x),\, x\} \;\; - \frac{1}{2 \cdot x^2} \;\; - W\Bigl(\tilde{X}(x)\Bigr) \cdot \Bigl(\frac{d\tilde{X}(x)}{dx}\Bigr)^2 \;\; = \;\; 0, \qquad (183)$$

where $W(x)$ is given by (177).

In fact one can find a closed exact *algebraic* expression for the "nome-like" series $\tilde{Q}(x)$. Recalling (131) one can write:

$$\tilde{Q}(x) \;\; = \;\; \exp\Bigl(\int \frac{dx}{F(x)}\Bigr) \;\; = \;\; x \cdot \frac{(1\, - 371\, x)^{371/2}}{(1\, - 373\, x)^{373/2}}$$

$$= \;\; x \;\; + 744\, x^2 \;\; + \frac{968689}{2}\, x^3 \;\; + 291714340\, x^4 \;\; + \;\; \cdots \qquad (184)$$

Taking into account the fact that $\tilde{X}(x)$ is the compositional inverse of $\tilde{Q}(x)$, one can rewrite (181) as a functional equation on the one-parameter series $y(a,\, x)$ given by (178):

$$a \cdot \tilde{Q}(x) \;\; = \;\; \tilde{Q}\Bigl(y(a,\, x)\Bigr). \qquad (185)$$

Since $\tilde{Q}(x)$ is an *algebraic function*, we see, from (185), that the one-parameter series $y(a,\, x)$, given by (178), *is actually an algebraic series for any value of* the parameter a (and not only N-th root of unity).

Using the *algebraic* expression of $\tilde{Q}(x)$, given in (184), one deduces that the series $y \, = \, y(a,\, x)$ is actually solution of:

$$a^2 \cdot x^2 \cdot \frac{(1\, - 371\, x)^{371}}{(1\, - 373\, x)^{373}} \;\; - y^2 \cdot \frac{(1\, - 371\, y)^{371}}{(1\, - 373\, y)^{373}} \;\; = \;\; 0. \qquad (186)$$

Taking into account the large degree in x or y of the (polynomial) condition (186), one should note that it can actually be quite difficult to get this (polynomial) equation from¶ a large series (178).

The series $\tilde{X}(x)$ is also an *algebraic series* $y \, = \, \tilde{X}(x)$, solution of:

$$x^2 \cdot (1\, - 373 \cdot y)^{373} \;\; - \;\; y^2 \cdot (1 - 371 \cdot y)^{371} \;\; = \;\; 0. \qquad (187)$$

Note that, even with a very large series (180), it is also quite hard, because of the high degree in y of (187), to find the algebraic expression (187), *even if it is really simple.*

Let us, now, introduce the series

$$y_2 \;\; = \;\; \tilde{X}\Bigl(\tilde{Q}(x)^2\Bigr) \;\; = \;\; x^2 \;\; + 1488\, x^3 \;\; + 1521481\, x^4 \;\; + 1301919152\, x^5$$

$$+ \frac{1996564263793}{2}\, x^6 \;\; + 708980642952488\, x^7 \;\; + \;\; \cdots \qquad (188)$$

† The same Schwarzian equations as (17) and (18) in subsection (1.4).

¶ Using, for instance, the command seriestoalgeq of gfun of Bruno Salvy.

This series (188) is an *algebraic* series, solution of $\tilde{Q}(y) = \tilde{Q}(x)^2$. The compositional inverse of series (188), is the series

$$y_{1/2} \; = \; \tilde{X}\Big(\tilde{Q}(x)^{1/2}\Big) \; = \; \omega \cdot x^{1/2} \; - 744\, x \; + {{1246199} \over {2}} \cdot \omega \cdot x^{3/2}$$
$$- \, 549660256\, x^2 \; + {{3995160282965} \over {8}} \cdot \omega \cdot x^{5/2} \; + \; \cdots \qquad (189)$$

where $\omega^2 = 1$. This series (189) is an algebraic series, solution of $\tilde{Q}(x) = \tilde{Q}(y)^2$. These two series, (188) and (189), are solutions of the Schwarzian equation with $W(x)$ given by (177).

Remark 9.1: The algebraic equations $\tilde{Q}(y) = \tilde{Q}(x)^2$ and $\tilde{Q}(y) = \tilde{Q}(x)^2$, and their corresponding algebraic series solutions (188) and (189), could be seen to be the "equivalent" of the modular equation (34), and its corresponding algebraic series solutions (36) and (37). However, one should note that the *modular equations*, like (34), are $x \leftrightarrow y$ symmetric, and, consequently, the modular equation (34) represents $q \rightarrow q^2$ *and* $q \rightarrow q^{1/2}$ *in the same time* (see series (36) *but also* (37)). In contrast, $\tilde{Q}(y) = \tilde{Q}(x)^2$ (resp. $\tilde{Q}(y) = \tilde{Q}(x)^2$) *breaks the* $x \leftrightarrow y$ *symmetry*. Therefore, the "equivalent" of the modular equation (34) is rather:

$$\Big(\tilde{Q}(y) \, - \, \tilde{Q}(x)^2\Big) \cdot \Big(\tilde{Q}(x) \, - \, \tilde{Q}(y)^2\Big) \; = \; 0. \qquad (190)$$

The one-parameter series (178) verifies the composition rule:

$$y\Big(a, y(a', \, x)\Big) \; = \; y(a\, a', \; x). \qquad (191)$$

The series (178) also verifies the relations:

$$F\Big(y(a, \, x)\Big) \; = \; F(x) \cdot {{\partial y(a, \, x)} \over {\partial x}} \; = \; a \cdot {{\partial y(a, \, x)} \over {\partial a}}. \qquad (192)$$

9.2. Truncation of the hypergeometric function $F(x)$*.*

The hypergeometric function $F(x)$ given by (102), expands as $x - 744\, x^2 - 393768\, x^3 + \cdots$ Let us consider a simple truncation of this hypergeometric function:

$$F(x) \; = \; x \; - 744\, x^2 \; - 393768\, x^3. \qquad (193)$$

From (135) one deduces:

$$W(x) \; = \; -{{1} \over {2}} \cdot {{1 \, + 2362608\, x^2 \, - 1171853568\, x^3 \, - 465159713472\, x^4} \over {x^2 \cdot \, (1 \, - 744\, x \, - 393768\, x^2)^2}}. \qquad (194)$$

The Schwarzian equation (108) with the previous $W(x)$, namely (194), has the following *one-parameter* solution-series:

$$y(a, x) \; = \; a \cdot x \; - 744 \cdot a \cdot (a-1) \cdot x^2 \; + 36 \cdot a \cdot (a-1) \cdot (9907\, a - 20845) \cdot x^3$$
$$- \, 80352 \cdot a \cdot (a-1)^2 \cdot (264\, a - 9379) \cdot x^4 \qquad (195)$$
$$- \, 648 \cdot a \cdot (a-1)^2 \cdot (250310357\, a^2 + 598043050\, a - 1207272939) \cdot x^5$$
$$+ {{482112} \over {5}} \cdot a \cdot (a-1)^3 \cdot (1944308192\, a^2 - 424834349\, a - 8498464743) \cdot x^6 \; + \; \cdots$$

This one-parameter series (195) is quite similar† to the one-parameter series (70). The series (195) actually verifies the composition rule:

$$y\Big(a, y(a', \, x)\Big) \; = \; y(a\, a', \; x). \qquad (196)$$

† The first three coefficients are actually the same.

Let us introduce the two limits

$$\tilde{Q}(x) \;=\; \lim_{a \rightarrow 0} {{y(a, \, x)} \over {a}} \;=\; x \;+744\,x^2 \;+750420\,x^3 \;+753621408\,x^4 \tag{197}$$
$$+\,782312864472\,x^5 \;+\; {{4097211834177216} \over {5}}\,x^6 \;+\; {{4331866321367059104} \over {5}}\,x^7 \;+\; \cdots$$

and:

$$\tilde{X}(x) \;=\; \lim_{a \rightarrow \infty} y\Bigl(a, \, {{x} \over {a}}\Bigr) \;=\; x \;-744\,x^2 \;+356652\,x^3 \;-21212928\,x^4 \tag{198}$$
$$-\,162201111336\,x^5 \;+\; {{937374311061504} \over {5}}\,x^6 \;-\; {{563689525139743392} \over {5}}\,x^7 \;+\; \cdots$$

Again these two series (197) and (198) verify the Schwarzian equations (182) and (183) but with $W(x)$ now given by (194). One verifies that the one-parameter series (195) *is actually of the form*:

$$y(a,x) \;=\; \tilde{X}\Bigl(a \cdot \tilde{Q}(x)\Bigr). \tag{199}$$

Again, from (199) and from the fact that $y(a, \, x) \, = \, x$ for $a \, = \, 1$, we see that the series (198) is actually the *compositional inverse* of the "nome-like" series (197):

$$y(1,x) \;=\; x \;=\; \tilde{X}\Bigl(\tilde{Q}(x)\Bigr). \tag{200}$$

The *one-parameter* series (195) is also solution of

$$a \cdot {{\partial y(a, \, x)} \over {\partial a}} \;=\; F(y(a, \, x)) \;=\; F(x) \cdot {{\partial y(a, \, x)} \over {\partial x}}, \tag{201}$$

and one can deduce, from (199), the following functional equation on the one-parameter series $y(a, \, x)$ given by (195):

$$\tilde{Q}\Bigl(y(a, \, x)\Bigr) \;=\; a \cdot \tilde{Q}(x). \tag{202}$$

Conversely, from (202), we get, recalling (137)

$${{dy} \over {F(y(a, \, x))}} \;=\; {{dx} \over {F(x)}} \;+\; {{da} \over {a}}, \tag{203}$$

which gives for a fixed

$$F(y(a, \, x)) \;=\; F(x) \cdot {{\partial y(a, \, x)} \over {\partial x}}, \tag{204}$$

and for x fixed:

$$a \cdot {{\partial y(a, \, x)} \over {\partial a}} \;=\; F(y(a, \, x)). \tag{205}$$

Let us introduce the series¶

$$y_2 \;=\; \tilde{X}\Bigl(\tilde{Q}(x)^2\Bigr) \;=\; x^2 \;+1488\,x^3 \;+2053632\,x^4 \;+2621653632\,x^5$$
$$+\,3244440682476\,x^6 \;+\; {{19627900112688192} \over {5}}\,x^7 \;+\; {{23401843163094440736} \over {5}}\,x^8$$
$$+\; {{1931791653412087472593 92} \over {35}}\,x^9 \;+\; \cdots \tag{206}$$

This series (206) is solution of the Schwarzian equation (108), with $W(x)$ given by (194), and is also solution of

$$2 \cdot F(y_2) \;=\; F(x) \cdot {{\partial y_2} \over {\partial x}}, \tag{207}$$

¶ Note that this series has the same first three coefficients than series (36).

i.e.

$$2 \cdot \Big(y_2 \; - 744\, y_2^2 \; - 393768\, y_2^3\Big) \; = \; \Big(x \; - 744\, x^2 \; - 393768\, x^3\Big) \cdot \frac{\partial y_2}{\partial x}, \qquad (208)$$

and one also has:

$$\tilde{Q}\Big(y_2(x)\Big) \; = \; \tilde{Q}(x)^2. \qquad (209)$$

Let us, now, introduce the *one-parameter* series

$$\begin{aligned} y_2^{(a)} \; &= \; y(a,\, y_2) \; = \; \tilde{X}\Big(a \cdot \tilde{Q}(x)^2\Big) \; = \; a \cdot x^2 \; + 1488\; a \cdot x^3 \\ &- 24 \cdot a \cdot (31\, a - 85599) \cdot x^4 \; - 35712 \cdot a \cdot (62\, a - 73473) \cdot x^5 \\ &+ 36 \cdot a \cdot (9907\, a^2 - 130673184\, a + 90254015568) \cdot x^6 \\ &+ \frac{160704}{5} \cdot a \cdot (49535\, a^2 - 262999040\, a + 122399922528) \cdot x^7 \; + \; \cdots \end{aligned} \qquad (210)$$

This one-parameter series (210) is solution of the Schwarzian equation (108), with $W(x)$ given by (194). It is also solution of

$$\tilde{Q}\Big(y_2^{(a)}(x)\Big) \; = \; a \cdot \tilde{Q}(x)^2, \qquad (211)$$

and also solution of

$$2 \cdot F(y_2^{(a)}) \; = \; F(x) \cdot \frac{\partial y_2^{(a)}}{\partial x} \; = \; 2 \cdot a \cdot \frac{\partial y_2^{(a)}}{\partial a}. \qquad (212)$$

where

$$F(x) \; = \; x \; - 744\, x^2 \; - 393768\, x^3 \; = \; x \cdot \Big(1 - p \cdot x\Big) \cdot \Big(1 - q \cdot x\Big), \qquad (213)$$

with:

$$p \; = \; 372 \; + 6 \cdot 14782^{1/2}, \qquad q \; = \; 372 \; - 6 \cdot 14782^{1/2}, \qquad (214)$$

Let us denote

$$\alpha \; = \; \frac{1}{2} \cdot \frac{p+q}{q-p} \; = \; -\frac{31}{14782} \cdot 14782^{1/2} \; = \; -0.25497 \; \cdots \qquad (215)$$

Following the previous calculations in subsection (9.1), one easily finds that the "nome-like" series (197) reads:

$$\begin{aligned} \tilde{Q}(x) \; &= \; \frac{x \cdot (1 - p \cdot x)^{p/(q-p)}}{(1 - q \cdot x)^{q/(q-p)}} \; = \; \frac{x}{\Big((1 - p \cdot x) \cdot (1 - q \cdot x)\Big)^{1/2}} \cdot \Big(\frac{1 - p \cdot x}{1 - q \cdot x}\Big)^{\alpha} \\ &= \; x \; + 744\, x^2 \; + 750420\; x^3 \; + 753621408\, x^4 \; + 782312864472\; x^5 \\ &+ \frac{4097211834177216}{5}\, x^6 \; + \frac{4331866321367059104}{5}\, x^7 \; + \; \cdots \end{aligned} \qquad (216)$$

This "nome-like" series (216) is *actually D-finite.* It is solution of the order-one linear differential operator ($\theta \, = \, x \cdot D_x$ is the homogeneous derivative):

$$\begin{aligned} \mathcal{L}_1 \; &= \; F(x) \cdot D_x \; - 1 \; = \; (x \; - 744\, x^2 \; - 393768\, x^3) \cdot D_x \; - 1 \\ &= \; (1 \; - 744\, x \; - 393768\, x^2) \cdot \theta \; - 1. \end{aligned} \qquad (217)$$

The radius of convergence of the "nome-like" series (216) is $1/p$, with p given by (214):

$$R \; = \; \frac{1}{p} \; = \; \frac{14782^{1/2}}{65628} \; - \frac{31}{32814} \; = \; 0.0009078632370 \cdots \qquad (218)$$

This "nome" series (216) is D-finite, with a finite radius of convergence, *but it is not globally bounded*. Note that $\tilde{X}(x)$ is only a *differentially algebraic function*.

Note that the order-one linear differential operator $\mathcal{L}_1$, given by (217), is *not globally nilpotent* [77]. The corresponding p-curvatures are null (or nilpotent that is the same for order-one linear differential operators) for the following primes:

$$3,\ 11,\ 13,\ 17,\ 23,\ 31,\ 47,\ 61,\ 73,\ 79,\ 89,\ 101,\ \cdots \tag{219}$$

but non-zero for the following primes:

$$5,\ 7,\ 19,\ 29,\ 37,\ 41,\ 43,\ 53,\ 59,\ 67,\ 71,\ \cdots \tag{220}$$

Note that, since $14782 = 2 \cdot 19 \cdot 389$, we could have expected that one does not see the transcendence of the "nome" mod. 19, the "nome" reducing to an algebraic function (see (214), (215)), and, thus, one could expect a zero p-curvature. This is not the case.

Note that the exponent of the "nome-like" series (216), at the singularity $x = 1/p$, is

$$\frac{p}{q-p} \;=\; -\frac{1}{2} \;-\frac{31}{14782^{1/2}} \;=\; -0.7549735291\ \cdots \tag{221}$$

which is not a rational number. This rules out the fact that the order-one linear differential operator (217) could be globally nilpotent [77].

Let us consider the simplest example of series $y(a, x)$, namely the (involutive) series (195) for $a = -1$:

$$\begin{aligned} y(-1, x) \;=\; & -x \;-1488\,x^2 \;-2214144\,x^3 \;-3099337344\,x^4 \;-4030574598144\,x^5 \\ & -\frac{23640158283604992}{5}\,x^6 \;-\frac{23310435220175683584}{5}\,x^7 \\ & -\frac{20590422517553304526848}{7}\,x^8 \;+\frac{12494610391145690921435136}{7}\,x^9 \;+\;\cdots \end{aligned} \tag{222}$$

Calculating the first fifty coefficients of this series, one can see that this (involutive) series is *not globally bounded*.

10. Comments and speculations on differentially algebraic series.

We have displayed miscellaneous series solutions of Schwarzian equations (and thus having a compositional property [30, 31]), which can be seen to be, or to generalize, *modular correspondences* [70]. We remark that we have the following situation: we have series depending on *one* parameter (sometimes *two parameters* for slightly "academical" examples like in subsection (8.1)), which reduce to series with *integer* coefficients for an *infinite set* of values of the parameter(s), namely the *integer values*‡. These one-parameter series are *generically, only differentially algebraic*, even for *integer* values of the parameter (where they are probably not even D-finite, see for instance (98)). In contrast, and remarkably, when the parameter is a N-th root of unity, the generically differentially algebraic one-parameter series become *algebraic functions*. We thus have an *infinite number* of *algebraic functions*.

It is interesting to note that a totally and utterly similar situation have been seen to occur in other very interesting situations in physics, or enumerative combinatorics.

‡ More generally, for *rational values* of the parameters we have globally bounded differentially algebraic series.

Along this line, *differentially algebraic* series with *integer* coefficients¶ exist, and correspond to remarkable solutions of *differentially algebraic* equations in physics, or enumerative combinatorics, like *λ-extensions of Ising correlation functions* [78, 79], or solutions of a differentially algebraic Tutte equation [80]. We have an *infinite set* of differentially algebraic series with *integer* coefficients that are *not* D-finite [80, 78, 79]. We also have the occurence of an *infinite number* of *algebraic series* for an *infinite set* of Tutte-Beraha values of the λ parameter. Note that these selected values can also be seen as N-th root of unity situation.

At first sight, these Tutte-Beraha examples [80], or λ-extension of correlation functions of the Ising model [78, 79], *are not related to Schwarzian equations* with their composition function properties†. Is it possible that such differentially algebraic series could also reduce (in a more or less involved way ...) to exact decompositions like $X\Big(\omega \cdot Q^n(x)\Big)$, that we found systematically throughout this paper, since many of the results of this paper are, in fact, consequences of such exact decompositions ?

• One motivation of this paper was to understand the very nature of the one-parameter series $y(a, x)$: we have seen that this series cannot be solution of an order-N linear differential operator (for some integer N independent of the parameter a) with coefficients polynomials in x and in the parameter a.

• The relation between the Schwarzian equations (such that $W(x) = -1/2/x^2 + \cdots$, see [30, 31]), and *modular correspondences* was also an important motivation. The solutions of the Schwarzian equations are larger that just the (infinite) set of "modular correspondences", *precisely* because of the occurrence of one-parameter series $y(a, x)$. Along this line we have first seen that the solution of the Schwarzian equations can actually correspond to series with *more than one parameter*. Modular correspondences are associated with modular curves and modular forms [14, 15, 16]. Consequently, another question was to know if one can generalize these concepts *beyond* the elliptic curves and modular forms framework.

We have also shown, with very simple (polynomial) examples for the function $F(x)$, that these structures can actually be generalized *far beyond* the elliptic curve (modular curve, Shimura curves, modular form, automorphic form) framework. Along this line, a first polynomial example (9.1) provides an example of *one-parameter series* $y(a, x)$, *algebraic for any value of the parameter*. We also found that the equivalent of the nome is a simple *algebraic* function (square root of a rational function). With that example one also understands why it can be extremely hard to see that some series are algebraic, even if the algebraic function to guess is of a quite simple form. Furthermore, a "truncated" example (9.2) shows that the "modular equation-like" series (see for instance (206), (222)) can actually be *non globally bounded*. The "nome-like" series is a *non globally bounded* but *still* D-finite, series (see (217)), the corresponding linear differential operator being *non globally nilpotent*.

11. Conclusion

This paper provides a simple, and pedagogical, illustration of exact non-linear symmetries in physics (exact representations of the renormalization group

¶ Not simply reducible to ratio of globally bounded D-finite series, or composition of globally bounded D-finite series.

† These λ-extension of Ising correlation functions are solutions of *Painlevé equations* [78, 79].

transformations like the *Landen transformation* for the square Ising model [32, 33], ...) and is a strong incentive to discover more differentially algebraic equations involving fundamental symmetries, and to develop more *differentially algebraic* series analysis in physics [80, 81], beyond examples like the square-lattice Ising model [81, 78, 79, 88, 89].

In this paper we first focused, essentially, on identities relating the *same* hypergeometric function with *two different* algebraic pullback transformations related by modular equations. This corresponds to the "classical" *modular forms* [19] (resp. automorphic forms) that emerged so many times in physics [46, 47, 72]: these algebraic transformations can be seen as simple illustrations of exact representations of the renormalization group of some Yang-Baxter integrable models [32, 33, 69]. These transformations are seen to be solutions of some Schwarzian relation.

The Schwarzian relation is seen to "encapsulate", in one differentially algebraic (Schwarzian) equation, all the *modular forms* and *modular equations* of the theory of elliptic curves. The Schwarzian condition can thus be seen as some quite fascinating "pandora box", which encapsulates an *infinite number* of highly remarkable modular equations, and a whole "universe" of *Belyi-maps*‡. It is important, however, to underline that these Schwarzian conditions are actually richer than just elliptic curves, and go beyond†† "simple" restrictions [87] to pullbacked ${}_2F_1$ hypergeometric functions. In a more general perspective, such Schwarzian conditions occur in Malgrange's pseudo-group approach [61, 62, 63, 68] of $\mathcal{D}$-enveloppes. At this level of mathematical abstraction, the question of a *modular correspondence interpretation* of these "Schwarzian" series was clearly an open question. This paper sheds some light on this open question. It sheds some light on the very nature of a one-parameter series solution of the Schwarzian equation, which is *not* generically a modular correspondence series, but *actually reduces* to an *infinite set of modular correspondence series* for an infinite set of (N-th root of unity) values of the parameter. This paper also provides (polynomial) examples that are very similar to modular correspondence series, but are far beyond the elliptic curves framework.

Acknowledgments: We would like to thank A. Bostan and G. Casale, for very fruitful discussions. We also thank A. Bostan for some p-curvature calculations.

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‡ Belyi-maps [82, 83, 84, 85, 86] are central to Grothendieck's program of "dessins d'enfants".
†† See the two Heun functions given by (164) in [71].

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