Infinite discrete symmetry group for the Yang–Baxter equations: spin models ☆

M.P. Bellon, J.-M. Maillard [1] and C.-M. Viallet [1]
Laboratoire de Physique Théorique et des Hautes Energies, Université de Paris, Tour 16, 1er étage, boîte 126, 4 Place Jussieu, F-75252 Paris Cedex 05, France

Received 25 February 1991; revised manuscript received 5 May 1991; accepted for publication 20 May 1991
Communicated by A.R. Bishop

We show that the star-triangle equation possesses an infinite discrete group of symmetry. This group is the Coxeter group $A_2^{(1)}$. It explains the presence of the spectral parameter in solutions of the equations. We describe a strategy for the resolution of the equations, and apply it to specific examples.

1. Introduction

The Yang–Baxter equations [1] and their higher dimensional generalizations are now considered as the defining relations of integrability. They are the "deus ex machina" in a number of domains of mathematics and physics (knot theory, quantum inverse scattering, S-matrix factorization, exactly solvable models in statistical mechanics, Bethe ansatz, quantum groups, chromatic polynomials and more awaited deformation theories). The appeal of these equations comes from their ability to *give global results from local ones*. For instance, they are a sufficient and, to some extent, necessary [2] condition for the commutation of families of transfer matrices of arbitrary size and even of corner transfer matrices. From the point of view of topology, one may understand these relations by considering them as the generation of a large set of discrete deformations of the lattice. This point of view underlies most studies in knot theory and statistical mechanics ($\mathbb{Z}$-invariance [3,4]).

It is a challenging problem to exhibit and classify all solutions of this highly overdetermined set of equations. Up to now, the quest for new solutions has largely proceeded by slight modifications or new avatars of the already known solutions. For instance, the six-vertex model has been revisited in an incredible number of ways in different domains of mathematical physics [#1]. It is rather easy to take different representations of a particular solution and quantum groups emerged as a way of writing Yang–Baxter equations independently of the representation [5].

We want to analyze the Yang–Baxter equations and their higher dimensional generalizations [6–9] without prejudice about what should be a solution, that is to say proceed by necessary conditions.

We will exhibit an infinite discrete group of transformations acting on the ingredients of the Yang–Baxter equations or their higher dimensional generalizations (tetrahedron equations).

These transformations act as an automorphy group of various quantities of interest in statistical mechanics (partition function, critical manifolds, phase diagram, ...), and are of great help for calculations, even outside of the domain of integrability [10].

What we show here is that *they form a group of*

☆ Work supported by CNRS.

[1] Also at: Research Institute for Theoretical Physics, University of Helsinki, Siltavuorenpenger 20C, SF-00170 Helsinki, Finland.

[#1] The six-vertex model emerges from the critical standard scalar Potts model and the Lieb–Temperley algebra, in the symmetric Ashkin–Teller model, in the S-matrix of the sine-Gordon model, in the XXZ quantum Hamiltonian. The list is not exhaustive.

symmetries of the equations defining integrability. They consequently appear as a group of automorphisms of the algebraic varieties parametrizing the solutions of the Yang–Baxter or tetrahedron equations. We will denote this group Aut.

When searching for new solutions, one should keep in mind the action of Aut globally on the whole parameter space.

The existence of Aut drastically constrains the varieties where solutions may be found. This group of symmetries is a powerful tool for burning the haystack and home towards the (integrable) needle. In the general case, it has infinite orbits and gives severe constraints on the algebraic varieties which parametrize the possible solutions (genus zero or one curves, algebraic surfaces which are not of the general type, ...). In the non-generic case, when Aut has finite order orbits, the algebraic varieties can be of general type, but the very finiteness condition allows for their determination.

In the framework of infinite group representations, it is crucial to recognize the essential difference between what these symmetry groups are for the Yang–Baxter equation and what they are for the higher dimensional tetrahedron and hyper-simplicial relations: the number of involutions generating our groups increases from 2 to 2^{d-1} when passing from two-dimensional to d-dimensional models and the group jumps from the semi-direct product $\mathbb{Z} \ltimes \mathbb{Z}_2$ to a much larger group, i.e. a group with an exponential growth with the length of the word.

It is worth recalling that for the Zamolodchikov solution of the tetrahedron relation [7,6], the partition function is the same as the one of the two-dimensional checkerboard Ising model. This example seems to indicate that three-dimensional integrability can only occur when the 2^{d-1} generators of the group satisfy additional relations allowing for a mere polynomial growth of the size, and possibly reducing to a semi-direct product of finite groups and $\mathbb{Z}$ factors.

The existence of Aut as a symmetry for the equations has the following consequence: we may say that solving the Yang–Baxter equation is equivalent to solving all its images by Aut. These images *generically tend to proliferate*, simply because Aut is infinite. Considering that the equations form an overdetermined set, it is easy to believe that the total set of equations is "less overdetermined" when the orbits of Aut are of finite order. One can therefore imagine that the best candidates for the integrability varieties are precisely the ones where the symmetry group possesses finite orbits: the exact solutions of Au-Yang et al. [11–13] seem to confirm this point of view [14,15].

A contrario, if one gets hold of an apparently isolated solution, the action of Aut will multiply it until building up, in experimentally not so rare cases, a continuous family of solutions from the original one. This is the solution to the so-called Baxterization problem (see ref. [16]). In all this, there is a subtle interplay between generic (infinite, possibly ergodic) and particular (finite) orbits of the symmetry group Aut.

We first show that the simplest example of the Yang–Baxter relation which is the star–triangle relation [1] has an infinite discrete group of symmetries generated by three involutions. These involutions are deeply linked with the so-called inversion relations [17–20].

We then draw the practical consequences of the existence of this symmetry on the resolution of the star–triangle equations. The examples we investigate are the five-state chiral Potts model, for which we recover the known results of refs. [21–25]. We also consider a particular six-state spin model.

In ref. [16], this analysis is extended to the Yang–Baxter, the tetrahedron and hyper-simplicial equations, for vertex models in any dimension. This could also be done in a similar manner for the "generalized star–triangle relation" of interaction round the face models [26,27].

2. The star–triangle relations

2.1. The setting

We consider a spin model with nearest neighbour interactions on a square lattice of size $M \times M'$, with periodic boundary conditions. The spin σ_i can take q values. The Boltzmann weight for an oriented bond $\langle ij \rangle$ will be denoted hereafter by $w(\sigma_i, \sigma_j)$. The weights $w(\sigma_i, \sigma_j)$ can be seen as the entries of a $q \times q$ matrix. In the following we will introduce a pictorial representation of the star–triangle relation. An arrow will be associated to the oriented bond $\langle ij \rangle$. The arrow from i to j indicates that the argument of the Boltzmann weight w is (σ_i, σ_j) rather than (σ_j, σ_i). This arrow will be relevant only for the so-called chiral models, that is to say that the $q\times q$ matrix describing w is not symmetric. An interesting class of $q\times q$ matrices has been extensively investigated in the last few years [11–13]: the general cyclic matrices. It is important to note that we *do not* restrict ourselves to this particular class of matrices. We will for instance use the following non-cyclic non-symmetric 6×6 matrix as another illustrative example:

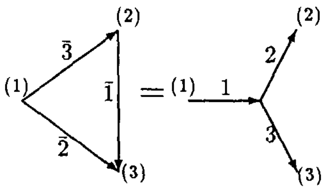
$$\begin{pmatrix} x & y & z & y & z & z \\ z & x & y & z & y & z \\ y & z & x & z & z & y \\ y & z & z & x & z & y \\ z & y & z & y & x & z \\ z & z & y & z & y & x \end{pmatrix} . \qquad (1)$$

2.2. The relations

We introduce the star–triangle equations both analytically [#2] and pictorially:

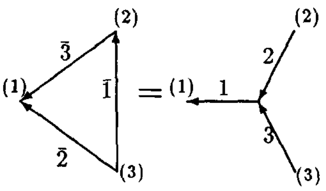
$$\sum_{\sigma} w_1(\sigma_1, \sigma) w_2(\sigma, \sigma_2) w_3(\sigma, \sigma_3)$$

$$= \lambda \bar{w}_1(\sigma_2, \sigma_3) \bar{w}_2(\sigma_1, \sigma_3) \bar{w}_3(\sigma_1, \sigma_2) . \qquad (2)$$



(st1.1)

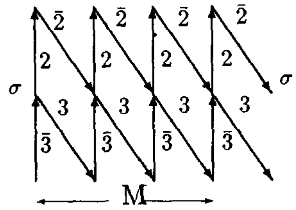
We give to the reader as an exercise to see that to satisfy eq. (2) *together with* the relation obtained by reversing all arrows



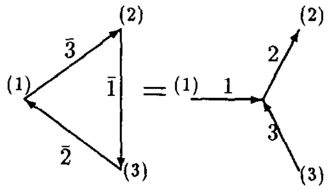
(st.1.2)

[#2] Since the w_i and $\bar{w}_i$ are homogeneous variables, there will always be a global multiplicative factor λ floating around in the star–triangle equations.

is a *sufficient* condition for the commutation of the diagonal transfer matrices of *arbitrary size M* with periodic boundary conditions $\mathbb{T}_M(w_2, \bar{w}_2)$ and $\mathbb{T}_M(\bar{w}_3, w_3)$:

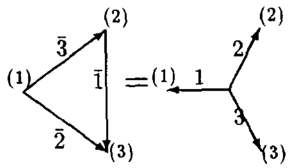


Note that for cyclic matrices [11–13] the star–triangle relations (st1.1) and (st1.2) give the same equation since one exchanges (st1.2) and (st1.1) by spin reversal. There are other star–triangle relations corresponding to other choices for the arrows on the six bonds of the star and triangle, namely



(st2.1)

and of course its associated twin (st2.2) obtained by reversal of all arrows or



(st3.1)

and its twin(st3.2). Of course, (1), (2) and (3) are on the same footing and therefore their permutation will give rise to equivalent star–triangle relations. One could obviously imagine many other choices for the arrows on the six bonds; however, they do not lead to the commutation of diagonal transfer matrices. We therefore have three systems of equations to study: (st1.1 and st1.2) or (st2.1 and st2.2) or (st.3.1 and st3.2).

For example, if the Boltzmann weights are given by the 6×6 matrix (1), (st1.1) will correspond to 20 equations, and (st1.2) gives the same equations. On the other hand, (st2.1) leads to 35 equations and (st2.2) to the same equations also, and finally (st3.1) and (st3.2) give the same system of 36 equations.

The spin σ_i may belong to some finite group (not necessarily Abelian, the Fisher–Griess group F1 would perfectly do), and the weights be functions of the group element $\sigma_i\sigma_j^{-1}$ [28]. This is the case for the model (1) with the group of permutations of three elements $\mathcal{S}_3$. It is easy to see that the number of equations for one star–triangle relation is not N^3 but at most N^2.

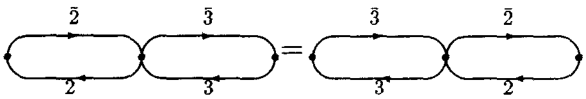
2.3. First necessary conditions

The interest of the star–triangle relations is that it implies the commutation of transfer matrices of arbitrary size M. This is what makes the model integrable. It is an extremely severe condition on the Boltzmann weights. It is an overdetermined set of equations.

This starts with size $M=1$, i.e., a two-site transfer matrix, the two sites being identified by the periodic boundary conditions. For relation (st1.1), this commutation means

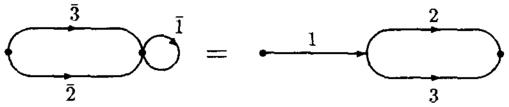
$$\sum_{\sigma'} w_2(\sigma',\sigma)\bar{w}_2(\sigma,\sigma')w_3(\sigma'',\sigma)\bar{w}_3(\sigma',\sigma'')$$
$$=\sum_{\sigma'} w_3(\sigma',\sigma)\bar{w}_3(\sigma,\sigma')w_2(\sigma'',\sigma)\bar{w}_2(\sigma',\sigma''),\qquad(3)$$

whose pictorial representation is



Other choices of arrow arrangement will lead to slightly different relations.

More simple constraints on the matrix of Boltzmann weights may be obtained straightforwardly from the star–triangle relation. For example if we pinch spin (2) and (3) in (st1.1), we get



i.e

$$\bar{w}_2(\sigma,\sigma'')\bar{w}_3(\sigma,\sigma'')\bar{w}_1(\sigma'',\sigma'')$$
$$=\sum_{\sigma'} w_1(\sigma,\sigma')w_2(\sigma',\sigma'')w_3(\sigma',\sigma'').\qquad(4)$$

These relations point towards a commutation condition on the matrices of Boltzmann weights themselves, although they are weaker. Notice that matrices of the form (1) do commute.

3. Infinite discrete symmetry group for the star–triangle relation

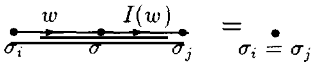
3.1. The inversion relation

Two distinct inverses act on the matrix of Boltzmann weights: the matrix inverse I and the dyadic (element by element) inverse J. We write down the inverse relations both analytically and pictorially:

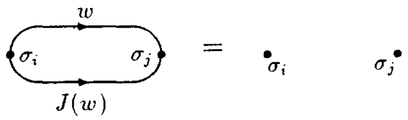
$$\sum_{\sigma} w(\sigma_i,\sigma)I(w)(\sigma,\sigma_j)=\mu\delta_{\sigma_i\sigma_j},\qquad(5)$$

$$w(\sigma_i,\sigma_j)J(w)(\sigma_i,\sigma_j)=1,\qquad(6)$$

where $\delta_{\sigma_i\sigma_j}$ denotes the usual Kronecker delta,



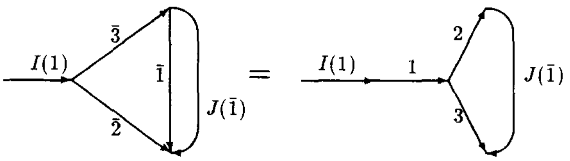
and

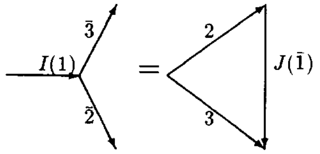


The two involutions I and J generate an infinite discrete group Γ (Coxeter group) isomorphic to the infinite dihedral group $\mathbb{Z}_2 \ltimes \mathbb{Z}$. The $\mathbb{Z}$ part of Γ is generated by IJ. In the parameter space of the model, that is to say some projective space $\mathbb{C}P_{n-1}$ (n homogeneous parameters), I and J are birational involutions. They give a *non-linear representation* of this Coxeter group by an infinite set of birational transformations [29]. It may happen that the action of Γ on specific points yields a finite orbit. This means that the representation of Γ identifies with the p-dihedral group $\mathbb{Z}_2 \ltimes \mathbb{Z}_p$. However this is not the generic situation, and these points often happen, remarkably enough, to lie on distinguished subvarieties of the parameter space [29].

3.2. The symmetries of the star–triangle relations

The two inversions I and J act on the star–triangle relation. We first give a pictorial representation of this action, starting from (st1.1) as an example;





(tst)

The transformed equation reads

$$\lambda \sum_{\sigma_1} I(w_1)(\tau, \sigma_1) \bar{w}_1(\sigma_2, \sigma_3)$$

$$\times \bar{w}_2(\sigma_1, \sigma_3) \bar{w}_3(\sigma_1, \sigma_2) J(\bar{w}_1)(\sigma_2, \sigma_3)$$

$$= \sum_{\sigma_1} \sum_{\sigma} I(w_1)(\tau, \sigma_1) w_1(\sigma_1, \sigma)$$

$$\times w_2(\sigma, \sigma_2) w_3(\sigma, \sigma_3) J(\bar{w}_1)(\sigma_2, \sigma_3) , \qquad (7)$$

or equivalently, using the definition of the inverses I and J (5), (6):

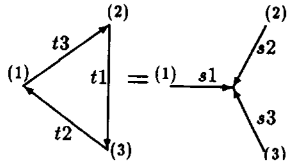
$$\lambda \sum_{\sigma_1} I(w_1)(\tau, \sigma_1) \bar{w}_2(\sigma_1, \sigma_3) \bar{w}_3(\sigma_1, \sigma_2)$$

$$= w_2(\tau, \sigma_2) w_3(\tau, \sigma_3) J(\bar{w}_1)(\sigma_2, \sigma_3) . \qquad (8)$$

We get an action on the space of solutions of the star–triangle relation.

If $(w_1, w_2, w_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ is a solution of eq. (2) (see picture (st1.1) for the specific arrangement of arrows), then $(I(w_1), \bar{w}_3, \bar{w}_2, J(\bar{w}_1), w_3, w_2)$ is also a solution of eq. (2), at the price of a permitted redefinition of λ. In this transformation, the weights w_1 and $\bar{w}_1$ play a special role.

At this point, it is better to formalize this action by introducing some notations. We may choose as a reference relation $\mathcal{ST}$ the symmetric configuration



$(\mathcal{ST})$

Any configuration may be obtained by reversing some arrows and permuting some bonds. With evident notations, we will denote by R_{s1}, R_{s2}, R_{s3}, R_{t1}, R_{t2}, R_{t3} the reversals of arrows, and by $P_{si,sj}$, $P_{si,tj}$, $P_{ti,tj}$ the permutations of bonds. Moreover I and J act on the bonds as I_{s1}, I_{s2}, ...

To illustrate this notation, configuration (st1.1) is obtained from the reference configuration by the action of $R_{s2}R_{s3}R_{t2}$, and we may write

$$(\mathrm{st1.1}) = R_{s2}R_{s3}R_{t2} \cdot (\mathcal{ST}) . \qquad (9)$$

The action of I and J described above (where (1) was playing a special role) identifies with the action of

$$\mathcal{K}_1 = R_{s2}R_{t3}I_{s1}J_{t1}P_{s2,t3}P_{s3,t2} . \qquad (10)$$

This action gives indeed (tst), that is to say

$$(\mathrm{tst}) = R_{s2}R_{s3}R_{t2}I_{s1}J_{t1}P_{s2,t3}P_{s3,t2} \cdot (\mathcal{ST}) . \qquad (11)$$

It is easy to check that $\mathcal{K}_1$ is an involution.

We may construct two similar involutions $\mathcal{K}_2$ and $\mathcal{K}_3$, obtained by cyclic permutation of the indices 1, 2, 3. We consequently have

$$\mathcal{K}_1 = R_{s2}R_{t3}I_{s1}J_{t1}P_{s2,t3}P_{s3,t2}, \quad \mathcal{K}_1^2 = 1 , \qquad (12)$$

$$\mathcal{K}_2 = R_{s3}R_{t1}I_{s2}J_{t2}P_{s3,t1}P_{s1,t3}, \quad \mathcal{K}_2^2 = 1 , \qquad (13)$$

$$\mathcal{K}_3 = R_{s1}R_{t2}I_{s3}J_{t3}P_{s1,t2}P_{s1,t1}, \quad \mathcal{K}_3^2 = 1 . \qquad (14)$$

If σ is the cyclic permutation $\sigma = \sigma_s \sigma_t$ with $\sigma_s = P_{s2,s3}P_{s1,s2}$ and $\sigma_t = P_{t2,t3}P_{t1,t2}$, the involutions $\mathcal{K}_i$ are related by

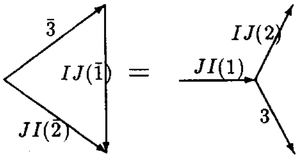
$$\mathcal{K}_2 = \sigma^2 \mathcal{K}_1 \sigma, \quad \mathcal{K}_3 = \sigma^2 \mathcal{K}_2 \sigma . \qquad (15)$$

The involutions $\mathcal{K}_i$ $(i = 1, 2, 3)$ verify the defining relations of the *Weyl group of an affine algebra of type* $A_2^{(1)}$ [30]:

$$(\mathcal{K}_1\mathcal{K}_2)^3 = (\mathcal{K}_2\mathcal{K}_3)^3 = (\mathcal{K}_3\mathcal{K}_1)^3 = 1 . \qquad (16)$$

We will denote by Aut the group generated by the three involutions $\mathcal{K}_i$ $(i = 1, 2, 3)$. We have constructed a non-linear representation of Aut.

This symmetry group contains an action of IJ. It may be obtained by successively operating with the previous involutions: first act with $\mathscr{K}_1$, then with $\mathscr{K}_3$, then operate with $\mathscr{K}_2$ and finally with $\mathscr{K}_1$. This sequence of operations, when used on relations (st1.1), yields first (tst), and then a sequence of relations ending with



(IJst1.1)

This sequence of transformations amounts to acting with the product

$$G_3 = \sigma \mathscr{K}_1 \mathscr{K}_2 \mathscr{K}_3 \mathscr{K}_1$$
$$= R_{s1} R_{s2} R_{t1} R_{t2} (JI)_{s1} (IJ)_{s2} (IJ)_{t1} (JI)_{t2} . \quad (17)$$

We may define similarly

$$G_2 = \sigma G_3 \sigma^2 = R_{s3} R_{s1} R_{t3} R_{t1} (JI)_{s3} (IJ)_{s1} (IJ)_{t3} (JI)_{t1} , \quad (18)$$

$$G_1 = \sigma G_2 \sigma^2 = R_{s3} R_{s2} R_{t3} R_{t2} (IJ)_{s3} (JI)_{s2} (JI)_{t3} (IJ)_{t2} . \quad (19)$$

We have the relations

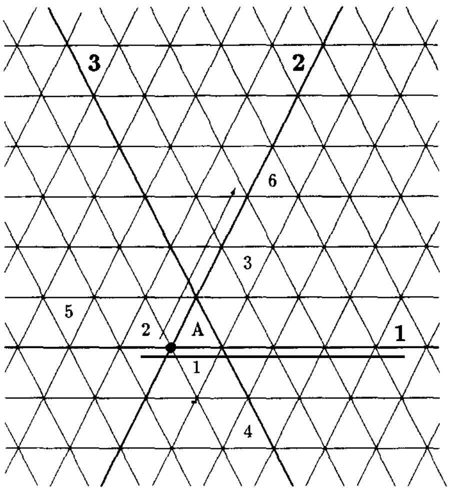
$$G_1 G_2 G_3 = 1 , \quad (20)$$

$$G_i G_j = G_j G_i \quad \forall\, i, j = 1, 2, 3 . \quad (21)$$

The symmetry group Aut is the semi-direct product of the Weyl group of an A_2 (finite dimensional simple of rank 2) Lie algebra by a bidimensional lattice translation group $\mathbb{Z} \times \mathbb{Z}$.

Note that we have constructed a *non-linear representation* of Aut by birational transformations on the parameter space $\mathbb{C}P_N$ (see refs. [29,31]).

The following figure shows the usual representation of Aut with reflections, in the root space of $A_2^{(1)}$. The involutions $\mathscr{K}_i$ ($i = 1, 2, 3$) are reflections around the straight lines 1, 2, 3. The fundamental alcove is the triangle denoted A. The numbers 1, ..., 6 appearing in other triangles are the successive images of A under the successive action of $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_3$, $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_3$. We see that the product $\mathscr{K}_3 \mathscr{K}_2 \mathscr{K}_1 \mathscr{K}_3 \mathscr{K}_2 \mathscr{K}_1$ is a translation of the lattice along axis 2, pictured by the arrow.



4. A strategy for the resolution of the star–triangle equation

It would be of interest to detail the action of Aut on the star–triangle relations for a general matrix of weights, but we will not do it here. We want to use the symmetry Aut both to furnish reasonable ansätze for the form of the matrix of Boltzmann weights, having in mind a reduction of the number of parameters, and to achieve the resolution of the equations. To do so we will choose specific forms for this matrix, preserved by the two basic inversions I and J. This permits the reduction of the number of parameters, without trivializing the action of Aut. We have at our disposal the "admissible" patterns introduced in refs. [29,32]. We will use two specific examples: the five-state chiral Potts model, and the six-state model introduced above (see eq. (1)).

Moreover we know that the solutions of the star–triangle relation (and more generally Yang–Baxter or (hyper)simplicial equations) lie on algebraic varieties in the space of parameters (homogeneous space) [33,34].

The symmetry group Aut acts on these varieties as a group of automorphisms. Since Aut is infinite, this constrains very strongly these algebraic varieties. A large proportion of the analysis of the star–triangle relations relies on the use of the symmetry Aut, and more precisely on the study of the orbits of Aut, or even of IJ.

In the two examples we shall consider, generic orbits of IJ lie on algebraic curves whose equations are known.

4.1. First solutions

There are always solutions of the star–triangle equations. We have one for example if we take

$$w_1(\sigma_i, \sigma_j) = \delta_{\sigma_i \sigma_j}, \quad \bar{w}_1(\sigma_i, \sigma_j) = 1 ,$$
$$w_2 = \bar{w}_3, \quad w_3 = \bar{w}_2 . \tag{22}$$

They are unfortunately a trivialization of some of the six weights entering the equations, and are singular points for the action of Aut. These solutions (which single out 1) are not a very good starting point for a perturbation, since we may not move the weights w_1 or $\bar{w}_1$. They have however been used for IRF models to construct a perturbative relation: the so-called differential star–triangle relation [35].

Other solutions named disorder solutions, for which a dimensional reduction of the model happens, are known [36,37]. The symmetry group Aut acts on these solutions, and generates an infinity of image varieties. The partition function is an automorphic function for the action of Γ, and is known exactly on the disorder varieties. The outcome is the very rich structure of the (multivalued) partition function on the image varieties, and the singularization of *critical and integrable* varieties (see the study of the standard scalar checkerboard Potts model [38]).

4.2. The star–triangle relation for the five-state chiral Potts model

The matrix of Boltzmann weights for the five-state chiral Potts model is the cyclic matrix

$$\begin{pmatrix} w(0) & w(1) & w(2) & w(3) & w(4) \\ w(4) & w(0) & w(1) & w(2) & w(3) \\ w(3) & w(4) & w(0) & w(1) & w(2) \\ w(2) & w(3) & w(4) & w(0) & w(1) \\ w(1) & w(2) & w(3) & w(4) & w(0) \end{pmatrix} . \tag{23}$$

The original detailed proof of the star–triangle relation for the chiral Potts model can be found in ref. [39]. The cyclicity of this matrix immediately reduces the number of equations, since one may shift all spins in the star–triangle relation by the same amount. Moreover any star–triangle relation and its twin, obtained by reversing all arrows, leads to the same set of equations.

4.2.1. The symmetric five-state model

We consider first the non-chiral model obtained by setting $w(1) = w(4)$, and $w(2) = w(3)$. The weight matrix (23) is then symmetric, and the previous distinctions between different star–triangle relations due to the presence of arrows disappears. The number of equations drops to 13.

In this model a linear pencil of algebraic curves (elliptic curves) emerges in the study of the orbits of the group Γ generated by I and J [29]. The fraction

$$\Delta = \frac{(u^2+v^2+3uv)(u-1)(v-1)}{2u^2v^2+2uv-(u^3+v^3)-uv(u+v)} , \tag{24}$$

where $u = w(1)/w(0)$ and $v = w(2)/w(0)$, is invariant by I and J. The curves in the pencil have as equation $\Delta = \text{const}$.

It is known that the pairs $(1, \bar{1})$ (respectively $(2, \bar{2})$ and $(3, \bar{3})$) play a symmetric role and have to lie on the same algebraic variety in $\mathbb{C}P_2 \times \mathbb{C}P_2$ [40]. We will assume that w_1 and $\bar{w}_1$ (respectively 2, 3) lie on the same algebraic curve. This finds its justification in our study of the orbits of the group Γ in $\mathbb{C}P_2 \times \mathbb{C}P_2$: the orbits of $(w_1, \bar{w}_1)$ under $IJ \times JI$ lie on curves *only if Δ is the same for w_1 and $\bar{w}_1$* [29,41]. We choose such a curve, that is to say a definite value of Δ.

If this is a generic curve, i.e. if IJ has infinite order, we may essentially bring $(w_1, w_2, w_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ to an "isotropic" point where $w_1 = w_2 = w_3$. Strictly speaking one would have to consider an infinite sequence which of course converges.

From such a definite solution (one point $(w_1, w_2, w_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$), the action of Aut yields more solutions.

The isotropic point verifies $\bar{w}_1 = \bar{w}_2 = \bar{w}_3$. Why go to this isotropic point? Simply because in this isotropic limit, the number of equations (and of unknowns) reduces considerably (five equations), 1, 2, and 3 being on the same footing in the star–triangle relations. The point is that we can solve this system explicitly.

We use the variables $u = w(1)/w(0)$, $v = w(2)/w(0)$, $\bar{u} = \bar{w}(1)/\bar{w}(0)$, $\bar{v} = \bar{w}(2)/\bar{w}(0)$. By elimination of $\bar{u}$ and $\bar{v}$, we find that u and v have to be a solution of

$$(1-4y-4y^2+y^3+y^4)(1+y+y^2+y^3+y^4)$$
$$\times(y^4+16y^3-19y^2-4y+1)(-1+4y-2y^2$$
$$-14y^3+12y^4+4y^5-16y^6+8y^7)$$
$$\times(4y^2+2y-1)^9(y-1)^{13}$$
$$\times(4y^7+4y^6+2y^5+6y^4-y^3+5y^2-y+1)y^{13}$$
$$\times(11y^3+12y^2+3y-1)$$
$$\times(4y^4+24y^3+14y^2-16y-1)=0\,, \tag{25}$$

and we calculate directly $\bar{u}$ and $\bar{v}$ from the star–triangle relations.

The factor $-1-2y^2-14y^3+4y+12y^4+4y^5-16y^6+8y^7$ corresponds to an illicit elimination. Among the other factors, a number correspond to various trivializations. The only relevant conditions are

$$1-4u+u^4-4u^2+u^3=0\,, \tag{26}$$

$$v^4+16v^3-19v^2-4v+1=0\,. \tag{27}$$

Eq. (26) (respectively (27)) has four real solutions $u_1, u_2, ..., v_4$:

$$u_1\approx-1.82709...,\quad u_2\approx0.20905...,$$
$$u_3\approx-1.33826...,\quad u_4\approx1.95629...,\tag{28}$$
$$v_1\approx-17.09739...,\quad v_2\approx0.15312...,$$
$$v_3\approx1.24987...,\quad v_4\approx-0.30560.... \tag{29}$$

The points we are looking at have as coordinates (u_i, v_i), $i=1, ..., 4$, with this definite correspondence between the roots. We also have the points obtained by the exchange $u\leftrightarrow v$. We can check that the points (u, v), and $(\bar{u}, \bar{v})$ are on the same curves $\Delta=\frac{1}{2}(3+\sqrt{5})$ for $i=1$, 2 or $\Delta=\frac{1}{2}(3-\sqrt{5})$ for $i=3$, 4. These orbits have two characteristic features:

(i) They decompose into hyperbolae. If we set $\omega=\exp(2i\pi/5)$ these hyperbolae are

$$(u^2-v)(\omega^2+\omega^3)-u(1-v)=0 \tag{30}$$

and its symmetric with respect to the line $u=v$, for $\Delta=\frac{1}{2}(3+\sqrt{5})$, and

$$(u^2-v)(\omega+\omega^4)-u(1-v)=0 \tag{31}$$

and its symmetric with respect to the line $u=v$, for $\Delta=\frac{1}{2}(3-\sqrt{5})$.

(ii) *All points on these curves have a finite orbit under the action of Γ*, since IJ is of order five.

Remark. This result eliminates the possibility of integrability for generic elliptic curves in the pencil, where IJ has infinite order.

The elliptic parametrization, and the addition law on the curves of this pencil will be given elsewhere [42]. The parametrization of the curves $\Delta=\frac{1}{2}(3\pm\sqrt{5})$ reduces to a rational one. On the hyperbola (30) it reads

$$u=\frac{t+\omega^2}{1+\omega^2 t},\quad \frac{v}{u}=\frac{t+\omega}{1+\omega t}\,. \tag{32}$$

The hyperbola (31) is obtained by the replacement $\omega\to\omega^{-1}$ in (32). We shall use in the sequel one of the four "isotropic" points, corresponding to

$$t_{\text{iso}}=\exp(16i\theta)\,, \tag{33}$$

$$\bar{t}_{\text{iso}}=\exp(38i\theta)\,, \tag{34}$$

with

$$\theta=\pi/30\,, \tag{35}$$

that is

$$u_{\text{iso}}=\frac{c(4)}{c(20)}=-1.827090...,$$

$$v_{\text{iso}}=\frac{c(4)}{c(20)}\frac{c(2)}{c(14)}=-17.097396...,$$

$$\bar{u}_{\text{iso}}=-\frac{c(7)}{c(1)}=-0.74723827...,$$

$$\bar{v}_{\text{iso}}=-\frac{c(7)}{c(1)}\frac{c(13)}{c(25)}=0.17939378...,$$

if we set $c(p)=\cos(p\theta)$. We use the parametrization (32), where a special role is given to u, and the obvious notation t_i, $\bar{t}_i$. With this parametrization of (1, 2, 3, $\bar{1}$, $\bar{2}$, $\bar{3}$) we have verified that the star–triangle equations become

$$t_i\bar{t}_i=-\omega^2\,, \tag{36}$$

$$t_1t_2t_3=\omega^4\,. \tag{37}$$

The one-parameter family of solutions we find here is the one of ref. [43]. It is interesting to recall that we have not presupposed any self-duality property.

The action of the group Aut reads

$$I: t_i\to\omega^4/t_i,\quad J: t_i\to1/t_i\,, \tag{38}$$

$$\mathscr{K}_1:(t_1,t_2,t_3,\bar{t}_1,\bar{t}_2,\bar{t}_3)$$
$$\to(\omega^4/t_1,\bar{t}_3,\bar{t}_2,1/\bar{t}_1,t_3,t_2)\,, \tag{39}$$
$$G_1:(t_1,t_2,t_3,\bar{t}_1,\bar{t}_2,\bar{t}_3)$$
$$\to(t_1,\omega t_2,\omega^4 t_3,\bar{t}_1,\omega^4\bar{t}_2,\omega\bar{t}_3)\,. \tag{40}$$

This form makes transparent the presence of a discrete bidimensional translation group $\simeq\mathbb{Z}\times\mathbb{Z}$ inside Aut, in a multiplicative form.

4.2.2. Infinitesimal resolution around the isotropic point

We come back to the chiral model, with five homogeneous parameters $w(k)$, $k=0$, 1, 2, 3, 4 and weight matrix (23).

The above (symmetric) isotropic point is a particular solution of the star–triangle relation for this model. We shall use the non-homogeneous variables $x(k)=w(k)/w(0)$, $k=1,\ldots,4$. We introduce the infinitesimal perturbation $X_i(k)$ of $x_i(k)$, and $\bar{X}_i(k)$ of $\bar{x}_i(k)$, with obvious notations.

The linearized star–triangle relations yield a homogeneous linear system for $X_i(k)$, $\bar{X}_i(k)$. This system is not only compatible, but it has a *four-dimensional space of solutions.*

The solutions verify

$$X_1(k)+X_2(k)+X_3(k)$$
$$=\bar{X}_1(k)+\bar{X}_2(k)+\bar{X}_3(k)=0\,,$$
$$k=1,\ldots,4\,. \tag{41}$$

If we introduce the symmetric and antisymmetric vectors $X^{\rm s}$ and $X^{\rm a}$ (respectively $\bar{X}^{\rm s}$ and $\bar{X}^{\rm a}$)

$$X^{\rm s}=\begin{pmatrix}1\\ s\\ s\\ 1\end{pmatrix},\quad X^{\rm a}=\begin{pmatrix}1\\ a\\ -a\\ -1\end{pmatrix},$$
$$\bar{X}^{\rm s}=\begin{pmatrix}1\\ \bar{s}\\ \bar{s}\\ 1\end{pmatrix},\quad \bar{X}^{\rm a}=\begin{pmatrix}1\\ \bar{a}\\ -\bar{a}\\ -1\end{pmatrix}, \tag{42}$$

with

$$s=[2-4c(2)-c(4)+c(6)$$
$$+7c(8)-7c(10)-c(12)-3c(14)]$$
$$\times[-2-4c(2)+8c(4)-c(6)$$
$$+2c(8)-4c(10)+c(12)-2c(14)]^{-1}\,,$$
$$a=[-2+4c(2)+c(4)-c(6)$$
$$-9c(8)+9c(10)+c(12)-c(14)]$$
$$\times[-6+2c(2)+8c(4)+17c(6)$$
$$-42c(8)+2c(10)+23c(12)+22c(14)]^{-1}\,,$$
$$\bar{s}=[-2+10c(2)-12c(4)+7c(6)$$
$$-7c(8)+3c(10)+4c(12)-2c(14)]$$
$$\times[10-15c(2)+11c(4)-9c(6)$$
$$+2c(10)+c(12)+4c(14)]^{-1}\,,$$
$$\bar{a}=[10-4c(2)-12c(4)+18c(6)$$
$$-26c(8)+14c(10)+8c(12)-8c(14)]$$
$$\times[11+20c(2)-60c(4)+68c(6)$$
$$-90c(8)+50c(10)+32c(12)-50c(14)]^{-1}\,,$$

with $c(p)=\cos(p\theta)$, i.e., numerically:

$$s\approx-58.28463\ldots,\quad a\approx-1.0308189\ldots\,,$$
$$\bar{s}\approx-1.834537\ldots,\quad \bar{a}\approx4.543390\ldots\,,$$

and set

$$X_i=s_iX^{\rm s}+a_iX^{\rm a}\,, \tag{43}$$
$$\bar{X}_i=\bar{s}_i\bar{X}^{\rm s}+\bar{a}_i\bar{X}^{\rm a},\quad i=1,2,3\,, \tag{44}$$

we get

$$s_1+s_2+s_3=a_1+a_2+a_3=0\,,$$
$$\bar{s}_i=-\alpha s_i,\quad \bar{a}_1=\beta(a_2-a_3)\,,$$
$$\bar{a}_2=\beta(a_3-a_1),\quad \bar{a}_3=\beta(a_1-a_2)\,, \tag{45}$$

with

$$\alpha=-\frac{1}{4\cos^2(\theta)}\approx-0.2527617250\ldots\,, \tag{46}$$
$$\beta=\tfrac{28}{31}-\tfrac{38}{31}c(1)+\tfrac{20}{93}c(3)+\tfrac{118}{93}c(7)$$
$$\approx0.0108158287\ldots \tag{47}$$

This proves [#3] the existence of a *four-parameter family of solutions* of the star–triangle equations containing the isotropic point. This family contains in particular the Fateev–Zamolodchikov solution where all weights w_i (respectively $\bar{w}_i$) belong to the hyperbola (30) of the curve $\Delta=\frac{1}{2}(3+\sqrt{5})$. The vectors X^s and $\bar{X}^s$ are tangent to that hyperbola. We know that IJ is of finite order on this curve (the order is five). We have the prejudice that the integrability surface is of the same nature, i.e. is a locus of points where $(IJ)^5=1$. Notice that such a locus is automatically invariant by both I and J. Such a surface is given by the two equations

$$\begin{aligned}A[&w(1)w(3)w(4)^2-3w(2)^2w(4)^2\\&+2w(1)w(2)w(4)w(0)+w(3)w(2)^2w(0)\\&-3w(1)w(3)^2w(0)+2w(4)w(2)w(3)^2]\\&-4w(2)^2w(4)^2-4w(1)w(3)^2w(0)\\&+3w(3)w(2)^2w(0)+3w(1)w(3)w(4)^2\\&+w(4)w(2)w(3)^2+w(1)w(2)w(4)w(0)=0\end{aligned} \tag{48}$$

and

$$\begin{aligned}A[&-2w(3)^2w(0)^2-2w(1)w(2)w(4)^2\\&-w(4)w(3)^2w(1)+3w(0)w(3)w(4)^2\\&+3w(2)w(1)w(3)w(0)-w(2)w(0)^2w(4)]\\&-w(1)w(2)w(4)^2+2w(2)w(0)^2w(4)\\&-w(0)w(3)w(4)^2+2w(4)w(3)^2w(1)\\&-w(3)^2w(0)^2-w(2)w(1)w(3)w(0)=0\,,\end{aligned} \tag{49}$$

with $A=\frac{1}{2}(-1\pm\sqrt{5})$. These surfaces cut the symmetric subset $w(1)=w(4)$, $w(2)=w(3)$, along the two hyperbolae (30) and (31) for $A=\frac{1}{2}(-1+\sqrt{5})$ and $A=-\frac{1}{2}(1+\sqrt{5})$ respectively. The vectors X^s, $\bar{X}^s$, X^a, $\bar{X}^a$ are tangent to this surface for $A=\frac{1}{2}(-1+\sqrt{5})$.

The consequences of eq. (45) on the commutation of transfer matrices $T_i=\mathbb{T}_M(\bar{w}_i, w_i)$ are the following: locally near the isotropic point T_i depends on $(s_i, a_i, \bar{s}_i, \bar{a}_i)$. The commutation of T_1 and T_2 is obtained by imposing relations (45). We first need $\bar{s}_1=-\alpha s_1$ which allows three parameters for T_1. At this point (45) fixes a_2 and $\bar{a}_2$ and the only free parameter for T_2 is s_2, giving a one-parameter family of commuting transfer matrices.

[#3] As a consequence of the implicit function theorem and the algebraicity of the solutions of the star–triangle equations.

4.3. *Analysis of a non-cyclic model*

We consider the model introduced above with weight matrix given by (1). The interest of this model is that there exists an algebraic invariant for the action of I and J, and consequently of the group Γ [29,44]. The orbits lie in elliptic curves [29,44]. The existence of this algebraic invariant makes the model a good candidate for integrability. The weight matrix being non-symmetric, we have to specify the arrangement of arrows in the star–triangle relations.

The presence of this true chirality (as opposed to what happens for the model (23) where a reversal of arrows amounts to a spin relabelling) breaks the symmetry between 1, 2, and 3. Eqs. (st1.1), (st2.1), and (st3.1) are three different systems of equations. By a direct investigation one sees that the only solutions, up to trivial ones, are obtained for $y=z$, that is to say when the model reduces to the standard scalar six-state Potts model.

5. Conclusion

We have described and used the infinite symmetry group of the Yang–Baxter relations for nearest neighbour spin models. It is important to stress two important facts about this symmetry.

The first is that it also exists for vertex models as well as for the higher dimensional generalizations of the Yang–Baxter equations as the tetrahedron equations [16].

The second one is that, as always in this field of integrability in statistical mechanics, one sees three faces of the subject.

There is a topological aspect, related for example to knot theory, for the Yang–Baxter relations have to do with deformations of knots [45].

There is also a purely algebraic aspect, already in the writing of the equations, but also in the finer description of their content (quantum inverse scattering, algebraic Bethe ansatz, quantum groups and all that [46]).

Finally there is an analytical aspect, as soon as for example elliptic curves and functions defined on them pop out.

All these three aspects were present from the early years of existence of the field, but their fascinating interrelations have been emerging only step by step.

The symmetry we have described permits such a progress. Indeed it has a very simple graphical representation and it bridges directly an algebraic symmetry and the analytical aspect, via the construction of the algebraic varieties of parameters

Acknowledgement

We would like to thank J. Avan, O. Babelon, M. Dubois-Violette, and M. Talon, for very stimulating discussions and comments. Two of us (JMM and CMV) would like to thank the Research Institute for Theoretical Physics (TFT) of the University of Helsinki for hospitality and support.

References

[1] R.J. Baxter, Exactly solved models in statistical mechanics (Academic Press, New York, 1981).
[2] P. Lochak and J.-M. Maillard, J. Math. Phys. 27 (1986) 593.
[3] R.J. Baxter, Philos. Trans. R. Soc. 289 (1978) 315.
[4] H. Au-Yang and J.H.H. Perk, Physica A 144 (1987) 44.
[5] M. Jimbo, Commun. Math. Phys. 102 (1986) 537.
[6] R.J. Baxter, Commun. Math. Phys. 88 (1983) 185.
[7] A.B. Zamolodchikov, Commun. Math. Phys. 79 (1981) 489.
[8] M.T. Jaekel and J.-M. Maillard, J. Phys. A 15 (1982) 1309.
[9] R.J. Baxter, Physica D 18 (1986) 321.
[10] D. Hansel, J.-M. Maillard, J. Oitmaa and M. Velgakis, J. Stat. Phys. 48 (1987) 69.
[11] H. Au-Yang, B.M. McCoy, J.H.H. Perk, S. Tang and M.L. Yan, Phys. Lett. A 123 (1987) 219.
[12] B.M. McCoy, J.H.H. Perk, S. Tang and C.-H. Sah, Phys. Lett. A 125 (1987) 9.
[13] R.J. Baxter, J.H.H. Perk and H. Au-Yang, Phys. Lett. A 128 (1988) 138.
[14] D. Hansel and J.-M. Maillard, Phys. Lett. A 133 (1988) 11.
[15] J. Avan, J.-M. Maillard, M. Talon and C.-M. Viallet, Int. J. Mod. Phys. B 4 (1990) 1743.
[16] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Phys. Lett. B 260 (1991) 87.
[17] Yu.G. Stroganov, Phys. Lett. A 74 (1979) 116.
[18] R.J. Baxter, J. Stat. Phys. 28 (1982) 1.
[19] M.T. Jaekel and J.-M. Maillard, J. Phys. A 15 (1982) 2241.
[20] J.-M. Maillard, J. Phys. (Paris) 46 (1984) 329.
[21] V.V. Bazhanov and Yu.G. Stroganoff, J. Stat. Phys. 59 (1990) 799.
[22] V.V. Bazhanov and R.M. Kashaev, Cyclic L-operators related with a 3-state R matrix, preprint RIMS-702, Kyoto, to be published in Commun. Math. Phys. (1990).
[23] V.V. Bazhanov, R.M. Kashaev, V.V. Mangazeev and Yu.G. Stroganoff, $Z_N^{\otimes n-1}$ generalization of the chiral Potts model, preprint IHEP-90-137, Pootvino, to be published in Commun. Math. Phys. (1990).
[24] E. Date, M. Jimbo, K. Miki and T. Miwa, New R matrices associated with cyclic representations of $U_q(A_2^{(2)})$, preprint RIMS-706 (1990).
[25] E. Date, M. Jimbo, K. Miki and T. Miwa, Generalized chiral Potts model and minimal cyclic representations of $U_q(\hat{gl}(n, C))$, preprint RIMS-715 (1990).
[26] R.J. Baxter, in: Proc. 1980 Enschede Summer School on Fundamental problems in statistical mechanics, V (North-Holland, Amsterdam, 1981).
[27] J.-M. Maillard and T. Garel, J. Phys. A 17 (1984) 1251.
[28] N.L. Biggs, Math. Proc. Cambridge Philos. Soc. 80, p. 429.
[29] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Integrable Coxeter groups, preprint LPTHE 91-4 and Helsinki HU TFT 91-4.
[30] V.G. Kac, Infinite dimensional Lie algebras, 2nd Ed. (Cambridge Univ. Press, Cambridge, 1985).
[31] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Higher dimensional mappings, preprint LPTHE 91-5 and Helsinki HU TFT 91-5.
[32] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Matrix patterns for integrability, in preparation.
[33] J.-M. Maillard, J. Math. Phys. 27 (1986) 2776.
[34] J.-M. Maillard, Exactly solvable models in statistical mechanics and automorphisms of algebraic varieties, in: Progress in statistical mechanics, Vol. 3, World Scientific Proc. 1988 Workshop in Statistical mechanics (Taiwan), ed. Chin-Kun Hu (World Scientific, Singapore, 1988).
[35] M. Jimbo and T. Miwa, Nucl. Phys. B 257 [FS14] (1985) 1.
[36] P. Rujan, J. Stat. Phys. 49 (1984) 139.
[37] M.T. Jaekel and J.-M. Maillard, J. Phys. A 18 (1985) 1229.
[38] M.T. Jaekel and J.-M. Maillard, J. Phys. A 17 (1984) 2079.
[39] H. Au-Yang and J.H.H. Perk, in: Advanced studies in pure mathematics, Vol. 19, eds. M. Jimbo, T. Miwa and A. Tsuchia (Academic Press, Kinokuniya, 1989) pp. 57–94.
[40] D. Hansel, J.-M. Maillard and P. Rujan, Int. J. Mod. Phys. B 3 (1989) 1539.
[41] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Higher dimensional mappings with exact properties, in preparation.
[42] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Mappings for the $\mathbb{Z}_N$ chiral Potts model, in preparation.
[43] V.A. Fateev and A.B. Zamolodchikov, Phys. Lett. A 92 (1982) 37.
[44] M.P. Bellon, J.-M. Maillard and C.-M. Viallet, Elliptic parametrization for a six-state spin model, in preparation.
[45] V.F.R. Jones, Sci. Am. 263 (1990) 52, and references therein.
[46] N.Yu. Reshetikhin, L.A. Takhtadzhyan and L.D. Faddeev, Leningrad Math. J. 1 (1990) 193.