Infinite discrete symmetry group for the Yang–Baxter equations. Vertex models

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We show that the Yang–Baxter equations for two-dimensional vertex models admit as a group of symmetry the infinite discrete group $A_2^{(1)}$. The existence of this symmetry explains the presence of a spectral parameter in solutions of the equations. We show that similarly, for three-dimensional vertex models and the associated tetrahedron equations, there also exists an infinite discrete group of symmetries. Although generalizing very naturally the previous one, this is a much bigger hyperbolic Coxeter group. We indicate how this symmetry should be used to resolve the Yang–Baxter equations and their higher-dimensional generalizations and initiate the study of a family of three-dimensional vertex models.

1. Introduction

The Yang–Baxter equations, which appeared twenty years ago [#1], have acquired a predominant role in the theory of integrable two-dimensional models in statistical mechanics [6,7] and field theory (quantum or classical). They have actually surpassed the borders of physics and have become fashionable in some parts of the mathematics literature. They support in particular the construction of quantum groups [8,9].

We show here that they possess an infinite discrete symmetry group. This group accounts for the existence of a spectral parameter and permits the so-called "baxterization" [10]. We then extend our results to the higher dimensional generalizations of the Yang–Baxter equations, namely the tetrahedron and hypersimplicial equations. The results we present here are the transposition to the vertex models of the symmetries of the star-triangle equations presented in ref. [11] (see also refs. [12,13]). These symmetries are built from the inversion relation, a transformation already widely used in statistical mechanics [14–18] and the symmetry group of the vertices (symmetry group of the square, of the cube, ...).

2. The Yang–Baxter relation for vertex models

We consider a vertex model on a two-dimensional square lattice of size $M \times M$ with periodic boundary conditions. To each bond is associated a variable with q possible states and a Boltzmann weight $w(i, j, k, l)$ is assigned to each vertex:

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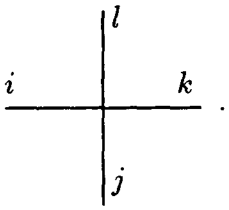
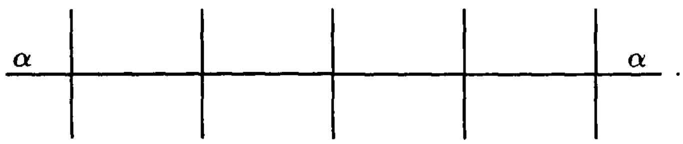
[#1] In fact, fifty years, Lars Onsager was totally aware of the key role played by the star–triangle relation in solving the two-dimensional Ising model, but he preferred to give an algebraic solution emphasizing Clifford algebras [1–5].



For each line configuration one can build the row-to-row transfer matrix with periodic boundary conditions,



The transfer matrix acts then on a q^M-dimensional space.

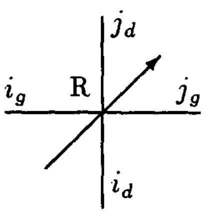
In order to write the Yang–Baxter relation, the q^4 homogeneous weights $w(i, j, k, l)$ are first arranged in a $q^2 \times q^2$ matrix R:

$$R^{ij}_{kl} = w(i, j, k, l) . \qquad (1)$$

The Yang–Baxter relation is a trilinear relation between three matrices $R(1, 2)$, $R(2, 3)$ and $R(1, 3)$:

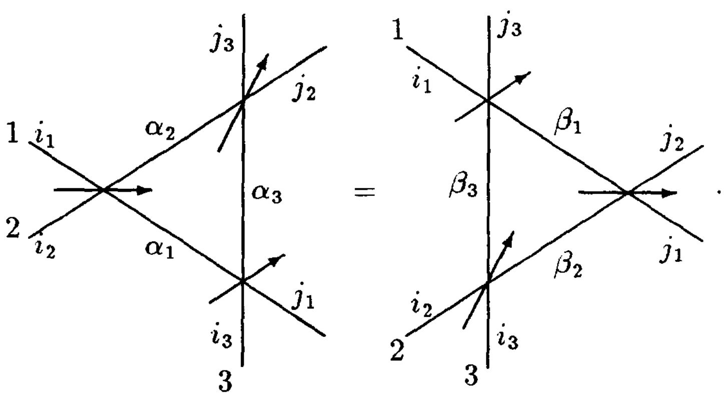
$$\sum_{\alpha_1,\alpha_2,\alpha_3} R^{i_1 i_2}_{\alpha_1 \alpha_2}(1, 2) R^{\alpha_1 i_3}_{j_1 \alpha_3}(1, 3) R^{\alpha_2 \alpha_3}_{j_2 j_3}(2, 3) = \sum_{\beta_1,\beta_2,\beta_3} R^{i_2 i_3}_{\beta_2 \beta_3}(2, 3) R^{i_1 \beta_3}_{\beta_1 j_3}(1, 3) R^{\beta_1 \beta_2}_{j_1 j_2}(1, 2) . \qquad (2)$$

The assignation (1) is arbitrary and we may specify it by complementing the vertex with an arrow and attributing numbers to the lines



$$= R^{i_g i_d}_{j_g j_d}(g, d).$$

With these rules the relation (2) has the following graphical representation:



(3)

The lines carry indices 1, 2, 3. We shall not get here into the arcanes of this relation, which appears in the theory of integrable models [9], the theory of the factorizable S matrix in two-dimensional field theory, the quantum inverse scattering method [19], and knot theory, and has been given a canonical meaning in terms of Hopf algebras [20] (quantum groups [8,9,21–23]) and the list is far from exhaustive. We however want to stress some of its characteristic features.

– The innocent look of these multilinear equations is fallacious since the system is largely overdetermined and the full solution is not known. The results we present here lead to a strategy for its resolution.

– The most powerful property of the Yang–Baxter equation is to produce *global* results from a *local property*: this relation on the local weights of the model yields the commutation of transfer matrices with periodic boundary conditions of *arbitrary size M* (and is actually to some extent a necessary condition for it [24]).

– Some especially interesting solutions depend on a continuous parameter called the spectral parameter. The presence of this parameter is fundamental for many applications in physics, as for example the Bethe ansatz method [25,5,19]. One has to realize that one of the main issues in the full resolution of (2) is precisely to describe what is this parameter and the algebraic variety on which it lives, although its presence may obscure the algebraic structures underlying the Yang–Baxter equation (the discovery of quantum groups was allowed by forgetting this parameter [23,8,26,9]). The problem of building up continuous families of solutions from an isolated one, known as the baxterization [10], is made straightforward by our study. Indeed our results explain the presence of the spectral parameter in the solution of the equation (see also ref. [12]).

3. Some algebra

3.1. Notations

The R-matrix appears naturally as a representation of an element of the tensor product $\mathscr{A}\otimes\mathscr{A}$ of some algebra $\mathscr{A}$ with itself. This algebra is a nice Hopf algebra in the context of quantum groups. We shall not dwell on this here but recall some simple operations on R.

In $\mathscr{A}\otimes\mathscr{A}$ we have a product inherited from the product in $\mathscr{A}$:

$$(a\otimes b)(c\otimes d)=ac\otimes bd\,. \tag{4}$$

R is an invertible element of $\mathscr{A}\otimes\mathscr{A}$ for this product and we shall denote the inverse for this product by $I(R)$:

$$R\cdot I(R)=I(R)\cdot R=1\otimes 1\,. \tag{5}$$

In terms of the representative matrix this reads

$$\sum_{\alpha,\beta} R^{ij}_{\alpha\beta}I(R)^{\alpha\beta}_{uv}=\delta^i_u\delta^j_v=\sum_{\alpha,\beta} I(R)^{ij}_{\alpha\beta}R^{\alpha\beta}_{uv}\,. \tag{6}$$

This is nothing else but the so-called *inversion relation* for vertex models [14,15,27–29]. On $\mathscr{A}\otimes\mathscr{A}$ we have a permutation operator σ:

$$\sigma(a\otimes b)=b\otimes a\,, \tag{7}$$

$$(\sigma R)^{ij}_{uv}=R^{ji}_{vu}\,,\quad \text{for the matrix } R\,. \tag{8}$$

Note that the representation of σ is just the conjugation by the permutation matrix P:

$$P^{ij}_{kl}=\delta_{il}\delta_{jk}\,, \tag{9}$$

$$\sigma R=PRP\,. \tag{10}$$

In the language of matrices we have a notation of transposition. Let us define partial transpositions t_g and t_d by

$$(t_g R)^{ij}_{uv} = R^{uj}_{iv} , \tag{11}$$

$$(t_d R)^{ij}_{uv} = R^{iv}_{uj} , \tag{12}$$

and the full transposition

$$t = t_g t_d = t_d t_g . \tag{13}$$

We shall in the sequel use another inversion J defined by

$$J = t_g I t_d = t_d I t_g , \tag{14}$$

or equivalently

$$\sum_{\alpha,\beta} R^{\alpha u}_{v\beta} J(R)^{\alpha i}_{j\beta} = \delta^i_u \delta^j_v = \sum_{\alpha,\beta} J(R)^{i\beta}_{\alpha j} R^{u\beta}_{\alpha v} \tag{15}$$

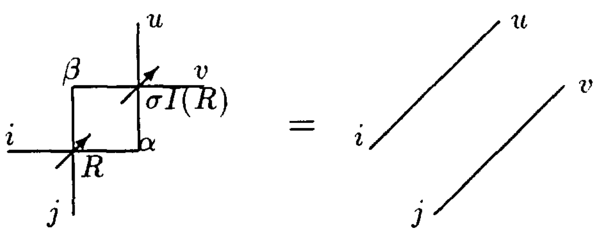
These operators verify straightforwardly

$$I^2 = J^2 = 1, \quad It = tI, \quad Jt = tJ, \quad \sigma^2 = t^2 = 1, \quad \sigma I = I\sigma, \quad \sigma J = J\sigma ,$$
$$(\sigma t_g)^2 = (\sigma t_d)^2 = t, \quad \sigma t_g \sigma t_d = 1 . \tag{16}$$

Note that the two inversions I and J do not commute. They generate an infinite discrete group Γ, the infinite dihedral group, isomorphic to the semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}_2$. This group is represented on the matrix elements by *birational transformations* [12,30,31]. Remark that for the *vertex models*, the birational transformations associated to the two involutions I and J are naturally related by collineations (this should be compared with the situation for nearest neighbour interaction spin models [12,32]).

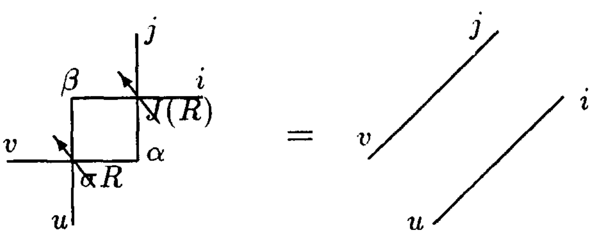
3.2. Graphical representation

Each of these operations has a graphical representation. For the inversion I or more precisely for σI it is



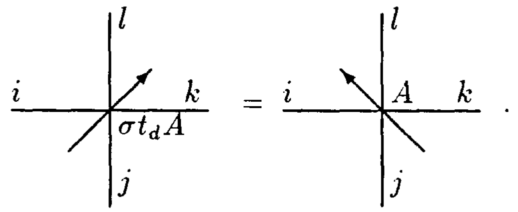
. (17)

The inversion J reads

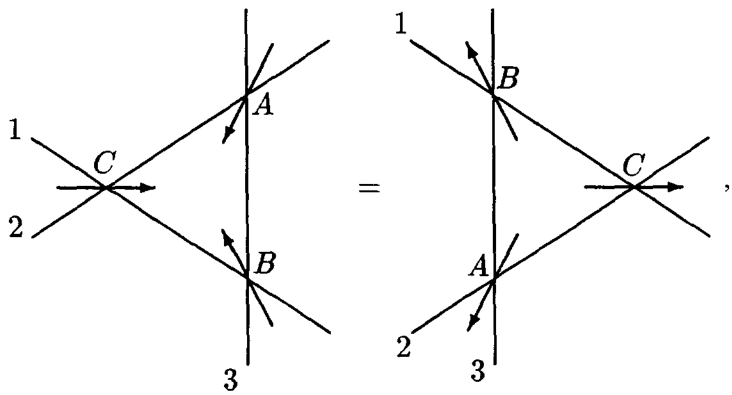


. (18)

The graphical representation mixes very well with the various operations introduced in section (3.1):



An immediate consequence is that we may picture the Yang–Baxter relation in a more symmetric way:



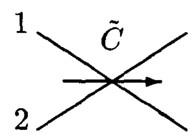
(19)

at the price of the redefinitions

$$A = tR(2,3)\ , \tag{20}$$

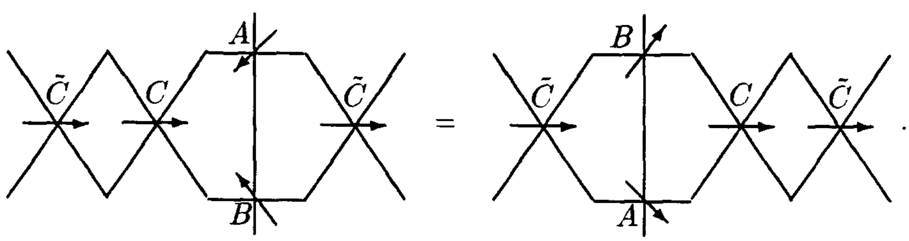
$$B = \sigma t_d R(1,3)\ , \tag{21}$$

$$C = R(1,2)\ . \tag{22}$$

We may bracket (19) with 

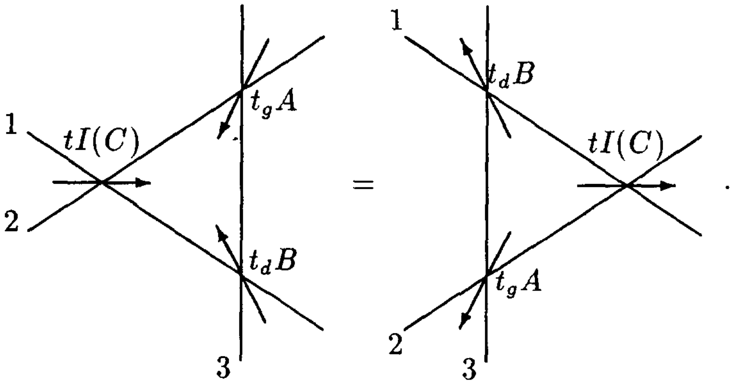
, where $\tilde{C} = \sigma I(C)$.

We get



(23)

that is to say



(24)

This relation is nothing but (19) after the redefinitions

$$A \to t_g A, \quad B \to t_d B, \quad C \to tIC. \tag{25}$$

We may denote by K_3 the operation (25). We have two other similar operations K_1 and K_2

$$K_1: A \to tIA, \quad B \to t_g B, \quad C \to t_d C,$$

$$K_2: A \to t_d A, \quad B \to tIB, \quad C \to t_g C.$$

The discrete group $\mathcal{A}$ut generated by the K_i's $(i=1, 2, 3)$ is a symmetry group of the Yang–Baxter equations. These generators K_i $(i=1, 2, 3)$ are involutions. We have

$$K_1 K_2: A \to It_g A, \quad B \to t_d IB, \quad C \to tC.$$

The K_i's satisfy the relation $(K_1K_2K_3)^2=1$. Actually, the operation $K_1K_2K_3$ is just the inversion I on R.

To make the structure of the group more transparent, let us introduce K_A, K_B and K_C, which are simply related to the K_i's by the transposition of two vertices:

$$K_A: A \to \sigma tIA, \quad B \to t_g \sigma C, \quad C \to \sigma t_g B,$$

$$K_B: A \to \sigma t_g C, \quad B \to \sigma tIB, \quad C \to t_g \sigma A,$$

$$K_C: A \to t_g \sigma B, \quad B \to \sigma t_g A, \quad C \to \sigma tIC.$$

It is easily verified that

$$K_A^2 = K_B^2 = K_C^2 = 1, \tag{26}$$

and

$$(K_A K_B)^3 = (K_B K_C)^3 = (K_C K_A)^3 = 1, \tag{27}$$

with no other relations. We recover the affine Coxeter group $A_2^{(1)}$ we already encountered in ref. [11].

A fundamental remark. Beware that, due to the different arrangement of indices, the relations we consider are not the Yang–Baxter equation that one considers in the study of quantum groups (shortly $RRR=RRR$) but rather its avatars ($RTT=TTR$) or even $ABC=CBA$. We will even in a forthcoming publication consider the $ABC=DEA$ relation which also leads to remarkable relations on the transfer matrices of arbitrary size. The meaning of these relations is detailed in the standard literature on integrable models [9] and quantum groups [21,22,23]. However we will show in the following that choices of the form of the matrices R will metamorphose the action of IJ and of similar products into a mere shift of the spectral parameter. This is the *core* of our strategy for the resolution of the Yang–Baxter equations and their higher dimensional generalizations.

Among the elements of the discrete group generated by the K_i's we have in particular

$$(K_1K_2)^2\colon A\to It_g It_g A = tIJA,\quad B\to t_d It_d IB = tJIB,\quad C\to C\,.$$

Since IJ is of infinite order, we have generated an *infinite discrete group* of symmetries. This is exactly the phenomenon that we described in ref. [12] for the star–triangle equations.

We have here a very powerful instrument for two purposes:

(1) Define adequate patterns for the matrix R [33]. Indeed if a set of relations among the entries of R are preserved by IJ (or at least by tIJ) this operation will be a transformation on the varieties associated to these relations. These varieties are of paramount importance in the resolution of the Yang–Baxter equations, since they are the varieties on which the spectral parameters lie.

(2) Permit the so-called baxterization of an isolated solution just acting with tIJ.

To illustrate point (1), we shall take in the next paragraph the example of the Baxter eight-vertex model [34,35], and we shall show subsequently how to introduce a spectral parameter for the solutions of the Yang–Baxter equations associated to sl(n) algebras.

4. The baxterization

4.1. Baxterization of the Baxter model

Consider the matrix of the symmetric eight-vertex model

$$R=\begin{pmatrix} a & 0 & 0 & d\\ 0 & b & c & 0\\ 0 & c & b & 0\\ d & 0 & 0 & a\end{pmatrix}. \tag{28}$$

Notice that this form is preserved by the operations I and J and that $tR=R$. The action of I is

$$a\to\frac{a}{a^2-d^2},\quad b\to\frac{b}{b^2-c^2},\quad c\to\frac{-c}{b^2-c^2},\quad d\to\frac{-d}{a^2-d^2}$$

and the action of J is

$$a\to\frac{a}{a^2-c^2},\quad b\to\frac{b}{b^2-d^2},\quad c\to\frac{-c}{a^2-c^2},\quad d\to\frac{-d}{b^2-d^2}\,.$$

We shall look at the solutions of the Yang–Baxter equations for matrices R of the form (28). The leading idea is to say that the parametrization of the solutions is just the parametrization of the algebraic varieties preserved by tIJ in the projective space $\mathbb{C}P_3$ of the entries (a, b, c, d). The remarkable fact is that not only these varieties exist but we may describe them completely. We use the visualization method we have already used [12,13] for spin models, that is to say just draw the orbits obtained by numerical iteration and look.

The problem of the baxterization is to introduce a spectral parameter into an isolated solution of the Yang–Baxter equations. We have solutions of this problem by acting with the symmetry group Γ of these equations.

This is best illustrated by fig. 1. This figure shows the orbit of point $*$ which is a matrix of the form (28). It is drawn by the iteration of IJ acting on the initial point $*$. The resulting points densify on the elliptic curve given by the intersection of the quadrics $\Delta_1=$const. and $\Delta_2=$const. (Clebsch's biquadratic), with Δ_1 and Δ_2 the Γ invariants

$$\Delta_1=\frac{a^2+b^2-c^2-d^2}{ab+cd},\quad \Delta_2=\frac{ab-cd}{ab+cd}\,. \tag{29}$$

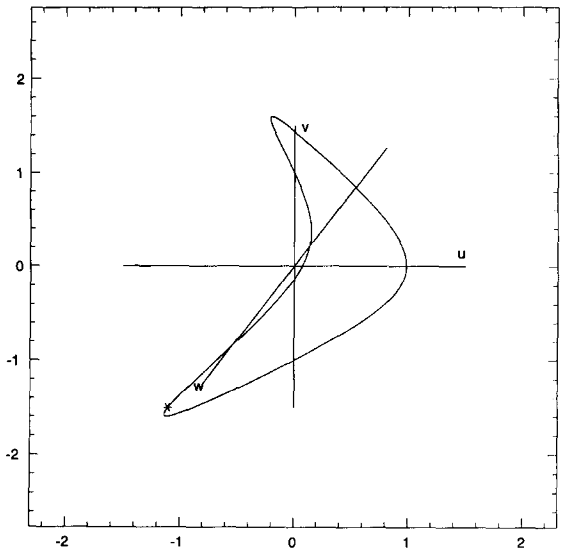


Fig. 1. Baxterization of the point *.

4.2. *Baxterization of the R matrix of $sl_q(n)$*

Another example corresponds to the baxterization of solutions associated to $sl(n)$ algebras [36]. There are special solutions generally denoted R_+ and R_-. For the simplest four-dimensional representation of the $sl(2)$ case, we have

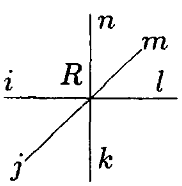
$$R_+ = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{30}$$

and a similar expression for R_- [36]. Looking for a family containing both R_+ and R_-, our baxterization procedure leads to the well known six-vertex model R-matrix $R = \lambda R_+ + 1/\lambda R_-$.

We leave it as an exercise for the reader to treat the $sl(3)$ case.

5. The tetrahedron equation

This equation is a generalization of the Yang–Baxter equation to three-dimensional vertex models [37–39]. In three dimensions a vertex is given by

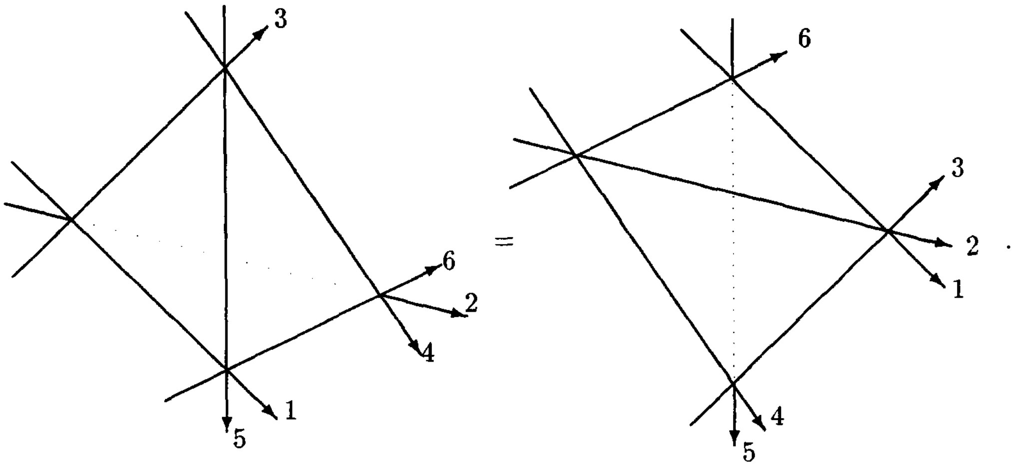


.

This vertex has a Boltzmann weight $w(i, j, k, l, m, n)$. Here again the weights may be arranged in a matrix of entries

$$R^{ijk}_{lmn} = w(i, j, k, l, m, n) \; . \tag{31}$$

The tetrahedron equation has a pictorial representation:



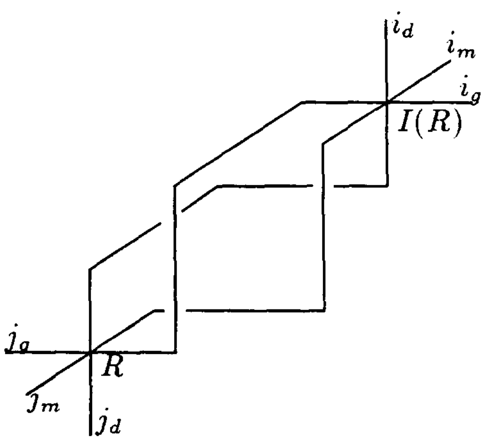
The algebraic form is

$$R_{123} R_{543} R_{516} R_{426} = R_{426} R_{516} R_{543} R_{123} \; . \tag{32}$$

We may here again introduce an inverse I,

$$\sum_{\alpha_g, \alpha_m, \alpha_d} (IR)^{i_g i_m i_d}_{\alpha_g \alpha_m \alpha_d} \cdot R^{\alpha_g \alpha_m \alpha_d}_{j_g j_m j_d} = \delta^{i_g}_{j_g} \delta^{i_m}_{j_m} \delta^{i_d}_{j_d} \, , \tag{33}$$

with the pictorial representation



$$= \delta^{i_g}_{j_g} \delta^{i_m}_{j_m} \delta^{i_d}_{j_d} .$$

We also introduce the partial transpositions t_g, t_m and t_d with

$$(t_g R)^{i_g i_m i_d}_{j_g j_m j_d} = R^{j_g i_m i_d}_{i_g j_m j_d} \, , \tag{34}$$

and similar definitions for t_m and t_d.

Generalizing the introduction of a more symmetric Yang–Baxter equation (19), we redefine

$$A = R_{123} , \quad B = t_d R_{543} , \quad C = t_g t_m R_{516} , \quad D = t R_{426} \, , \tag{35}$$

where t is the full transposition $t_g t_m t_d$. Eq. (32) then takes the more symmetric form

$$\sum_{s_1,\dots,s_6} A^{i_1i_2i_3}_{s_1s_2s_3} B^{i_5i_4j_3}_{s_5s_4s_3} C^{j_5j_1i_6}_{s_5s_1s_6} D^{j_4j_2j_6}_{s_4s_2s_6} = \sum_{r_1,\dots,r_6} D^{r_4r_2r_6}_{i_4i_2i_6} C^{r_5r_1r_6}_{i_5i_1j_6} B^{r_5r_4r_3}_{j_5j_4i_3} A^{r_1r_2r_3}_{j_1j_2j_3} . \tag{36}$$

We may multiply the previous equation by $(IA)^{u_1u_2u_3}_{i_1i_2i_3}$ and $(tIA)^{v_1v_2v_3}_{j_1j_2j_3}$ and sum over (i_1, i_2, i_3) and (j_1, j_2, j_3). This amounts to a bracketing of the tetrahedron equations by two times the same vertex, in a procedure trivially generalizing the one for the Yang–Baxter equation (23). We recover (36) with A, B, C and D transformed by

$$K_1: A\to tIA, \quad B\to t_d B, \quad C\to t_m C, \quad D\to t_m D . \tag{37}$$

We have in a similar way the operations

$$K_2: A\to t_d A, \quad B\to tIB, \quad C\to t_g C, \quad D\to t_g D ,$$

$$K_3: A\to t_g A, \quad B\to t_g B, \quad C\to tIC, \quad D\to t_d D ,$$

$$K_4: A\to t_m A, \quad B\to t_m B, \quad C\to t_d C, \quad D\to tID .$$

Each of these four operations is an involution. They satisfy various relations, for instance $(K_1K_2K_3K_4)^2=1$. *The K_i's generate a group $\mathcal{A}ut_3$ which is a symmetry group of the tetrahedron equations.* This group is "monstrous" since the number of elements of length smaller than l is of exponential growth with respect to l, unlike the case of the affine Coxeter groups (as $A_2^{(1)}$ for the Yang–Baxter equation) where this number is of polynomial growth. It is also a symmetry group for the three-dimensional vertex model *even if* [16] the model does not satisfy the tetrahedron equation. The operations playing a role similar to the one of I and J in the two-dimensional Yang–Baxter equations are the *four* involutions

$$I, \quad J=t_g I t_m t_d, \quad K=t_m I t_d t_g, \quad L=t_d I t_g t_m . \tag{38}$$

In order to precise the algebraic structure of the group Γ_3 generated by I, J, K and L, it is simpler to consider as generators two of the partial transpositions t_g and t_d, I and the full transposition t. The third partial transposition can be recovered as the product tt_gt_d and t commutes with all other generators and so contributes a mere $\mathbb{Z}_2$ factor in the group. We are thus considering the Coxeter group generated by three involutions t_g, t_d and I, with two of them commuting. This is represented by the following Dynkin diagram:

For this group again the number of elements of length smaller than l is greater than $2^{l/2}$. This is in fact a *hyperbolic* Coxeter group [40].

6. A three-dimensional model

Our strategy for finding solutions of the tetrahedron equations is to seek for patterns of the Boltzmann weights of the three-dimensional vertex compatible with the symmetry group Γ_3. By this we mean that its form should be preserved by Γ_3.

6.1. The model

We will therefore consider a simple model where i, j, k, l, m and n take only two values $+1$ and -1. The matrix (31) is an 8×8 matrix. We will require that its pattern is invariant under the inverse I [33] and the various partial transpositions t_g, t_m and t_d. We aim at having (see remark 1 below) a generalization of the Baxter eight-vertex model and we impose the following restrictions:

$$w(i,j,k,l,m,n)=w(-i,-j,-k,-l,-m,-n)\ , \tag{39}$$

$$w(i,j,k,l,m,n)=0 \quad \text{if} \quad ijklmn=-1\ . \tag{40}$$

These constraints amount to saying that the 8×8 matrix is the direct product of two times the same 4×4 matrix. It is further possible to impose that this matrix is symmetric since, in this case, $t_g R$ (and any other partial transpose) is also symmetric. Let us introduce the following notations for the entries of the 4×4 block of the R matrix:

$$\begin{pmatrix} a & d_1 & d_2 & d_3 \\ d_1 & b_1 & c_3 & c_2 \\ d_2 & c_3 & b_2 & c_1 \\ d_3 & c_2 & c_1 & b_3 \end{pmatrix} . \tag{41}$$

The four rows and columns of this matrix correspond to the states $(+,+,+)$, $(+,-,-)$, $(-,+,-)$ and $(-,-,+)$ for the triplets (i,j,k) or (l,m,n). The R-matrix can be completed by spin reversal, according to the rule (39). t_g (respectively t_m, t_d) simply exchanges c_1 and d_1 (respectively c_2 and d_2, c_3 and d_3) and I acts as the inversion of this 4×4 matrix.

Here two preliminary remarks are in order.

Remark 1. The tetrahedron equation allows for the commutation of plane-to-plane transfer matrices of arbitrary width and depth, and in particular, with depth 1. This amounts to saying that row-to-row transfer matrices of the two-dimensional model deduced by taking the trace on one of the three axes commute ($R^{ij}_{lm}=\sum_k R^{ijk}_{lmk}$). In the particular case we consider, (39), (40), (41), this leads to an eight-vertex model, with the homogeneous variables of the Baxter model a, b, c, d given by

$$a\to a+b_3,\quad b\to b_1+b_2,\quad c\to 2c_3,\quad d\to 2d_3\ . \tag{42}$$

The communication of these deduced row-to-row transfer matrices implies that the integrability varieties are subvarieties of the intersection of the six quadrics $\Delta_1(a+b_3, b_1+b_2, 2c_3, 2d_3)=\text{const.}$ $\Delta_2(a+b_3, b_1+b_2, 2c_3, 2d_3)=\text{const.}$ and similar expressions for the two other axes. The observation that the integrability varieties of d-dimensional models are subvarieties of those of the $(d-1)$-dimensional models obtained by this partial trace procedure is quite general and not restricted to model (39), (40), (41).

Remark 2. There exist gauge-like transformations (weak-graph duality) on the matrix R^{ijk}_{lmn} which amount to performing some particular conjugation on the matrix [41,42]. In view of this symmetry, and having in mind to find variables with a good behaviour with respect to the inversion I, it is interesting to consider the coefficients of the characteristic polynomial of the matrix R.

6.2. Algebraic invariants

For heuristic reasons we consider first the eight-vertex model. In terms of the four eigenvalues λ_i of the 4×4 matrix (28), the algebraic invariants of the Baxter model Δ_1 and Δ_2 (29) are given by any two ratios of the three roots of the polynomial

$$(x-\lambda_1\lambda_2-\lambda_3\lambda_4)(x-\lambda_1\lambda_3-\lambda_2\lambda_4)(x-\lambda_1\lambda_4-\lambda_3\lambda_2)=x^3-\sigma_2 x^2+(\sigma_1\sigma_3-4\sigma_4)x-(\sigma_4\sigma_1^2+\sigma_3^2-4\sigma_2\sigma_4)\ . \tag{43}$$

These invariants correspond to the breaking of the $\mathcal{S}_4$ permutation symmetry of the eigenvalues down to C_{4v}. This very example is deeply related to the Galois theory of the solvability of a polynomial [43].

For our three-dimensional mode, the characteristic polynomial is the square of the one of the 4×4 matrix

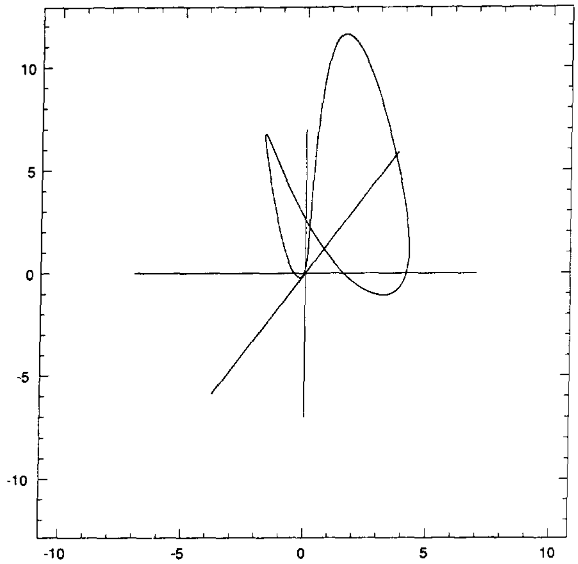


Fig. 2. Orbit of a point symmetric under the exchange 2↔3.

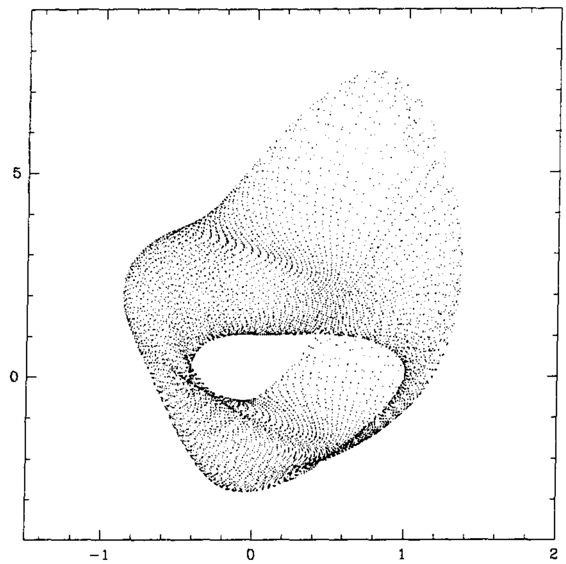


Fig. 3. Orbit of a generic point.

(41), the coefficients of which are

$$\sigma_1^{(3d)} = a + b_1 + b_2 + b_3 , \tag{44}$$

$$\sigma_2^{(3d)} = a(b_1 + b_2 + b_3) + b_1 b_2 + b_2 b_3 + b_3 b_1 - (c_1^2 + c_2^2 + c_3^2 + d_1^2 + d_2^2 + d_3^2) , \tag{45}$$

$\vdots$

Since $\sigma_2^{(3d)}$ is invariant by t_g, t_m and t_d and takes a simple factor (the inverse of the determinant) under the action of I, the variety $\sigma_2^{(3d)} = 0$ *is invariant under* Γ_3. Given the hugeness of the group Γ_3, it is already an

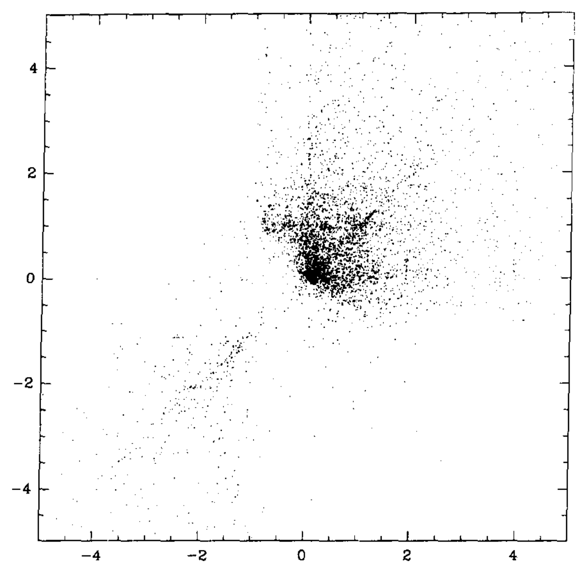


Fig. 4. Random orbit of the same generic point.

astonishing fact to have such a covariant expression. We shall give elsewhere a more extensive study of the invariants of Γ_3.

6.3. Orbits of Γ_3

To have some flavour of the possible (integrable?) algebraic varieties invariant under Γ_3, we study its orbits [12,13]. We start with the study of the subgroup generated by some infinite order element namely *IJ*. This element gives a special role to axis 1. With an initial point symmetric under the exchange of 2 and 3, we get remarkably a *curve*!! (see fig. 2). Other starting points lead to orbits lying on higher dimensional varieties (see fig. 3). However, what we are really interested in are the orbits of the *whole* Γ_3 group. The size of this group prevents us from studying exhaustively the full set of group elements of a given length even for quite small values of this length. We have nevertheless explored the group by a random construction of typical elements of increasingly large length (see fig. 4). We will give elsewhere a more extensive study of these orbits.

7. Conclusion

We have exhibited an infinite discrete symmetry group for the Yang–Baxter equations for *vertex* models. This group is the Coxeter group $A_2^{(1)}$ which is the semi-direct product of $\mathbb{Z}\times\mathbb{Z}$ by some finite group. The same group has already been found as a symmetry group of the star–triangle relation [11].

As happened there, the symmetry is responsible for the presence of the spectral parameter. In other words, the discrete symmetry gives rise to a continuous one (see ref. [11]).

A similar study for the generalized star–triangle relation of the interaction around a face (IRF) model, sketched in ref. [27], can be performed rigorously along the same lines, leading to the same result.

For three-dimensional vertex models, the symmetry group, though generalizing very naturally the previous group (generated by four involutions with similar relations) is drastically different: it is so "large" that the chances are quite small that it leaves enough room for any invariant integrability varieties. It is not useless to recall the unique non-trivial known solution of the tetrahedron equations [37,38,39]. For this model the group $\mathscr{A}ut_3$ does not have a free action. The three axes are not on the same footing, so that we do not have a "true" three-dimensional symmetry (two-dimensional checkerboard models coupled together).

Is there still any hope for a three-dimensional exactly solvable model with genuine three-dimensional symmetry? The group of symmetries we have described gives the best line of attack to this problem.

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