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The perimeter generating functions of three-choice, imperfect, and one-punctured staircase polygons*

M Assis[1], **M van Hoeij**[2] **and J-M Maillard**[3,4]

[1] MASCOS, School of Mathematics and Statistics, University of Melbourne, Carlton, VIC, Australia

[2] Florida State University, Department of Mathematics, 1017 Academic Way, Tallahassee, FL 32306-4510, USA

[3] LPTMC, UMR 7600 CNRS, Université de Paris 6, Tour 23, 5ème étage, case 121, 4 Place Jussieu, F-75252 Paris Cedex 05, France

E-mail: michael.assis@unimelb.edu.au, hoeij@mail.math.fsu.edu and maillard@lptmc.jussieu.fr

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Abstract

We consider the isotropic perimeter generating functions of three-choice, imperfect, and one-punctured staircase polygons, whose 8th order linear Fuchsian ODEs are previously known. We derive simple relationships between the three generating functions, and show that all three generating functions are joint solutions of a common 12th order Fuchsian linear ODE. We find that the 8th order differential operators can each be rewritten as a direct sum of a direct product, with operators no larger than 3rd order. We give closed-form expressions for all the solutions of these operators in terms of ${}_2F_1$ hypergeometric functions with rational and algebraic arguments. The solutions of these linear differential operators can in fact be expressed in terms of two modular forms, since these ${}_2F_1$ hypergeometric functions can be expressed with two, rational or algebraic, pullbacks.

Keywords: staircase polygons, three-choice polygons, imperfect staircase polygons, punctured staircase polygons, modular forms, hypergeometric functions, desingularization

Mathematics Subject Classification: 03D05, 11Yxx, 33Cxx, 34Lxx, 34Mxx, 34M55, 39-04, 68Q70

* Dedicated to A J Guttmann on the occasion of his 70th birthday.
[4] Author to whom any correspondence should be addressed.

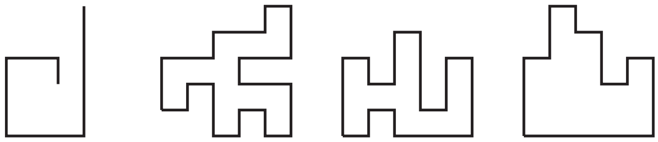

Figure 1. From left to right, examples of a spiral (a directed walk), a general self-avoiding polygon (SAP), a column convex polygon, and a histogram (column convex).

1. Introduction

Self-avoiding walks (SAWs) and self-avoiding polygons (SAPs) have long been studied in enumerative combinatorics as models of percolation, polymers, surface roughness, and more [1], although both their generating functions remain unsolved to this day. SAPs can be considered to be embeddings of simple closed curves into a regular lattice, typically the hypercubic lattice. Several classes of SAWs and SAPs have been solved by imposing either convexity or directedness constraints, or both. Convexity is defined with respect to an angle: the SAP is said to be convex with respect to an angle if for any line at that angle there are at most two intersections of the SAP with the line. Horizontal and vertical convexity are common constraints. Examples of a directed walk, a generic SAP, and column convex SAPs are shown in figure 1. Within a class, walks and polygons usually have the same growth constant, also known as the connective constant, although recently prudent polygons have been shown to be exponentially sparse among prudent walks [2].

The study of SAPs and its sub-categories involves the search for exact expressions of their generating functions as a function of various parameters of interest. These include the perimeter, width, height, site perimeter, left and right corners, and area, and for certain classes of SAPs, a generating function has been found which include all of these parameters explicitly, e.g. [3]. Among the known generating functions, rational, algebraic, *D*-finite, non-*D*-finite, and natural boundaries have been derived (see [4] for a good review). Furthermore, among still unsolved classes, it is possible to prove results concerning the nature of the unsolved generating function. For example, the anisotropic perimeter generating function for the full SAPs class has been proven to not be a *D*-finite function in [5]. The wide variety of types of functions which arise in the study of SAPs offers an intriguing source of knowledge for what constitutes exact solutions in statistical mechanics.

Among known perimeter generating functions are rational functions, algebraic functions, q-series, and natural boundaries [4], and quite generically the nature of the isotropic and anisotropic perimeter generating functions are the same. In the case of column-convex but not row-convex SAPs, the area generating functions are simpler than the corresponding perimeter generating function, being rational functions [4]. However, all known cases of area-perimeter generating functions involve q-series [4].

Three-choice and one-punctured staircase polygons are two classes of SAPs well studied in the literature [6, 7], known to be *D*-finite functions [8, 9] but whose perimeter generating functions have resisted closed-form solutions [10]. We here provide hypergeometric solutions to the operators appearing in their linear ODEs. It has long been suspected that their generating functions are related to each other [7–9, 11], and indeed we here show that they are equal up to the sum of an algebraic factor. Our hypergeometric solutions constitute the first example of a SAP generating function which is *D*-finite but not algebraic.

We begin by reviewing the literature of staircase, three-choice, and punctured staircase polygons in section 2, followed by an analysis of the linear differential operators of the three-

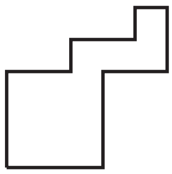

Figure 2. Example of a staircase polygon, which is also a directed and convex SAP.

choice and punctured staircase polygon linear ODEs in section 3. We then provide solutions for the linear differential operators in section 4 and explore hypergeometric and modular function identities of the solutions in section 5. We end with a discussion of generalizations of the results in section 6, followed by conclusions in section 7.

2. Known results

2.1. Staircase polygons

Staircase polygons are polygons formed from two SAWs that both start at the origin, move using only north or east steps (sometimes south and east steps [11]) and only intersect once again at their common endpoint. An example of a staircase polygon is shown in figure 2. Even though they have a long history in enumerative combinatorics and are among the most well studied classes of SAPs, their literature can be difficult to navigate, with numerous erroneous references, many independent proofs, and multiple equivalent names. Viewed in terms of their area, they are often called parallelogram polyominoes. In [12] they are also called skew Ferrers diagrams, defined as the difference between two Ferrers diagrams. While it is unstated in [12], it is clear from [13] that only the connected skew Ferrers diagrams are being considered in [12], such that indeed they correspond to staircase polygons. Finally, viewed in terms of two vicious walkers which start at the origin and end at their only other common point, they have also been called two-chain watermelons, or two-watermelons in [14].

Staircase polygons are examples of convex and directed SAPs and all typical quantities of interest are known exactly for them. Jack Levine appears to be the first to have published a proof of the isotropic and anisotropic perimeter generating functions in 1959 [15]. Nevertheless, his paper seems to have been largely neglected in the literature. Pólya in 1969 published the formula for the isotropic perimeter generating function, stated without proof but with reference to a diary entry from 1938 [16]. Other independent proofs have appeared, in 1984 [17] and in 1987 [18] for the isotropic perimeter. The site perimeter is a relevant quantity in percolation theory, and for staircase polygons it can be computed from the perimeter and the number of corners the polygon has. The perimeter-corner generating function is given in [19].

The inclusion of area in the generating function began with Pólya in 1969 [16], who provided an expression for the area-perimeter generating function, stated without proof. An expression for the area generating function was first proven in 1974 [20], followed by various proofs of the area-perimeter generating function as a continued fraction [21, 22], as well as a ratio of q-Bessel functions [22, 23]. The area-width generating function was given in [12], while the area-perimeter-left/right height generating function was given in [24]. The most general generating function, enumerated by area, perimeter, width, height, and left/right corners, was given in [3].

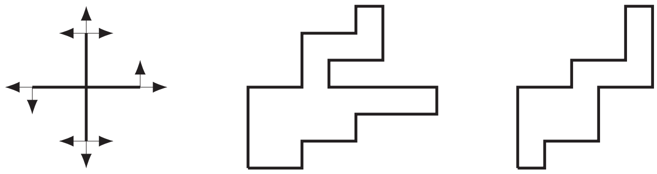

Figure 3. From left to right, Manna's three-choice walk possible turn directions after proceeding one step from the origin, an imperfect staircase polygon, and a staircase polygon.

Here we collect a few expressions. The isotropic perimeter generating function of half-perimeter n is related to the Catalan numbers C_n

$$P^{\mathrm{S}} = \sum_{n=2}^{\infty} \binom{2n}{n} \cdot \frac{x^n}{(4n-2)} = \sum_{n=1}^{\infty} C_n \cdot x^{n+1}$$
$$= \frac{1-2x-\sqrt{1-4x}}{2} = \frac{4x^2}{(1+\sqrt{1-4x})^2}. \tag{1}$$

We note the single square root singularity at $x = 1/4$.

The anisotropic perimeter generating function, in terms of h horizontal and v vertical perimeter, is given by

$$P(x, y) = \sum_{h,v \geqslant 1} \binom{h+v-1}{h} \binom{h+v-1}{v} \cdot \frac{x^h y^v}{(h+v-1)}, \tag{2}$$

$$= \frac{1}{2} \cdot (1 - x - y - \sqrt{1 - 2x - 2y + x^2 + y^2 - 2xy}), \tag{3}$$

where the previous result is obtained by setting $y = x \to \sqrt{x}$. The anisotropic perimeter generating function satisfies the simple algebraic equation

$$P = (P + x) \cdot (P + y), \tag{4}$$

and the inversion relation

$$P(x, y) - x \cdot P\left(\frac{1}{x}, \frac{y}{x}\right) = 1 - x. \tag{5}$$

Finally, we note the following functional equation for the isotropic half-perimeter x and area q generating function [25]

$$P(x, q) = \frac{q \cdot x^2}{1 - 2q \cdot x - P(q\, x, q)}. \tag{6}$$

2.2. Three-choice and imperfect staircase polygons

Three-choice SAWs were defined by Manna in 1984 [26] as SAWs where right-handed turns are disallowed after travelling in the east or west directions, and in 1993 their SAP equivalents were considered [6]. See figure 3 for a graphical definition of three-choice walks. There are two classes of three-choice polygons, the usual staircase polygons and imperfect staircase polygons whose perimeter can be broken up into four directed paths, examples of which are also shown in figure 3. Depending on the authors, 'three-choice polygons' can either mean both classes, or only the imperfect staircase polygons, e.g. [7]. A polynomial time algorithm

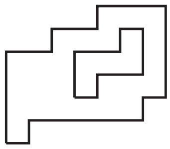

Figure 4. Example of a one-punctured staircase polygon.

for the enumeration of three-choice polygons by isotropic perimeter was given in [27], which hinted at its solvability. In that same work, it was shown using the theory of algebraic languages, that the perimeter generating function is not algebraic. Nevertheless, in [9] its 8th order linear ODE was found from a long series expansion, so that it is a *D*-finite transcendental function. The singularity closest to the origin on the positive real axis is at $x = 1/4$, the same location as for staircase polygons.

From the analysis of its series expansion [28], it is also expected that the anisotropic perimeter generating function is both solvable and *D*-finite (see page 85 of [10] for the mention, see [4] of [10], of an unpublished proof that it is *D*-finite). Furthermore, the generating function for the area and anisotropic perimeter was shown in [7] to satisfy self-reciprocity and inversion relations. The anisotropic generating function for imperfect staircase polygons satisfies the following inversion relation [29]

$$P^{\mathrm{I}}(x, y) + x^2 \cdot P^{\mathrm{I}}\left(\frac{1}{x}, \frac{y}{x}\right) = \frac{x^2 - 1}{2} \cdot \left(\frac{1 - 2x^2 - y^2 - x^2y^2 + x^4 + (x^2 - 1) \cdot \sqrt{\Delta}}{\sqrt{\Delta}}\right), \tag{7}$$

where

$$\Delta = (1 + x + y)(1 + x - y)(1 - x + y)(1 - x - y). \tag{8}$$

The full three-choice polygon perimeter generating function series reads

$$P^{\mathrm{T}} = 4x^2 + 12x^3 + 42x^4 + 152x^5 + 562x^6 + 2108x^7 + 7986x^8 + \cdots, \tag{9}$$

and the subset of only imperfect staircase polygon perimeter generating function series reads

$$P^{\mathrm{I}} = x^4 + 6x^5 + 29\,x^6 + 130x^7 + 561x^8 + 2368x^9 + 9855x^{10} + \cdots \tag{10}$$

2.3. Punctured staircase polygons

Punctured staircase polygons are staircase polygons with holes in the shape of a smaller staircase polygons whose perimeter does not share any vertices with the outer perimeter. We give an example of a one-punctured staircase polygon in figure 4. We here only consider one-punctured staircase polygons and below use ‘punctured staircase polygons’ synonymously with one-punctured staircase polygons. Punctured staircase polygons were first considered in [7], where the generating function for the area and anisotropic total perimeter were shown to satisfy self-reciprocity and inversion relations. In [11], a polynomial time algorithm was given for the enumeration of the total isotropic perimeter generating function for one to three holes, hinting at its solvability. In that same work, the exact generating functions for punctured staircase polygons with holes of perimeter 4 and 6 were found. Subsequently in [8], the total perimeter generating function was found to satisfy an 8th order linear ODE by consideration of a large series expansion. The singularity closest to the origin on the positive real axis is at

$x = 1/4$, coinciding with the location of the staircase polygon and three-choice SAP singularities.

In [11], it was noticed that all of the differential approximant exponents for the three-choice and punctured staircase polygons were equal, and in [7] it was seen that the inversion relation for their area and anisotropic perimeter generating functions were similar. Furthermore, in [8, 9] it was noted that the same transfer matrix can be used to enumerate the perimeter generating functions of both three-choice and punctured staircase polygons, subject simply to different boundary conditions. It therefore does not come as a surprise below that we find an exact algebraic relationship relating these generating functions.

We note that the exact perimeter generating function for staircase polygons with holes in the shape of 90°-rotated staircase polygons has been given and proven in [30] as an algebraic function, the solution of a 4th order linear ODE. It appears that there is no relation between the rotated-punctured generating function and the punctured perimeter generating function considered here.

The punctured staircase polygons total perimeter generating function series reads

$$\begin{aligned} P^{\mathrm{P}} = x^8 \; + \; 12\, x^9 + \; 94\, x^{10} + \; 604\, x^{11} \; + \; 3463\, x^{12} \\ + \, 18\,440 x^{13} \; + \; 93\,274 x^{14} + \; \cdots \end{aligned} \tag{11}$$

3. Differential operator structures

In the following, we denote the order of operators by subscripts. We also denote with $\oplus$ the direct sum $O_n = O_m \oplus O_p$ of two order-m and order-p linear differential operators O_m and O_p, such that all solutions of O_m and O_p are solutions of O_n. The direct sum structure means that the two operators O_m, O_p are two possible right-factors of O_n, namely $O_n = \tilde{O}_m \cdot O_p = \tilde{O}_p \cdot O_m$. Conversely, forming the operator O_n from lower order operators O_m, O_p amounts to taking the $\mathrm{LCLM}(O_m, O_p)$ of the two operators, where LCLM stands for least common left multiple (see [31] for more details).

The 8th order linear differential operator L_8^{P} denotes the operator annihilating the perimeter generating function of punctured staircase polygons. This linear differential operator has the following product and direct sum decomposition, where the product structure is of a different form than in [8]

$$L_8^{\mathrm{P}} = I_3 \cdot I_1 \cdot I_2 \cdot \bar{I}_1 \cdot \tilde{I}_1 = J_7 \oplus J_1 = (K_3 \cdot K_2 \cdot K_1 \cdot N_1) \oplus J_1. \tag{12}$$

Similarly, for the case of the three-choice staircase polygon perimeter generating function with imperfect staircase polygons included, we have the following 8th order operator product and direct sum decomposition

$$\begin{aligned} L_8^{\mathrm{T}} &= L_3 \cdot L_2 \cdot L_1 \cdot \bar{L}_1 \cdot \tilde{L}_1 = M_6 \oplus M_1 \oplus \bar{M}_1 \\ &= (N_3 \cdot N_2 \cdot N_1) \oplus M_1 \; \oplus \bar{M}_1. \end{aligned} \tag{13}$$

And finally, the corresponding 8th order operator for only imperfect staircase polygons also decomposes as follows

$$\begin{aligned} L_8^{\mathrm{I}} &= Q_3 \cdot Q_2 \cdot Q_1 \cdot \bar{Q}_1 \cdot \tilde{Q}_1 = M_6 \oplus R_1 \oplus \bar{R}_1 \\ &= (N_3 \cdot N_2 \cdot N_1) \oplus \; R_1 \oplus \; \bar{R}_1. \end{aligned} \tag{14}$$

Comparing L_8^{T} and L_8^{I}, both have direct sum decompositions into the *same* 6th order linear differential operator $M_6 = N_3 \cdot N_2 \cdot N_1$, and simple first order operators. As a consequence, these two linear differential operators are homomorphic (up to 7th order intertwinners).

In the product form, operators of the same order for the three cases are homomorphic [31] to each other, for instance $I_3 \simeq L_3 \simeq Q_3$ and $I_2 \simeq L_2 \simeq Q_2$. That is, $I_3 \cdot V_2 = W_2 \cdot L_3$ for intertwinner operators V_2, W_2 of second order. The solutions of the operator M_6 which appears both in L_8^{T} and L_8^{I} are also solutions of the operator J_7 of L_8^{P}. The J_7 and M_6 are homomorphic to each other with sixth order intertwining operators. The K_2 and N_2 operators are homomorphic to each other with first order intertwining operators, and the K_3 and N_3 operators are homomorphic to each other with second order intertwining operators in one direction, and first order intertwining operators in the other direction. Not surprisingly, L_8^{P} is homomorphic to L_8^{T} and L_8^{I} (again with 7th order intertwinners).

The operator M_6 has three solutions analytic at $x = 0$, and J_7 has four analytic solutions. The operators N_3, N_2, N_1 have the following form (D_x denotes, here, and elsewhere in the paper, the differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$)

$$N_3 = D_x^3 + \frac{p_{24}}{x \cdot (1 + 4x) \cdot p_{23}} \cdot D_x^2 + \frac{2\, p_{31}}{x^2 \cdot (1 + 4x) \cdot p_6\, q_{24}} \cdot D_x + \frac{2\, p_{37}}{x^3 \cdot (1 + 4x) \cdot p_6^2\, q_{24}}, \tag{15}$$

$$N_2 = D_x^2 - \frac{2\, p_7}{x \cdot (1 - 4x) \cdot p_6} \cdot D_x - \frac{2\, p_9}{x \cdot (1 - 4x)^2 (1 + x + 7x^2) \cdot p_6}, \tag{16}$$

$$N_1 = D_x + \frac{4}{1 - 4x}. \tag{17}$$

where the polynomials p_j, q_j of order j are given in appendix A. The solution of N_1 is $(1 - 4x)$.

Since the L_8^{T} and L_8^{I} operators are quite similar, as noted previously, it is not surprising that their LCLM only has order 10 instead of the generic 16th order expected of the LCLM of two 8th order operators. Similarly, that is the case among any pair of the operators L_8^{T}, L_8^{I}, L_8^{P}. To some extent this explains that the LCLM of all three operators, which encapsulates the generating functions of three-choice, imperfect, and punctured staircase polygons, is only of 12th order, half of the expected order. See appendix B for details of the LCLM structures. Note that any linear combination of the form

$$A_0 + \frac{A_1}{(1 - 4x)} + A_2 \cdot (1 - 4x) + A_3 \cdot (1 - 4x)^2 + \frac{A_4}{\sqrt{1 - 4x}} + A_5 \cdot \sqrt{1 - 4x} + A_6 \cdot (1 - 4x)^{3/2}, \tag{18}$$

is actually solution of the 12th order LCLM of the three 8th order linear differential operators L_8^{T}, L_8^{I}, and L_8^{P}.

3.1. Equivalence of generating functions

From [27], the relationship between the three-choice and imperfect staircase polygons is known to be

$$\frac{1}{2} P^{\mathrm{T}} - P^{\mathrm{I}} = x \cdot \frac{\mathrm{d}P^{\mathrm{S}}}{\mathrm{d}x} = \left(\frac{x}{\sqrt{1 - 4x}} - x \right), \tag{19}$$

where P^{S} is the staircase polygon generating function in (1).

The right-hand side of (19) is a solution of a 2nd order operator which is the LCLM of two simple first order operators. From equation (19) one immediately deduces that P^{T} is a solution of the LCLM of L_8^{I} and of this 2nd order operator, yielding the result that the LCLM of P^{T} and P^{I} is of 10th order as seen in appendix B. Conversely, the calculations of appendix B provide a means to deduce the relationship of equation (19).

Similarly, from the LCLM of P^{I} and P^{P} in appendix B, one can find the following algebraic relationship

$$P^{\mathrm{I}} + P^{\mathrm{P}} = -\frac{x^2}{2} \cdot \frac{\mathrm{d}P^{\mathrm{S}}}{\mathrm{d}x} + \frac{x^3}{1-4x} \tag{20}$$

$$= \frac{x}{2} \cdot \left(x - \frac{x}{\sqrt{1-4x}} \right) + \frac{x^3}{1-4x}. \tag{21}$$

Using the two relationships (19) and (20), we can deduce the following relationship

$$P^{\mathrm{P}} + \frac{1}{2} P^{\mathrm{T}} = -\frac{x \cdot (x-2)}{2} \cdot \frac{\mathrm{d}P^{\mathrm{S}}}{\mathrm{d}x} + \frac{x^3}{1-4x} \tag{22}$$

$$= \frac{x \cdot (x-2)}{2} \cdot \left(x - \frac{x}{\sqrt{1-4x}} \right) + \frac{x^3}{1-4x}. \tag{23}$$

While the LCLMs of the operators L_8^{T}, L_8^{I}, L_8^{P} provide a proof of the above relationships, we still have not found a direct combinatorial derivation or interpretation of relationships (20)–(23).

Note that by eliminating $\mathrm{d}P^{\mathrm{S}}/\mathrm{d}x$ between (19)–(23) and thereby eliminating all square roots, we find the following very simple relationship among P^{T}, P^{I}, P^{P}

$$P^{\mathrm{P}} + \left(\frac{x}{4}\right) \cdot P^{\mathrm{T}} + \left(1 - \frac{x}{2}\right) \cdot P^{\mathrm{I}} = \frac{x^3}{1-4x}. \tag{24}$$

Again, we have not yet found a combinatorial explanation of (24).

4. Results

The solutions to all three generating functions can be most simply expressed in terms of the solutions in equation (18), which are the seven first order solutions of $L_{12}^{\mathrm{TIP}} = \mathrm{LCLM}(L_8^{\mathrm{T}}, L_8^{\mathrm{I}}, L_8^{\mathrm{P}})$, plus solutions of $M_6 = N_3 \cdot N_2 \cdot N_1$. We here focus on the solutions of the linear differential operator L_8^{I}, since the solutions of the other two generating functions are easily related to the solution of L_8^{I} by relationships (19) and (20).

The generating function for imperfect staircase polygons is given as the sum of algebraic and transcendental functions

$$P^{\mathrm{I}} = \frac{1}{60} \cdot (P_{\mathrm{alg}}^{\mathrm{I}} + P_{\mathrm{trans}}^{\mathrm{I}}), \tag{25}$$

where the algebraic part $P_{\mathrm{alg}}^{\mathrm{I}}$ is actually a series with *integer* coefficients

$$P_{\mathrm{alg}}^{\mathrm{I}} = \frac{135}{8} + \frac{59}{4} \cdot (1-4x) + \frac{15}{8} \cdot (1-4x)^2 - \frac{85}{16} \cdot \frac{1}{\sqrt{1-4x}} - \frac{105}{8} \cdot \sqrt{1-4x} - \frac{65}{16} \cdot (1-4x)^{3/2}, \tag{26}$$

$$= 11 - 34x - 70x^3 - 265x^4 - 1020x^5 - 3920x^6 - 15\,060x^7 - 57\,915x^8 + \ldots, \tag{27}$$

and where the transcendental part $P^{\rm I}_{\rm trans}$

$$P^{\rm I}_{\rm trans} = -11 + 34x + 70x^3 + 325x^4 + 1380x^5 + 5660x^6 + 22\,860x^7 + 91\,575x^8 + \ldots, \tag{28}$$

is a solution of $N_3 \cdot\ N_2 \cdot\ N_1$ and can be decomposed as the linear combination

$$P^{\rm I}_{\rm trans} = -\frac{19}{2} \cdot {\rm Sol}_2 - \frac{3}{2} \cdot {\rm Sol}_3, \tag{29}$$

of the two regular solutions Sol_2 and Sol_3

$$N_2 \cdot N_1({\rm Sol}_2) = 0, \tag{30}$$

$$N_3 \cdot\ N_2 \cdot\ N_1({\rm Sol}_3) = 0, \tag{31}$$

corresponding to the series expansions with *integer* coefficients

$${\rm Sol}_2 = 1 - 2x + 3x^2 + 4x^3 + 13x^4 + 36x^5 + 95x^6 + 246x^7 + 588x^8 + \ldots, \tag{32}$$

$${\rm Sol}_3 = 1 - 10x - 19x^2 - 72x^3 - 299x^4 - 1148x^5 - 4375x^6 - 16\,798x^7 + \ldots. \tag{33}$$

The series Sol_2, Sol_3 are the unique series, up to an overall factor, multiplying the largest logarithmic power of the formal solutions of $N_2 \cdot\ N_1$ and $N_3 \cdot\ N_2 \cdot\ N_1$, respectively, as seen in equation (C.1) of appendix C.

It is remarkable to observe that the linear combination of rational coefficients from $(1/60)P^{\rm I}_{\rm alg}$, $(19/120)\mathrm{Sol}_2$ and $(3/120)\mathrm{Sol}_3$ actually gives the integer series corresponding to $P^{\rm I}$. As will be seen below, N_2 and N_3 are of a quite different nature, so that it is rather surprising that a solution of $N_3 \cdot N_2 \cdot N_1$ is precisely able to compensate the $P^{\rm I}_{\rm alg}/60$ series in order to generate the integer series of $P^{\rm I}$.

4.1. Exact N_2 solution

The second order linear differential operator N_2 has the following solution as a ${}_2F_1$ hypergeometric function with a rational cubic pullback, which can be found, for example, using the program hypergeomdeg3 described in [32]

$${\rm Sol}(N_2) = \frac{1\ + x\ + 7\,x^2}{18 \cdot\ x \cdot\ (1-x)^2 \cdot (1-4x)^{3/2}} \cdot \mathcal{S}, \tag{34}$$

$$= 1 + 7x + 28x^2 + 122\,x^3 + 500\,x^4 + 1997\,x^5 + 7899\,x^6 + 30\,996\,x^7 + 120\,774\,x^8 + 468\,035\,x^9 + 1805\,351\,x^{10} + 6932\,732\,x^{11} + \cdots, \tag{35}$$

where

$$\mathcal{S} = (1-x)(1-4x)(1 + 45\,x^2 + 44\,x^3) \cdot\ \frac{{\rm d}\mathcal{H}}{{\rm d}x} + 18 \cdot (1 - 3\,x - 13\,x^2) \cdot\ \mathcal{H}, \tag{36}$$

with

$$\mathcal{H} = {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], \frac{27x^3}{(1-x)^3}\right). \tag{37}$$

This solution can also be expressed as the following sum of two contiguous ${}_2F_1$ hypergeometric functions

$$\mathrm{Sol}(N_2) = \frac{1}{(1-x)^2 \cdot (1-4x)^{3/2}} \cdot \left[(x + 45x^3 + 44x^4) \cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [2], \frac{27x^3}{(1-x)^3}\right) \right.$$
$$\left. + (1 + x + 7x^2) \cdot (1 - 3x - 13x^2) \cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], \frac{27x^3}{(1-x)^3}\right) \right]. \tag{38}$$

As a sum of two contiguous ${}_2F_1$ hypergeometric functions, we can wonder whether a different contiguous basis exists for $\mathrm{Sol}(N_2)$ which gives smaller algebraic pre-factors. We have made use of the Maple procedure contiguous2f1.mpl developed by Vidūnas [33], but we have been unable to find a simpler contiguous basis.

While the form of the pullback in (37) does not reveal the physical singularity at $x = 1/4$, the standard hypergeometric identity

$${}_2F_1([a, b], [c], z)) = (1 - z)^{-a} \cdot {}_2F_1\left([a, c - b], [c], \frac{z}{z - 1}\right), \tag{39}$$

changes the pullback to

$$z = \frac{27x^3}{(1-x)^3} \longrightarrow \frac{z}{z-1} = -\frac{27x^3}{(1-4x)(1+x+7x^2)}, \tag{40}$$

from which we see that the physical singularity is mapped to $z = \infty$ and where we also see the appearance of the unphysical singularities at the roots of polynomial $1 + x + 7x^2$, which were already observed in [8].

4.2. Exact N_3 solution

The solution to the operator N_3 is much more involved and the path to discovering its solution is not obvious. The discovery of the solution starts with seeing that the exterior square of N_3 has a rational solution, which means that this 3rd order operator is homomorphic to the *symmetric square* of a second order operator. Constructing $\bar{N}_3$ by conjugating N_3 by the square root of this rational function, we used the 'conic program' described in [34] and available at [35] in order to find a second order linear differential operator $\bar{V}_2$, such that its *symmetric square* is, not equal, but *homomorphic* to $\bar{N}_3$. This second order linear differential operator $\bar{V}_2$ reads

$$\bar{V}_2 = D_x^2 + \frac{p_{36}}{(1-4x)(1+4x)(1+4x^2) \cdot p_{33}} \cdot D_x$$
$$- 2 \cdot \frac{q_{36}}{x \cdot (1 - 4\,x)\,(1 + 4\,x)(1 + 4\,x^2) \cdot p_{33}}, \tag{41}$$

where the polynomials p_{33}, p_{36} and q_{36} are given in appendix A. The quite large degree 33 polynomial p_{33} corresponds to *apparent singularities*. To go further it is crucial to get rid of p_{33} and to find a 2nd order linear differential operator homomorphic to $\bar{V}_2$ with none of these apparent singularities. This corresponds to the so-called *desingularization* of a Fuchsian linear differential operator. Note that in general it is *not easy* to get rid of the apparent singularities *without introducing new ones*. We are not interested in such a 'partial' desingularization, but in a 'complete' desingularization, which is, in general, *not always possible* without increasing the order. One looks for an operator equivalence that keeps the exponents of the new linear differential operator at the true singularities in a fairly narrow range: to 'minimize' the number of apparent singularities one needs to 'maximize' the sum of all the exponents at all the true regular singularities[5]. In this particular case, we have been able to find such a 2nd order

[5] The technical details on how to implement these briefly sketched ideas will be found in a forthcoming paper [36].

Fuchsian linear differential operator V_2, homomorphic to $\bar{V}_2$, with no apparent singularities[6]. This much simpler second order operator V_2 reads

$$(1-4x)^2 \cdot (1+4x)^2 \cdot (1+4x^2)^2 \cdot x^2 \cdot V_2$$
$$= (1-4x)^2 \cdot (1+4x)^2 \cdot (1+4x^2)^2 \cdot x^2 \cdot D_x^2$$
$$+ (192x^4+24x^2-1) \cdot (1-4x) \cdot (1+4x) \cdot (1+4x^2) \cdot x \cdot D_x$$
$$+ 16128x^8+3280x^6+532x^4-16x^2+1. \tag{42}$$

These two operators, $\bar{V}_2$ and V_2, are homomorphic with first order intertwinners

$$V_2 \cdot A_1 = B_1 \cdot \bar{V}_2, \qquad C_1 \cdot V_2 = \bar{V}_2 \cdot D_1, \tag{43}$$

where[7] the first order intertwiners A_1, B_1, C_1, D_1 are of the form

$$A_1 = \frac{x \cdot (1-4x)(1+4x)(1+4x^2)}{p_{33}} \cdot (\tilde{p}_{15} \cdot D_x - 2 \cdot \tilde{q}_{14}), \tag{44}$$

$$B_1 = \frac{1}{p_{33}} \cdot \left(\tilde{p}_{15} \cdot (1-4x)(1+4x)(1+4x^2) \cdot x \cdot D_x + \frac{\tilde{p}_{52}}{p_{33}}\right), \tag{45}$$

$$C_1 = \tilde{p}_{15} \cdot \left(D_x - \frac{\mathrm{d}\ln(1/U)}{\mathrm{d}x}\right) + \frac{\tilde{p}_{47}}{p_{33}}, \tag{46}$$

$$D_1 = \tilde{p}_{15} \cdot \left(D_x - \frac{\mathrm{d}\ln(x \cdot U)}{\mathrm{d}x}\right) + 2 \cdot \tilde{p}_{14}. \tag{47}$$

where U denotes the algebraic function $\sqrt{(1-16x^2)(1+4x^2)}$, and where the polynomials $\tilde{p}_{14}$, $\tilde{q}_{14}$, $\tilde{p}_{15}$, $\tilde{p}_{47}$ and $\tilde{p}_{52}$ are polynomials with integer coefficients of degree 14, 14, 15, 47 and 52 respectively. Note that $\tilde{p}_{14}$ or $\tilde{q}_{14}$ *do not* correspond to derivatives of $\tilde{p}_{15}$. Therefore the first order intertwiners A_1 and D_1 *are not Fuchsian operators*, since one does not have a logarithmic derivative. Along this line it is obvious that B_1 and C_1 are also *not Fuchsian operators*: these four order-one intertwiners, corresponding to the homomorphisms the two *Fuchsian* order-two operators, $\bar{V}_2$ and V_2 , are *not themselves Fuchsian*.

The 2nd order operator V_2 being homorphic to $\bar{V}_2$, and the 3rd order operator N_3 being homorphic to the symmetric square of $\bar{V}_2$, one finds straightforwardly that the 3rd order operator N_3 is homomorphic to the symmetric square of the (much simpler) 2nd order operator V_2

$$N_3 \cdot T_2 = W_2 \cdot \mathrm{Sym}(V_2)^2, \tag{48}$$

where the intertwinner T_2 is a 2nd order operator of the form

$$T_2 = \frac{4\,p_{10}}{3x^3 \cdot (1-4x)^2(1+x+7x^2) \cdot p_6} \cdot D_x^2$$
$$+ \frac{4\,p_{14}}{3x^4 \cdot (1-4x)^2(1+x+7x^2)(1-12x^2-64x^4) \cdot p_6} \cdot D_x$$
$$+ \frac{16\,p_{18}}{3x^5 \cdot (1-4x)^2(1+x+7x^2)(1-24x^2+16x^4+1536x^6+4096x^8) \cdot p_6}, \tag{49}$$

with polynomial coefficients $p_j(x)$ of order j defined in appendix A.

[6] This much simpler second order operator can be obtained from van Hoeij's program reduceorder available here [37].

[7] Note that the 3rd order operators $V_2 \cdot A_1$ and $C_1 \cdot V_2$ in (43) still have p_{33} (and another polynomial $\tilde{p}_{15}$, see below) as apparent singularities.

The relevant solution of V_2 is given by

$$\mathrm{Sol}(V_2) = \mathcal{S}(x,\, U)$$
$$= x \cdot\; U \cdot \left(\frac{13\,-\,28\,x^2\;-\,12\,U}{(1\,+\,20\,x^2)^2}\right)^{1/4} \cdot\; {}_2F_1\left(\left[\frac{1}{8},\,\frac{3}{8}\right],\,[1],\,\frac{4096\,x^{10}}{(1\,-\,4\,x^2\,+\,U)^4}\right), \tag{50}$$

$$= x \cdot\; U \cdot\; \mathrm{Heun}\left(-\frac{1}{4},\,\frac{1}{16},\,\frac{3}{8},\,\frac{5}{8},\,1,\,\frac{1}{2},\,-4\,x^2\right), \tag{51}$$

$$= x\,-\,5\,x^3\,-\,\frac{95}{2}\,x^5\,-\,\frac{655}{2}\,x^7\,-\,\frac{27\,365}{8}\,x^9\,-\,\frac{305\,131\,x^{11}}{8}\,-\,\frac{7365\,195}{16}x^{13}$$
$$-\,\frac{92\,787\,415}{16}\,x^{15}\,-\,\frac{9671\,421\,805}{128}\,x^{17}\,-\,\frac{129\,164\,164\,935}{128}\,x^{19}\,+\,\cdots, \tag{52}$$

with

$$U\,=\,+\sqrt{(1\,-\,16\,x^2)\,(1\,+\,4\,x^2)}\,. \tag{53}$$

Finding the solution of V_2 in (50) as a ${}_2F_1$ hypergeometric function with an *algebraic* pullback is highly non-trivial. It can be found, for example, using the new Maple procedure hypergeometricsols written by Erdal Imamoglu and available at [38]. The result found from the program contains a different algebraic pre-factor as well as a different algebraic pullback, which can then be simplified to the form (50) found above[8].

One can of course imagine that considering the other branch of the square root in (53) gives an alternate expression. This is actually the case, changing U into $-U$ gives, in fact, the same solution *up to a* $-5^{-1/2}$ *factor*, in the following alternative form

$$\mathrm{Sol}(V_2) = -\,5^{-1/2}\,\cdot\;\mathcal{S}(x,\,-U)$$
$$= x \cdot\; U \cdot \left(\frac{13\,-\,28\,x^2\;+\,12\,U}{5^2\,\cdot\,(1\,+\,20\,x^2)^2}\right)^{1/4} \cdot\; {}_2F_1\left(\left[\frac{1}{8},\,\frac{3}{8}\right],\,[1],\,\frac{4096\,x^{10}}{(1\,-\,4\,x^2\,-\,U)^4}\right). \tag{54}$$

Note that the two pullbacked hypergeometric functions in (50) and (54) are actually two different series with *integer coefficients*

$${}_2F_1\left(\left[\frac{1}{8},\,\frac{3}{8}\right],\,[1],\,\frac{4096\,x^{10}}{(1\,-\,4\,x^2\,+\,U)^4}\right) = 1\,+\,12\,x^{10}\,+\,240\,x^{12}\,+\,4200\,x^{14}$$
$$+\,67\,200\,x^{16}\;+\,1040\,700\,x^{18}\,+\,15\,830\,388\,x^{20}\;+\,238\,737\,720\,x^{22}\,+\,\cdots, \tag{55}$$

while

$${}_2F_1\left(\left[\frac{1}{8},\,\frac{3}{8}\right],\,[1],\,\frac{4096\,x^{10}}{(1\,-\,4\,x^2\,-\,U)^4}\right) = 1\,+\,12\,x^2\,-\,12\,x^4\;+\,744\,x^6$$
$$-\,2700\,x^8\;+\,115\,140\,x^{10}\,-\,782\,520\,x^{12}\;+\,24\,418\,920\,x^{14}\,-\,238\,316\,940\,x^{16}$$
$$+\,6113\,609\,700\,x^{18}\;-\,74\,768\,429\,700\,x^{20}\;+\,1698\,621\,342\,600\,x^{22}\,+\,\cdots \tag{56}$$

The series solution (52) is not a series with integer coefficients but it is globally bounded [43] so that it can be recast into a series with integer coefficients by changing $x \to 2x$. In fact, the square of the series solution (52) is a series with integer coefficients, since the square of the algebraic pre-factor is a series with integer coefficients.

[8] Similarly, the solution of N_2 in (34) can be found in an alternative form through the hypergeometricsols program.

The solution of N_3 is straightforwardly found by applying the second order intertwinner operator T_2 to the square of the solution of V_2

$$\mathrm{Sol}(N_3) = T_2(\mathrm{Sol}(V_2)^2). \tag{57}$$

Using the *Clausen identity* [39]

$${}_2F_1\left(\left[\frac{1}{8}, \frac{3}{8}\right], [1], z\right)^2 = {}_3F_2\left(\left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right], [1, 1], z\right), \tag{58}$$

one can rewrite the quadratic expression (57) of the ${}_2F_1$ in (50) in terms of the pullbacked ${}_3F_2$ and its derivatives

$${}_3F_2\left(\left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right], [1, 1], \frac{4096\, x^{10}}{(1 - 4\, x^2 + U)^4}\right). \tag{59}$$

We further note that using the following hypergeometric identity

$${}_2F_1\left(\left[\frac{1}{8}, \frac{3}{8}\right], [1], \frac{16\, z^2 \cdot (1 - z)}{(2 - z)^4}\right) = \left(\frac{2 - z}{2}\right)^{1/2} \cdot {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{2}\right], [1], z\right), \tag{60}$$

it is possible to transform the ${}_2F_1$ hypergeometric function in $\mathrm{Sol}(N_3)$ to the complete elliptic integrals of the first and second kinds $K(z)$, $E(z)$.

In appendix D we consider the problem of integrating $\mathrm{Sol}(N_3)$ back through $N_2 \cdot N_1$ in order to find Sol_3 of (31).

4.3. Singularity analysis

The nearest singularity on the positive real axis for the generating functions is at $x = 1/4$. It was already known from [8, 9]) that the singularity at $x = 1/4$ has a square root divergence as well as a logarithmic singularity. The square root divergence has contributions from both the algebraic and transcendental parts of the generating function solution in (25). From the Heun function form of Sol_3 in (51) we can see clearly that Sol_3 only contributes corrections to the square root singularity of the algebraic part. Therefore, the sole contribution to the logarithmic singularity at $x = 1/4$ comes from the hypergeometric solution Sol_2 in (34) with (40), which also contributes to the square root singularity.

Note that the singularities at $1 + 4x = 0$, $1 + 4x^2 = 0$, $1 + x + 7x^2 = 0$, analyzed in [8, 9]) only emerge from the transcendental part of the solution in (25).

5. Modular forms and hypergeometric identities

5.1. The solutions of N_2 as modular forms

The solution (34) of N_2 in terms of (37) can actually be seen to be associated with a *modular form*.

Let us consider the *modular curve*

$$\begin{aligned} &10\,077\,696 \cdot\ C^3D^3 + 377\,9136 \cdot C^2D^2 \cdot\ (C + D) \\ &\quad + 472\,392 \cdot\ C\,D \cdot\ (C^2 - 87\,CD + D^2) + 19\,683 \cdot\ (C + D) \cdot\ (C^2 + 440\,CD + D^2) \\ &\quad - 59\,049 \cdot (C^2 - 87\,CD + D^2) + 59\,049 \cdot (C + D) - 19\,683 = 0, \end{aligned} \tag{61}$$

which is a genus-zero curve with the simple rational parameterization

$$C = \left(\frac{3\,x}{1 - x}\right)^3, \qquad D = \left(\frac{1 - 4\,x}{1 + 5\,x}\right)^3, \tag{62}$$

such that

$$D(x) = C\left(\frac{1 - 4\,x}{4 + 11\,x}\right), \qquad C(x) = D\left(\frac{1 - 4\,x}{4 + 11\,x}\right). \tag{63}$$

Along this line, introducing N_2^p as the $(1 - 4\,x)/(4 + 11\,x)$ pullback of the linear differential operator N_2, one sees that the symmetric square of N_2^p and of N_2 are actually homomorphic.

With C and D given by (62), and thus related by the modular curve (61), one has the following non-trivial identity on the *same* ${}_2F_1$ hypergeometric function with the two different pullbacks $1 - D$ and C

$${}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], 1 - D\right) = \frac{1 + 5\,x}{1 - x} \cdot {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], C\right), \tag{64}$$

namely

$${}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], \frac{27\,x \cdot (1 + x + 7\,x^2)}{(1 + 5\,x)^3}\right) = \frac{1 + 5\,x}{1 - x} \cdot\ {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], \frac{27\,x^3}{(1 - x)^3}\right).$$

This relation is (after the change of variable $x \rightarrow 3\,x/(1 - x)$), nothing but *Ramanujan's Cubic transformation* (see corolory 2.4 page 97 of [40] and (2.23) in [41])

$$(1 + 2\,x) \cdot\ {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], x^3\right) = {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], 1 - \left(\frac{1 - x}{1 + 2\,x}\right)^3\right). \tag{65}$$

This relation is also, up to a simple change of variables, the relation on page 44, table 18, fifth line in Maier's paper [42]

$$\begin{aligned} &{}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], \frac{x \cdot\ (x^2 + 9\,x + 27)}{(x + 3)^3}\right) \\ &\quad = 3 \cdot\ \frac{x + 3}{x + 9} \cdot\ {}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], \frac{x^3}{(x + 9)^3}\right). \end{aligned} \tag{66}$$

Such non-trivial identities on the *same* ${}_2F_1$ hypergeometric function with the *two different pullbacks* related by a *modular curve*, show the emergence of a *modular form* (see Maier's paper [42]).

5.2. The solutions of N_3 as modular forms

The solution of N_3 is given in terms of the solution of V_2 given in either the form (50) or (54), through (57). The fact that the same solution series (52) can be expressed *in two different ways*, (50) or (54), corresponds to a *quite non-trivial identity*, namely

$$\mathcal{S}(x,\, U) = -5^{-1/2} \cdot \ \mathcal{S}(x,\, -U), \tag{67}$$

between the same hypergeometric function but with two different algebraic pullbacks. Such non-trivial identity *actually corresponds to a modular form* (a covariance with respect to the isogenies associated with the *modular curve* [43, 48]).

These two *algebraic* pullbacks

$$A = \frac{4096\, x^{10}}{(1 - 4\, x^2 - U)^4}, \qquad B\frac{4096\, x^{10}}{(1 - 4\, x^2 \ + U)^4}, \tag{68}$$

are related by the *genus-zero* modular curve

$$\begin{aligned}
&722\,204\,136\,308\,736 \cdot \ A^4B^4 \cdot \ (625A^2 \ + 1054AB \ + 625B^2) \\
&\quad + 13\,931\,406\,950\,400 \cdot \ A^3B^3 \cdot (A + B) \cdot \ (5A^2 - 43786\, AB \ + 5B^2) \\
&\quad + 1343\,692\,800 \cdot \ A^2B^2 \cdot [3\,(A^4 + B^4) \\
&\quad - 15\,066\,308\, AB \cdot \ (A^2 + B^2) + 114\,938\,242\, A^2B^2] \\
&\quad + 103\,680 \cdot \ AB \cdot \ (A + B) \cdot [A^4 + B^4 \\
&\quad + 45\,004\,444\, AB \cdot \ (A^2 + B^2) + 35\,527\,135\,712\, 6A^2B^2] \\
&\quad + 204\,800\, AB \cdot \ (A + B)(6137A^2 \ + 847\,562\,510\, AB \ + 6137B^2) \\
&\quad - 6553\,600 \cdot \ AB \cdot \ (863A^2 - 3718\,702AB + 863B^2) + 8724\,152\,320 \cdot \ AB \cdot \ (A + B) \\
&\quad + [A^6 + B^6 - 65\,094\,150\, AB \cdot \ (A^4 + B^4) - 13\,453\,926\,179\,834\,900 \cdot \ A^3B^3 \\
&\quad + 98\,471\,158\,056\,975\, A^2B^2 \cdot \ (A^2 + B^2)] \\
&\quad - 42\,94\,967\,296 \cdot \ AB = 0.
\end{aligned} \tag{69}$$

This genus-zero modular curve can be seen as corresponding to the elimination of the x variable between the two 'auxilliary equations' (see (70) in [43])

$$\begin{aligned}
&(1 \ + 20\, x^2)^4 \cdot \ A^2 + 256\, x^2 \cdot \ (224\, x^8 - 400\, x^6 - 50\, x^4 + 20\, x^2 - 1) \cdot \ A \\
&\quad + 65536 \cdot \ x^{12} = 0,
\end{aligned} \tag{70}$$

$$\begin{aligned}
&(1 \ + 20\, x^2)^4 \cdot \ B^2 + 256\, x^2 \cdot \ (224\, x^8 - 400\, x^6 - 50\, x^4 + 20\, x^2 - 1) \cdot \ B \\
&\quad + 65536 \cdot \ x^{12} = 0.
\end{aligned} \tag{71}$$

These two auxiliary equations are actually, and surprisingly, *genus-one* curves rather than genus zero: this is a consequence of the fact that the exact expression of the two algebraic pullbacks in (70) and (71) requires a square root, namely U. On the other hand, the modular curve (69) that one would expect to be a genus-one curve *is in fact a genus-zero curve*.

We note that the pullback of the ${}_2F_1([1/8,\, 3/8],\, [1],\, z)$ hypergeometric function in the solution of N_3 can be rewritten as

$$\frac{4096\, x^{10}}{(1 - 4\, x^2 - U)^4} = 16\, x^2 \cdot \left(\frac{1 - 4\, x^2 + U}{1 + 20\, x^2}\right)^4, \tag{72}$$

which can be rewritten in an alternative way which is linear in U

$$\begin{aligned}
&\frac{128\, x^2 \cdot \ (1 \ - 20\, x^2 \ + 50\, x^4 \ + 400\, x^6 - 224\, x^8)}{(1 \ + 20\, x^2)^4} \\
&\quad + U \cdot \ \frac{128\, x^2 \cdot \ (1 - 4\, x^2)\,(1 \ + 2\, x^2)\,(1 \ - 12\, x^2)}{(1 \ + 20\, x^2)^4}.
\end{aligned} \tag{73}$$

From this rewriting of the pullback, it is tempting to see the singularity $1 + 20x^2 = 0$ of the pullback as a singularity of the function. This is not the case, as can be seen in appendix E.

5.2.1. Another parametrization. If one recalls the definition of the square root variable U in (53) and the previous expressions for the pullback A, in (72), one remarks that all these expressions are, in fact, functions of $X = x^2$. The definition (53) of U corresponds to a *rational curve* $U^2 - (1 - 16\,X)(1 + 4\,X) = 0$, which can be parametrized as follows

$$U = \frac{5}{4} \cdot \frac{t^2 - 64}{t^2 + 64}, \qquad X = -\frac{1}{32} \cdot \frac{(3\,t - 8)\,(t - 24)}{t^2 + 64}, \tag{74}$$

yielding the following rational parametrization of A and B for the genus-zero modular curve (69)

$$A(t) = -\frac{1}{2}\frac{(3\,t - 8)(t - 24)^5}{(7\,t - 8)^4\,(t^2 + 64)}, \quad B(t) = -\frac{1}{2}\frac{(3\,t - 8)^5\,(t - 24)}{(t - 56)^4\,(t^2 + 64)} = A\left(\frac{64}{t}\right). \tag{75}$$

Performing the change of variable $t = 24 + u$, one has the alternative parametrization

$$U = \frac{5}{4} \cdot \frac{(u + 32)\,(u + 16)}{u^2 + 48\,u + 640}, \qquad X = -\frac{1}{32} \cdot \frac{(64 + 3\,u) \cdot u}{u^2 + 48\,u + 640}, \tag{76}$$

$$A(u) = -\frac{1}{2}\,\frac{(64 + 3\,u) \cdot u^5}{(160 + 7\,u)^4\,(u^2 + 48\,u + 640)},$$

$$B(u) = -\frac{1}{2}\,\frac{(64 + 3\,u)^5 \cdot u}{(u - 32)^4\,(u^2 + 48\,u + 640)}.$$

Rewriting the solution of N_3 in terms of the ${}_3F_2$ hypergeometric function (59) amounts to considering the following ${}_3F_2$ identity

$$\begin{aligned}(160 - 5\,u) \cdot {}_3F_2\left(\left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right], [1, 1], A(u)\right)\\ = (160 + 7\,u) \cdot {}_3F_2\left(\left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right], [1, 1], B(u)\right),\end{aligned} \tag{77}$$

which corresponds, using the Clausen identity (58), to the ${}_2F_1$ identity

$$\begin{aligned}(160 - 5\,u)^{1/2} \cdot {}_2F_1\left(\left[\frac{1}{8}, \frac{3}{8}\right], [1], A(u)\right)\\ = (160 + 7\,u)^{1/2} \cdot {}_2F_1\left(\left[\frac{1}{8}, \frac{3}{8}\right], [1], B(u)\right).\end{aligned} \tag{78}$$

5.2.2. Infinite order symmetry on a Heun function. The occurrence of modular forms corresponds to identities like (67) or (78), relating the same ${}_2F_1$ hypergeometric function with two different pullbacks, which are related by a modular curve (69). These infinite order symmetries of the ${}_2F_1$ hypergeometric functions corresponds to *isogenies* [43, 48] *of the elliptic curves* which amount to multiplying or dividing the ratio of the two periods of an elliptic curve by an integer N.

The solution of $\mathrm{Sol}(V_2)$, expressed in terms of a ${}_2F_1$ hypergeometric function (50), can also be expressed as a simple Heun function (51). One can thus expect an *infinite order symmetry* identity on this Heun function. The identity reads

$$\mathcal{A}_1(X) \cdot \text{ Heun}\left(-\frac{1}{4}, \frac{1}{16}, \frac{3}{8}, \frac{5}{8}, 1, \frac{1}{2}, -4\,X\right)$$
$$= \mathcal{A}_2(Y) \cdot \text{Heun}\left(-\frac{1}{4}, \frac{1}{16}, \frac{3}{8}, \frac{5}{8}, 1, \frac{1}{2}, -4\,Y\right), \tag{79}$$

where X and Y are related by a *genus-one* curve $P(X, Y) = 0$ given in appendix F, and $\mathcal{A}_1(X)$ and $\mathcal{A}_2(X)$ are two algebraic expressions also given in appendix F. If the expression of the solution of $\text{Sol}(V_2)$ in terms of Heun function looks (artificially) simpler, the representation of the infinite order isogeny symmetries is more involved, since we do not have a rational parametrization of $P(X, Y) = 0$.

5.3. N_2 versus N_3

We can attempt to find a relationship between the ${}_2F_1$ hypergeometric functions appearing in the solutions of N_2 and N_3 (via V_2). In order to achieve that goal let us rather try to reduce both their corresponding ${}_2F_1$ hypergeometric functions with a pullback, ${}_2F_1([1/3, 2/3], [1], r(x))$ and ${}_2F_1([1/8, 3/8], [1], s(x))$, to a standard [42] ${}_2F_1([1/12, 5/12], [1], t(x))$ form. This can indeed be done, according to the two identities below

$${}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], x\right) = \frac{1}{Q^{1/4}} \cdot \; {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], -\frac{64x \cdot (x-1)^3}{(8x+1)^3}\right), \tag{80}$$

where

$$Q = 5 - 4\sqrt{1 - 4\,x \cdot (1-x)}, \tag{81}$$

and

$${}_2F_1\left(\left[\frac{1}{8}, \frac{3}{8}\right], [1], -\frac{4x}{(1-x)^2}\right)$$
$$= \left(\frac{1-x}{1-4x}\right)^{1/4} \cdot \; {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], -\frac{27x}{(1-4x)^3}\right). \tag{82}$$

Using the first identity (80) on the ${}_2F_1$ hypergeometric function (37), occurring in the solution (34) of the second order operator N_2 yields

$${}_2F_1\left(\left[\frac{1}{3}, \frac{2}{3}\right], [1], \frac{27x^3}{(1-x)^3}\right)$$
$$= \frac{(1-x)^{3/4}}{(1+5\,x)^{1/4} \cdot (1 - 8\,x + 43\,x^2)^{1/4}} \cdot \; {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], \mathcal{P}_2\right), \tag{83}$$

where the pullback $\mathcal{P}_2$ reads

$$\mathcal{P}_2 = \frac{1728\,x^3 \cdot (1-4\,x)^3\,(1 + x + 7\,x^2)^3}{(1-x)^3\,(1+5\,x)^3\,(1 - 8\,x + 43\,x^2)^3}. \tag{84}$$

Similarly, using the second identity (82) on the solution $\text{Sol}(V_2)$ given by (50) occurring in the solution of the 3rd order operator N_3 yields a rewriting of $\text{Sol}(V_2)$ in terms of a pullbacked hypergeometric function ${}_2F_1([1/12, 5/12], [1], \mathcal{P}_3)$.

The elimination of x between these two pullbacks $\mathcal{P}_2$ and $\mathcal{P}_3$ yields an involved polynomial relation $P(\mathcal{P}_2, \mathcal{P}_3) = 0$, where the polynomial P is the sum of 1665 monomials of degree 36 in $\mathcal{P}_2$ and 48 in $\mathcal{P}_3$. In other words, the solutions of $\text{Sol}(N_2)$ and $\text{Sol}(N_3)$ and their corresponding modular forms are far from being simply related.

6. Towards generalizations of the results

There are several ways in which the results of this paper could be extended. From [28], it is known that the *anisotropic perimeter* generating functions for three-choice and imperfect staircase polygons have a simple structure. Since for all known closed-form solutions it has been shown [4] that the anisotropic perimeter generating functions are simple extensions of their isotropic counterparts, one could expect that the anisotropic versions of the generating functions in this paper could be simple extensions of the hypergeometric functions appearing in the solutions. It may be that only the arguments (pullbacks) of the ${}_2F_1$ hypergeometric functions become two-variable rational or algebraic functions. Another plausible extension would be generalizations to two-variable hypergeometric functions, such as Appell or Horn functions [44].

A second generalization of the results would be to consider the *area-perimeter* generating function. All known results for area-perimeter generating functions involve q-series [4]. In [8, 9] conjectured forms for the area-perimeter generating functions are proposed for three-choice and imperfect staircase polygons, and one-punctured staircase polygons, respectively. The conjectures involve q-Bessel functions with algebraic pre-factors. Alternatively, based on the hypergeometric results above, it is reasonable to propose the appearance of q-hypergeometric functions, also called basic hypergeometric functions [45]. We note that they have already appeared in the SAP area generating function of prudent polygons in [46].

Finally, it is possible to consider the effect of increasing the number of punctures for punctured staircase polygons. In [11], the effect of increasing the number of punctures was considered: it was found that as the number of punctures increases, the perimeter generating function critical exponent increases by $3/2$ per puncture, while the area generating function critical exponent increases by 1 per puncture. In both cases, the critical point was found to be unchanged by a finite number of punctures. However, in [47], it was found that once the number of punctures is allowed to be unbounded, the perimeter generating function has a zero radius of convergence. Considering our ${}_2F_1$ hypergeometric function representation of the one-punctured perimeter generating function, a simple scenario that could explain these properties, would be that going from one to n punctures, the ${}_2F_1$ hypergeometric functions are of the form ${}_2F_1([a + 3n/2, b], [c], \mathcal{P}(x))$. Under this scenario, for finite n, there is a critical exponent increase of $3/2$ per puncture, while the critical point remains unchanged, and as $n \to \infty$, corresponding to an unbounded number of punctures, the hypergeometric function will become a confluent hypergeometric function with the critical point mapping to the confluent irregular singularity at infinity, whose series will have zero radius of convergence, in agreement with what was found in [47].

7. Conclusions

We have demonstrated for the first time a non-algebraic, D-finite perimeter generating function for SAPs, given in terms of ${}_2F_1$ hypergeometric functions, and we have provided simple relationships between the generating functions of three-choice, imperfect, and one-punctured staircase polygons. We have expressed the generating functions as a sum of algebraic and transcendental parts, each of which is a series in integer coefficients up to an overall factor of $1/60$. We have been able to fully analyze the solutions of their 8th order linear differential operators since they, up to the semi-direct product, reduce to a 3rd order, a 2nd order, and first order operators. We have found that the 2nd order operator has *modular* *form* solutions which can be rewritten as a ${}_2F_1$ hypergeometric function with *two possible pullbacks*. Similarly we have found that the 3rd order operator is homomorphic to the symmetric square of an 2nd order operator which also has solutions in terms of another *modular form* which, again, can be expressed as a ${}_2F_1$ hypergeometric function with two possible pullbacks. In that case, these two pullbacks are related by a genus-zero modular curve. These two *modular forms* are not simply related, as can be seen when one rewrites them in terms of a common ${}_2F_1([1/12, 5/12], [1], \mathcal{P}(x))$ hypergeometric functions for respective $\mathcal{P}(x)$ pullbacks. All these exact results for the three perimeter generating functions illustrate, one more time [43, 48], the emergence in enumerative combinatorics and lattice statistical mechanics of (quite non-trivial) *modular forms*. The emergence of modular forms is often a consequence of the fact that the functions one considers in enumerative combinatorics and lattice statistical mechanics, can also be written as n-fold integrals and are, in fact, diagonal of rational functions [43, 49]. One can reasonably conjecture that the generating functions analyzed here are actually diagonal of rational functions.

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Appendix A. Operator polynomial definitions

$$p_6 = 6874x^6 - 2913x^5 + 660x^4 - 230x^3 + 60x^2 + 6x - 2, \tag{A.1}$$

$$p_7 = 13\,748x^7 - 1341x^6 + 1047x^5 - 1510x^4 + 600x^3 + 42x^2 - 34x + 3, \tag{A.2}$$

$$\begin{aligned} p_9 = {} & 96\,236x^9 + 35\,756x^8 + 29\,198x^7 - 30\,049x^6 \\ & + 10\,841x^5 - 9226x^4 + 2819x^3 - 913x^2 + 224x - 21, \end{aligned} \tag{A.3}$$

$$\begin{aligned} p_{10} = {} & 577\,416x^{10} + 11\,494x^9 - 265\,110x^8 - 104\,347x^7 + 14\,641x^6 - 17\,865x^5 \\ & + 11\,006x^4 - 2990x^3 + 582x^2 - 46x - 6, \end{aligned} \tag{A.4}$$

$$\begin{aligned} p_{14} = {} & 258\,682\,368x^{14} + 5149\,312x^{13} - 116\,080\,384x^{12} - 56\,911\,264x^{11} \\ & - 11\,623\,368x^{10} - 16\,910\,078x^9 + 7172\,550x^8 - 2518\,103x^7 \\ & + 1020\,461x^6 - 167\,793x^5 + 12\,300x^4 + 3408x^3 - 2168x^2 + 118x + 16, \end{aligned} \tag{A.5}$$

$$\begin{aligned} p_{18} = {} & 9312\,565\,248x^{18} + 185\,375\,232x^{17} - 4282\,264\,448x^{16} \\ & - 2497\,630\,752x^{15} - 973\,632\,640x^{14} - 1001\,053\,992x^{13} \\ & + 183\,496\,040x^{12} - 202\,323\,868x^{11} + 81\,320\,436x^{10} \\ & - 20\,237\,208x^9 + 7144\,232x^8 - 1005\,079x^7 - 46\,763x^6 \\ & + 31\,581x^5 - 13\,467x^4 - 853x^3 + 717x^2 - 36x - 5, \end{aligned} \tag{A.6}$$

$$
\begin{aligned}
p_{23} = {} & 22\,984\,790\,671\,360x^{23} - 14\,160\,990\,742\,528x^{22} + 47\,196\,432\,034\,304x^{21} \\
& - 40\,184\,041\,956\,352x^{20} + 23\,871\,790\,843\,776x^{19} - 12\,862\,188\,171\,584x^{18} \\
& + 4334\,321\,680\,992x^{17} - 808\,339\,934\,032x^{16} + 70\,414\,369\,000x^{15} \\
& + 59\,364\,489\,644x^{14} - 38\,533\,903\,096x^{13} + 4418\,397\,469x^{12} \\
& + 2623\,726\,024x^{11} - 1386\,913\,106x^{10} + 512\,965\,024x^{9} \\
& - 144\,171\,921x^{8} + 17\,788\,918x^{7} + 2272\,607x^{6} \\
& - 949\,665x^{5} + 63\,356x^{4} + 6516x^{3} - 426x^{2} - 28x - 2,
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
p_{24} = {} & 1103\,269\,952\,225\,280x^{24} - 228\,305\,326\,678\,016x^{23} + 2331\,485\,726\,244\,864x^{22} \\
& - 1249\,271\,454\,269\,440x^{21} + 549\,381\,083\,516\,928x^{20} - 225\,290\,952\,722\,816x^{19} \\
& - 39\,003\,496\,295\,360x^{18} + 46\,746\,500\,840\,896x^{17} - 10\,249\,554\,621\,312x^{16} \\
& + 1973\,847\,887\,848x^{15} + 157\,900\,491\,180x^{14} - 637\,108\,022\,672x^{13} \\
& + 233\,984\,558\,200x^{12} - 24\,645\,390\,372x^{11} - 4177\,273\,140x^{10} \\
& + 2621\,821\,288x^{9} - 942\,904\,492x^{8} + 195\,411\,966x^{7} \\
& + 1609\,130x^{6} - 6956\,791x^{5} + 515\,168x^{4} \\
& + 60\,240x^{3} - 2676x^{2} - 256x - 20,
\end{aligned} \tag{A.8}
$$

$$
\begin{aligned}
q_{24} = {} & 9193\,916\,268\,5440x^{24} - 79\,628\,753\,641\,472x^{23} + 202\,946\,718\,879\,744x^{22} \\
& - 207\,932\,599\,859\,712x^{21} + 135\,671\,205\,331\,456x^{20} - 75\,320\,543\,530\,112x^{19} \\
& + 30\,199\,474\,895\,552x^{18} - 7567\,681\,417\,120x^{17} + 1089\,997\,410\,032x^{16} \\
& + 167\,043\,589\,576x^{15} - 213\,500\,102\,028x^{14} + 56\,207\,492\,972x^{13} \\
& + 6076\,506\,627x^{12} - 8171\,378\,448x^{11} + 3438\,773\,202x^{10} \\
& - 1089\,652\,708x^{9} + 215\,327\,593x^{8} - 8698\,490x^{7} \\
& - 6071\,267x^{6} + 1203\,089x^{5} - 37\,292x^{4} \\
& - 8220x^{3} + 314x^{2} + 20x + 2,
\end{aligned} \tag{A.9}
$$

$$
\begin{aligned}
p_{31} = {} & 42\,659\,311\,790\,230\,732\,800x^{31} - 29\,563\,777\,128\,137\,269\,248x^{30} \\
& + 109\,223\,631\,121\,278\,908\,416x^{29} - 104\,250\,789\,052\,923\,003\,904x^{28} \\
& + 56\,305\,599\,211\,721\,642\,496x^{27} - 24\,632\,274\,570\,479\,903\,488x^{26} \\
& + 2681\,522\,191\,975\,403\,520x^{25} + 4954\,045\,657\,465\,530\,112x^{24} \\
& - 3002\,212\,360\,825\,142\,752x^{23} + 813\,437\,846\,722\,409\,936x^{22} \\
& - 111\,297\,129\,389\,473\,336x^{21} - 52\,910\,191\,450\,930\,076x^{20} \\
& + 54\,606\,842\,707\,567\,716x^{19} - 20\,972\,722\,051\,528\,144x^{18} \\
& + 4010\,775\,763\,371\,668x^{17} - 56\,829\,134\,814\,870x^{16} \\
& - 339\,637\,124\,743\,736x^{15} + 184\,268\,988\,549\,260x^{14} \\
& - 53\,836\,053\,904\,996x^{13} + 8015\,086\,193\,990x^{12} \\
& - 65\,165\,405\,654x^{11} - 228\,234\,905\,736x^{10} \\
& + 60\,861\,828\,179x^{9} - 9657\,526\,198x^{8} \\
& + 348\,474\,792x^{7} + 171\,633\,727x^{6} - 20\,489\,803x^{5} \\
& - 752\,308x^{4} + 120\,494x^{3} + 5118x^{2} + 196x - 46,
\end{aligned} \tag{A.10}
$$

$$
\begin{aligned}
p_{33} = {} & 598\,165\,554\,871\,664\,640\,x^{33} + 2237\,833\,352\,545\,566\,720\,x^{32} \\
& + 2923\,793\,972\,548\,599\,808\,x^{31} + 4898\,487\,216\,451\,354\,624\,x^{30} \\
& + 3066\,215\,578\,973\,962\,240\,x^{29} + 3807\,302\,765\,273\,284\,608\,x^{28} \\
& + 1548\,493\,928\,510\,070\,784\,x^{27} + 993\,756\,825\,730\,510\,848\,x^{26} \\
& + 330\,665\,809\,850\,894\,336\,x^{25} + 30\,479\,967\,060\,547\,584\,x^{24} \\
& - 10\,206\,189\,043\,353\,856\,x^{23} - 12\,653\,592\,201\,109\,760\,x^{22} \\
& - 11\,185\,832\,980\,157\,184\,x^{21} + 210\,438\,316\,943\,104\,x^{20} \\
& + 61\,265\,169\,248\,832\,x^{19} + 26\,357\,534\,470\,800\,x^{18} \\
& + 181\,882\,051\,733\,304\,x^{17} - 22\,437\,164\,475\,672\,x^{16} \\
& - 13\,063\,730\,138\,481\,x^{15} + 1615\,865\,720\,985\,x^{14} \\
& - 1270\,457\,215\,869\,x^{13} + 262\,962\,192\,538\,x^{12} \\
& + 171\,533\,661\,840\,x^{11} - 44\,224\,719\,936\,x^{10} \\
& - 5369\,927\,527\,x^{9} + 2555\,317\,932\,x^{8} \\
& + 150\,451\,837\,x^{7} - 51\,686\,841\,x^{6} \\
& - 5506\,775\,x^{5} + 562\,261\,x^{4} + 119\,245\,x^{3} \\
& + 6577\,x^{2} + 201\,x + 3,
\end{aligned}
\tag{A.11}
$$

$$
\begin{aligned}
p_{36} = {} & 1186\,760\,460\,865\,382\,645\,760\,x^{36} + 4296\,640\,036\,887\,488\,102\,400\,x^{35} \\
& + 5656\,257\,186\,120\,920\,465\,408\,x^{34} + 9610\,563\,099\,027\,778\,306\,048\,x^{33} \\
& + 6331\,246\,887\,273\,737\,748\,480\,x^{32} + 7968\,414\,685\,458\,358\,861\,824\,x^{31} \\
& + 3417\,201\,107\,002\,357\,972\,992\,x^{30} + 2613\,021\,963\,777\,068\,236\,800\,x^{29} \\
& + 880\,949\,926\,005\,413\,642\,240\,x^{28} + 234\,438\,363\,912\,752\,283\,648\,x^{27} \\
& + 39\,705\,299\,093\,018\,075\,136\,x^{26} - 33\,621\,804\,577\,702\,641\,664\,x^{25} \\
& - 24\,563\,052\,057\,588\,912\,128\,x^{24} - 3677\,799\,503\,014\,345\,728\,x^{23} \\
& - 2383\,201\,063\,097\,856\,256\,x^{22} + 353\,349\,079\,985\,541\,632\,x^{21} \\
& + 422\,742\,538\,805\,020\,416\,x^{20} - 18\,935\,528\,677\,020\,992\,x^{19} \\
& + 22\,888\,292\,241\,850\,368\,x^{18} - 3272\,140\,352\,378\,880\,x^{17} \\
& - 6181\,103\,422\,702\,752\,x^{16} + 779\,365\,487\,308\,732\,x^{15} \\
& + 111\,813\,502\,211\,919\,x^{14} - 10\,554\,167\,286\,006\,x^{13} \\
& + 34\,694\,255\,695\,001\,x^{12} - 6950\,573\,977\,656\,x^{11} \\
& - 2354\,238\,734\,992\,x^{10} + 643\,662\,074\,352\,x^{9} \\
& + 58\,104\,579\,207\,x^{8} - 23\,471\,784\,508\,x^{7} \\
& - 1309\,856\,379\,x^{6} + 330\,362\,442\,x^{5} + 30\,382\,891\,x^{4} \\
& - 2170\,504\,x^{3} - 357\,735\,x^{2} - 13\,190x - 201,
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
q_{36} = {} & 4898\,975\,894\,398\,933\,401\,600\,x^{36} + 17\,161\,731\,218\,095\,095\,152\,640\,x^{35} \\
& + 21\,883\,875\,427\,370\,219\,339\,776\,x^{34} + 35\,975\,336\,852\,527\,465\,365\,504\,x^{33} \\
& + 2304\,246\,2618\,293\,310\,717\,952\,x^{32} + 27\,819\,843\,458\,704\,542\,793\,728\,x^{31} \\
& + 11\,698\,815\,400\,453\,787\,467\,776\,x^{30} + 8612\,877\,073\,871\,460\,311\,040\,x^{29} \\
& + 2839\,184\,277\,919\,494\,885\,376\,x^{28} + 762\,150\,624\,816\,330\,670\,080\,x^{27}
\end{aligned}
$$

$$
\begin{aligned}
&+ 129\,975\,590\,943\,822\,506\,496\,x^{26} - 89\,860\,876\,746\,914\,479\,616\,x^{25}\\
&- 66\,319\,997\,468\,290\,851\,840\,x^{24} - 10\,529\,561\,536\,853\,716\,224\,x^{23}\\
&- 6419\,230\,746\,695\,990\,912\,x^{22} + 860\,563\,513\,487\,440\,608\,x^{21}\\
&+ 974\,313\,641\,274\,336\,048\,x^{20} + 10\,482\,129\,303\,048\,704\,x^{19}\\
&+ 70\,108\,534\,257\,870\,090\,x^{18} - 6926\,151\,459\,159\,618\,x^{17}\\
&- 11\,166\,062\,802\,273\,588\,x^{16} + 850\,683\,965\,895\,387\,x^{15}\\
&- 39\,984\,544\,775\,712\,x^{14} + 21\,618\,692\,208\,399\,x^{13}\\
&+ 63\,135\,026\,727\,396\,x^{12} - 10\,612\,700\,119\,674\,x^{11}\\
&- 3345\,898\,146\,854\,x^{10} + 785\,954\,523\,741\,x^{9}\\
&+ 58\,318\,133\,424\,x^{8} - 25\,700\,455\,352\,x^{7}\\
&- 726\,830\,868\,x^{6} + 322\,300\,149\,x^{5} + 17\,524\,916\,x^{4}\\
&- 1750\,194\,x^{3} - 160\,296\,x^{2} - 3802x - 30,
\end{aligned}
\tag{A.13}
$$

$$
\begin{aligned}
p_{37} = {}& 136\,845\,384\,314\,821\,493\,391\,360x^{37} - 105\,679\,883\,821\,940\,306\,952\,192x^{36}\\
&+ 391\,986\,252\,374\,413\,297\,836\,032x^{35} - 399\,101\,140\,153\,885\,695\,805\,440x^{34}\\
&+ 264\,914\,197\,644\,249\,574\,493\,184x^{33} - 202\,222\,045\,944\,023\,129\,525\,760x^{32}\\
&+ 114\,104\,519\,102\,302\,216\,106\,752x^{31} - 43\,668\,075\,787\,265\,613\,729\,792x^{30}\\
&+ 25\,076\,614\,970\,145\,903\,635\,968x^{29} - 19\,991\,047\,450\,482\,347\,090\,016x^{28}\\
&+ 12\,192\,274\,657\,696\,530\,720\,432x^{27} - 5894\,005\,459\,795\,198\,246\,136x^{26}\\
&+ 2451\,165\,042\,003\,983\,275\,604x^{25} - 794\,981\,526\,560\,526\,083\,280x^{24}\\
&+ 152\,372\,293\,756\,401\,144\,616x^{23} + 8253\,142\,081\,467\,241\,688x^{22}\\
&- 20\,500\,019\,699\,204\,933\,934x^{21} + 10\,371\,074\,800\,492\,484\,800x^{20}\\
&- 3606\,002\,718\,331\,668\,490x^{19} + 926\,435\,638\,472\,444\,976x^{18}\\
&- 166\,886\,391\,470\,186\,702x^{17} + 14\,211\,777\,436\,985\,726x^{16}\\
&+ 3766\,364\,585\,838\,030x^{15} - 2146\,994\,035\,077\,380x^{14}\\
&+ 493\,421\,682\,837\,626x^{13} - 44\,957\,551\,610\,202x^{12}\\
&- 4718\,419\,878\,437x^{11} + 1904\,660\,277\,428x^{10}\\
&- 282\,375\,704\,850x^{9} + 24\,803\,569\,832x^{8}\\
&+ 178\,483\,388x^{7} - 316\,167\,306x^{6} + 15\,725\,362x^{5}\\
&+ 663\,472x^{4} + 136\,032x^{3} - 9204x^{2} - 1348x + 36.
\end{aligned}
\tag{A.14}
$$

Appendix B. LCLMs of the 8th order operators of P^{T}, P^{I}, P^{P}

The LCLM of the operators L_8^{T} and L_8^{I} produces a 10th order l e following form

$$
\begin{aligned}
\mathrm{LCLM}(L_8^{\mathrm{T}}, L_8^{\mathrm{I}}) &= L_{10}^{\mathrm{TI}} = L_3^{(1)} \cdot L_2^{(1)} \cdot L_1^{(1)} \cdot L_1^{(2)} \cdot D_x^3\\
&= (N_3 \cdot N_2 \cdot N_1) \oplus L_1^{(3)} \oplus L_1^{(4)} \oplus L_1^{(5)} \oplus D_x,
\end{aligned}
\tag{B.1}
$$

where the three $L_1^{(j)}$ in the direct sum have, respectively, the solutions

$$
x^2, \frac{x}{\sqrt{1-4x}}, \quad \frac{(9 + 26\,x^2)}{\sqrt{1-4x}}. \tag{B.2}
$$

Note that as a consequence of the direct sum structure, L_{10}^{TI} has very simple algebraic solutions which can be written in the following form for arbitrary constants A_j

$$A_0 \, + A_1 \, \cdot \, x + A_2 \, \cdot \, x^2 \, + A_3 \, \cdot \, \frac{x}{\sqrt{1 - 4x}} \, + A_4 \, \cdot \, \frac{(9 \, + 26\, x^2)}{\sqrt{1 - 4x}}, \tag{B.3}$$

The LCLM of the operators L_8^{T} and L_8^{P} produces a 10th order operator of the following form

$$\begin{aligned} &\mathrm{LCLM}(L_8^{\mathrm{T}}, L_8^{\mathrm{P}}) \\ &= L_{10}^{\mathrm{TP}} = L_3^{(2)} \cdot \, L_1^{(6)} \cdot \, L_1^{(7)} \cdot L_2^{(2)} \cdot \, L_1^{(8)} \cdot \, L_1^{(9)} \cdot \, L_1^{(10)} \\ &= (K_3 \cdot K_2 \cdot N_1) \oplus L_1^{(11)} \oplus L_1^{(12)} \oplus L_1^{(13)} \oplus L_1^{(14)}. \end{aligned} \tag{B.4}$$

The $L_1^{(j)}$ operators in the direct sum have, respectively, the following solutions

$$\frac{(3 - x^2)}{\sqrt{1 - 4x}}, \quad \frac{(3 - 2x)}{\sqrt{1 - 4x}}, \frac{(1 \, + r_1 x + r_2\, x^2)}{(1 - 4x)}, \quad \frac{x\, (1 + r_3\, x + r_4\, x^2)}{(1 - 4x)}, \tag{B.5}$$

where the r_i are rational numbers with large integer numerators and denominators. Therefore any linear combination of solutions (B.5) of $L_1^{(j)}$ together with the $(1 - 4x)$ solution of N_1 is a solution of L_{10}^{TP}.

The LCLM of the linear differential operators L_8^{I} and L_8^{P} produces a 10th order linear differential operator of the following form

$$\begin{aligned} \mathrm{LCLM}(L_8^{\mathrm{I}}, L_8^{\mathrm{P}}) &= L_{10}^{\mathrm{IP}} = L_3^{(3)} \cdot L_1^{(15)} \cdot L_1^{(16)} \cdot L_2^{(3)} \cdot L_1^{(17)} \cdot \, L_1^{(18)} \cdot N_1 \\ &= [L_3^{(4)} \cdot ((L_2^{(4)} \cdot N_1) \oplus L_1^{(19)})] \oplus L_1^{(20)} \oplus L_1^{(21)} \oplus L_1^{(22)}. \end{aligned} \tag{B.6}$$

The three first order operators $L_1^{(20)}$–$L_1^{(22)}$ have the following solutions

$$\frac{(9 - 34x)}{\sqrt{1 - 4x}}, \qquad \frac{x^2}{\sqrt{1 - 4x}}, \qquad s_0 + s_1\, x + s_2\, x^2, \tag{B.7}$$

where the s_j are very large integers. The solution of $L_1^{(19)}$ is of the form

$$\frac{r_0 \, + r_1\, x \, + r_2\, x^2 \, + r_3\, x^3}{1 - 4x}, \tag{B.8}$$

where the r_j are quite large integers.

As a consequence, any linear combination of solutions of the form

$$\begin{aligned} &A_0 \, \cdot \, (1 \, - 4x) \, + A_1 \, \cdot \, (1 + \, 2x^2) + A_2 \, \cdot \, \frac{x \, \cdot \, (1 - 9x^2)}{(1 - 4x)} \\ &\quad + A_3 \, \cdot \, \frac{(9 - 34x)}{\sqrt{1 - 4x}} \, + A_4 \, \cdot \, \frac{x^2}{\sqrt{1 - 4x}}. \end{aligned} \tag{B.9}$$

are solutions of L_{10}^{IP}, where we have simplified two of the solutions by appropriate linear combinations of the solutions of $L_1^{(19)}$ and $L_1^{(22)}$.

Finally, the LCLM of the three linear differential operators L_8^{T}, L_8^{I}, and L_8^{P} produces a 12th order operator of the following form

$$\begin{aligned} \mathrm{LCLM}(L_8^{\mathrm{T}}, L_8^{\mathrm{I}}, L_8^{\mathrm{P}}) &= L_{12}^{\mathrm{TIP}} = L_3^{(5)} \cdot L_2^{(5)} \cdot L_1^{(23)} \cdot L_1^{(24)} \cdot L_1^{(25)} \cdot L_1^{(26)} \cdot D_x^3 \\ &= L_3^{(6)} \cdot [(L_2^{(6)} \cdot L_1^{(27)}) \oplus L_1^{(28)} \oplus L_1^{(29)} \oplus L_1^{(30)} \oplus L_1^{(31)} \oplus L_1^{(32)} \oplus L_1^{(33)}], \end{aligned} \tag{B.10}$$

which has the following seven first order operator solutions

$$A_0 + \frac{A_1}{(1-4x)} + A_2 \cdot (1-4x) + A_3 \cdot (1-4x)^2 + \frac{A_4}{\sqrt{1-4x}} + A_5 \cdot \sqrt{1-4x} + A_6 \cdot (1-4x)^{3/2}. \tag{B.11}$$

Appendix C. Formal power series

The formal solutions of the 8th order linear operator annihilating P^{T} have the following form at $x = 0$

$$S_1 = 1 - 4x, \qquad S_2 = x^2, \qquad S_3 = \frac{9 - 58x + 26x^2}{9\sqrt{1-4x}},$$
$$S_4 = \sum_{n=0}^{\infty} c_n^{(4)} x^n, \qquad S_5 = \sum_{n=-1}^{\infty} c_n^{(5)} x^n + \ln(x) \cdot \sum_{n=0}^{\infty} c_n^{(4)} x^n,$$
$$S_6 = \sum_{n=0}^{\infty} c_n^{(6)} x^n, \qquad S_7 = \sum_{n=0}^{\infty} c_n^{(7)} x^n + \ln(x) \cdot \sum_{n=0}^{\infty} c_n^{(6)} x^n,$$
$$S_8 = \sum_{n=0}^{\infty} c_n^{(8)} x^n + 2\ln(x) \cdot \sum_{n=0}^{\infty} c_n^{(7)} x^n + \ln^2(x) \cdot \sum_{n=0}^{\infty} c_n^{(6)} x^n, \tag{C.1}$$

where both $c_n^{(4)}$ and $c_n^{(6)}$ are integers sequences. Note that only $c_n^{(5)}$ starts at $n = -1$, and also note the factor of *2* in the second term in S_8. The series $c_n^{(6)}$ is determined uniquely from the $\ln^2(x)$ terms, and likewise, the series $c_n^{(4)}$ is determined uniquely as the series multiplying the logarithm in the logarithmic solution of the operator $N_2 \cdot N_1$.

C.1. Series at infinity

Around $y = 1/x = 0$, the series solutions of L_8^{T}

$$S_1 = 1 - 4y^{-1}, \qquad S_2 = y^{-2}, \qquad S_3 = \frac{9 - 58y^{-1} + 26y^{-2}}{\sqrt{1-4y^{-1}}},$$
$$S_4 = \sum_{n=0}^{\infty} d_n^{(4)} y^n + \ln(x) \cdot (1 - 4y^{-1}), \qquad S_5 = y^{-1/2} \cdot \sum_{n=0}^{\infty} d_n^{(5)} y^n,$$
$$S_6 = y^{1/2} \cdot \sum_{n=0}^{\infty} d_n^{(6)} y^n, \qquad S_7 = y^{3/2} \cdot \sum_{n=0}^{\infty} d_n^{(7)} y^n, \qquad S_8 = y^{5/2} \cdot \sum_{n=0}^{\infty} d_n^{(8)} y^n. \tag{C.2}$$

Appendix D. Integrating N_3 back through $N_2 \cdot N_1$

Introducing the wronskian of N_2

$$W(N_2) = \frac{2 - 6x - 60x^2 + 230x^3 - 660x^4 + 2913x^5 - 6874x^6}{x^3 \cdot (1 - 4x)^4}, \tag{D.1}$$

and recalling the solution of N_1, $\mathrm{Sol}(N_1) = 1 - 4x$, as well as the solutions $\mathrm{Sol}(N_2)$ in (34) or (38) of N_2 and the solution $\mathrm{Sol}(N_3)$ of N_3, the solution Sol_3 of (31) entering into $P_{\mathrm{trans}}^{\mathrm{I}}$ can be written as

$$\mathrm{Sol}(N_1) \cdot \Bigg(-11 + \int \frac{\mathrm{Sol}(N_2)}{\mathrm{Sol}(N_1)} \cdot (-10$$
$$+ 90 \cdot \int \Bigg(\int \frac{W}{\mathrm{Sol}(N_2)^2} \cdot \int \Bigg(\frac{\mathrm{Sol}(N_2) \cdot \mathrm{Sol}(N_3)}{W(N_2)} \cdot \,\mathrm{d}x\Bigg) \cdot \,\mathrm{d}x\Bigg) \cdot \,\mathrm{d}x\Bigg) \cdot \,\mathrm{d}x\Bigg). \qquad \text{(D.2)}$$

Appendix E. Apparent singularities in Sol(N_3)

Let us try to understand in the solutions (50) and (54) the denominator of the pullbacks, $(1 - 4\,x^2 - U)$ and $(1 - 4\,x^2 + U)$, as well as the two expressions of the numerator of the pre-factors, namely $(13 - 28\,x^2 - 12\,U)$ and $(13 - 28\,x^2 + 12\,U)$.

If one performs the resultant of these expressions with the definition of U^2, namely $(1 - 12\,x^2 - 64\,x^4 - U^2) = 0$, one gets respectively $4\,x^2 \cdot (1 + 20\,x^2)$ and $25 \cdot (1 + 20\,x^2)^2$. Therefore, let us consider the values of x such that $1 + 20\,x^2 = 0$. At these points one sees that $1 - 4\,x^2$ is equal to $+6/5$, that $U = \pm\,6/5$, and that $(13 - 28\,x^2)/12$ is equal to $+6/5$.

Typically, in the neighborhood of $1 + 20\,x^2 = 0$, namely for $x \simeq i/(2 \cdot 5^{1/2}) + \epsilon$, we have

$$(1 + 20\,x^2)^2 \simeq -80 \cdot \epsilon^2 + 160\,i\,5^{1/2}\,\epsilon^3 + \cdots, \qquad \text{(E.1)}$$

$$(13 - 28\,x^2 - 12\,U) \simeq -\frac{625}{9} \cdot \epsilon^2 + \frac{18125}{162}\,i\,5^{1/2} \cdot \epsilon^3 + \cdots, \qquad \text{(E.2)}$$

$$(1 - 4\,x^2 - U)^4 \simeq \frac{25}{81} \cdot \epsilon^4 + \cdots, \qquad \text{(E.3)}$$

$$\frac{4096\,x^{10}}{(1 - 4\,x^2 - U)^4} \simeq -\frac{324}{78125} \cdot \epsilon^{-4} + \cdots, \qquad \text{(E.4)}$$

so that the expression

$$x \cdot U \cdot \left(\frac{13 - 28\,x^2 - 12\,U}{(1 + 20\,x^2)^2}\right)^{1/4} \cdot {}_2F_1\left(\left[\frac{1}{8}, \frac{3}{8}\right], [1], \frac{4096\,x^{10}}{(1 - 4\,x^2 - U)^4}\right), \qquad \text{(E.5)}$$

behaves, up to a complex constant, like

$${}_2F_1\left(\left[\frac{1}{8}, \frac{3}{8}\right], [1], -\frac{324}{78125} \cdot \epsilon^{-4} + \cdots\right)$$
$$= \left(\frac{78125}{324} \cdot \epsilon^4 + \cdots\right) \cdot {}_2F_1\left(\left[\frac{1}{8}, \frac{1}{8}\right], \left[\frac{3}{4}\right], -\frac{78125}{324} \cdot \epsilon^4 + \cdots\right), \qquad \text{(E.6)}$$

which is analytic.

Therefore $1 + 20x^2$ corresponds to an apparent singularity which is nevertheless necessary to write the modular form as a ${}_2F_1$ hypergeometric function.

Likewise, if one looks at $(1 - A)$, where A is given in (68), it can be recast as

$$4 \cdot (1 - 4x^2 + U)^{-4} \cdot [(2x-1)(2x+1) \cdot U + 32x^5 + 24x^4 + 10x^2 - 1] \times [(2x-1)(2x+1) \cdot U - 32x^5 + 24x^4 + 10x^2 - 1]. \tag{E.7}$$

The elimination of U in the two factors in the numerator yield, besides $x = 0$,

$$16x^3 + 12x^2 + 8x - 1 = 0, \quad 16x^3 - 12x^2 + 8x + 1 = 0. \tag{E.8}$$

Through a similar procedure as above, the roots of (E.8) can also be shown to be apparent singularities.

Appendix F. Infinite order symmetry on the Heun function (51)

Let us denote $X = x^2$. Let us consider the *genus-one* algebraic curve

$$\begin{aligned}
&40\,960\,000\, X^8Y^8 \cdot (625\, X^2 + 1054\, XY + 625\, Y^2)\\
&+ 16\,384\,000\, X^7Y^7 \cdot (X + Y) \cdot (625\, X^2 + 1402\, XY + 625\, Y^2)\\
&+ 409\,600\, X^6Y^6 \cdot (4375\, X^4 + 19\,831\, X^3Y + 31\,956\, X^2Y^2 + 19\,831\, XY^3 + 4375\, Y^4)\\
&+ 40\,960\, X^5Y^5 \cdot (X + Y) \cdot \mathcal{P}_5 + 512\, X^4Y^4 \cdot \mathcal{P}_4 + 12\,800 \cdot X^3Y^3 \cdot (X + Y) \cdot \mathcal{P}_3\\
&+ 1600\, X^2Y^2 \cdot \mathcal{P}_2 + 160\, X\, Y \cdot (X + Y) \cdot \mathcal{P}_1\\
&- 200\, XY \cdot (X + Y) \cdot \mathcal{Q}_1 - 50\, XY \cdot \mathcal{R}_1\\
&+ 20\, XY \cdot (X + Y) \cdot (X^4 + 2\, X^3Y + X^2Y^2 + 2\, XY^3 + Y^4)\\
&- XY \cdot (X^4 + X^3Y + X^2Y^2 + XY^3 + Y^4)\\
&+ X^{10} + 226\, X^9Y + 136\,451\, X^8Y^2 - 1049\,824\, X^7Y^3 - 1268\,099\, X^6Y^4\\
&- 1254\,150\, X^5Y^5 - 1268\,099\, X^4Y^6 - 1049\,824\, X^3Y^7\\
&+ 136\,451\, X^2Y^8 + 226\, XY^9 + Y^{10} = 0,
\end{aligned} \tag{F.1}$$

where

$$\begin{aligned}
\mathcal{P}_5 &= 4375 \cdot (X^4 + Y^4) + 777 \cdot (X^3Y + XY^3) - 2000 \cdot X^2Y^2,\\
\mathcal{P}_4 &= 21875 \cdot (X^6 + Y^6) - 99\,848 \cdot (X^5Y + XY^5)\\
&\quad - 4066\,848 \cdot (X^4Y^2 + X^2Y^4) - 6920\,598 \cdot X^3Y^3,\\
\mathcal{P}_3 &= 35 \cdot (X^6 + Y^6) - 168 \cdot (X^5Y + XY^5)\\
&\quad - 4876 \cdot (X^4Y^2 + X^2Y^4) - 37\,398 \cdot X^3Y^3,\\
\mathcal{P}_2 &= 7 \cdot (X^8 + Y^8) + 133 \cdot (X^7Y + XY^7) + 531\,59(X^6Y^2 + X^2Y^6)\\
&\quad + 92\,442 \cdot (X^5Y^3 + X^3Y^5) + 102\,662\, X^4Y^4,\\
\mathcal{P}_1 &= X^8 + Y^8 + 85 \cdot (X^7Y + XY^7) - 42\,039 \cdot (X^6Y^2 + X^2Y^6)\\
&\quad + 61\,670 \cdot (X^5Y^3 + X^3Y^5) - 39\,482 \cdot X^4Y^4,\\
\mathcal{Q}_1 &= 2 \cdot (X^6 + Y^6) - 143 \cdot (X^5Y + XY^5) + 254 \cdot (X^4Y^2 + X^2Y^4) - 149\, X^3Y^3,\\
\mathcal{R}_1 &= X^6 + Y^6 - 70 \cdot (X^5Y + XY^5) - 60 \cdot (X^4Y^2 + X^2Y^4) - 60\, X^3Y^3.
\end{aligned} \tag{F.2}$$

One can write Y in (F.1) as a series expansion in X, namely

$$
\begin{aligned}
Y = X^5 &+ 20\,X^6 + 350\,X^7 + 5600\,X^8 + 86\,725\,X^9 + 1319\,200\,X^{10} \\
&+ 19\,894\,850\,X^{11} + 298\,777\,600\,X^{12} + 4479\,731\,850\,X^{13} + 67\,155\,693\,600\,X^{14} \\
&+ 1007\,421\,693\,450\,X^{15} + 15\,130\,465\,630\,600\,X^{16} \\
&+ 227\,576\,601\,943\,225\,X^{17} + 3428\,478\,377\,045\,600\,X^{18} \\
&+ 51\,737\,085\,633\,726\,100\,X^{19} \\
&+ 782\,050\,723\,102\,305\,200\,X^{20} + 11\,841\,094\,422\,935\,733\,850\,X^{21} \\
&+ 179\,579\,006\,419\,196\,877\,600\,X^{22} + 2727\,744\,732\,078\,726\,781\,850\,X^{23} \\
&+ 41\,496\,463\,049\,656\,511\,818\,600\,X^{24} + 632\,194\,923\,237\,727\,485\,070\,075\,X^{25} \\
&+ 9644\,872\,198\,249\,006\,185\,042\,100\,X^{26} \\
&+ 147\,340\,024\,316\,081\,333\,011\,633\,850\,X^{27} \\
&+ 2253\,708\,185\,187\,840\,469\,204\,115\,600\,X^{28} \\
&+ 34\,514\,542\,785\,442\,406\,208\,079\,674\,225\,X^{29}.
\end{aligned}
$$

We have the following identity on the pullbacks

$$
\begin{aligned}
&\frac{4096\,X^5}{[1 - 4\,X + \sqrt{(1 - 16\,X)\,(1 + 4\,X)}\,]^4} \\
&\quad = \frac{4096\,Y^5}{[1 - 4\,Y - \sqrt{(1 - 16\,Y)\,(1 + 4\,Y)}\,]^4} \\
&\quad = 256\,X^5 + 5120\,X^6 + 89\,600\,X^7 + 1433\,600\,X^8 + 22\,201\,600\,X^9 \\
&\qquad + 337\,689\,600\,X^{10} + 5092\,057\,600\,X^{11} + 76\,458\,905\,600\,X^{12} + \cdots,
\end{aligned}
\tag{F.3}
$$

from which one deduces from (50), the following *infinite order automorphism identity on a Heun function*

$$
\begin{aligned}
&\mathcal{A}_1(X)\cdot\ \mathrm{Heun}\Big(-\frac{1}{4},\,\frac{1}{16},\,\frac{3}{8},\,\frac{5}{8},\,1,\,\frac{1}{2},\,-4\,X\Big) \\
&\quad = \mathcal{A}_2(Y)\cdot\ \mathrm{Heun}\Big(-\frac{1}{4},\,\frac{1}{16},\,\frac{3}{8},\,\frac{5}{8},\,1,\,\frac{1}{2},\,-4\,Y\Big),
\end{aligned}
\tag{F.4}
$$

where

$$
\mathcal{A}_1(X) = \left[\frac{(1 + 20\,X)^2\cdot\ (1 - 12\,X - 64\,X^2)}{(1 - 16\,X)^2\cdot\ (1 + 4\,X)^2\cdot\ [13 - 28\,X - 12\,\sqrt{(1 - 16\,X)\,(1 + 4\,X)}\,]}\right]^{1/4},
$$

$$
\mathcal{A}_2(Y) = \left[\frac{25\cdot\ (1 + 20\,Y)^2\cdot\ (1 - 12\,Y - 64\,Y^2)}{(1 - 16\,Y)^2\cdot\ (1 + 4\,Y)^2\cdot\ [13 - 28\,Y + 12\,\sqrt{(1 - 16\,Y)\,(1 + 4\,Y)}\,]}\right]^{1/4}.
$$

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