

Painlevé versus Fuchs

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Abstract

The sigma form of the Painlevé VI (PVI) equation contains four arbitrary parameters and generically the solutions can be said to be genuinely 'nonlinear' because they do not satisfy linear differential equations of finite order. However, when there are certain restrictions on the four parameters, there exist one-parameter families of solutions which do satisfy (Fuchsian) differential equations of finite order. We study this phenomenon of Fuchsian solutions to the Painlevé equation with a focus on the particular PVI equation which is satisfied by the diagonal correlation function $C(N, N)$ of the Ising model. We obtain Fuchsian equations of order $N + 1$ for $C(N, N)$ and show that the equation for $C(N, N)$ is equivalent to the N th symmetric power of the equation for the elliptic integral E . We show that these Fuchsian equations correspond to rational algebraic curves with an additional Riccati structure and we show that the Malmquist Hamiltonian p, q variables are rational functions in complete elliptic integrals. Fuchsian equations for off-diagonal correlations $C(N, M)$ are given which extend our considerations to discrete generalizations of Painlevé.

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1. Introduction

The correlation functions of the Ising model were first calculated by Kaufman and Onsager [1] in terms of determinants whose elements are certain hypergeometric functions. For this reason,

it follows from a theorem on holonomic functions [2] that they must satisfy linear ordinary differential equations. However, these correlations also have a remarkable connection with nonlinear equations as well. The first such result was the expression as $T \rightarrow T_c$ of the scaled correlation function in terms of a Painlevé III (PIII) function by Wu, McCoy, Tracy and Barouch [3] in 1976. Subsequently, in 1980 it was shown for arbitrary fixed T by Jimbo and Miwa [4] that the diagonal correlation $C(N, N)$ is given in terms of a PVI function and by McCoy, Wu [5] and Perk [6] that the correlation at a general position $C(M, N)$ and its 'dual' $C^*(M, N)$ satisfy some remarkable quadratic identities or double recursions which are discrete generalizations of the Painlevé ODEs.

The Painlevé representation of the correlation functions is by now well known but, curiously enough, almost nothing is known about the corresponding linear equations beyond the fact that the diagonal correlation function $C(1, 1)$ is a particular case of the hypergeometric function. In this paper, we will study these linear equations for the Ising correlation functions and the much more general question of when solutions of the PVI equation will satisfy Fuchsian differential equations.

The most general four-parameter-dependent sigma form of Painlevé VI can be written as [7, 8]

$$\zeta'(t(t-1)\zeta'')^2 + (2\zeta'(t\zeta' - \zeta) - \zeta'^2 - v_1v_2v_3v_4)^2 = (\zeta' + v_1^2)(\zeta' + v_2^2)(\zeta' + v_3^2)(\zeta' + v_4^2) \quad (1)$$

with

$$\zeta = t(t-1)\frac{d \ln \tau}{dt} + K_1t + K_2, \quad (2)$$

where

$$K_1 = v_1v_2 - v_1v_3 - v_2v_3 \quad (3)$$

and

$$K_2 = -\frac{1}{2}(v_1v_2 - v_1v_3 - v_1v_4 - v_2v_3 - v_2v_4 + v_3v_4). \quad (4)$$

This is a second-order nonlinear equation which allows branch points only at the three points $t = 0, 1, \infty$ and locally near these singularities the function τ has, following Jimbo's expansions [9], an expansion of the form

$$\tau = x^{p_j} \sum_{k=-\infty}^{\infty} x^{k^2+k\alpha} \sum_{n=0}^{\infty} \eta^{-k} \cdot a_j(n, k; \alpha) \cdot x^n, \quad (5)$$

where x is the local variable at $t = 0, 1, \infty$, and two boundary conditions for the second-order PVI equation specified by α and η will in general be different at the three singularities. The coefficients $a_j(n, k; \alpha)$ depend on the value of $j = 0, 1, \infty$ and satisfy $a_j(n, -k, \alpha) = a_j(n, k, -\alpha)$ and we note that

$$p_0 = \{\alpha^2 - (v_1 + v_2 - v_3 - v_4)^2\}/4, \quad (6)$$

$$p_1 = \{\alpha^2 - (v_1 + v_2 - v_3 + v_4)^2\}/4, \quad (7)$$

$$p_\infty = \alpha^2/4 + K_1. \quad (8)$$

Comparison of (5) with the well-known expansion of Jimbo [9] reveals that many of the coefficients in Jimbo's expansion vanish identically. Several $a_j(n, k, \alpha)$ are explicitly given in section 2.1.

In general, the local expansion (5) has an infinite number of confluent singularities which indicates that it cannot satisfy a *linear* differential equation. Therefore, even though the most

general solution of the PVI equation cannot satisfy a *linear* equation, the specific boundary conditions which specify the solution to be the physical diagonal correlation function of the Ising model will allow a Fuchsian equation of *order generically greater than 2* to be satisfied.

In this paper, we study this phenomenon of the existence of boundary conditions for which solutions of certain PVI equations satisfy Fuchsian differential equations⁶. There are several ways in which this phenomenon may occur. One way is that conditions can be found on the four parameters v_k and on α such that the general local expansions at $t = 0, 1, \infty$ degenerate by having the coefficients $a_j(n, k; \alpha)$ all vanish if k is sufficiently large. This will give a one-parameter family of solutions which has only a finite number of confluent singularities. We study this mechanism in detail in section 2.1. However, there may also exist one-parameter families which cannot be obtained from the two-parameter families (5) by specialization. An example of this is given in section 2.4.

For concreteness, we will consider in detail the specific PVI equation for the diagonal Ising correlation obtained by Jimbo and Miwa [4]:

$$(t(t-1)\sigma'')^2 = N^2 \cdot ((t-1)\sigma' - \sigma)^2 - 4\sigma'((t-1)\sigma' - \sigma - 1/4)(t\sigma' - \sigma) \tag{9}$$

which is obtained from (1) by setting

$$v_1 = v_4 = N/2, \quad v_2 = (1-N)/2, \quad v_3 = (1+N)/2, \tag{10}$$

$$\sigma = \zeta + N^2 t/4 - 1/8. \tag{11}$$

The diagonal $C_N = C(N, N)$ is related to σ for $T > T_c$ by

$$\sigma(t) = t(t-1) \cdot \frac{d}{dt} \log(C_N) - 1/4, \quad \text{with } t = (\sinh(2J_v/kT) \cdot \sinh(2J_h/kT))^2 < 1, \tag{12}$$

where $C_N \simeq t^{N/2}$ when $t \rightarrow 0$, and for $T < T_c$ by

$$\sigma(t) = t(t-1) \cdot \frac{d}{dt} \log(C_N) - t/4, \quad \text{with } t = (\sinh(2J_v/kT) \cdot \sinh(2J_h/kT))^{-2} < 1, \tag{13}$$

where $C_N \simeq 1$ when $t \rightarrow 0$, and where the variable J_v (J_h) is the Ising model vertical (horizontal) coupling constant. The detailed specification of the behaviour of $\sigma(t)$ near $t = 0$ needed to uniquely specify C_N as the diagonal Ising correlation function is sketched in section 2.4 (see for instance equation (53)). Do note that since all the calculations of this paper are systematically checked with high-temperature expansions when available, we introduce a variable t which is the inverse of that of Jimbo and Miwa [4]. For integer N , equation (9) is in the class of so-called ‘classical’ equations [8] which are known to generate Toeplitz determinants whose elements are hypergeometric functions [7, 8, 11]. We present the Fuchsian equations satisfied by C_N for small values of N in section 2. These equations have remarkable structure and in section 2.2 we show that the associated $(N+1)$ -order differential operators are homomorphic to the N th symmetric power of the second-order differential operator associated with the elliptic integral E . In section 3, we present an algebraic formulation of the Fuchsian equations for $C(N, N)$ by studying the Riccati formulation of solutions to PVI for $N = 1, 2$ which are related to differential structures on certain rational curves. In section 4, we extend our considerations to the discrete generalization of Painlevé VI, namely a quadratic double

⁶ For a warm-up on Painlevé VI, sigma form of Painlevé VI, and on the question of the holonomic solutions inside Painlevé VI, we recommend two magnificent papers in French, one by Garnier [10] and the other one by Okamoto [7] (see also in English [11]).

recursion on the two-point correlation functions $C(N, M)$ together with their dual $C^*(N, M)$. We will show that these structures can be generalized, *mutatis mutandis*, to $C(N, M)$ s. $C(N, M)$ s are also solutions of Fuchsian linear ODEs, with a quadratic increasing order. The associated differential operators are now homomorphic to direct sums of N th symmetric power of the second-order differential operator associated with the complete elliptic integral E . $C(N, M)$ s are actually sums of several homogeneous polynomials in the complete elliptic integrals E and K . This is a consequence of various remarkable simplifications in the ‘discrete Painlevé’ double recursions, like the fact that algebraic or rational expressions become polynomials by remarkable factorizations and by the occurrence of perfect squares. Combining these various results together, one has some quite curious and fascinating alchemical wedding between complete elliptic integrals, rational curves and discrete generalizations of Painlevé VI (and Hirota–Bäcklund transformations). The confrontation between the nonlinear Painlevé world and the linear Fuchsian world (Painlevé versus Fuchs) yields the emergence of quite interesting structures not only of differential nature but also of *algebraic geometry* nature. We finally see in section 5 that, in the case of $C(N, N)$ holonomic solutions, the p and q Malmquist’s variables corresponding to the Hamiltonian structure of the sigma form of Painlevé VI are remarkably *rational expressions* of E and K , and even *rational expressions* of E/K . We have the same result for the σ and σ' variables. These last results are in complete agreement with the previously mentioned results, namely the rational character of the algebraic curves corresponding to the existence of holonomic solutions $C(N, N)$ s for the sigma form of Painlevé VI and the existence of simple Riccati equations for the uniformizing parameter.

The number of new exact results we have obtained being quite large and the explicit formulae for some of these results being quite cumbersome, we will just sketch here these new exact results, giving the simplest formulae. More exhaustive formulae will be given in forthcoming publications.

2. Solutions of sigma form of Painlevé VI and Fuchsian linear ODEs

We consider, from now on, the isotropic square Ising model and the high-temperature regime, i.e. $t = s^4$ where $s = \sinh(2J/kT)$. The introduction of these two variables, t and s , may look a bit redundant: the variable t is well suited to write our results on diagonal correlations functions, while the variable s is clearly better suited for non-diagonal correlations. The results for the low-temperature regime are similar. The diagonal two-point correlation functions of the square Ising model $C(N, N)$ and its dual $C^*(N, N)$ can be calculated from Toeplitz determinants [1, 12, 13]:

$$C(N, N) = \det(a_{i-j}), \quad 1 \leq i, j \leq N, \quad (14)$$

$$C^*(N, N) = (-1)^N \det(a_{i-j-1}), \quad 1 \leq i, j \leq N, \quad (15)$$

where a_n s read in terms of ${}_2F_1$ hypergeometric function:

$$\begin{aligned} a_n &= -\frac{(-1/2)_{n+1}}{(n+1)!} t^{n/2+1/2} \cdot {}_2F_1(1/2, n+1/2; n+2; t), & n \geq -1, \\ a_n &= -\frac{(1/2)_{-n-1}}{(-n-1)!} t^{-n/2-1/2} \cdot {}_2F_1(-1/2, -n-1/2; -n; t), & n \leq -1, \end{aligned} \quad (16)$$

where $(\alpha)_n$ is the usual Pochhammer symbol.

The diagonal two-point correlation functions of the square Ising model $C(N, N)$ and $C^*(N, N)$ being given by the Toeplitz determinant (14) whose entries are solutions of linear second-order differential equations, they are necessarily solutions of a linear differential equation, with order $N! \cdot 2^N$ as an upper bound for generic entries of the determinant.

Since the diagonal two-point correlation functions of the square Ising model $C(N, N)$ are given by the determinants (14), it is straightforward to obtain a sufficiently large number of series coefficients and to get the linear differential equations satisfied by these series [14–16]. Denoting by D_t the derivative with respect to the variable t , the first linear differential operators L_{NN} corresponding to $C(N, N)$ are

$$L_{11} = D_t^2 + \frac{1}{t} \cdot D_t + \frac{1}{4(t-1)t^2}, \tag{17}$$

$$L_{22} = D_t^3 + 2 \frac{(t-2)}{(t-1)t} \cdot D_t^2 - \frac{1}{(t-1)t^2} \cdot D_t - \frac{1}{2} \frac{t+2}{t^3(t-1)^2}, \tag{18}$$

$$L_{33} = D_t^4 + 2 \frac{(t-5)}{(t-1)t} \cdot D_t^3 + \frac{1}{2} \frac{(41-11t-2t^2)}{t^2(t-1)^2} \cdot D_t^2 + \frac{1}{2} \frac{(2t^2+2t-5)}{t^3(t-1)^2} \cdot D_t + \frac{9}{16} \frac{15+13t+4t^2}{(t-1)^3t^4}, \tag{19}$$

$$L_{44} = D_t^5 - \frac{20}{t(t-1)} \cdot D_t^4 + \frac{(113+7t-2t^2)}{(t-1)^2t^2} \cdot D_t^3 - \frac{1}{2} \frac{(322+95t-9t^2-16t^3)}{(t-1)^3t^3} \cdot D_t^2 + \frac{(97+40t-10t^2-12t^3)}{(t-1)^3t^4} \cdot D_t - 4 \frac{32+33t+20t^2+5t^3}{(t-1)^4t^5}, \tag{20}$$

$$L_{55} = D_t^6 - 5 \frac{(t+7)}{t(t-1)} \cdot D_t^5 + \frac{1}{4} \frac{(52t^2+483t+1617)}{(t-1)^2t^2} \cdot D_t^4 - \frac{1}{2} \frac{(4t^3+370t^2+1707t+3503)}{(-1+t)^3t^3} \cdot D_t^3 - \frac{1}{16} \frac{(1552t^4-1016t^3-13191t^2-29618t-29855)}{t^4(t-1)^4} \cdot D_t^2 + \frac{5}{16} \frac{(720t^4+640t^3-2175t^2-6912t-8801)}{(t-1)^4t^5} \cdot D_t + \frac{25784t^4+3428t^3+6921t^2+8650t+7865}{64(t-1)^5t^6}, \tag{21}$$

$$L_{66} = D_t^7 - 14 \frac{(4+t)}{(t-1)t} \cdot D_t^6 + 14 \frac{(81+39t+7t^2)}{t^2(t-1)^2} \cdot D_t^5 - \frac{N_4}{(t-1)^3t^3} \cdot D_t^4 + \frac{N_3}{t^4(t-1)^4} \cdot D_t^3 + \frac{N_2}{(t-1)^5t^5} \cdot D_t^2 - \frac{1}{4} \frac{N_1}{t^6(t-1)^5} \cdot D_t - \frac{9}{2} \frac{N_0}{(t-1)^6t^7}, \tag{22}$$

where

$$\begin{aligned} N_4 &= 10162 + 7059t + 2411t^2 + 376t^3, \\ N_3 &= 37973 + 35162t + 17893t^2 + 5116t^3 + 500t^4, \\ N_2 &= -28706 - 55327t - 46180t^2 - 21437t^3 - 3358t^4 + 1736t^5, \\ N_1 &= -390548 - 402496t - 240997t^2 - 63239t^3 + 24152t^4 + 25088t^5, \\ N_0 &= 23814 + 26839t + 24583t^2 + 16599t^3 + 7345t^4 + 1620t^5. \end{aligned} \tag{23}$$

These operators are of order $N + 1$ and are irreducible. We further note that, in contrast to the Fuchsian equations for the n -particle contributions $\chi^{(n)}$ s of the susceptibility of the Ising model [14–17], the Fuchsian differential equations satisfied by $C(N, N)$ s have *no apparent singularities*. The linear differential operators, L_{NN}^* , for $C^*(N, N)$ s are obtained by the change t into $1/t$ in the previous differential operators.

These *Fuchsian* differential equations (17)–(22) for $C(N, N)$ s have the following general form:

$$\sum_{i=2}^{N+1} t^i (t-1)^{i-1} P_i^{(N)}(t) \cdot D_t^i + t(t-1) P_1^{(N)}(t) \cdot D_t + P_0^{(N)}(t) = 0, \quad (24)$$

where $P_i^{(N)}(t)$ is a polynomial in t of degree $N + 1 - i$ for $i = 2, \dots, N + 1$ and $P_1^{(N)}(t)$ and $P_0^{(N)}(t)$ are of degree $N - 1$.

The only singular points of (24) are the three regular singular points $t = 0, \infty, 1$. From the indicial equation of the differential equations for the first L_{NN} s, we infer the remarkably simple expressions of the critical exponents $\rho^{(1)}, \rho^{(\infty)}$ and $\rho^{(0)}$ at, respectively, the regular singular points $t = 1, t = \infty$ and $t = 0$:

$$\rho_n^{(1)} = (n-1)^2, \quad (25)$$

$$\begin{aligned} \rho_n^{(\infty)} &= \frac{5}{8} + \frac{3}{4}N + \frac{1}{4}n^2 - \frac{1}{4}(2N+3) \cdot n - \frac{(-1)^n}{4}n + \frac{(-1)^n}{8}(2N+3), \\ \rho_n^{(0)} &= -\frac{1}{8} + \frac{3}{4}N + \frac{1}{4}(n+1)(n+2) - \frac{1}{2}(N+3) \cdot n + \frac{(-1)^n}{4}(n+1) - \frac{(-1)^n}{8}(2N+5), \end{aligned} \quad (26)$$

where $n = 1, 2, \dots, N + 1$.

2.1. Local solutions at $t = 0, 1, \infty$

It is of interest to compare the local expansion (5) of the PVI equation with the exponents of the Fuchsian equations⁷. For concreteness, we concentrate on $t = 1^-$ which corresponds to $T = T_c^+$ in the Ising model. We have the following coefficients in (5) valid for $0 < \alpha < 1$:

$$a_1(0, 0; \alpha) = 1, \quad (27)$$

$$\begin{aligned} a_1(0, -1; \alpha) = a_1(0, 1; -\alpha) &= \frac{1}{16\alpha^2(1-\alpha)^2} (\alpha - v_1 - v_2 - v_3 + v_4)(\alpha - v_1 - v_2 \\ &+ v_3 - v_4)(\alpha - v_1 + v_2 - v_3 - v_4)(\alpha + v_1 - v_2 - v_3 - v_4), \end{aligned} \quad (28)$$

$$\begin{aligned} a_1(1, 0; \alpha) &= -\frac{\alpha^2}{8} + \frac{1}{2}(-v_1v_2 + v_1v_3 + v_1v_4 + v_2v_3 + v_2v_4 - v_3v_4) \\ &+ \frac{1}{8\alpha^2}(v_1 + v_2 + v_3 - v_4)(v_1 + v_2 - v_3 + v_4) \\ &\times (v_1 - v_2 + v_3 + v_4)(v_1 - v_2 - v_3 - v_4), \end{aligned} \quad (29)$$

$$\begin{aligned} a_1(0, -2; \alpha) = a_1(0, 2; -\alpha) &= \frac{a_1(0, -1; \alpha)^2}{265(\alpha-1)^2(\alpha-2)^4(\alpha-3)^2} \\ &\times [(\alpha-2)^2 - (v_1 + v_2 + v_3 - v_4)^2][(\alpha-2)^2 - (v_1 + v_2 - v_3 + v_4)^2] \\ &\times [(\alpha-2)^2 - (v_1 - v_2 + v_3 + v_4)^2][(\alpha-2)^2 - (-v_1 + v_2 + v_3 + v_4)^2]. \end{aligned} \quad (30)$$

⁷ Recall that, for the Ising case and for $T > T_c$, $\tau = t^{1/4}C_N$ (resp. $\tau = C_N$ for $T < T_c$).

For the Ising case (10), this reduces to

$$p_1 = \alpha^2/4, \tag{31}$$

$$a_1(0, -1; \alpha) = a_1(0, 1; -\alpha) = \frac{\alpha - 2N}{16\alpha}, \quad a_1(1, 0; \alpha) = \frac{(1 - \alpha^2)}{8}, \tag{32}$$

$$a_1(0, -2; \alpha) = a_1(0, 2; -\alpha) = \frac{a_1(0, -1; \alpha)^2 \cdot ((\alpha - 2)^2 - (2N)^2)}{256(\alpha - 2)^2}. \tag{33}$$

When used in (5) these expressions will reproduce the $N + 1$ exponents of L_{NN} at $t = 1$ where, in the limit $\alpha \rightarrow 0$, the terms in (5) with $x^{k^2 \pm k\alpha}$ become $(t - 1)^{k^2} \ln^k(t - 1)$. We see from (33) that, when $\alpha = 0$, $a_1(n, \pm 2; 0) = 0$ for $N = 1$ which is consistent with the fact that $C(1, 1)$ satisfies a second-order linear differential equation. We have carried the expansion to order $(t - 1)^{12}$. In particular, we have obtained the coefficient of $(t - 1)^9 \ln^3 |t - 1|$ and have verified that it vanishes for $N = 1, 2$ and have obtained all terms in the expansion of $C(N, N)$ given in [18].

More generally, the condition that there exists a value of α such that $a_j(n, k; \alpha) = 0$ for all k sufficiently large is a condition necessary for τ function of the PVI equation to satisfy a linear differential equation of finite order and the series

$$\sum_{n=0}^{\infty} a_j(n, k; \alpha) \cdot x^{k^2+k\alpha+p_j+n} \tag{34}$$

will be solutions to the Fuchsian equation. For example, one condition for a second-order Fuchsian equation is $a_1(0, 1; \alpha) = 0, a_1(0, -2; \alpha) = 0$, which are satisfied if, respectively,

$$\alpha = -v_1 - v_2 + v_3 - v_4, \quad \alpha - 2 = -v_1 + v_2 - v_3 - v_4, \tag{35}$$

implying $v_2 - v_3 = -1$, which is the restriction Forrester and Witte [8] needed for a solution of PVI to satisfy a hypergeometric equation. This condition implies that the τ functions are determinants of hypergeometric functions. We thus see that, at order $x^{p+4+(1-\alpha)}$, the local expansion provides a necessary condition for the reduction of a one-parameter family of solutions to PVI to a solution of a second-order linear differential equation. By examining the vanishing of $a(0, k; \alpha)$ for higher values of k , necessary conditions for the existence of one-parameter families satisfying higher order linear differential equations will be obtained. Similar necessary conditions can be obtained from the local expansions at $t = 0, \infty$.

2.2. The Fuchsian differential operators as N th symmetric power

The most profound and surprising structure of the solutions of PVI which satisfy Fuchsian equations is, however, not seen in these local expansions and, thus, it is important to observe that the operators L_{NN} given in (17)–(22) for $C(N, N)$ have the remarkable property that they are equivalent⁸ to the N th symmetric power⁹ of L_{11} :

$$A_N \cdot L_{NN} = \text{Sym}^N(L_{11}) \cdot R_N. \tag{36}$$

The first A_N and R_N intertwinners read for $N = 2$:

$$A_2 = t^2 D_t^2 + \frac{1}{4} \frac{(31t - 23)t}{t - 1} \cdot D_t + \frac{3}{4} \frac{15t - 7}{t - 1}, \tag{37}$$

⁸ For the equivalence of differential operators see e.g. [19–21].

⁹ For the definition of the symmetric power of differential operators see e.g. [21–23].

$$R_2 = t^2 \cdot D_t^2 + \frac{3}{4}t \cdot D_t - \frac{1}{4} \frac{3t - 5}{t - 1}. \quad (38)$$

We have calculated exactly these intertwinners up to $N = 6$ but the expressions are too large to be given here. As a consequence of this property (36), the differential Galois group of L_{NN} is not a $SL(N + 1, C)$ group as we could expect at first sight, but an $SL(2, C)$ group in the symmetric power representation. We expect that this property extends much more generally to other solutions of the general four-parameter PVI which satisfy Fuchsian equations.

Another consequence of (36) is that the solutions of this order $N + 1$ differential operator L_{NN} are actually homogeneous polynomials of degree N in the two solutions of L_{11} , see [21–23].

Let us now introduce the two elliptic integrals

$$K = {}_2F_1(1/2, 1/2; 1; s^4), \quad E = {}_2F_1(1/2, -1/2; 1; s^4) \quad (39)$$

and the second-order linear differential operator for E (D_s denotes the derivative with respect to s):

$$L_E = D_s^2 + \frac{D_s}{s} - 4 \frac{s^2}{s^4 - 1}. \quad (40)$$

This operator actually identifies with L_{11}^* .

One can easily show that the second-order linear differential operator L_{11} (associated with $C(1, 1)$ and written in the variable s) and the second-order linear differential operator L_E are equivalent:

$$\left(\frac{s^4 - 1}{s} \cdot D_s + 6s^2 \right) \cdot L_{11} = L_E \cdot \left(\frac{s^4 - 1}{s} \cdot D_s - 2/s^2 \right). \quad (41)$$

More generally, one can show in the s variable that the L_{NN} 's are actually equivalent to the L_{NN}^* 's. Since K can be simply expressed in terms of E and its first derivative, $C_{N,N}$ s are thus solutions of an operator which is homomorphic to $\text{Sym}^N(L_E)$:

$$\tilde{A}_N \cdot L_{NN} = \text{Sym}^N(L_E) \cdot \tilde{R}_N \quad (42)$$

or

$$L_{NN} \cdot B_N = S_N \cdot \text{Sym}^N(L_E), \quad (43)$$

where the intertwinners B_N and S_N (or \tilde{A}_N and \tilde{R}_N) are linear differential operators of order N . In fact, beyond $C(N, N)$, relations (36), (42), (43) relate *all* solutions of L_{NN} to $\text{Sym}^N(L_E)$. From (43) one can easily deduce that the diagonal two-point correlation functions $C(N, N)$ can be deduced as the action of a linear differential operator of order N on the N th power of the complete elliptic E :

$$C(N, N) = B_N(E^N). \quad (44)$$

The expressions of the intertwinners B_N can also be retrieved from the determinantal expressions of $C(N, N)$ and the relations between the hypergeometric functions a_n, a_{n-1} (see (14), (16)) and its derivative. This gives a general method to obtain the differential operators $L_{N,N}$.

2.3. $C(N, N)$ s as a homogeneous polynomial of the complete elliptic integrals E and K

The property (36) or (43) can be illustrated by considering the specific solution $C(N, N)$ of the $(N + 1)$ -order differential equations L_{NN} . The matrix elements a_n of the Toeplitz determinant representation may all be expressed as linear combinations of the elliptic integrals E and K

and, thus, $C(N, N)$ will be given as polynomials in these functions and this is in agreement with the previous relation (44). For low orders, these polynomials have been presented by Ghosh and Shrock [24]. For example,

$$C(2, 2) = \frac{1}{3s^4} \cdot (3(s^4 - 1)^2 K^2 + 8(s^4 - 1) \cdot EK - (s^4 - 5) \cdot E^2),$$

$$C(3, 3) = \frac{4}{135s^{10}} \cdot P_3(E, K),$$

where

$$P_3(E, K) = (33s^4 - 1)(s^4 - 1)^3 \cdot K^3 + 3(s^8 + 48s^4 - 1)(s^4 - 1)^2 \cdot EK^2 - 3(s^4 - 1)(s^{12} + 3s^8 - 69s^4 + 1) \cdot E^2K - (1 + 21s^8 - 96s^4 + 10s^{12}) \cdot E^3. \tag{45}$$

We note that these expressions are, respectively, quadratic and cubic *homogeneous* polynomials in E and K . We have obtained similar expressions for all $C(N, N)$ and $C^*(N, N)$ for $N = 4, 5, 6, \dots, 21$, and relation (44) gives similar relations for *any* values of N . They are a *homogeneous* polynomial of degree N in the complete elliptic integrals¹⁰ E and K , with simple rational coefficients (a polynomial in s with integer coefficients divided by some power of s). From a physics viewpoint, one should note that the particular rational coefficients one gets in front of the monomials $E^k \cdot K^{N-k}$ are far from being arbitrary as a general formula such as (44) could suggest. These coefficients are such that, for instance, the linear differential equation for $C(N, N)$ s has no apparent singularities. Furthermore, the contribution associated with the various monomials $E^k \cdot K^{N-k}$ clearly has poles (s^{-10} or s^{-4} in the previous example (45)). These coefficients are also ‘fine-tuned’ in such a way that, for instance, these various poles cancel together, in order to give an expression with a well-defined high-temperature series expansion (series at $s = 0$). We have many other remarkable properties corresponding to the behaviour of $C(N, N)$ s near $s = 1$ or $s = \infty$.

2.4. Non-trivial disentangling of solutions of linear Fuchsian ODEs near $t = 0$

Let us make here a comment on the existence of surprisingly simple hypergeometric solutions of the N -dependent sigma form (9) of Painlevé VI. Consider the second-order differential operator

$$L_h = D_t^2 + \left(\frac{1}{t} + \frac{1}{2(t-1)} \right) \cdot D_t - \frac{1}{4} \frac{N^2}{t^2} + \frac{1}{16(t-1)^2} \tag{46}$$

which has regular singularities at $t = 0, t = 1$ and $t = \infty$ with, respectively, the critical exponents $(\pm N/2), (1/4, 1/4)$ and $(1/4 \pm N/2)$.

It can be verified that *any linear combination* of the two solutions of (46) satisfies the N -dependent sigma form (9) of the Painlevé VI equation *for arbitrary N* , not *necessarily an integer*. For instance, when N is *not an integer*, one has the following two solutions of (9):

$$\sigma = t(t-1) \frac{d \ln \tau}{dt} - \frac{1}{4}, \quad \text{where } \tau = f_+ + \lambda \cdot f_-, \tag{47}$$

where f_{\pm} are the two independent solutions of (46):

$$f_{\pm} = t^{\pm N/2} \cdot (1-t)^{1/4} \cdot {}_2F_1([1/2, 1/2 \pm N], [1 \pm N], t). \tag{48}$$

¹⁰ This result can also be found in equations (2.16)–(2.19) of [25], which also show very explicitly that $C(N, N)$ is a homogeneous polynomial of E and K of degree N for all N (something that is already, albeit less explicit, in the appendix of [12]).

When the parameter N is an integer (and only in this case), that is to say in the Ising case we are interested in, the second-order differential operator L_h is, after conjugation by $(1 - s^4)^{1/4}$ (or in the t variable conjugation by $(1 - t)^{1/4} t^{N/2}$), equivalent to L_E ; when N is an integer, one solution is given above in terms of a hypergeometric function analytic at $t = 0$, and the other one has a logarithmic singularity at $t = 0$ (and similarly for $t = 1$ and $t = \infty$).

At first sight, the existence of such ‘additional’ solutions should not be seen as a surprise: we certainly expect the solutions of the N -dependent sigma form of Painlevé VI that are also, at the same time, solutions of a linear (Fuchsian) ODE to be a quite complicated ‘stratified’ space. However, let us focus on the series expansion at $t = 0$ of the analytic solution of (46), which simply reads

$$\begin{aligned} h_N &= \frac{1}{4^N} \frac{\Gamma(2N+1)}{\Gamma(N+1)^2} \cdot f_+ \\ &= c_0(N) \cdot t^{N/2} + c_1(N) \cdot t^{N/2+1} + c_2(N) \cdot t^{N/2+2} + \dots \end{aligned} \quad (49)$$

The coefficients $c_k(N)$ in the series expansion of (49) read

$$c_k(N) = \frac{1}{4^N} \frac{\Gamma(2N+1)}{\Gamma(N+1)^2} \frac{(-1/4)_k}{k!} {}_3F_2([1/2, 1/2+N, -k], [1+N, 5/4-k], 1). \quad (50)$$

Let us now consider the series expansion of the diagonal correlation functions $C(N, N)$:

$$C(N, N) = d_0(N) \cdot t^{N/2} + d_1(N) \cdot t^{N/2+1} + d_2(N) \cdot t^{N/2+2} + \dots, \quad (51)$$

where $d_0(N)$, $d_1(N)$ and $d_2(N)$ read, respectively,

$$\frac{\Gamma(2N+1)}{\Gamma(N+1)\Gamma(N+1)} \frac{1}{4^N}, \quad \frac{\Gamma(2N+1)}{\Gamma(N+1)\Gamma(N+2)} \frac{N}{4^{N+1}}, \dots \quad (52)$$

One has the following result, that may look quite surprising at first sight: the coefficients $c_k(N)$ of solution (49) and the coefficients $d_k(N)$ of the diagonal two-point correlation functions $C(N, N)$, solution of the order $N+1$ Fuchsian ODE, are identical up to $k = 3N/2 + 1$:

$$C(N, N) - h_N = \frac{1}{16} \left(\frac{(1/2)_N \cdot ((3/2)_N)^2}{\Gamma(N+2)\Gamma(N+3)^2} \right) \cdot t^{3N/2+2} + \dots \quad (53)$$

The coefficient in (53) in front of $t^{3N/2+2}$ can be seen as the initial condition defining¹¹ $C(N, N)$. Seeking for conditions allowing solutions of the sigma form of Painlevé VI to be also (the log-derivative of) solutions of linear Fuchsian differential equations, this difficulty to disentangle, near $t = 0$, a solution of a second-order differential equation and a solution of linear Fuchsian differential equations of arbitrary $N+1$ order seems to indicate that series analysis such as (5) may not be the easiest approach to take into account such subtle¹² fine-tuning: we need a less analytical and more ‘global’ algebraic approach.

3. Algebraic viewpoint of the Fuchsian differential equations

The existence of $C(N, N)$ as solutions common to the sigma form of the Painlevé VI equation and to linear Fuchsian differential equations can be addressed in an effective algebraic geometry approach of differential equations as introduced explicitly by Ritt [27, 28]. This approach amounts, when working with various linear and nonlinear differential equations, to introducing

¹¹ For $T < T_c$, one has $C(N, N) - (1-t)^{1/4} = 1/4 \cdot ((1/2)_N (3/2)_N) / ((N+1)!)^2 \cdot t^{N+1} + \dots$, the coefficient in front of t^{N+1} corresponding to the initial condition defining $C(N, N)$ in the low-temperature regime [26].

¹² Cauchy’s theorem does not apply to PVI at $t = 0$ or $t = 1$. As a consequence, even with given boundary conditions (a large set of first terms in the series), there can be ‘branching’ in the series computation. These subtle ‘branching’ series calculations will be addressed elsewhere.

as many variables as the number of derivatives of the function we study. The analysis of the compatibility between these various linear and nonlinear differential equations will correspond to considering an algebraic variety given by various polynomial relations on these variables. These relations can be studied from the algebraic viewpoint (parametrization when the genus is 0 or 1, birational transformations¹³, singularity analysis, blow-up, etc). The very last step, recalling that the various introduced variables are not independent but can be deduced from each other by successive derivation, provides further constraints. In other words, a set of differential equations is seen as an algebraic variety plus some differential structure on top of it.

Let us show how this algebraic viewpoint of differential equations works in our (subtle) compatibility problem of the sigma form of Painlevé VI and the Fuchsian linear ODEs of arbitrary order $N + 1$. The correlation function $C(1, 1)$ satisfies a second-order linear differential equation which can be written in a Riccati form in terms of $\sigma(t)$ and $\sigma'(t)$. More generally, the $(N + 1)$ -order Fuchsian linear ODE satisfied by $C(N, N)$ s can be written in a ‘generalized Riccati form [29, 30]’ in terms of $\sigma(t)$, $\sigma'(t)$ and its successive derivatives $\sigma^{(n)}(t)$ up to $n = N$ (where $\sigma(t)$ is deduced from $C(N, N)$ by the logarithmic derivative relation (12)). Similarly, the sigma form of Painlevé VI equation (9) is not seen as a nonlinear ODE, but as a polynomial relation between the three variables $\sigma(t)$, $\sigma'(t)$ and $\sigma''(t)$.

Introducing the variables $S_0 = \sigma(t)$, $S_1 = \sigma'(t)$, $S_2 = \sigma''(t)$, etc, the third-order Fuchsian linear ODE for $C(2, 2)$ yields a ‘generalized Riccati form’ which is a polynomial relation between S_0 , S_1 and S_2 :

$$64t^2(t - 1)^2S_2 - 16t(8t + 5)(t - 1) \cdot S_1 + 192t(t - 1)S_0S_1 + 64S_0^3 - 16(16t + 1)S_0^2 + 4(32t^2 + 16t - 21)S_0 + 45 = 0. \tag{54}$$

The elimination of the variable S_2 between this ‘generalized Riccati form’ and (9) seen as a polynomial relation between the three variables S_0 , S_1 and S_2 yields an algebraic relation between $S_0 = \sigma(t)$ and $S_1 = \sigma'(t)$ which reads

$$(4S_0 - 3)(64S_0^3 - 16(16t + 1)S_0^2 + 4(64t^2 - 16t - 21) \cdot S_0 + 45) - 32t(4S_0 - 3)(t - 1)(8t - 1 - 4S_0) \cdot S_1 + 256t^2(t - 1)^2S_1^2 = 0, \tag{55}$$

which is compatible with (54) and (9). This can be checked by eliminating S_2 between the derivative of (55) and (54) or (9) to get again (55) or directly by plugging a series expansion or an exact expression of $C(2, 2)$ in (55).

Seen as a relation between S_0 and S_1 (the variable t is considered as a simple parameter), the algebraic curve (55) is actually a *rational curve*. It can thus be parametrized in terms of two rational functions:

$$\begin{aligned} S_0 &= \frac{3}{4} \frac{A_2 \cdot u^2 + A_1 \cdot u + A_0}{B_2 \cdot u^2 + B_1 \cdot u + B_0}, \\ S_1 &= \frac{3}{t} \cdot \frac{(\alpha_1 \cdot u + \alpha_0) \cdot (C_3 \cdot u^3 + C_2 \cdot u^2 + C_1 \cdot u + C_0)}{(B_2 \cdot u^2 + B_1 \cdot u + B_0)^2}, \end{aligned} \tag{56}$$

¹³ At this step, it is worth recalling that Bäcklund transformations are actually *birational transformations* in ‘some’ variables.

where

$$\begin{aligned}
 \alpha_1 &= -6t - 3 + 8t^2, & \alpha_0 &= 4 \cdot (1 - 2t), \\
 A_0 &= -176 + 48t - 320t^2 + 256t^3, \\
 A_1 &= 120 + 184t - 144t^2 + 768t^3 - 512t^4, \\
 A_2 &= 9 - 57t + 24t^2 + 76t^3 - 448t^4 + 256t^5, \\
 B_0 &= 192t^2 - 272t - 112, & B_1 &= -8(3t + 1)(16t^2 - 26t - 3), \\
 B_2 &= 45 + 51t - 168t^2 - 260t^3 + 192t^4, \\
 C_0 &= 1088 + 384t + 2624t^2 + 1280t^3 - 1536t^4, \\
 C_1 &= -1296 - 2816t + 688t^2 - 7776t^3 - 3840t^4 + 4608t^5, \\
 C_2 &= 108 + 1848t + 636t^2 - 3328t^3 + 8304t^4 + 4416t^5 - 4608t^6, \\
 C_3 &= +189 + 36t - 1323t^2 + 210t^3 + 2460t^4 - 2792t^5 - 1856t^6 + 1536t^7.
 \end{aligned} \tag{57}$$

In the spirit of the ‘algebraic viewpoint of differential equations’ [27, 28], having performed the algebraic geometry calculations we had in mind, we now recall that there is some differential structure on this *rational curve* by imposing that the variable S_1 is actually the derivative with respect to t of the variable S_0 :

$$S_1 = \frac{dS_0}{dt} = \frac{\partial S_0}{\partial u} \cdot \frac{du}{dt} + \frac{\partial S_0}{\partial t}, \tag{58}$$

yielding, after some quite nice simplifications, that $\frac{du}{dt}$ is not a rational expression of u , as one could expect at first sight, but a quadratic polynomial in u , which gives a simple Riccati form:

$$\begin{aligned}
 16t(t-1)(6t^2-5t-9) \cdot \frac{du}{dt} &= (63 - 135t - 120t^2 - 140t^3 + 192t^4) \cdot u^2 \\
 &+ 8(15 + 51t + 46t^2 - 60t^3) \cdot u - 272 - 112t + 192t^2
 \end{aligned} \tag{59}$$

that can easily be associated with a linear second-order differential equation bearing on some function F :

$$v = \frac{1}{F} \cdot \frac{dF}{dt} = -\frac{1}{16} \frac{192t^4 - 140t^3 - 120t^2 - 135t + 63}{t(-1+t)(6t^2-5t-9)} \cdot u. \tag{60}$$

Similar calculations can be performed for $N = 3$; the generalized Riccati form for the Fuchsian linear ODE of order 4 is now a polynomial relation of the form

$$S_3 = P(S_0, S_1, S_2; t), \tag{61}$$

where P is a polynomial of the three variables S_0, S_1 and S_2 , the coefficients being rational function (with integer coefficients) in the variable t seen as a parameter. In order to combine¹⁴ this generalized Riccati form (61) with (9) for $N = 3$, we need, in order to perform eliminations of variables (ideal of polynomials), to rewrite (9), the sigma form of Painlevé VI taken for $N = 3$ as a relation between σ, σ' and σ'' and $\sigma^{(3)}$ as well. This is easily obtained by performing the derivative of (9) with respect to t , thus getting a polynomial relation between σ, σ' and σ'' and $\sigma^{(3)}$. Considering this last polynomial relation and the generalized Riccati form (61), we can easily eliminate $S_3 = \sigma^{(3)}$, getting a new polynomial relation on S_0, S_1 and S_2 . We can, now, eliminate S_2 between this new polynomial relation and (9) for $N = 3$ which is also a polynomial relation on S_0, S_1 and S_2 , in order to get, finally, a polynomial relation on $S_0 = \sigma$ and $S_1 = \sigma'$ only. This final relation reads

$$\begin{aligned}
 4096t^3(t-1)^3 \cdot S_1^3 + 256t^2(t-1)^2 Q_2 \cdot S_1^2 - 16t(t-1) Q_1 \cdot S_1 \\
 - (45 - 8(2t+7)S_0 + 16S_0^2) \cdot Q_0 = 0,
 \end{aligned} \tag{62}$$

¹⁴ Or, in mathematical wording, to calculate the ideal of these two differential equations.

where

$$\begin{aligned}
 Q_2 &= 48S_0^2 - 8(22t + 13) \cdot S_0 + 55 + 448t + 64t^2, \\
 Q_1 &= -768S_0^4 + 256(22t + 13)S_0^3 - 32(376t^2 + 584t + 125)S_0^2 \\
 &\quad + 16(384t^3 + 1984t^2 + 766t + 25)S_0 + 1125 + 2880t - 25\,920t^2, \\
 Q_0 &= 1575 + 16(576t^3 - 110t - 145 - 96t^2)S_0 \\
 &\quad - 32(56t - 9 + 264t^2)S_0^2 + 256(10t + 3)S_0^3 - 256S_0^4.
 \end{aligned}
 \tag{63}$$

Similar calculations (of ideal of differential equations seen as ideal of polynomials), can be performed, mutatis mutandis, for $N = 4, 5$ and 6 . These eliminations yield polynomial relations in t , $S_0 = \sigma$ and $S_1 = \sigma'$ of the form

$$\sum_{i=0}^{i=N} t^i (t - 1)^i P_i(S_0, t) \cdot S_1^i = 0,
 \tag{64}$$

where $P_i(S_0, t)$ s are polynomials in t and $S_0 = \sigma$, of degree $2i$ in S_0 . Again, these relations (64) seen as algebraic curves in S_0 and S_1 (t being seen as a parameter) are *rational curves*. From the previous remark that $C(N, N)$ s are homogeneous polynomials of E and K , one can easily deduce that $S_0 = \sigma$ and $S_1 = \sigma'$ are rational expressions of the ratio $r = E/K$ (or E'/E).

Now, similarly to the previous calculations, recalling that the variable S_1 is the derivative with respect to t of the variable S_0 , one also finds Riccati equations similar to (59) for the uniformizing parameter u :

$$\frac{du}{dt} = \beta_2(t) \cdot u^2 + \beta_1(t) \cdot u + \beta_0(t),
 \tag{65}$$

where $\beta_0(t), \beta_1(t)$ and $\beta_2(t)$ are quite simple rational expressions of t , the Riccati equation (65) having only $t = 0, t = 1$ and $t = \infty$ as regular singularities. The calculations are too large to be given here and will be detailed in a forthcoming publication.

Note that, in such a ‘global’ Riccati algebraic approach, one has to be careful because of the existence of many singular¹⁵ solutions of (9) corresponding to algebraic τ functions:

$$\begin{aligned}
 \sigma &= t(t - 1) \cdot \frac{d}{dt} \log(\tau) - 1/4 \\
 \tau &= t^\alpha \cdot (1 - t)^\beta, \quad (4\beta - 1)^2 N^2 + 16\beta(4\alpha + 1)(\alpha + \beta) = 0,
 \end{aligned}
 \tag{66}$$

like, for instance, (α, β) being $(-N/2, -1/4 \cdot N/(N - 1)), (-1/8 \cdot (4N^2 + 1), N^2)$ or $(-1/4, 1/4)$, and especially $(N/2, -1/4 \cdot N/(N + 1))$ which corresponds to a series expansion with leading order similar to (49).

4. Generalization to non-diagonal correlation functions $C(N, M)$

Most of the results, previously displayed, can be generalized to the non-diagonal correlation functions $C(N, M)$ of the square Ising model. $C(N, M)$ s are also given by determinants (see [12]) whose entries are holonomic quantities solutions of linear differential equations of order 3. $C(N, M)$ s are thus holonomic solutions of linear differential equations. At first sight, the growth of the order of the corresponding differential operators should also be exponential in N and M .

We found that the order of these linear differential operators is, again, not growing exponentially with N and M but has a *quadratic growth order* and depends on the parity of

¹⁵ We use here the terminology of singular solutions of differential equations [27, 28].

$M - N$. For all the Fuchsian linear differential operators we have obtained (N and $M \leq 6$), the order can be reproduced by

$$q = \frac{1}{8} \cdot (M + N + 2) \cdot (4 + (3 - (-1)^{M-N}) \cdot |M - N|). \tag{67}$$

These linear differential operators L_{NM} are too large to be given explicitly here. Let us just give one of them, namely the linear differential operator L_{12} , corresponding to the simplest non-diagonal (and non-horizontal or vertical like $C(0, N)$ or $C(N, 0)$) two-point correlation function. The linear differential operator L_{12} reads

$$L_{12} = D_s^5 + \frac{5(2s^2 + 3)D_s^4}{s(1 + s^2)} + \frac{q_3 \cdot D_s^3}{s^2(1 + s)^2(1 - s)^2(1 + s^2)^2} + \frac{q_2 \cdot D_s^2}{s^3(1 + s)^3(1 - s)^3(1 + s^2)^3} + \frac{q_1 \cdot D_s}{s^4(1 + s)^3(1 - s)^3(1 + s^2)^4} + \frac{q_0}{s^5(1 + s)^3(1 - s)^3(1 + s^2)^5}, \tag{68}$$

where the polynomials q_i read

$$\begin{aligned} q_3 &= 13s^8 + 30s^6 - 78s^4 - 50s^2 + 53, \\ q_2 &= 5s^{12} - 7s^{10} + 34s^8 - 128s^6 - 65s^4 - 97s^2 + 2, \\ q_1 &= -5s^{14} + 2s^{12} - 67s^{10} - 118s^8 - 816s^6 + 157s^4 - 76s^2 - 101, \\ q_0 &= -192s^{10} + 1840s^8 - 453s^6 + 127s^4 - 15s^2 - 27. \end{aligned} \tag{69}$$

Let us comment on the remarkable simplifications we encountered when computing $C(N, M)$ s from the quadratic double recursions (discrete generalizations of Painlevé equations) they satisfy [18] together with $C^*(N, M)$ s. From the expressions of $C(N, N)$ s as a homogeneous polynomial in E and K , and the expressions of $C(0, 1)$, we can obtain $C(N, M)$ and $C^*(N, M)$, step by step using this quadratic double recursion [18]. At first sight, these $C(N, M)$ s should be given as rational expressions of E and K and, in some cases, as roots of quadratic polynomials with polynomial expressions in E and K . Remarkably, as a consequence of factorizations and simplifications in the numerator and denominator of these rational expressions, and the occurrence of a perfect square in the case of roots of quadratic polynomials, $C(N, M)$ s are *actually* always given by polynomial expressions in E and K , that are no longer homogeneous polynomials, but *sums* of homogeneous polynomials¹⁶, as the following example shows¹⁷:

$$C(1, 3) = \frac{1}{3s^6} \cdot (P_1 + P_3), \tag{70}$$

$$\begin{aligned} P_1 &= 2(s^4 - 1)(s^2 + 1)s^2 \cdot K - s^2(s^2 + 1)(s^4 + 3s^2 - 2) \cdot E, \\ P_3 &= (6s^2 - 1 + 11s^4) \cdot E^3 + (s^4 - 1)(7s^4 + 12s^2 - 3) \cdot K E^2 \\ &\quad + (s^4 - 1)(s^2 + 3)(s^4 + 2s^2 - 1)(s^2 - 1) \cdot E K^2 + (s^4 - 1)^2(s^2 - 1)^2 \cdot K^3. \end{aligned}$$

The two linear and cubic components $P_1/3s^6$ and $P_3/3s^6$ are, respectively, solutions of the

¹⁶ This result can also be found in equations (3.22)–(3.35) of [31], which first show that $C(N - 1, N)$ is a homogeneous polynomial of E, K and the complete elliptic integral of the third kind Π_1 in the anisotropic case. Together with information from Montroll, Potts and Ward [12] (note e.g. equation (A.19)) this means that the same statement holds for $C(N - k, N)$ for $k = 2, \dots, N$. Reduction of Π_1 in the isotropic case then shows that $C(N - k, N)$ is an inhomogeneous polynomial of E and K .

¹⁷ Our results on the expressions of $C(N, M)$ s are in agreement with those given, for N and $M \leq 4$, in [32, 33].

two linear differential operators:

$$\begin{aligned}
 L_1 &= D_s^2 - \frac{(3s^4 - 7s^2 + 14)}{s(s^2 + 1)(s^2 - 2)} \cdot D_s + 4 \frac{11s^4 - 9s^2 + 4}{s^2(s^2 + 1)^2(s^2 - 2)(-1 + s^2)}, \\
 L_3 &= D_s^4 - 2 \cdot \frac{A_3}{(s^2 - 1)s \cdot N} \cdot D_s^3 + \frac{A_2}{s^2(s^4 - 1)^2 \cdot N} \cdot D_s^2 \\
 &\quad + \frac{A_1}{s^3(s^4 - 1)^2 \cdot N} \cdot D_s + \frac{A_0}{s^4(s^4 - 1)^3 \cdot N},
 \end{aligned} \tag{71}$$

$$\begin{aligned}
 N &= s^{12} + 5s^{10} + 14s^8 + 54s^6 + 49s^4 + 13s^2 - 1, \\
 A_3 &= 3s^{14} + 15s^{12} + 44s^{10} + 98s^8 + 383s^6 + 415s^4 + 133s^2 - 11, \\
 A_2 &= 19s^{20} + 121s^{18} + 248s^{16} - 408s^{14} - 974s^{12} + 2546s^{10} \\
 &\quad + 9597s^8 + 11\,440s^6 + 6521s^4 + 1277s^2 - 147, \\
 A_1 &= -27s^{20} - 161s^{18} + 240s^{16} + 5576s^{14} + 17\,854s^{12} + 28\,590s^{10} \\
 &\quad + 30\,491s^8 + 19\,360s^6 + 8799s^4 + 1931s^2 - 333, \\
 A_0 &= -1792s^{20} - 13\,136s^{18} - 37\,568s^{16} - 52\,256s^{14} - 48\,848s^{12} \\
 &\quad - 32\,576s^{10} - 20\,720s^8 - 1568s^6 + 1600s^4 - 688s^2 + 192,
 \end{aligned} \tag{72}$$

which are homomorphic to the first and third symmetric powers of the linear differential operator L_E :

$$\begin{aligned}
 L_3 \text{ equiv. } \text{Sym}^3(L_E), \quad \text{that is } L_3 \cdot Q_3 &= W_3 \cdot \text{Sym}^3(L_E), \\
 L_1 \text{ equiv. } L_E, \quad \text{that is } L_1 \cdot Q_1 &= W_1 \cdot L_E,
 \end{aligned} \tag{73}$$

where Q_3 and W_3 (resp. Q_1 and W_1) are linear differential operators of order 3 (resp. 1). The order 6 linear differential operator corresponding to $C(1, 3)$, that is the LCLM of L_1 and L_3 , is homomorphic to the LCLM of L_E and $\text{Sym}^3(L_E)$:

$$L_1 \oplus L_3 \text{ equiv. } L_E \oplus \text{Sym}^3(L_E). \tag{74}$$

Also note that for the horizontal, or vertical, correlations ($N = 0$ or $M = 0$) one also has a homogeneous polynomial of E and K of degree 0. Let us consider for instance the simple correlation $C(0, 1)$:

$$C(0, 1) = 1/2 \frac{\sqrt{1+s^2}}{s} + 1/2 \frac{(s-1)(s+1)\sqrt{1+s^2}}{s} \cdot K. \tag{75}$$

The first term (of degree 0 in E and K) is solution of an order 1 linear differential operator l_0 , whereas the second term is solution of an order 2 linear differential operator l_1 :

$$\begin{aligned}
 l_0 &= Ds + \frac{I}{s(I+s^2)}, \\
 l_1 &= Ds^2 + \frac{(s^2-3)Ds}{s(s^2-1)} + \frac{2s^6+9s^4+4s^2+1}{(1+s^2)^2s^2(s^2-1)^2}.
 \end{aligned} \tag{76}$$

Up to a conjugation by $(1+s^2)^{1/2}$, the order 2 linear differential operator l_1 is an operator homomorphic to L_E :

$$(1+s^2)^{-1/2} \cdot l_1 \cdot (1+s^2)^{1/2} = Ds^2 + \frac{(-4s^2+3s^4-3)}{(1+s^2)s(s^2-1)} \cdot Ds + \frac{s^6-s^4+7s^2+1}{(s^2-1)^2(1+s^2)s^2} \tag{77}$$

with $(1+s^2)^{-1/2} \cdot l_1 \cdot (1+s^2)^{1/2}$ equiv. L_E . One actually finds that $C(0, 1)$ is solution of the third-order operator direct sum of l_0 and l_1 and is thus equivalent (up to conjugation by $(1+s^2)^{1/2}$) to the direct sum of l_0 and L_E .

From the fact that $C(N, M)$ s are *actually* always given by polynomial expressions sums of homogeneous polynomials in E and K , one easily deduces that the corresponding linear differential operators L_{NM} are homomorphic to direct sums of symmetric products of the second-order linear differential operator (40), yielding generalizations of (36):

$$L_{NM} \text{ equiv. } \bigoplus_m \text{Sym}^m(L_E), \quad (78)$$

where for $N - M$ odd, m is running as $N, N + 1, N + 2, \dots, M$ and for $N - M$ even, as $N, N + 2, N + 4, \dots, M$, and where $\text{Sym}^m(L_E) = l_0$ when $m = 0$.

This structure is a consequence of the fact that $C(N, M)$ s are given by *polynomial expressions* in E and K , instead of the *rational or algebraic expressions* in E and K which one could expect at first sight from the discrete Painlevé double recursions. This corresponds to quite remarkable identities and simplifications (factorizations, occurrence of perfect squares). From a less nonlinear and more ‘Fuchsian’ linear viewpoint, an explanation is the following. The non-diagonal $C(N, M)$ s are determinants of holonomic functions, hence they are holonomic themselves. On the other hand, they are rational (or even algebraic expressions in E and K). Now, because the Galois group of L_E is $SL(2, C)$, results from [21, 34] show that expressions in E and K which are holonomic will have to be polynomial.

Again, one can check that all these linear differential operators L_{NM} are Fuchsian differential operators with only three regular singular points $t = 0, t = 1, t = \infty$. This is a straight consequence of the fact that these L_{NM} s can be built as linear differential operators having polynomial solutions in E and K and, thus, they inherited the three regular singular points $t = 0, t = 1, t = \infty$ from the complete elliptic integrals E and K , and from the fact that the coefficients of the monomials $E^i \cdot K^j$ are extremely simple rational expressions with no singularity except poles at $s = 0$ (polynomial in s divided by powers of s).

The results we got on the non-diagonal correlation functions $C(N, M)$ are too numerous, and require too much space, to be given here (even if the final result is remarkably simple and elegant). However, one sees the emergence of quite fascinating structures relating an infinite set of Fuchsian linear differential operators depending on two integers N and M (L_{NM} s), with some quadratic double recursions that are nothing but discrete generalizations of Painlevé, these structures being themselves closely linked with complete elliptic integrals.

5. Bäcklund transformation and Malmquist Hamiltonian structure

Let us recall that since the work of Malmquist [35] it has been known that the Painlevé VI equation can be obtained from Hamilton equations

$$p' = \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad q' = \frac{dq}{dt} = \frac{\partial H}{\partial p} \quad (79)$$

with

$$t(t-1) \cdot H = q(q-1)(q-t)p^2 - Q(q) \cdot p + (n_3 - n_1)(n_3 - n_2)(q-t), \quad (80)$$

where

$$Q(q) = (n_3 + n_4)(q-1)(q-t) + (n_3 - n_4)q(q-t) - (n_1 + n_2)(q-1)q.$$

With this structure, it follows that p is a rational function of t, q and q' . The Hamiltonian is the t -logarithmic derivative of the function $\tau(t)$. The correlation functions $C(N, N)$ being solutions of the sigma form of Painlevé VI, one may find how the expressions of the two variables p and q (for which the Bäcklund transformations are *birational*) in the restricted case $n_1 = N/2, n_2 = (1-N)/2, n_3 = (1+N)/2$ and $n_4 = N/2$ appear in terms of the elliptic integrals K and E . Considering the diagonal correlation function $C(2, 2)$ taken as

$\tau(t) = t^{1/4}C(2, 2)$, one might expect, at first sight, to obtain the variables p and q as algebraic expressions in terms of E and K (and t). Remarkably, one obtains the surprising result that the two variables p and q are *actually rational expressions*¹⁸ of E and K . For $N = 2$, one thus gets two solutions, the simplest one being

$$p = -\frac{((t + 1)E + (t - 1)K)N_p^{(1)} \cdot N_p^{(2)}}{2t(2E + (t - 1)K)D_p^{(1)}D_p^{(2)}}, \quad q = -\frac{t(2E + (t - 1)K) \cdot N_q}{((t + 1)E + (t - 1)K)N_p^{(1)}}, \quad (81)$$

$$\begin{aligned} N_p^{(1)} &= -(9t - 1)(t - 1)^2 \cdot K^2 - 2(17t - 1)(t - 1) \cdot EK + (1 + t^2 - 34t) \cdot E^2, \\ N_p^{(2)} &= -(t - 1)K^2 - 2EK + E^2, \\ D_p^{(1)} &= -3K^2(t - 1)^2 - 8(t - 1)EK + (-5 + t)E^2, \\ D_p^{(2)} &= -K^2(t - 1)^2 + 2(t - 1)^2EK + (5t - 1)E^2, \\ N_q &= -(3t - 11)(t - 1)^2 \cdot K^2 + 2(t - 1)(3t^2 - t + 14) \cdot EK + (17t^2 - 2t + 17) \cdot E^2. \end{aligned} \quad (82)$$

One notes the homogeneous occurrence, in terms of degree, of E and K in these relations. The variables p and q have the rational parametrization of an algebraic curve. Obviously, the uniformization parameter similar to that introduced in section 3 can be chosen as the ratio $u = E/K$ (or E'/E) of the two elliptic integrals. One can then deduce that the parameter u is a solution of a Riccati differential equation. These results generalize straightforwardly to all p, q associated with $C(N, N)$ s leading, remarkably, to *rational functions* of E and K and yielding rational parametrization for the corresponding algebraic curves between p and q . We have the same results in the variables σ and σ' . The expressions of the Bäcklund transformation corresponding to changing N into $N + 1$ in terms of the variables p, q will be analysed elsewhere.

6. Conclusion

The phenomenon of the existence of a one-parameter family of solutions to the Painlevé VI equation has been presented in this paper by the study of the specific PVI equation which is satisfied by $C(N, N)$, the diagonal correlation function of the Ising model. However, the existence of such linear equations is a much larger phenomenon and certainly holds for all PVI equations where the difference of any two of the parameters v_j is an integer because, in that case, there is a class of solutions which can be written as finite-dimensional determinants whose elements are hypergeometric functions.

Even though the existence of these Fuchsian differential equations follows from the general theorem on holonomic functions, the specific form and properties of these equations are tedious to obtain. However, the expressions obtained for small N (via series computations) have been sufficient to guess the structure that is proved in sections 2 and 3. Moreover, using these initial computations, it has been possible to make a remarkably simple conjecture for the exponents which is in complete agreement with the local expansion of the Painlevé VI equation at its singular points and this conjecture puts restrictions on the coefficients in the differential equations.

In this paper, we have obtained the Fuchsian equations by starting with the PVI equation. However, the question can be reversed and we can ask what are the conditions on the Fuchsian equations which will lead to PVI equations. For second-order Fuchsian equations, it would be sufficient to require that the exponents at the singularities agree with the exponents allowed

¹⁸ Formulae expressing p and q as ratios of tau functions can be found in equations (5.42), (5.43) of [36].

by the local expansions of PVI. But for higher order equations the exponents do not fully specify the Fuchsian equation. The extra parameters which need to be specified are referred to as accessory parameters and only very specific accessory parameters will lead to Fuchsian solutions of PVI. The needed restrictions on these parameters are not known.

The more general version of this is the question of determining whether or not a specific set of solutions to a Fuchsian equation will also satisfy some nonlinear equation (not necessarily PVI). This is in some sense the original question asked by Jimbo and Miwa [4] and this is particularly important because, for $C(N, N)$, the nonlinear PVI is much simpler than the linear equations $L_{N,N}$. It was found in [14–17] that the three- and four-particle contributions to the susceptibility of the Ising model, χ , satisfy Fuchsian equations whose structure appears rather complicated and the question may be asked whether these functions, or their sum χ , can also satisfy a nonlinear equation which might be simpler in appearance.

Finally, we remark that perhaps the most interesting discovery in this paper is that the operator L_{NN} is equivalent to the N th symmetric power of the operator L_E . This property extends to the operator L_h (which is isomorphic to L_E up to a conjugation). One might wonder whether all solutions of the sigma form of Painlevé VI that are also solutions of linear differential equations would be produced from symmetric powers of L_E by intertwiners.

References

- [1] Kaufman B and Onsager L 1949 *Phys. Rev.* **76** 1244
- [2] Stanley R P 1980 *Eur. J. Comb.* **1** 175–88
- [3] Wu T T, McCoy B M, Tracy C A and Barouch E 1976 *Phys. Rev. B* **13** 316–74
- [4] Jimbo M and Miwa T 1980 *Proc. Japan Acad. A* **56** 405
Jimbo M and Miwa T 1981 *Proc. Japan Acad. A* **57** 347
- [5] McCoy B M and Wu T T 1980 *Phys. Rev. Lett.* **45** 675
- [6] Perk J H H 1980 *Phys. Lett. A* **79** 3–5
- [7] Okamoto K 1979 *Japan. J. Math.* **5**
- [8] Forrester P J and Witte N S 2004 *Nagoya Math. J.* **174** 29–114
- [9] Jimbo M 1982 *Publ. RIMS* **18** 1137–61
- [10] Garnier R 1912 *Ann. l'Ecole Normale Supérieure* **29** 1–126
- [11] Okamoto K 1987 *Ann. Mat. Pura Appl.* **146** 337–81
- [12] Montrol E W, Potts R B and Ward J C 1963 *J. Math. Phys.* **4** 308–22
- [13] McCoy B M and Wu T T 1973 *The Two-Dimensional Ising Model* (Cambridge: Harvard University Press)
- [14] Zenine N, Boukraa S, Hassani S and Maillard J M 2004 *J. Phys. A: Math. Gen.* **37** 9651–68 (Preprint [math-ph/0407060](https://arxiv.org/abs/math-ph/0407060))
- [15] Zenine N, Boukraa S, Hassani S and Maillard J M 2005 *J. Phys. A: Math. Gen.* **38** 1875–99 (Preprint [hep-ph/0411051](https://arxiv.org/abs/hep-ph/0411051))
- [16] Zenine N, Boukraa S, Hassani S and Maillard J M 2005 *J. Phys. A: Math. Gen.* **38** 4149–73 (Preprint [cond-mat/0502155](https://arxiv.org/abs/cond-mat/0502155))
- [17] Zenine N, Boukraa S, Hassani S and Maillard J-M 2005 *J. Phys. A: Math. Gen.* **38** 9439–74 (Preprint [hep-th/0506214](https://arxiv.org/abs/hep-th/0506214))
- [18] Orrick W P, Nickel B G, Guttman A J and Perk J H H 2001 *J. Stat. Phys.* **102** 795–841
- [19] Singer M F 1996 *Appl. Algebra Eng. Commun. Comput.* **7** 77–104
- [20] van Hoeij M 1996 *Proc. ISSAC '96* (New York: ACM) pp 219–25
- [21] van der Put M and Singer M F 2003 *Galois Theory of Linear Differential Equations* (Grundlehren der Mathematischen Wissenschaften vol 328) (Berlin: Springer)
- [22] Singer M F and Ulmer F 1995 Necessary conditions for Liouvillian solutions of (third order) linear differential equations *Appl. Algebra Eng. Commun. Comput.* **6** 1–22
- [23] van Hoeij M and Weil J-A 1997 An algorithm for computing invariants of differential Galois groups *J. Pure Appl. Algebra* **117–118** 353–79
- [24] Ghosh R K and Shrock R E 1984 *Phys. Rev. B* **30** 3790–94
- [25] Au-Yang H and Perk J H H *MathPhys Odyssey 2001: Integrable Models and Beyond* ed M Kashiwara and T Miwa (Basle: Birkhauser) pp 23–48

-
- [26] Ghosh R K 2005 On the low temperature series expansion for the diagonal correlation functions in the two-dimensional Ising model *Preprint* [cond-mat/0505166](#)
- [27] Ritt J F 1966 *Differential Algebra* (New York: Dover)
- [28] Ritt J F 1932 *Differential Equations from the Algebraic Viewpoint* (Washington, DC: American Mathematical Society)
- [29] Ince E 1956 *Ordinary Differential Equations* (New York: Dover)
- [30] Poole E G C 1960 *Introduction to the Theory of Linear Differential Equations* (New York: Dover)
- [31] Au-Yang H and Perk J H H *MathPhys Odyssey 2001: Integrable Models and Beyond* ed M Kashiwara and T Miwa (Basle: Birkhauser) pp 1–21
- [32] Shrock R E and Ghosh R K 1985 *Phys. Rev. B* **31** 1486–9
- [33] Ghosh R K and Shrock R E 1985 *J. Stat. Phys.* **38** 473–82
- [34] Singer M F 1986 *Trans. Am. Math. Soc.* **295** 753–63
- [35] Malmquist J 1922 *Ark. Mat. Astron. Fys.* **18** 1–89
- [36] Forrester P J and Witte N S 2003 *Nonlinearity* **16** 1919–44