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Inversion relations and symmetry groups for Potts models on the triangular lattice

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Abstract. Inversion relations are obtained for the standard scalar q-state triangular Potts model with two- and three-spin interactions, generalizing previously known results for two-spin interaction models. It is shown that these inversion relations generate a group of symmetries of the model which is naturally represented in terms of birational transformations in a four-dimensional parameter space. This group of birational transformations is generically a very large one, namely a hyperbolic Coxeter group. In this framework of very large groups of symmetries, a remarkable situation pops out: the one for which q corresponds to Tutte–Beraha numbers.

1. *Introduction*

There generally exist two different approaches to the exact determination of critical conditions (manifolds) of lattice models. First of all, many criticality conditions in algebraic forms have been shown to be related to some 'integrability' (Yang–Baxter equations or star–triangle relations) of the model, the algebraicity being a consequence of the integrability. However, a notable exception exists. This is the Potts model on the triangular lattice with two- and three-spin interactions in alternate lattice faces [1–3]. While its critical variety [3] is algebraic, the model is *not* integrable along this variety (except for zero three-spin interactions) [1]. As a consequence, the usefulness of integrability in connection with criticality in this model is limited.

Another approach to the determination of critical variety comes from the analysis of *inversion relations* [4–7] and the symmetry groups generated by these relations [8–10]. However, such studies in the past have often been restricted, for spin models, to edge interactions. Here, we extend the inversion relation to triangular Potts models with non-zero three-spin interactions, for which the integrability approach fails, and use it to analyse the related symmetry groups. The corresponding symmetry groups will be seen to yield remarkable birational representations of hyperbolic Coxeter groups. Remarkably, the *Tutte–Beraha numbers* [11] also pop out from this analysis: additional relations among the generators of the 'Coxeter' group occur.

We first briefly recall in section 2 some results on the nearest-neighbour interaction standard scalar Potts models, including the known inversion relations and symmetry groups. We then generalize, in section 3, this inversion relation to include three-spin interactions. The group of symmetries generated by these inversions is next obtained in section 4. Finally, in section 5 we briefly discuss how the analysis might be extended to other lattices, such as the chequer-board and Kagomé lattices.

2. Review of duality and inversion relations

In this section we recall the previously known duality and inversion relation for the triangular Potts model.

The partition function of the triangular Potts model with two- and three-spin interaction in alternate, say up-pointing, triangles reads

$$Z = \sum_{\{\sigma_i\}} \prod_{\langle i,j\rangle} e^{K_1\delta_{ij}} \prod_{\langle j,k\rangle} e^{K_2\delta_{jk}} \prod_{\langle k,i\rangle} e^{K_3\delta_{ki}} \prod_{\Delta} e^{K\delta_{ijk}} \tag{1}$$

where $\delta_{ij} = \delta(\sigma_i, \sigma_j)$, $\delta_{ijk} = \delta(\sigma_i, \sigma_j)\delta(\sigma_j, \sigma_k)$, the first three products are taken over the edge-interaction Boltzmann weights along the three lattice directions and the last product is over all *up-pointing* triangles of the three-spin interaction Boltzmann weight, and where the sum is over all spin configurations.

In this framework, we introduce the notations

$$\begin{aligned} x_i &= e^{K_i} \qquad i = 1, 2, 3 \\ x &= e^K \\ y &= x\, x_1 x_2 x_3 - (x_1 + x_2 + x_3) + 2\,. \end{aligned} \tag{2}$$

Then, there exists a duality transformation D which reads [1,2]

$$D: \quad \begin{aligned} x_i &\to x_i^* = 1 + q(x_i - 1)/y \\ x &\to x^* = \frac{x_1^* + x_2^* + x_3^* - 2 + q^2/y}{x_1^* x_2^* x_3^*} \\ y &\to y^* = q^2/y\,. \end{aligned} \tag{3}$$

The duality transformation D is an involution (that is, $D^2 = \text{identity}$).

The simplest self-dual varieties obtained from (3) read

$$y = q \tag{4}$$

and

$$y = -q\,. \tag{5}$$

It has been shown that (4) is a critical variety for ferromagnetic interactions [3].

In the case of pure edge interactions, namely $K = 0$, the triangular Potts model possesses the following inversion relation [12–14]:

$$\mathcal{I}_1: \quad x_1 \to 2 - q - x_1 \quad x_2 \to 1/x_2 \quad x_3 \to 1/x_3 \tag{6}$$

and $\mathcal{I}_2$, $\mathcal{I}_3$ obtained by cyclic permutations of the indices. The symmetry group Γ_{triang} generated by $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$ can easily be studied when introducing the following convenient variables [5]:

$$y_i = \frac{x_i - q_+}{x_i - q_-} \qquad i = 1, 2, 3 \tag{7}$$

where $q_\pm$ are the roots of the second-order equation $t^2 + (q-2)\,t + 1 = 0$. In terms of these variables, transformation $\mathcal{I}_1$ reads

$$(y_1, y_2, y_3) \rightarrow \left(\frac{1}{y_1}, \frac{q_+^2}{y_2}, \frac{q_+^2}{y_3}\right). \tag{8}$$

Let us also consider the (generically) infinite-order generators of Γ_{triang}:

$$\mathcal{J}_1 = \mathcal{I}_2\mathcal{I}_3 \qquad \mathcal{J}_2 = \mathcal{I}_3\mathcal{I}_1 \qquad \mathcal{J}_3 = \mathcal{I}_1\mathcal{I}_2\,. \tag{9}$$

For instance, $\mathcal{J}_1$ reads

$$(y_1, y_2, y_3) \rightarrow \left(y_1, \frac{y_2}{q_+^2}, q_+^2\, y_3\right). \tag{10}$$

From (10), one sees directly that the $\mathcal{J}_i$'s do commute. Elements of the 'infinite' part of Γ_{triang} thus read

$$\mathcal{J}_1^{n_1}\mathcal{J}_2^{n_2}\mathcal{J}_3^{n_3} \qquad \{n_1, n_2, n_3\} \in \mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}\,. \tag{11}$$

As a consequence of the property $\mathcal{J}_1\mathcal{J}_2\mathcal{J}_3 =$ identity, Γ_{triang} is, up to a semidirect product by a finite group, isomorphic to $\mathbf{Z}\times\mathbf{Z}$ [12, 14].

Moreover, one easily gets the algebraic varieties invariant under Γ_{triang}:

$$y_1 y_2 y_3 = q_+^2 \qquad y_1 y_2 y_3 = -q_+^2\,. \tag{12}$$

In fact, the first variety is nothing but the self-dual variety (4) for $x = 1$, also known as the ferromagnetic critical variety.

3. A new inversion relation

We now generalize the inversion relation when there are non-zero three-spin interactions on the up-pointing triangles.

Starting from a triangular face of the lattice with interactions x_1, x_2, x_3, and x, the Boltzmann factor $w(\alpha, \beta, \gamma)$ reads

$$w(\alpha, \beta, \gamma) = x_1^{\delta_{\alpha\beta}} x_2^{\delta_{\beta\gamma}} x_3^{\delta_{\gamma\alpha}} x^{\delta_{\alpha\beta\gamma}} \tag{13}$$

where α, β, γ are the spin states of the three sites.

An inversion relation is one in which the new interactions x_1', x_2', x_3', x' and the associated Boltzmann factor w' satisfy the relation

$$\sum_\beta w(\alpha, \beta, \gamma)\, w'(\beta, \alpha', \gamma) = \lambda_1\, \delta_{\alpha,\alpha'}\,. \tag{14}$$

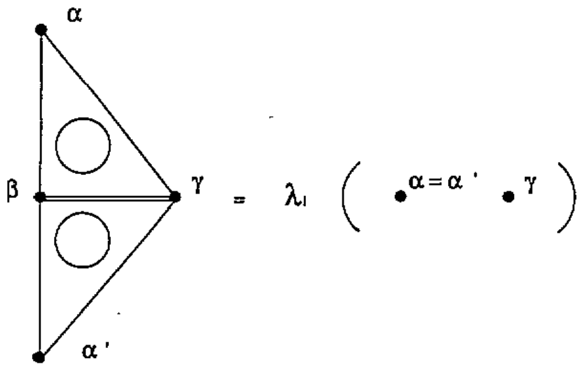

Figure 1. Schematic diagram representing the inversion transformation (14). The circles inside the triangles indicate three-spin interactions.

Here w' is the same as w with interactions x_i', x' replacing x_i, x. The situation is shown in figure 1. Note here that direction 1 is chosen to be the direction of inversion.

The Boltzmann weight $w(\alpha, \beta, \gamma)$ is invariant under a common shift of each spin state α, β, γ. Therefore, we may take $\gamma = 0$ and represent the Boltzmann weight by a $q \times q$ matrix W whose entries are $w(\alpha, \beta, 0)$:

$$\mathbf{W} = \begin{pmatrix} xx_1x_2x_3 & x_3 & x_3 & \cdots & x_3 \\ x_2 & x_1 & 1 & \cdots & 1 \\ x_2 & 1 & x_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ x_2 & 1 & \cdots & 1 & x_1 \end{pmatrix}. \tag{15}$$

Then, condition (14) becomes the following matrix equation:

$$\mathbf{W} \cdot \mathbf{W}' = \lambda_1 \mathbf{I}_q \tag{16}$$

where $\mathbf{I}_q$ is the $q \times q$ identity matrix, and $\mathbf{W}'$ is obtained from $\mathbf{W}$ with x_i, x replaced by x_i', x'.

Using a $\mathbf{Z}_{q-1}$ Fourier transform, the $q \times q$ matrix $\mathbf{W}$ can be block-diagonalized into one 2×2 block and $(x_1 - 1)\,\mathbf{I}_{q-2}$. Then, one can easily obtain x_i', x' in terms of x_i, x. This leads to

$$\lambda_1 = \frac{(1 - x_1)[xx_1(x_1 + q - 2) - (q - 1)]}{(xx_1 - 1)} \tag{17}$$

and

$$\begin{aligned} x' &= \frac{(xx_1 - 1)^2(x_1 + q - 2)}{[xx_1^2 + (q-3)xx_1 - (q-2)](x_1 - 1)} \\ x_1' &= -\frac{xx_1^2 + (q-3)xx_1 - (q-2)}{xx_1 - 1} \\ x_2' &= \frac{x_1 - 1}{x_3(xx_1 - 1)} \\ x_3' &= \frac{x_1 - 1}{x_2(xx_1 - 1)}. \end{aligned} \tag{18}$$

The birational transformation

$$\mathcal{I}_1 : \quad \{x_1, x_2, x_3, x\} \to \{x'_1, x'_3, x'_2, x'\} \tag{19}$$

together with the transformations $\mathcal{I}_2$ and $\mathcal{I}_3$ obtained by cyclic permutation of indices corresponding to inversions in directions 2 and 3 respectively, is a new inversion relation generalizing (6). Note that we have reversed the roles of x'_2 and x'_3 in the inversion relation (19) so that it reduces to (6) upon putting $x = 1$.

Alternatively, one can rewrite (13) as

$$w(\alpha, \beta, \gamma) = 1 + w_1\,\delta_{\alpha\beta} + w_2\,\delta_{\beta\gamma} + w_3\,\delta_{\gamma\alpha} + y\,\delta_{\alpha\beta\gamma} \tag{20}$$

where

$$w_i = x_i - 1 \qquad i = 1, 2, 3 \tag{21}$$

and y is given by (2). Substituting (20) and a similar expression for w' into (14), setting the coefficients of the Kronecker deltas equal after carrying out the summation over β, one obtains the conditions

$$\begin{aligned}
&\lambda_1 = w_1 w'_1 \\
&q + w_1 + w_2 + w'_1 + w'_3 + w_2 w'_3 = 0 \\
&w_1 w'_3 + q w_3 + w_3 w'_1 + y(1 + w'_3) = 0 \\
&w'_1 w_2 + q w'_2 + w'_2 w_1 + y'(1 + w_2) = 0 \\
&y(w'_1 + w'_2) + y'(w_1 + w_3) + q w_3 w'_2 + y y' = 0
\end{aligned} \tag{22}$$

from which one recovers (17) and (18).

4. Analysis of the new symmetry group

Analogous to discussions given in section 2, the inversion transformations $\mathcal{I}_i$'s generate a symmetry group of the model. Again, one considers the (generically) infinite order generators of this group, namely, the $\mathcal{J}_i$'s defined in (9). Since the $\mathcal{I}_i$'s are involutions, property $\mathcal{J}_1\mathcal{J}_2\mathcal{J}_3 =$ identity still holds, but the $\mathcal{J}_i$'s do not commute as in the $x = 1$ case. Hence the symmetry group of the model is now a free group, with two generators $\mathcal{J}_1$ and $\mathcal{J}_2$. This amounts to considering the symmetry group as a 'Coxeter' group, defined by its generators and relations among the generators (the $\mathcal{I}_i$'s, or the $\mathcal{J}_i$'s,...).

At this point, let us recall that the critical manifold needs to be compatible with all the symmetries of the model. As a consequence, the algebraic variety (4), which is critical, needs to be stable under this 'huge' Coxeter group. A direct calculation reveals that (4) is actually *stable under the whole free group of symmetries* of the model. This analysis thus offers an alternate way of locating the critical variety (4). Moreover, this group is so huge that there probably exists no other stable algebraic variety† and hence no other algebraic critical condition.

† We thank C M Viallet for providing an answer to this question ruling out the existence of other stable varieties of degree 4 or less.

The above results are valid for arbitrary q. We now consider particular values of q, the Tutte–Beraha numbers [11], for which additional relations among the generators occur. In the $x = 1$ case, one directly sees from (10) that the $\mathcal{J}_i$'s become finite-order transformations of order N, when $q_{\pm}^2 = e^{2\pi k i/N}$, $k = 1, 2, \ldots, N-1$, or equivalently $q = 2 - 2\cos(k\pi/N)$. *Amazingly, this situation also holds for* $x \neq 1$. However, when $x \neq 1$, it appears that the $\mathcal{J}_i$s do not commute and the group still grows in an 'arborescent' way with the exceptions of $N = 1, 2, 3$. For $N = 1$ ($q = 0$ or $q = 4$) and for $N = 2$ ($q = 2$), the symmetry group degenerates into a finite group. For $N = 3$ ($q = 1$ or $q = 3$), the symmetry group is, up to a semidirect product by a finite group, isomorphic to $\mathbf{Z} \times \mathbf{Z}$, recovering the situation for $x = 1$ and arbitrary q. This particular case will be analysed elsewhere in a forthcoming publication.

5. Inversion relation for the Kagomé lattice

Let us consider here another lattice, with a $2\pi/3$-rotation symmetry: the Kagomé lattice.

The most general Kagomé lattice can be regarded as a chequer-board interaction-round-face (IRF) model with the elementary cell shown in figure 2. The Boltzmann weights of this cell read

$$W(\alpha, \beta, \alpha', \beta') = \sum_{\sigma} e^{K_1 \delta_{\alpha,\beta}} e^{K_2 \delta_{\beta,\sigma}} e^{K_3 \delta_{\alpha,\sigma}} e^{K_1' \delta_{\alpha',\beta'}} e^{K_2' \delta_{\beta',\sigma}} e^{K_3' \delta_{\alpha',\sigma}} . \qquad (23)$$

It should be noted that the role played by the special direction 1 in (23) can also be replaced by directions 2 or 3.

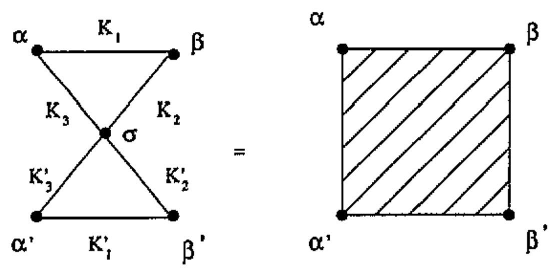

Figure 2. A unit cell of the Kagomé lattice viewed as a chequer-board lattice.

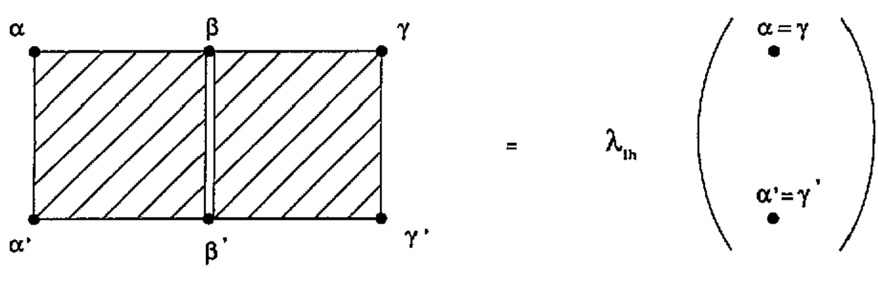

Figure 3. Schematic diagram representing the inversion transformation (24).

The inversion relation I_{1h}, for this particular choice of direction shown schematically in figure 3, reads

$$\sum_{\alpha',\beta'} W(\alpha,\beta,\alpha',\beta')\, W'(\beta,\gamma,\beta',\gamma') = \lambda_{1h}\, \delta_{\alpha,\gamma}\, \delta_{\alpha',\gamma'} . \tag{24}$$

Here, the Boltzmann weight is best represented by a $q^2 \times q^2$ matrix. Straightforward calculation shows that the inversion relation I_{1h} does exist†. It is tempting to consider a similar inversion relation in the 'vertical' direction. However, a detailed calculation shows that the $q^2 \times q^2$ 'vertical' matrix Boltzmann weight is not invertible, since its rank is q.

Similarly one can also introduce transformations I_{2h} and I_{3h} corresponding, respectively, to the 'horizontal' inversion transformations in directions 2 and 3. The analysis of these transformations and their group action in a larger parameter space, including the six-dimensional Kagomé model, will be discussed elsewhere.

6. Conclusion

We have obtained the inversion relation for the two- and three-site interaction q-state standard Potts model on the triangular lattice. The symmetry group generated by this inversion relation has been seen to be generically a very large one (a free group). The critical variety of this model has been shown to be invariant under this large group. When q is a Tutte–Beraha number, the representations of these symmetry groups are drastically modified.

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† However, W' may no longer be of the form given by (23).