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Ising model susceptibility: Fuchsian differential equation for $\chi^{(4)}$ and its factorization properties

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Abstract

We give the Fuchsian linear differential equation satisfied by $\chi^{(4)}$, the 'four-particle' contribution to the susceptibility of the isotropic square lattice Ising model. This Fuchsian differential equation is deduced from a series expansion method introduced in two previous papers and is applied with some symmetries and tricks specific to $\chi^{(4)}$. The corresponding order ten linear differential operator exhibits a large set of factorization properties. Among these factorizations one is highly remarkable: it corresponds to the fact that the two-particle contribution $\chi^{(2)}$ is actually a solution of this order ten linear differential operator. This result, together with a similar one for the order seven differential operator corresponding to the three-particle contribution, $\chi^{(3)}$, leads us to a conjecture on the structure of all the n-particle contributions $\chi^{(n)}$.

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1. Introduction

The magnetic susceptibility of square lattice Ising model, can be written [1] as an infinite sum

$$\chi(T) = \sum_{n=1}^{\infty} \chi^{(n)}(T) \tag{1}$$

of individual contributions, with the odd (respectively even) n corresponding to high (respectively low) temperature case. These individual contributions are $(n-1)$-dimensional integrals [2–7], and are seen as successive n-particle contributions to the susceptibility [1].

To get an understanding of the analytical structure of χ, two approaches are usually taken nowadays. One approach taking into account a fundamental nonlinear symmetry, namely nonlinear Painlevé difference equations [8–13]. provides a series expansion for the whole susceptibility χ. With this method coefficients of χ were recently generated [8].

The second approach considers the individual n-particle excitations as given by $(n-1)$-dimensional integrals. Isotropic series coefficients are generated [6–8]. This latter method allows one to seek the differential equations satisfied by the $\chi^{(n)}$, since they are D-finite in contrast to the whole susceptibility χ for which there are strong indications that it has a natural boundary in the complex plane of the variable $s = sh(2K)$, where $K = J/kT$ is the usual Ising model coupling constant. This has been shown for the isotropic case [6] and for the anisotropic case [14–16]. Such a function cannot be D-finite.

The understanding of the magnetic susceptibility may then require knowledge of each (or some) of the individual contributions. This knowledge can be in the form of a closed expression, as is the case for $\chi^{(1)}$ and $\chi^{(2)}$ or in the form of a differential equation as found for $\chi^{(3)}$ [17, 18]. The last case was far from being obvious and has required the building of an original method of expansion. The use of a remarkable formula allowed us to give the series expansion in the temperature variable (or a closely related variable) where the $(n-1)$-dimensional integrals have been fully performed.

In this paper, we continue to use this expansion method to tackle the next individual contribution, namely, $\chi^{(4)}$. We should note that, although, the method is general and applicable for high or low temperature and for any n, some of the tricks and tools used may be specific for a given $\chi^{(n)}$. In section 2, we present the basic features of the expansion method that allow us to obtain the fully integrated $\chi^{(4)}$ as four sums of products of four hypergeometric functions, without any numerical approximation. In section 3, we give the homogeneous Fuchsian linear differential equation satisfied by $\chi^{(4)}$. Section 4 contains some remarkable algebraic properties of this differential equation. Finally, section 5 contains our conclusions.

2. Fully integrated $\tilde{\chi}^{(4)}$ expansion

2.1. The expansion method

Let us focus on the fourth contribution to the susceptibility χ defined by the triple integral as given in [7]

$$\begin{aligned} \chi^{(4)} &= (1-s^{-4})^{1/4} \cdot \tilde{\chi}^{(4)} \\ \tilde{\chi}^{(4)} &= \int_0^{2\pi} \frac{\mathrm{d}\phi_1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\phi_2}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\phi_3}{2\pi} \cdot \tilde{y}_1\tilde{y}_2\tilde{y}_3\tilde{y}_4 \cdot R^{(4)} \cdot H^{(4)} \end{aligned} \tag{2}$$

with

$$R^{(4)} = \frac{1+\tilde{x}_1\tilde{x}_2\tilde{x}_3\tilde{x}_4}{1-\tilde{x}_1\tilde{x}_2\tilde{x}_3\tilde{x}_4} \tag{3}$$

$$H^{(4)} = \frac{1}{4!}\prod_{i<j} \frac{\tilde{x}_i\tilde{x}_j}{(1-\tilde{x}_i\tilde{x}_j)^2} \cdot (Z_i - Z_j)^2 \tag{4}$$

$$Z_n = \exp(\mathrm{i}\phi_n), \qquad n = 1, \ldots, 4 \tag{5}$$

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 = 0 \tag{6}$$

$$\begin{aligned} \tilde{x}_n &= \frac{s}{1+s^2 - s\cos\phi_n + \sqrt{(1+s^2-s\cos\phi_n)^2 - s^2}}, \\ \tilde{y}_n &= \frac{s}{\sqrt{(1+s^2-s\cos\phi_n)^2-s^2}}, \qquad n = 1, \ldots, 4. \end{aligned} \tag{7}$$

Instead of the variable s, we found it more suitable to use $w = \frac{1}{2}s/(1+s^2)$ which has, by construction, Kramers–Wannier duality invariance ($s \leftrightarrow 1/s$) and thus allows us to deal with both limits (high and low temperature, small and large s) on an equal footing [17, 18]. It is also convenient to consider the scaled variables

$$x_n = \frac{\tilde{x}_n}{w} = \frac{2}{1 - 2w\cos\phi_n + \sqrt{(1-2w\cos\phi_n)^2 - 4w^2}},$$
$$y_n = \frac{\tilde{y}_n}{2w} = \frac{1}{\sqrt{(1-2w\cos\phi_n)^2 - 4w^2}} \tag{8}$$

which behave like $1 + O(w)$ at small w.

As performing the integrals in (2) is highly non-trivial, we apply the 'expansion method' previously described in [17, 18], where the key ingredient was the Fourier expansion of yx^n, a quantity appearing in any $\chi^{(p)}$.

This remarkable formula, for yx^n, that carries only *one summation index* reads

$$yx^n = a(0,n) + 2\sum_{k=1}^{\infty} w^k a(k,n)\cos k\phi$$
$$= \sum_{k=-\infty}^{\infty} A(k,n)Z^k = \sum_{k=-\infty}^{\infty} A(k,n)Z^{-k} \tag{9}$$

with

$$A(k,n) = A(-k,n) = w^{|k|}a(|k|,n) \tag{10}$$

where $a(k,n)$ is a *non-terminating* hypergeometric series that reads

$$a(k,n) = \binom{m}{k}$$
$$\times\, {}_4F_3\left(\frac{(1+m)}{2}, \frac{(1+m)}{2}, \frac{(2+m)}{2}, \frac{(2+m)}{2}; 1+k, 1+n, 1+m; 16w^2\right) \tag{11}$$

where $m = k+n$. Note that $a(k,n) = a(n,k)$.

The integrand of $\tilde{\chi}^{(4)}$ is expanded in the various variables x_j, instead of the variable w. In this framework, with the help of the Fourier expansion (9), the angular integration becomes straightforward as was shown in [18] for the $\tilde{\chi}^{(3)}$ case (see below for the $\tilde{\chi}^{(4)}$ case).

2.2. *Calculation of* $\tilde{\chi}^{(4)}$

With $H^{(4)}$ taken as in (4), $\tilde{\chi}^{(4)}$ will be given by *five* summations on products of four hypergeometric functions. This is shown in appendix A. This route is feasible for any $\chi^{(n)}$ and does not use any symmetry or tricks specific to the considered $\chi^{(n)}$.

In the following, we use alternatively a simplified form of $H^{(4)}$ equivalent, for integration purposes, to (4) such that $\tilde{\chi}^{(4)}$ will be expressed by only *four* summations (i.e., one summation less compared to expression (A.30) given in appendix A). Let us just sketch the salient steps of this calculation. The details are left to appendices B, C and D.

With the help of the key relation

$$(Z_i - Z_j)\frac{\tilde{x}_i\tilde{x}_j}{1-\tilde{x}_i\tilde{x}_j} = -(\tilde{x}_i - \tilde{x}_j)\frac{Z_iZ_j}{1-Z_iZ_j} \tag{12}$$

the quantity $H^{(4)}$ becomes[4]:

$$H^{(4)} = \frac{1}{4!}\prod_{i<j}\frac{\tilde{x}_i - \tilde{x}_j}{1-\tilde{x}_i\tilde{x}_j}\cdot\prod_{i<j}\frac{Z_i - Z_j}{1-Z_iZ_j}. \tag{13}$$

[4] We have used the constraint (6) in the form $\prod_i Z_i = 1$.

Using the symmetry of the rest of the integrand in the angular variables, the quantity $H^{(4)}$ can be written (see appendix B) as[5]

$$H^{(4)} \equiv -\frac{1}{8}(Z_1 - Z_2)(Z_3 - Z_4)\frac{Z_1 Z_2}{(1 - Z_1 Z_2)^2} \cdot \left(P^{(x)}_{12\text{–}34} - (1 \leftrightarrow 3) - (2 \leftrightarrow 3)\right) \tag{14}$$

where

$$P^{(x)}_{12\text{–}34} = (\tilde{x}_1 - \tilde{x}_2)(\tilde{x}_3 - \tilde{x}_4) \cdot \left(\frac{\tilde{x}_1\tilde{x}_2}{1 - \tilde{x}_1\tilde{x}_2}\frac{\tilde{x}_3\tilde{x}_4}{1 - \tilde{x}_3\tilde{x}_4} + \frac{(\tilde{x}_1\tilde{x}_2)^2}{1 - \tilde{x}_1\tilde{x}_2} + \frac{(\tilde{x}_3\tilde{x}_4)^2}{1 - \tilde{x}_3\tilde{x}_4}\right). \tag{15}$$

Taking expression (14) for $H^{(4)}$, expanding the integrand in the x_j variables, one notes that the integrand depends only on combinations of the form

$$\left(\prod_{i=1}^{4} y_i \cdot x_i^{n_i}\right) \cdot (Z_1 - Z_2)(Z_3 - Z_4)\frac{Z_1 Z_2}{(1 - Z_1 Z_2)^2} \tag{16}$$

which have simple integration rules (see appendix C) and, thus, the problem of angular integrations is settled.

Finally, the expansion method described above, together with the form (14) of $H^{(4)}$, allow us to obtain $\tilde{\chi}^{(4)}(w)$ as a *fully integrated expansion* in the form (see appendix D)

$$\begin{aligned}\tilde{\chi}^{(4)} = 16w^{16} \cdot \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} w^{8m+4k+4n+2j} \cdot (2m+1)(2m+2k+1) \\ \times (1 + \theta(j-1)) \cdot (1 + \theta(k-1)) \\ \times \frac{1}{2}(V(m, m+k, n, n+j) + V(m, m+k, n+j, n))\end{aligned} \tag{17}$$

where $\theta(x)$ is the step function defined as

$$\theta(x) = \begin{cases} 1, & x \geqslant 0 \\ 0, & x < 0 \end{cases} \tag{18}$$

and

$$\begin{aligned}V(m, k, n, j) = {} & d(m, m+1; j+k+2) \cdot d(n+m+2, n+m+3; k) \\ & + d(m, m+1; k) \cdot d(n+m+2, n+m+3; j+k+2) \\ & + d(m, n+m+3; j+k+2) \cdot d(m+1, n+m+2; k) \\ & + d(m, n+m+3; k) \cdot d(m+1, n+m+2; j+k+2) \\ & - d(m, n+m+2; j+k+2) \cdot d(m+1, n+m+3; k) \\ & - d(m, n+m+2; k) \cdot d(m+1, n+m+3; j+k+2)\end{aligned} \tag{19}$$

with

$$d(n_1, n_2; k) = a(n_1, k+1)a(n_2, k) - a(n_1, k)a(n_2, k+1). \tag{20}$$

2.3. Series generation

Note that the summand in equation (17) depends only on combinations of $C_d(n_1, n_2, k, j)$ which is the coefficient of w^{2j} in the expansion of $d(n_1, n_2; k)$. It is given by

$$\begin{aligned}C_d(n_1, n_2, k, j) = \sum_{i=0}^{j}(C_a(n_1, k+1, i)C_a(n_2, k, j-i) \\ - C_a(n_1, k, i)C_a(n_2, k+1, j-i))\end{aligned} \tag{21}$$

[5] Throughout this paper the notation $\equiv$ stands for equality for integration purposes.

where $C_a(n,k,i)$ is the coefficient of w^{2i} in the expansion of $a(n,k)$ which reads

$$C_a(n,k,i) = \binom{n+k+2i}{i}\binom{n+k+2i}{i+k}. \tag{22}$$

From our integrated form of $\tilde{\chi}^{(4)}$, the generation of series coefficients becomes straightforward. *Recall that* (17) *is already integrated*, and, thus, the computing time to obtain the series coefficients comes from the evaluation of the sums. For the forms used, this time is of order N^7. Improvements can be made by optimal data storage in order to avoid repeated summation evaluations. Actually, we have found it more efficient to store the coefficients C_d.

We have been able to generate, from formal calculations, a long series of coefficients from expression (17) up to order[6] 432:

$$\frac{\tilde{\chi}^{(4)}(w)}{16w^{16}} = 1 + 64w^2 + 2470w^4 + 74724w^6 + \cdots + O(w^{434}) \tag{23}$$

Note that our formal calculation program has been rewritten by J Dethridge[7] into an optimized C++ program that can give the series in few hours using very little memory.

3. The Fuchsian differential equation satisfied by $\tilde{\chi}^{(4)}$

It is clear from expression (17) that $\tilde{\chi}^{(4)}$ is even[8] in w. We thus introduce, in the following, the variable $x = 16w^2$. With our long series, and with a dedicated program, we have succeeded in obtaining the differential equation for $\tilde{\chi}^{(4)}$ that is given by (with $x = 16w^2$)

$$\sum_{n=1}^{10} a_n(x) \cdot \frac{\mathrm{d}^n}{\mathrm{d}x^n} F(x) = 0 \tag{24}$$

with

$$\begin{aligned}
&a_{10} = -512x^6(x-4)(1-x)^6 P_{10}(x), \\
&a_9 = 256(1-x)^5 x^5 P_9(x), && a_8 = -384(1-x)^4 x^4 P_8(x), \\
&a_7 = 192(1-x)^3 x^3 P_7(x), && a_6 = -96(1-x)^2 x^2 P_6(x), \\
&a_5 = 144(1-x)x P_5(x), && a_4 = -72P_4(x), \qquad a_3 = -108P_3(x) \\
&a_2 = -54P_2(x) && a_1 = -27P_1(x)
\end{aligned} \tag{25}$$

where $P_{10}(x), P_9(x) \ldots, P_1(x)$ are polynomials of degree respectively 17, 19, 20, 21, 22, 23, 24, 23, 22 and 21 given[9] in appendix E.

With (25), the differential equation needs 242 unknowns to be found (counting the polynomial in front of the derivative of order 0, which is identically null). Our series expansion for $\tilde{\chi}^{(4)}/16w^{16}$, having only 217 terms in the variable $x = 16w^2$, is, thus, not long enough to let the differential equation be found. This calls for some comments.

The differential equation (24) is of *minimal order*. It is obvious that one can obtain other differential equations of *greater order*. As explained in [18] (see section 4), before the differential equation built from a series expansion pops out, the singularities computed as

[6] More precisely we obtained this series up to order 368 with a 2 Giga-memory computer, up to order 390 with a 3 Giga-memory computer, and up to order 432 with a 8 Giga-memory computer of the stix laboratory at the Ecole Polytechnique (medicis platform for formal calculations).

[7] Private communication.

[8] This is also the case for any $\tilde{\chi}^{(2n)}$ (see for instance [7]).

[9] We thank one of the referees, as well as A J Guttmann and I Jensen, for detecting the possible existence of a misprint respectively in the submitted version of the paper and in the file released in the electronic arXiv (a minus sign misprint in the last coefficient of polynomial P_3).

Table 1. Critical exponents for each regular singular point. P is the maximum power of the logarithmic terms for each singularity. x_P is any of the 17 roots of $P_{10}(x)$.

x-singularity	s-singularity	Critical exponents in x	P
0	$0, \infty$	$8, 3, 3, 2, 2, 1, 1, 0, 0, -1/2$	1
1	± 1	$3, 2, 1, 1, 0, 0, 0, 0, -1, -3/2$	3
4	$\pm\frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}$	$8, 7, 13/2, 6, 5, 4, 3, 2, 1, 0$	0
∞	$\pm\mathrm{i}$	$5/2, 3/2, 3/2, 1/2, 1/2, 1/2, 1/2, 0, -1/2, -1/2$	3
x_P, 17 roots	s_P, 68 roots	$10, 8, 7, 6, 5, 4, 3, 2, 1, 0$	0

roots of the polynomial in front of the highest derivative reach stabilized numerical values as the degrees of the polynomials and/or the order gets higher. Even the minimal multiplicity of the singularity can be seen. This leads us to take the following form for the $\tilde{\chi}^{(4)}$ differential equation:

$$\sum_{i=0}^{q} R_i(x) \cdot x^i (1-x)^i \cdot \frac{\mathrm{d}^i}{\mathrm{d}x^i} F(x) = 0 \qquad (26)$$

where only the physical singularities are explicitly included. Note that the other (non-apparent) singularity can also be included. If the differential equation can be identified with the number of series coefficients at hand, the unnecessary terms will factor out. A compromise has to be found with respect to how long the series is. Now, there are many ways to choose the degrees of the polynomials $R_i(x)$. We have taken

$$\deg(R_i(x)) = \mu + q - i, \qquad i = 0, 1, \ldots, q \qquad (27)$$

in order to have the point at infinity as a *regular singular point*[10].

If the differential equation of order q and degree μ exists, the series of $\tilde{\chi}^{(4)}/16w^{16}$ should have at least N terms, with:

$$N = \mu \cdot (q+1) + \tfrac{1}{2} q \cdot (q+3). \qquad (28)$$

Staying below the above hyperbola, we have obtained, at $q = 11$ and $\mu = 9$ which requires 185 terms in the series, two linearly independent differential equations that satisfy the remaining 30 terms of the series. The combination of these two differential equations gives (24) which has been checked to be of minimal order. Note the fact that besides the Fuchsian differential equation of minimal order with an apparent polynomial, there are *other differential equations of higher order that require less terms in the series to be identified*. We plan to report on this feature elsewhere.

The singularities of this differential equation $x = 0, 1, 4$ and $x = \infty$ are all regular singular points. The roots of the polynomial $P_{10}(x)$ are *apparent singularities*. One notes, in contrast, to $\chi^{(3)}$ that no new singularity is found besides the known physical and non physical singularities (i.e., Nickel's [6, 7]). The critical exponents of all these singular points are given in table 1 with the maximum power of logarithmic terms in the solutions. As was the case for $\chi^{(3)}$, these logarithmic terms appear due to the multiple roots of the indicial equation. A noteworthy remark is the occurrence of logarithmic terms up to the power 3 for $\chi^{(4)}$ to be compared with the power 2 for $\chi^{(3)}$ at the singular points $w = \pm 1/4$ and $w = \infty$.

It is worth recalling the Fuchsian-relation on Fuchsian type equations. Denoting by $x_1, x_2, \ldots, x_m, x_{m+1} = \infty$, the regular singular points of a Fuchsian-type equation of order q

[10] This feature can easily be seen, making the change of variable $t = 1/x$ and looking for the necessary condition for $t = 0$ to be a regular singular point from which (27) is deduced.

and $\rho_{j,1}, \ldots, \rho_{j,q}$ $(j = 1, \ldots, m+1)$ the q roots of the *indicial equation* [28, 29] corresponding to each regular singular point w_j, the following Fuchsian relation [28, 29] holds:

$$\sum_{j=1}^{m+1} \sum_{k=1}^{q} \rho_{j,k} = \frac{(m-1)q(q-1)}{2}. \qquad (29)$$

The number of regular singular points here is $m + 1 = 21$ corresponding respectively to the 17 roots of P_{10}, the $x = 0, 1, 4$, regular singular points and the point at infinity $x = \infty$. The Fuchsian relation (29) is actually satisfied here with $q = 10$, $m = 20$.

Considering the Fuchsian relation (29), but now in the variable s, one first remarks that the $\rho_{j,k}$ exponents are *the same* as those in the x variable for the 68 roots of the apparent polynomial, as well as for the four $x = 4$ singularities (namely $s = \pm\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$), but *are multiplied by a factor two* for all the other ones. This is obvious from the definition of $x = 4s^2/(1+s^2)^2$ at $x = 0$ and $x = \infty$, and this can be seen for $x = 1$ from

$$1 - x = \left(\frac{1-s^2}{1+s^2}\right)^2. \qquad (30)$$

The Fuchsian relation is actually satisfied in the s variable, but now, with $q = 10$, $m = 77$.

The singular behaviours of the solutions of the differential equation can be read easily from table 1. Near $x = 4$, they are $t^{13/2}$, where $t = 4 - x$. Near $x = \infty$, they are $t^{-1/2}\log^k(t)$, (with $k = 0, 1$), $t^{1/2}\log^k(t)$ (with $k = 0, 1, 2, 3$). $t^{3/2}\log^k(t)$ (with $k = 0, 1$) and $t^{5/2}$, where $t = 1/x$. Near $x = 1$, they are $t^{-3/2}$, t^{-1}, $\log^k(t)$ (with $k = 1, 2, 3$) and $t\log(t)$, where $t = 1 - x$. Let us note that these behaviours are for the *general* solution of the differential equation (24).

As far as the *physical* solution is concerned, the dominant singular behaviours at $x = 4$ (namely $t^{13/2}$) and $x = \infty$ (namely $t^{-1/2}\log(t)$) are present in the physical solution $\tilde{\chi}^{(4)}$ confirming [7, 8]. At the ferromagnetic point ($x = 1$ which also corresponds to the anti-ferromagnetic point for $\chi^{(4)}$), with the dominant behaviour $t^{-3/2}$, the growth of the coefficients would be $(3/2)_N/N! \sim \sqrt{N}$. Since this is not the case, the coefficients of the series, for large values of N, behaving like $C_N \sim 0.2544 \times 10^{-4}$, this $t^{-3/2}$ singular behaviour will not contribute to the physical solution. Only the *subdominant singular behaviours will be present*. We will return to these points in a forthcoming publication that will give the amplitudes of the dominant and subdominant singular behaviours contributing in the physical solution $\tilde{\chi}^{(4)}$ near each singular point of the differential equation.

4. Properties of the Fuchsian differential equation (24)

From table 1, and from the formal solutions around the singular points, it is easy to find the following simple solutions of (24):

$$\mathcal{S}_0(x) = \text{constant} \qquad (31)$$

$$\mathcal{S}_1(x) = \frac{8 - 12x + 3x^2}{8(1-x)^{3/2}}, \qquad (32)$$

$$\mathcal{S}_2(x) = \frac{2 - 6x + x^2}{2(1-x)\sqrt{x}}. \qquad (33)$$

These solutions correspond to solutions of some differential operators of order one. Let us call these order one differential operators respectively L_0, L_1 and L_2.

A remarkable finding *is the following solution of the Fuchsian differential equation* (24):

$$\mathcal{S}_3(x) = \tfrac{1}{64}x^2 \cdot {}_2F_1\left(\tfrac{5}{2}, \tfrac{3}{2}; 3; x\right) \qquad (34)$$

which is nothing but the two-particle contribution to the magnetic susceptibility, i.e., $\tilde{\chi}^{(2)}$ associated with an operator of order two, N_0 ($N_0(\mathcal{S}_3) = 0$). We will come back to this point later.

The second solution of the order-two operator N_0 is given in terms of the MeijerG function [19]:

$$\tilde{\mathcal{S}}_3(x) = \frac{\pi}{2}\,\text{MeijerG}\,([[],[1/2,3/2]],[[2,0],[]],x) \tag{35}$$

which can also be written as

$$\tilde{\mathcal{S}}_3(x) = \mathcal{S}_3(x)\log(x) + B(x) \tag{36}$$

with

$$B(x) = \frac{1}{12\pi}\cdot\sum_{k=0}^{\infty} x^k \frac{\mathrm{d}}{\mathrm{d}k}\Big(\frac{\Gamma(k-1/2)\Gamma(k+1/2)}{\Gamma(k-1)\Gamma(k+1)}\Big). \tag{37}$$

With these five solutions corresponding to three differential operators of order one, and one differential operator of order two, it is easy to construct 24 factorizations of $\mathcal{L}_{10}$, the differential operator corresponding to the Fuchsian differential equation (24), which can be written[11] as

$$\mathcal{L}_{10} = O_5 \cdot G(N). \tag{38}$$

$G(N)$ is a shorthand notation of a differential operator of order 5, factorizable in one operator of order two and three operators of order 1. $G(N)$ has 24 different factorizations involving eight differential operators of order two and 24 operators of order one. The differential operator O_5 factorizes as $M_1 \cdot L_{24}$, i.e., one operator of order four and one operator of order one. Let us give two examples of the 24 factorizations[12]:

$$\mathcal{L}_{10} = M_1 \cdot L_{24} \cdot N_4 \cdot L_{12} \cdot L_3 \cdot L_0 \tag{39}$$

$$\mathcal{L}_{10} = M_1 \cdot L_{24} \cdot L_{13} \cdot N_6 \cdot L_3 \cdot L_0. \tag{40}$$

This large number of factorizations [22, 31, 21] induces the occurrence of intertwiners[13]. In the examples above, one has $N_4 \cdot L_{12} = L_{13} \cdot N_6$. Seeking a similar relation for $L_{24} \cdot N_4$, one finds $N_9 \cdot L_{25}$. This last factorization introduces six factorizations that we denote as

$$\mathcal{L}_{10} = M_1 \cdot N_9 \cdot G(L). \tag{41}$$

$G(L)$ is a notation for an operator of order four, factorizable as four operators of order one.

One factorization of $G(L)$ reads

$$G(L) = L_{25} \cdot L_{12} \cdot L_3 \cdot L_0. \tag{42}$$

This differential operator $G(L)$ that factorizes $\mathcal{L}_{10}$ at right, obviously has $\mathcal{S}_0, \mathcal{S}_1$ and $\mathcal{S}_2$ as solutions. The fourth solution (of the order four differential operator $G(L)$) can be obtained by order reduction. It reads

$$\mathcal{S}_4(x) = \frac{4(x-2)\sqrt{4-x}}{x-1} + 16\log\frac{x}{(2+\sqrt{4-x})^2} + 16\mathcal{S}_1(x)\cdot\log g(x) - 16\sqrt{x}\mathcal{S}_2(x)\cdot{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{x}{4}\right) \tag{43}$$

[11] All the operators are such that the coefficient in front of the highest derivative is +1.

[12] We denote by L the operators of order 1, by N, the operators of order 2 and by M the operators of order 4 (see appendix F).

[13] We thank Jacques-Arthur Weil for useful comments on the equivalence of linear differential operators [20] (see equation (5) in [21]).

with

$$g(x) = \frac{1}{x}((8 - 9x + 2x^2) + 2(2 - x) \cdot \sqrt{(1 - x)(4 - x)}). \tag{44}$$

To get more factorizations, we need to obtain simple solutions of $\mathcal{L}_{10}^*$, *the adjoint of the differential operator* $\mathcal{L}_{10}$. There is no solution of $\mathcal{L}_{10}^*$ corresponding to an order-one operator, however, *we have been able to find a solution corresponding to an operator of order two* (denoted N_8^*). This solution[14] of N_8^* is a combination of elliptic integrals with polynomials of quite large degrees and reads

$$\mathcal{S}_1^*(x) = \frac{x(1-x)^6(4-x)}{3840\,000 \cdot P_{10}(x)} \cdot (q_1(x)K(x) + q_2(x)E(x)) \tag{45}$$

where

$$K(x) = {}_2F_1\big(\tfrac{1}{2}, \tfrac{1}{2}; 1; x\big), \qquad E(x) = {}_2F_1\big(-\tfrac{1}{2}, \tfrac{1}{2}; 1; x\big) \tag{46}$$

and

$$\begin{aligned} q_1(x) = (1-x)(&47\,352\,014\,438\,400 - 246\,257\,318\,625\,280x + 275\,880\,211\,382\,272x^2 \\ &+ 68\,328\,139\,784\,192x^3 + 645\,943\,284\,072\,448x^4 - 2774\,821\,715\,853\,312x^5 \\ &+ 3217\,221\,650\,489\,344x^6 - 683\,914\,539\,437\,568x^7 \\ &- 2042\,467\,767\,948\,624x^8 + 3083\,863\,919\,521\,506x^9 \\ &- 2746\,206\,480\,894\,969x^{10} + 1558\,224\,994\,851\,490x^{11} \\ &- 347\,705\,392\,468\,761x^{12} - 145\,625\,559\,012\,638x^{13} \\ &+ 117\,842\,186\,745\,065x^{14} - 30\,744\,722\,745\,590x^{15} \\ &+ 3089\,482\,306\,025x^{16} + 18\,651\,488\,480x^{17} \\ &- 21\,574\,317\,760x^{18} + 821\,760\,000x^{19}) \end{aligned} \tag{47}$$

$$\begin{aligned} q_2(x) = &-47\,352\,014\,438\,400 + 269\,933\,325\,844\,480x - 390\,130\,367\,987\,712x^2 \\ &+ 27\,877\,970\,018\,304x^3 - 580\,571\,855\,978\,496x^4 + 3135\,069\,528\,473\,600x^5 \\ &- 4488\,375\,407\,386\,624x^6 + 1736\,922\,901\,371\,392x^7 \\ &+ 2600\,277\,912\,748\,368x^8 - 5123\,144\,341\,863\,018x^9 \\ &+ 5224\,617\,790\,090\,830x^{10} - 3547\,418\,998\,359\,865x^{11} \\ &+ 1453\,586\,336\,314\,895x^{12} - 273\,126\,255\,420\,088x^{13} \\ &- 8194\,519\,962\,996x^{14} + 11\,308\,926\,014\,655x^{15} \\ &- 1235\,672\,485\,785x^{16} - 60\,982\,101\,700x^{17} \\ &+ 15\,765\,744\,320x^{18} - 607\,865\,600x^{19} - 3840\,000x^{20}. \end{aligned} \tag{48}$$

This order 2 operator N_8 completes the factorization scheme of $\mathcal{L}_{10}$, the differential operator of order ten. Let us recap the factorizations

$$\mathcal{L}_{10} = N_8 \cdot M_2 \cdot G(L) \qquad \mathcal{L}_{10} = M_1 \cdot N_9 \cdot G(L) \qquad \mathcal{L}_{10} = M_1 \cdot L_{24} \cdot G(N). \tag{49}$$

$G(L)$ is a differential operator of order four, factorizable in order one operators. It has six different factorizations involving 13 differential operators of order one. $G(N)$ is an operator of order five, factorizable in one operator of order two and three operators of order one. $G(N)$ has 24 different factorizations involving eight differential operators of order two, and 24 order one operators with 12 appearing in $G(L)$.

[14] We did not look for the second solution of N_8^*, our purpose being the factorization of $\mathcal{L}_{10}$.

All these 36 factorizations (49) are given in appendix F. Appendix F shows that these 36 factorizations can be considered as *only one* factorization up to a set of equivalence symmetries [20–26].

From the ten solutions of the differential equation (24), six solutions are given explicitly, $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_4$ (this last being a solution of an order four operator) and two solutions $(\mathcal{S}_3, \tilde{\mathcal{S}_3})$ corresponding to the differential equation of $\tilde{\chi}^{(2)}$. The remaining four solutions are those of the operator of order eight, $M_2 \cdot G(L)$.

From the 36 factorizations shown in appendix F, those six of the form $N_8 \cdot M_2 \cdot G(L)$ are of the most importance. Their occurrence allows us to get the contribution α of $\tilde{\chi}^{(2)}$ in the physical solution $\tilde{\chi}^{(4)}$ from $M_2 \cdot G(L)(\tilde{\chi}^{(4)} - \alpha\tilde{\chi}^{(2)}) = 0$. This contribution is obtained easily and gives

$$\tilde{\chi}^{(4)} = \tfrac{1}{3}\tilde{\chi}^{(2)} + \Phi_4 \qquad (50)$$

where Φ_4 is solution of the order eight differential operator $M_2 \cdot G(L)$.

Recall that the same situation occurred for the differential equation of order seven for $\tilde{\chi}^{(3)}$. One can see that the rational solution $w/(1-4w)$ occurring[15] in the differential equation of order seven for $\tilde{\chi}^{(3)}$ is nothing but $\tilde{\chi}^{(1)}$. With this remark we can rewrite a decomposition of $\tilde{\chi}^{(3)}$ we gave in [17, 18] as follows:

$$\tilde{\chi}^{(3)} = \tfrac{1}{6}\tilde{\chi}^{(1)} + \Phi_3 \qquad (51)$$

where Φ_3 is solution of the order six differential operator (noted L_6 in [17, 18]).

5. Comments and speculations

Denoting $\mathbf{L}_4$ the order ten differential operator associated with the ordinary differential equation satisfied by $\tilde{\chi}^{(4)}$, and more generally, $\mathbf{L}_n$ the differential operators associated with the ordinary differential equation satisfied by $\tilde{\chi}^{(n)}$, one has

$$\mathbf{L}_4(\tilde{\chi}^{(4)}) = \mathbf{L}_4(\tilde{\chi}^{(2)}) = 0. \qquad (52)$$

Furthermore, one can see that the rational solution $w/(1-4w)$, occurring in the differential equation of order seven for $\tilde{\chi}^{(3)}$, is nothing but $\tilde{\chi}^{(1)}$ and thus one has

$$\mathbf{L}_3(\tilde{\chi}^{(3)}) = \mathbf{L}_3(\tilde{\chi}^{(1)}) = 0. \qquad (53)$$

Both relations come from the factorizations of the differential operators corresponding to $\tilde{\chi}^{(3)}$ and $\tilde{\chi}^{(4)}$.

At this point, it is tempting to make some conjectures generalizing relations (52), (53) and relations (50), (51).

From (52) and (53) the conjecture is

$$\mathbf{L}_{2n+1}(\tilde{\chi}^{(2n+1)}) = \mathbf{L}_{2n+1}(\tilde{\chi}^{(1)}) = 0, \qquad \mathbf{L}_{2n}(\tilde{\chi}^{(2n)}) = \mathbf{L}_{2n}(\tilde{\chi}^{(2)}) = 0 \qquad (54)$$

meaning that the differential operator of $\chi^{(1)}$ (respectively $\chi^{(2)}$) right divides the differential operator corresponding to $\chi^{(2n+1)}$ (respectively $\chi^{(2n)}$).

One stronger conjecture is to expect the same situation as in (50), (51) occurring in the higher particle contributions, i.e.,

$$\tilde{\chi}^{(2n)} = \alpha_{2n}\tilde{\chi}^{(2)} + \Phi_{2n}, \qquad \tilde{\chi}^{(2n+1)} = \alpha_{2n+1}\tilde{\chi}^{(1)} + \Phi_{2n+1} \qquad (55)$$

where Φ_{2n} (respectively Φ_{2n+1}) is the solution of a differential operator that right divides the differential operator $\mathbf{L}_{2n}$ (respectively $\mathbf{L}_{2n+1}$) and is not divisible by the differential operator

[15] This corresponds to the solution S_1 given in [17, 18] which is $S_1 = \tilde{\chi}^{(1)}/2$.

$\mathbf{L}_2$ (respectively $\mathbf{L}_1$). In this situation, it is easy to obtain the numbers α_n which give the contribution of $\tilde{\chi}^{(1)}$ and $\tilde{\chi}^{(2)}$ in the higher $\tilde{\chi}^{(n)}$ as explained for $\tilde{\chi}^{(4)}$ (see text before (50)).

Let us note that both conjectures are free from any constraint due to the singularities that occur in the differential equations, since any $\tilde{\chi}^{(2n+1)}$ (respectively $\tilde{\chi}^{(2n)}$) has the singularities occurring in $\tilde{\chi}^{(1)}$ (respectively $\tilde{\chi}^{(2)}$).

A much stronger conjecture is to expect any $\chi^{(m)}$ to be 'embedded' in any $\chi^{(n)}$ where m divides n with same odd-even parity.

$$\mathbf{L}_n(\tilde{\chi}^{(n)}) = \mathbf{L}_n(\tilde{\chi}^{(m)}) = 0, \qquad \mathbf{L}_n = \mathbf{L}_n^{(m)} \cdot \mathbf{L}_m. \tag{56}$$

This means that $\mathbf{L}_n$ might be built from the *least common left multiple* (lclm) of the differential operators associated with the $\mathbf{L}_m$ where m divides the integer n respecting even–odd parity. The fact that n should be multiple of m is due to the non-physical singularities appearing in the differential equations. For instance, no 'embedding' like $\chi^{(3)}$ being solution of the differential equations of $\chi^{(5)}$ has to be expected. Whether $\chi^{(3)}$ is a solution of the differential equation satisfied by, e.g., $\chi^{(9)}$ is not ruled out, the (non-apparent) singularities of the first being (non-apparent) singularities of the last. But, then, the new singularities discovered [17] for $\chi^{(3)}$ have to occur in the differential equation of $\chi^{(9)}$.

Similarly one can expect, with this last conjecture, the fol lowing relations for $\mathbf{L}_{12}$ and $\mathbf{L}_{15}$ (associated with $\tilde{\chi}^{(12)}$ and $\tilde{\chi}^{(15)}$):

$$\begin{aligned} &\mathbf{L}_{12}(\tilde{\chi}^{(12)}) = \mathbf{L}_{12}(\tilde{\chi}^{(6)}) = \mathbf{L}_{12}(\tilde{\chi}^{(4)}) = \mathbf{L}_{12}(\tilde{\chi}^{(2)}) = 0 \\ &\mathbf{L}_{15}(\tilde{\chi}^{(15)}) = \mathbf{L}_{15}(\tilde{\chi}^{(5)}) = \mathbf{L}_{15}(\tilde{\chi}^{(3)}) = \mathbf{L}_{15}(\tilde{\chi}^{(1)}) = 0 \end{aligned} \tag{57}$$

and thus

$$\begin{aligned} &\mathbf{L}_{12} = \mathbf{L}_{12}^{(6)} \cdot \mathbf{L}_6 = \mathbf{L}_{12}^{(4)} \cdot \mathbf{L}_4 = \mathbf{L}_{12}^{(2)} \cdot \mathbf{L}_2 \\ &\mathbf{L}_{15} = \mathbf{L}_{15}^{(5)} \cdot \mathbf{L}_5 = \mathbf{L}_{15}^{(3)} \cdot \mathbf{L}_3 = \mathbf{L}_{15}^{(1)} \cdot \mathbf{L}_1. \end{aligned} \tag{58}$$

For instance, finding the differential operator satisfied by $\tilde{\chi}^{(15)}$ would enable us to see if the differential operator $\mathbf{L}_{15}$ actually has some relation with the least common left multiple of $\mathbf{L}_1$, $\mathbf{L}_3$ and $\mathbf{L}_5$.

6. Conclusion

Considering the isotropic Ising square lattice model susceptibility, we extended our 'expansion method' (that allowed us to find the differential equation satisfied by $\chi^{(3)}$), to the four-particle contribution to the susceptibility, namely $\chi^{(4)}$. We first obtained $\chi^{(4)}$ as a *fully integrated multisum on products of four hypergeometric functions*, and obtained a long series for $\chi^{(4)}$. From this long series, we gave the Fuchsian differential equation of order ten satisfied by $\chi^{(4)}$. This differential equation has a rich structure in terms of factorizations.

We have given in closed form six, of the ten solutions of the differential equation (24), namely, $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_4$, $\mathcal{S}_3$ and $\tilde{\mathcal{S}}_3$. The remaining four solutions are those of an operator of order eight.

One of these solutions is highly remarkable: *it is actually the two-particle contribution* $\chi^{(2)}$. *A similar situation also occurred for the differential equation of the three particle contribution* $\chi^{(3)}$, *which actually had* $\chi^{(1)}$ *as solution.*

In general, for all the $\tilde{\chi}^{(n)}$, it is tempting to expect $\tilde{\chi}^{(1)}$ to be a solution of the differential equation satisfied by the $\tilde{\chi}^{(2n+1)}$ and $\tilde{\chi}^{(2)}$ to be a solution of the differential equation satisfied by the $\tilde{\chi}^{(2n)}$.

Beyond, one can contemplate much stronger conjectures corresponding to further 'embedding', namely $\tilde{\chi}^{(m)}$ being solution of the differential equation of $\tilde{\chi}^{(n)}$, when n is a multiple of m, n and m having the same parity.

A confirmation of these various embeddings and conjectures is crucial because it corresponds to a *global structure* of the $\tilde{\chi}^{(n)}$ *and thus of* χ *itself*. We will investigate this feature in a future publication.

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Appendix A

In this appendix, we show that, with $H^{(4)}$ taken as in (4), i.e., not the alternative simplified form we use in this paper, our expansion method applied to $\tilde{\chi}^{(4)}$ produces *five* summations on products of *four* hypergeometric functions. Note that this method is applicable to any $\chi^{(n)}$.

Let us write the product $R^{(4)} \cdot H^{(4)}$ as

$$R^{(4)} \cdot H^{(4)} = T^{(4)} \cdot A^{(4)} \tag{A.1}$$

with

$$T^{(4)} = R^{(4)} \cdot \prod_{i<j} \frac{\tilde{x}_i \tilde{x}_j}{(1 - \tilde{x}_i \tilde{x}_j)^2} \tag{A.2}$$

$$A^{(4)} = \frac{1}{4!} \cdot \prod_{i<j} (Z_i - Z_j)^2. \tag{A.3}$$

By standard expansion of the quantity $T^{(4)}$, defined by (A.2), in the variables $\tilde{x}_i$, one obtains

$$\begin{aligned} T^{(4)} = & \sum_{p=0}^{\infty} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \sum_{i_3=0}^{\infty} \sum_{i_4=0}^{\infty} \sum_{i_5=0}^{\infty} \sum_{i_6=0}^{\infty} (1 + \theta(p-1)) \cdot \prod_{k=1}^{6} (i_k + 1) \\ & \times \tilde{x}_1^{p+i_1+i_2+i_3+3} \tilde{x}_2^{p+i_1+i_4+i_5+3} \tilde{x}_3^{p+i_2+i_4+i_6+3} \tilde{x}_4^{p+i_3+i_5+i_6+3} \end{aligned} \tag{A.4}$$

where $\theta(x)$ is the step function given in (18). Defining

$$\begin{aligned} n_1 &= p + i_1 + i_2 + i_3, \qquad & n_2 &= p + i_1 + i_4 + i_5 \\ n_3 &= p + i_2 + i_4 + i_6, \qquad & n_4 &= p + i_3 + i_5 + i_6, \end{aligned} \tag{A.5}$$

one can solve (A.5) in the indices (i_3, i_4, i_5, i_6), to obtain

$$i_3 = -(p + i_1 + i_2) + n_1 \tag{A.6}$$

$$i_4 = -(p + i_1 + i_2) + \tfrac{1}{2}(n_1 + n_2 + n_3 - n_4) \tag{A.7}$$

$$i_5 = i_2 - \tfrac{1}{2}(n_1 - n_2 + n_3 - n_4) \tag{A.8}$$

$$i_6 = i_1 - \tfrac{1}{2}(n_1 + n_2 - n_3 - n_4). \tag{A.9}$$

All these indices should be *integers*, inducing constraints on the n_i and limitations on the summations of the remaining indices (i.e., p, i_1, i_2). With this change of summation indices, $T^{(4)}$ becomes

$$T^{(4)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty} C(n_1, n_2, n_3, n_4) \cdot \tilde{x}_1^{n_1+3}\tilde{x}_2^{n_2+3}\tilde{x}_3^{n_3+3}\tilde{x}_4^{n_4+3} \tag{A.10}$$

and, since $T^{(4)}$ is completely symmetric in the variables $\tilde{x}_i$, the coefficient $C(n_1, n_2, n_3, n_4)$ is completely symmetric in the n_i indices and is given by summation on the remaining indices p, i_1 and i_2:

$$\begin{aligned} C(n_1, n_2, n_3, n_4) = {} & \sum_{p=0}^{\infty}\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} (1 + \theta(p-1)) \cdot \prod_{k=1}^{6}(i_k + 1) \\ & \times \theta(i_3) \cdot \theta(i_4) \cdot \theta(i_5) \cdot \theta(i_6) \cdot \sigma(n_1 + n_2 + n_3 + n_4) \\ & \times \theta\left(\frac{1}{2}(n_1 + n_2 + n_3 - n_4)\right) \end{aligned} \tag{A.11}$$

where theta functions (18) of indices (i_3, i_4, i_5, i_6) take place to keep track of the fact that these indices should be positive integers. The symbol $\sigma(n)$ defined as

$$\sigma(n) = \tfrac{1}{2}(1 + (-1)^n) \tag{A.12}$$

comes from the fact that i_3, i_4, i_5, i_6 are integers, due to the right most terms at right-hand side of (A.7)–(A.9), the n_i should verify $n_1 + n_2 + n_3 + n_4 =$ eveninteger. The argument in the last theta in (A.11) comes from (A.7).

Let us define the index q and its upper limit of summation (see (A.6), (A.7)):

$$q = p + i_1 + i_2 \tag{A.13}$$

$$q_0 = \min\left(n_1, \frac{n_1 + n_2 + n_3 - n_4}{2}\right). \tag{A.14}$$

Furthermore, the coefficient $C(n_1, n_2, n_3, n_4)$ being symmetric in all its arguments, it is sufficient to compute it in the case where

$$n_1 \leqslant n_2 \leqslant n_3 \leqslant n_4. \tag{A.15}$$

The coefficient $C(n_1, n_2, n_3, n_4)$ now becomes

$$\begin{aligned} C(n_1, n_2, n_3, n_4) = {} & \theta\left(\frac{n_1 + n_2 + n_3 - n_4}{2}\right) \cdot \sigma(n_1 + n_2 + n_3 + n_4) \\ & \times \sum_{q=0}^{q_0}\sum_{p=0}^{q}\left((1 + \theta(p-1)) \cdot \sum_{i_1=0}^{q-p}\left(\prod_{k=1}^{6}(i_k + 1)\right)\right) \end{aligned} \tag{A.16}$$

where the indices i_3, i_4, i_5, i_6 are given by (A.6)–(A.9) and $i_2 = q - p - i_1$. The three summations can be performed and one obtains

$$C(n_1, n_2, n_3, n_4) = \theta\left(\frac{n_1 + n_2 + n_3 - n_4}{2}\right) \cdot \sigma(n_1 + n_2 + n_3 + n_4) \times \binom{q_0 + 4}{4} Q(n_1, n_2, n_3, n_4, q_0) \tag{A.17}$$

with the polynomial $Q(n_1, n_2, n_3, n_4, q_0)$ given by

$$Q(n_1, n_2, n_3, n_4, q_0) = \frac{1}{8} Q_0(n_1, n_2, n_3, n_4) + \frac{q_0}{2520} Q_1(n_1, n_2, n_3, n_4) + \frac{q_0^2}{7560} Q_2(n_1, n_2, n_3, n_4) + \frac{q_0^3}{1890} Q_3(n_1, n_2, n_3, n_4) + \frac{q_0^4}{540}(92 - 81n_1 - 9n_2 - 9n_3 + 63n_4) + \frac{4q_0^5}{135} \tag{A.18}$$

with

$$Q_0(n_1, n_2, n_3, n_4) = (1 + n_1)(n_1 + n_2 - n_3 - n_4 - 2) \times (n_1 - n_2 + n_3 - n_4 - 2)(2 + n_1 + n_2 + n_3 - n_4) \tag{A.19}$$

$$\begin{aligned} Q_1(n_1, n_2, n_3, n_4) = {} & -2216 + 2472n_1 + 1194n_1^2 - 1281n_1^3 + 126n_1^4 \\ & - 48n_2 + 1392n_1n_2 - 987n_1^2n_2 + 126n_1^3n_2 + 774n_2^2 + 441n_1n_2^2 \\ & - 126n_1^2n_2^2 + 147n_2^3 - 126n_1n_2^3 - 48n_3 + 1392n_1n_3 - 987n_1^2n_3 \\ & + 126n_1^3n_3 - 1548n_2n_3 - 882n_1n_2n_3 + 252n_1^2n_2n_3 - 147n_2^2n_3 \\ & + 126n_1n_2^2n_3 + 774n_3^2 + 441n_1n_3^2 - 126n_1^2n_3^2 - 147n_2n_3^2 \\ & + 126n_1n_2n_3^2 + 147n_3^3 - 126n_1n_3^3 - 2568n_4 - 708n_1n_4 \\ & + 2709n_1^2n_4 - 378n_1^3n_4 - 288n_2n_4 + 1134n_1n_2n_4 - 252n_1^2n_2n_4 \\ & - 147n_2^2n_4 + 126n_1n_2^2n_4 - 288n_3n_4 + 1134n_1n_3n_4 - 147n_3^2n_4 \\ & + 294n_2n_3n_4 - 252n_1n_2n_3n_4 - 252n_1^2n_3n_4 + 126n_1n_2n_4^2 \\ & + 126n_1n_3^2n_4 - 486n_4^2 - 1575n_1n_4^2 + 378n_1^2n_4^2 - 147n_2n_4^2 \\ & - 147n_3n_4^2 + 126n_1n_3n_4^2 + 147n_4^3 - 126n_1n_4^3 \end{aligned} \tag{A.20}$$

$$\begin{aligned} Q_2(n_1, n_2, n_3, n_4) = {} & -2992 - 4446n_1 + 6534n_1^2 - 1449n_1^3 - 1926n_2 \\ & - 819n_1^2n_2 - 270n_2^2 + 945n_1n_2^2 + 315n_2^3 - 1926n_3 + 2808n_1n_3 \\ & - 819n_1^2n_3 + 540n_2n_3 - 1890n_1n_2n_3 - 315n_2^2n_3 + 2808n_1n_2 \\ & - 270n_3^2 + 945n_1n_3^2 - 315n_2n_3^2 + 315n_3^3 + 594n_4 - 315n_3n_4^2 \\ & - 8532n_1n_4 + 3213n_1^2n_4 - 1728n_2n_4 + 1134n_1n_2n_4 + 315n_4^3 \\ & - 315n_2^2n_4 - 1728n_3n_4 + 1134n_1n_3n_4 + 630n_2n_3n_4 \\ & - 315n_3^2n_4 + 1998n_4^2 - 2079n_1n_4^2 - 315n_2n_4^2 \end{aligned} \tag{A.21}$$

$$\begin{aligned} Q_3(n_1, n_2, n_3, n_4) = {} & 520 - 1215n_1 + 495n_1^2 - 207n_2 + 144n_1n_2 \\ & + 144n_1n_3 + 270n_2n_3 - 135n_3^2 + 801n_4 - 738n_1n_4 \\ & - 108n_3n_4 + 243n_4^2 - 135n_2^2 - 207n_3 - 108n_2n_4. \end{aligned} \tag{A.22}$$

From the definition of $\tilde{\chi}^{(4)}$ in (2), and from (A.1), (A.10), one gets

$$\tilde{\chi}^{(4)} = 16w^{16} \cdot \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty} w^{n_1+n_2+n_3+n_4} C(n_1, n_2, n_3, n_4) \cdot M(n_1, n_2, n_3, n_4). \qquad \text{(A.23)}$$

The expansion method on the variables x_i introduces some summations free of any angular dependence. These summations appear in the coefficient $C(n_1, n_2, n_3, n_4)$ which is a kind of 'geometrical factor' that appears for all the $\chi^{(n)}$ with $n \geqslant 4$.

At this step the integrations have not yet been performed. They are contained in

$$M(n_1, n_2, n_3, n_4) = \int_0^{2\pi} \frac{\mathrm{d}\phi_1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\phi_2}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\phi_3}{2\pi} \left(\prod_{i=1}^{4} y_i \cdot x_i^{n_i+3} \right) \cdot A^{(4)}. \qquad \text{(A.24)}$$

Due to its structure, it is obvious that $M(n_1, n_2, n_3, n_4)$ is completely symmetric in the indices n_i. We write $M(n_1, n_2, n_3, n_4)$ as an integral over the four angles $(\phi_1, \phi_2, \phi_3, \phi_4)$, by introducing Dirac delta function $\delta(\phi_1 + \phi_2 + \phi_3 + \phi_4)$ as a Fourier expansion that reads

$$2\pi\,\delta(\phi_1 + \phi_2 + \phi_3 + \phi_4) = \sum_{k=-\infty}^{\infty} (Z_1 Z_2 Z_3 Z_4)^k. \qquad \text{(A.25)}$$

$M(n_1, n_2, n_3, n_4)$ thus becomes

$$M(n_1, n_2, n_3, n_4) = \sum_{k=-\infty}^{\infty} M(n_1, n_2, n_3, n_4; k) \qquad \text{(A.26)}$$

where

$$M(n_1, n_2, n_3, n_4; k) = \int_0^{2\pi} \frac{\mathrm{d}\phi_1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\phi_2}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\phi_3}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\phi_4}{2\pi}$$
$$\times \left(\prod_{i=1}^{4} y_i \cdot x_i^{n_i+3} \right) A^{(4)} \cdot (Z_1 Z_2 Z_3 Z_4)^k. \qquad \text{(A.27)}$$

The Fourier expansion (9) implies the following integration rule:

$$\int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} (y x^m) \cdot Z^j = A(m, j). \qquad \text{(A.28)}$$

The calculation of (A.27) thus becomes straightforward and *does not induce any summation*. The quantity $M(n_1, n_2, n_3, n_4; k)$ comes out as a huge expression with 201 terms, each of them being a product of four hypergeometric functions. Due to definition (A.26) where k runs from $-\infty$ to ∞, and to the symmetry of (A.26) under the permutation of the n_i, these 201 terms reduce to only 16 terms and $M(n_1, n_2, n_3, n_4; k)$ reads

$$\begin{aligned}
M(n_1, n_2, n_3, n_4; k) \equiv{}& A(n_1, k)A(n_2, k)A(n_3, k)A(n_4, k) \\
&- A(n_1, k)A(n_2, k)A(n_3, k)A(n_4, k+4)A(n_1, k)A(n_2, k)A(n_3, k+2) \\
&\times A(n_4, k+2) + 2A(n_1, k)A(n_2, k)A(n_3, k+1)A(n_4, k+3) \\
&- 3A(n_1, k)A(n_2, k+1)A(n_3, k+1)A(n_4, k+2) \\
&- A(n_1, k)A(n_2, k)A(n_3, k+3)A(n_4, k+5) \\
&+ A(n_1, k)A(n_2, k)A(n_3, k+4)A(n_4, k+4) \\
&- 2A(n_1, k)A(n_2, k+1)A(n_3, k+3)A(n_4, k+4) \\
&+ 2A(n_1, k)A(n_2, k+1)A(n_3, k+2)A(n_4, k+5) \\
&+ 2A(n_1, k)A(n_2, k+2)A(n_3, k+3)A(n_4, k+3)
\end{aligned}$$

$$- A(n_1,k)A(n_2,k+2)A(n_3,k+2)A(n_4,k+4)$$
$$+ A(n_1,k)A(n_2,k+2)A(n_3,k+4)A(n_4,k+6)$$
$$- A(n_1,k)A(n_2,k+2)A(n_3,k+5)A(n_4,k+5)$$
$$- A(n_1,k)A(n_2,k+3)A(n_3,k+3)A(n_4,k+6)$$
$$+ 2A(n_1,k)A(n_2,k+3)A(n_3,k+4)A(n_4,k+5)$$
$$- A(n_1,k)A(n_2,k+4)A(n_3,k+4)A(n_4,k+4) \qquad \text{(A.29)}$$

which is equal to (A.27) for summation purposes.

Finally, collecting (A.23), (A.26), $\tilde{\chi}^{(4)}$ can be written as

$$\tilde{\chi}^{(4)} = 16w^{16} \cdot \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty} w^{n_1+n_2+n_3+n_4} C(n_1,n_2,n_3,n_4) \cdot M(n_1,n_2,n_3,n_4) \qquad \text{(A.30)}$$

Appendix B

In this appendix, we give the explicit derivation of expression (14) given in section 2.

Considering $H^{(4)}$ as in (13), we define the following relations:

$$\prod_{i<j} \frac{\tilde{x}_i - \tilde{x}_j}{1 - \tilde{x}_i \tilde{x}_j} = P^{(x)}_{12\text{–}34} - (1 \leftrightarrow 3) - (2 \leftrightarrow 3) \qquad \text{(B.1)}$$

$$\prod_{i<j} \frac{Z_i - Z_j}{1 - Z_i Z_j} = P^{(Z)}_{12\text{–}34} - (1 \leftrightarrow 3) - (2 \leftrightarrow 3) \qquad \text{(B.2)}$$

where $P^{(x)}_{12\text{–}34}$ is given in (15). Using the constraint (6), $P^{(Z)}_{12\text{–}34}$ reads

$$P^{(Z)}_{12\text{–}34} = \tilde{A}^{(4)} + (Z_2 - Z_1)(Z_3 - Z_4)(1 + Z_1 Z_2 + Z_3 Z_4) \qquad \text{(B.3)}$$

with

$$\widehat{A}^{(4)} = (Z_2 - Z_1)(Z_3 - Z_4) \cdot \frac{Z_1 Z_2}{(1 - Z_1 Z_2)^2}. \qquad \text{(B.4)}$$

Note that, while $H^{(4)}$ and $R^{(4)}$ are completely symmetric under the permutation of the angles ϕ_i, the right-hand side of (B.1), (B.2) are anti-symmetric under the same transformation.

From (13), (B.1), (B.2) and using the symmetry of the rest of the integrand (namely $R^{(4)}$ times the product over y_i variables), we can write $H^{(4)}$ as

$$H^{(4)} \equiv \tfrac{1}{8} P^{(Z)}_{12\text{–}34} \cdot \big(P^{(x)}_{12\text{–}34} - (1 \leftrightarrow 3) - (2 \leftrightarrow 3)\big). \qquad \text{(B.5)}$$

We obtain $H^{(4)}$, given in (14), from (B.5) with the first term at the right-hand side of (B.3). Again using the property of complete symmetry of the integrand, that part of (B.5) with the second term at the right-hand side of (B.3) can be written as

$$-\tfrac{1}{8} P^{(x)}_{12\text{–}34} \cdot ((Z_1 - Z_2)(Z_3 - Z_4)(1 + Z_1 Z_2 + Z_3 Z_4) - (1 \leftrightarrow 3) - (2 \leftrightarrow 3)) \qquad \text{(B.6)}$$

and *is identically null*.

Appendix C

In this appendix, we give the explicit integration rules corresponding to the form (16) given in the main text. For this purpose, let us consider

$$J(n_1,n_2,n_3,n_4) = \int_0^{2\pi} \frac{\mathrm{d}\phi_1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\phi_2}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\phi_3}{2\pi} \left(\prod_{i=1}^{4} y_i x_i^{n_i} \right) \cdot \widehat{A}^{(4)} \qquad \text{(C.1)}$$

where $\widehat{A}^{(4)}$ is given in (B.4). Expression $J(n_1, n_2, n_3, n_4)$ has the following properties:

$$J(n, n, n_3, n_4) = J(n_1, n_2, n, n) = 0$$
$$J(n_1, n_2, n_3, n_4) = -J(n_2, n_1, n_3, n_4) = J(n_3, n_4, n_1, n_2) \tag{C.2}$$

where the last identity comes from the fact that

$$\frac{Z_1 Z_2}{(1 - Z_1 Z_2)^2} = \frac{Z_3 Z_4}{(1 - Z_3 Z_4)^2}. \tag{C.3}$$

Changing the integration variables from (ϕ_1, ϕ_2, ϕ_3) to $(\phi = \phi_1 + \phi_2, \phi_2, \phi_3)$, $\widehat{A}^{(4)}$, defined in (B.4), becomes

$$\widehat{A}^{(4)} = \frac{1}{(1 - Z)^2} \cdot \left[\left(\frac{1}{Z_2 Z_3} + Z_2 Z_3 \right) \cdot Z - \frac{Z_2}{Z_3} - \frac{Z_3}{Z_2} Z^2 \right] \tag{C.4}$$

with $Z = Z_1 Z_2$. Taking $\widehat{A}^{(4)}$ in the form (C.4), expanding $y_1 x_1^{n_1}$ and $y_4 x_4^{n_4}$, and using the Fourier expansion (9), the integration over ϕ_2 and ϕ_3 is straightforward[16]. Some standard manipulations give

$$J(n_1, n_2, n_3, n_4) = -\int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \cdot \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} D(n_1, n_2; i) D(n_3, n_4; j) \cdot \frac{Z^{j-i+1}}{(1 - Z)^2} \tag{C.5}$$

where

$$D(n_1, n_2; k) = A(n_1, k + 1) A(n_2, k) - A(n_1, k) A(n_2, k + 1). \tag{C.6}$$

From the definition (10) of $A(n, k)$, one can easily show that

$$D(n_1, n_2; -k) = -D(n_1, n_2; k - 1). \tag{C.7}$$

With this property (C.7), and after some manipulations, $J(n_1, n_2, n_3, n_4)$ reads

$$J(n_1, n_2, n_3, n_4) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} D(n_1, n_2; i) D(n_3, n_4; j) N(i, j) \tag{C.8}$$

where

$$N(i, j) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} Z^{-j-i} \frac{1 - Z^{2i+1}}{1 - Z} \frac{1 - Z^{2j+1}}{1 - Z} \tag{C.9}$$

which, by expansion in powers of Z and integration, simply reads

$$N(i, j) = 2 \min(i, j) + 1. \tag{C.10}$$

Finally, taking the form (C.8), together with (C.10), and using the definition (10) of $A(n, k)$, we obtain

$$J(n_1, n_2, n_3, n_4) = \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} w^{2j+4k+2} (2k + 1) (1 + \theta(j - 1))$$
$$\times (d(n_1, n_2; k) d(n_3, n_4; j + k) + d(n_1, n_2; j + k) d(n_3, n_4; k)) \tag{C.11}$$

where $d(n_1, n_2; k)$ is defined in (20).

[16] see the integration rule (A.28).

Appendix D

In this appendix, we give all the necessary details of the derivation of the result (17) given in the text. We consider $\tilde{\chi}^{(4)}$ in the form (2), with $H^{(4)}$ given in (14).

The first step is to consider the expansion of the product $R^{(4)} \cdot P^{(x)}_{12-34}$. Let us give the expansions, in the x_i variables, for the first two terms in the brackets in (15):

$$R^{(4)} \cdot \frac{\tilde{x}_1\tilde{x}_2}{1-\tilde{x}_1\tilde{x}_2}\frac{\tilde{x}_3\tilde{x}_4}{1-\tilde{x}_3\tilde{x}_4} = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} (\tilde{x}_1\tilde{x}_2)^m\,(\tilde{x}_3\tilde{x}_4)^l\,\theta(m-1)\theta(l-1)\,(2\min(m,l)-1) \tag{D.1}$$

$$R^{(4)} \cdot \frac{(\tilde{x}_1\tilde{x}_2)^2}{1-\tilde{x}_1\tilde{x}_2} = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} (\tilde{x}_1\tilde{x}_2)^m\,(\tilde{x}_3\tilde{x}_4)^l\,\theta(m-l-2)\theta(m-2)\,(1+\theta(l-1)). \tag{D.2}$$

$R^{(4)} \cdot P^{(x)}_{12-34}$ can be expanded as

$$R^{(4)} \cdot P^{(x)}_{12-34} = (x_1-x_2)\cdot(x_3-x_4)\cdot\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} w^{2m+2l+2}\,(x_1x_2)^m\,(x_3x_4)^l\cdot c(m,l) \tag{D.3}$$

with

$$\begin{aligned} c(m,l) &= \theta(m-1)\theta(l-1)\cdot(2\min(m,l)-1) + \theta(m-l-2)\theta(m-2)\cdot(1+\theta(l-1)) \\ &\quad +\theta(l-m-2)\theta(l-2)\cdot(1+\theta(m-1)). \end{aligned} \tag{D.4}$$

Considering $\tilde{\chi}^{(4)}$ in the form (2), with $H^{(4)}$ given by (14), taking the expansion of $R^{(4)} \cdot P^{(x)}_{ij-kl}$ in the form (D.3), performing the angular integration using definition (C.1), and collecting all terms, gives

$$\tilde{\chi}^{(4)} = 16w^6\cdot\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{1}{2}w^{2m+2p}c(m,p)I(m,p) \tag{D.5}$$

where

$$I(m,p) = J(m,m+1,p,p+1) - J(m,p,m+1,p+1) - J(m,p+1,p,m+1). \tag{D.6}$$

Using identities (C.2) for expression $J(n_1,n_2,n_3,n_4)$, one has

$$I(p,m) = I(m,p) \qquad I(m,m) = I(m,m+1) = 0. \tag{D.7}$$

The symmetry of $c(m,p)$, together with relations (D.7), allow us, after some manipulations, to write $\tilde{\chi}^{(4)}$ as

$$\tilde{\chi}^{(4)} = 16w^{10}\cdot\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} w^{4m+2n}(2m+1)\cdot I(m,n+m+2) \tag{D.8}$$

where we have used the following identity $c(m,n+m+2) = 2m+1$.

From the previous equation, taking definition (D.6) together with expression (C.11), one obtains

$$\tilde{\chi}^{(4)} = 16w^{12}\cdot\sum_{m,k,n,j=0}^{\infty} w^{4m+4k+2n+2j}(2m+1)\,(2k+1)\,\frac{1}{2}\big(1+\theta(j-1)\big)\cdot V(m,k,n,j-2) \tag{D.9}$$

where $V(m,k,n,j)$ is defined in (19), with the constraints:

$$V(m,k,n,-2) = V(m,k,n,-1) = 0. \tag{D.10}$$

Thus, $\tilde{\chi}^{(4)}$ reads

$$\tilde{\chi}^{(4)} = 16w^{16} \cdot \sum_{m,k,n,j=0}^{\infty} w^{4m+4k+2n+2j}(2m+1)\cdot(2k+1)\cdot V(m,k,n,j). \tag{D.11}$$

The last simplification comes from the following identity on $V(m, k, n, j)$:

$$V(k,m,j,n) = V(m,k,n,j) \tag{D.12}$$

which is obtained with the help of the symmetry of $a(n, k)$, namely $a(n, k) = a(k, n)$.

Finally, using the general identity

$$\sum_{p_1=0}^{\infty}\sum_{p_2=0}^{\infty} s(p_1,p_2) = \sum_{p=0}^{\infty}\sum_{i=0}^{\infty}(1+\theta(i-1))\,\frac{1}{2}\,(s(p+i,p)+s(p,p+i)) \tag{D.13}$$

and identity (D.12), expression (D.11) for $\tilde{\chi}^{(4)}$ can, after some manipulation, be written in the form (17) displayed in the text.

Appendix E

$$\begin{aligned} P_{10}(x) = {} & 192\,598\,769\,664\,000 - 943722\,860\,380\,160x + 1154\,055\,263\,764\,480x^2 \\ & + 223\,612\,469\,147\,648x^3 + 498\,965\,092\,419\,008x^4 - 4709\,824\,359\,388\,336x^5 \\ & + 6098\,813\,329\,440\,179x^6 - 2687\,506\,699\,337\,617x^7 \\ & - 752\,969\,324\,018\,818x^8 + 1919\,011\,581\,320\,339x^9 \\ & - 1526\,656\,430\,056\,013x^{10} + 660\,280\,621\,356\,073x^{11} \\ & - 134\,468\,923\,815\,612x^{12} + 7980\,003\,107\,181x^{13} \\ & + 671\,991\,155\,376x^{14} - 17\,122\,807\,680x^{15} \\ & - 5111\,390\,208x^{16} + 43\,929\,600x^{17} \end{aligned} \tag{E.1}$$

$$\begin{aligned} P_{9}(x) = {} & 39\,290\,149\,011\,456\,000 - 297\,447\,362\,118\,287\,360x + 785\,045\,342\,889\,902\,080x^2 \\ & - 738\,039\,510\,467\,911\,680x^3 + 115\,430\,038\,068\,948\,224x^4 \\ & - 1014\,554\,971\,757\,285\,568x^5 + 3530\,827\,584\,228\,091\,380x^6 \\ & - 4191\,676\,869\,770\,939\,784x^7 + 2038\,314\,404\,276\,922\,993x^8 \\ & + 302\,242\,744\,021\,242\,129x^9 - 1228\,406\,406\,705\,470\,056x^{10} \\ & + 1043\,856\,642\,401\,535\,101x^{11} - 506\,114\,460\,611\,418\,219x^{12} \\ & + 139\,123\,852\,417\,110\,463x^{13} - 18\,756\,086\,615\,529\,738x^{14} \\ & + 585\,977\,956\,065\,027x^{15} + 80\,289\,868\,813\,296x^{16} \\ & + 338\,076\,239\,232x^{17} - 554\,882\,826\,240x^{18} + 4524\,748\,800x^{19} \end{aligned} \tag{E.2}$$

$$\begin{aligned} P_{8}(x) = {} & -197\,221\,140\,135\,936\,000 + 2237\,384\,185\,085\,952\,000x - 9353\,251\,554\,131\,968\,000x^2 \\ & + 17\,417\,661\,939\,469\,352\,960x^3 - 12\,764\,493\,479\,941\,255\,168x^4 \\ & + 5721\,656\,503\,410\,538\,176x^5 - 31\,513\,487\,740\,861\,948\,240x^6 \\ & + 74\,327\,124\,872\,294\,722\,259x^7 - 74\,793\,556\,916\,527\,926\,194x^8 \\ & + 31\,202\,576\,844\,037\,093\,699x^9 + 8393\,285\,642\,528\,898\,691x^{10} \\ & - 21\,583\,664\,946\,799\,604\,164x^{11} + 16\,831\,944\,005\,767\,359\,920x^{12} \\ & - 7706\,219\,890\,490\,209\,273x^{13} + 2036\,277\,808\,796\,847\,523x^{14} \end{aligned}$$

$$- 267\,301\,496\,692\,115\,881x^{15} + 8265\,260\,736\,796\,220x^{16} + 1095\,242\,827\,496\,832x^{17} + 4127\,794\,059\,264x^{18} - 7320\,844\,222\,464x^{19} + 58\,514\,227\,200x^{20} \tag{E.3}$$

$$\begin{aligned} P_7(x) = {} & 764\,231\,918\,026\,752\,000 - 16\,971\,026\,878\,221\,516\,800x \\ & + 115\,553\,546\,628\,494\,786\,560x^2 - 357\,128\,015\,448\,284\,200\,960x^3 \\ & + 527\,925\,844\,930\,277\,396\,480x^4 - 327\,328\,761\,204\,753\,779\,200x^5 \\ & + 264\,654\,998\,297\,650\,108\,784x^6 - 1129\,192\,842\,855\,079\,731\,378x^7 \\ & + 2154\,212\,025\,640\,162\,544\,967x^8 - 1923\,633\,718\,567\,453\,943\,246x^9 \\ & + 708\,245\,350\,538\,757\,849\,333x^{10} + 257\,335\,760\,183\,269\,925\,607x^{11} \\ & - 530\,877\,971\,609\,916\,541\,978x^{12} + 387\,828\,430\,484\,350\,338\,476x^{13} \\ & - 169\,278\,351\,316\,199\,852\,707x^{14} + 43\,213\,759\,197\,299\,807\,177x^{15} \\ & - 5540\,993\,509\,581\,072\,863x^{16} + 171\,042\,063\,471\,920\,404x^{17} \\ & + 21\,596\,554\,335\,396\,864x^{18} + 67\,975\,771\,296\,768x^{19} \\ & - 139\,524\,297\,965\,568x^{20} + 1091\,386\,982\,400x^{21} \end{aligned} \tag{E.4}$$

$$\begin{aligned} P_6(x) = {} & 2366\,653\,681\,631\,232\,000 + 15\,657\,511\,469\,396\,787\,200x \\ & - 418\,375\,361\,108\,283\,228\,160x^2 + 2413\,740\,972\,333\,717\,913\,600x^3 \\ & - 6304\,389\,540\,169\,208\,659\,968x^4 + 7885\,479\,774\,610\,742\,167\,552x^5 \\ & - 4194\,762\,998\,349\,204\,963\,264x^6 + 4787\,870\,532\,372\,577\,669\,360x^7 \\ & - 18\,347\,979\,159\,424\,299\,125\,323x^8 + 30\,953\,188\,609\,834\,801\,509\,491x^9 \\ & - 25\,453\,291\,295\,484\,088\,365\,791x^{10} + 8618\,062\,554\,065\,795\,648\,668x^{11} \\ & + 3501\,102\,053\,489\,313\,155\,231x^{12} - 6497\,943\,932\,928\,673\,199\,052x^{13} \\ & + 4524\,991\,429\,929\,970\,477\,637x^{14} - 1899\,642\,804\,937\,798\,765\,560x^{15} \\ & + 470\,903\,276\,066\,137\,118\,684x^{16} - 59\,187\,343\,449\,427\,131\,707x^{17} \\ & + 1841\,781\,629\,480\,835\,082x^{18} + 217\,686\,233\,501\,214\,240x^{19} \\ & + 505\,738\,155\,562\,752x^{20} - 1358\,525\,988\,175\,872x^{21} + 10\,379\,070\,873\,600x^{22} \end{aligned} \tag{E.5}$$

$$\begin{aligned} P_5(x) = {} & -4240\,254\,512\,922\,624\,000 + 62\,906\,863\,287\,743\,283\,200x \\ & - 176\,844\,924\,247\,909\,335\,040x^2 - 926\,585\,736\,920\,055\,152\,640x^3 \\ & + 6926\,618\,061\,828\,909\,694\,976x^4 - 17\,270\,267\,632\,607\,743\,877\,120x^5 \\ & + 19\,088\,006\,038\,191\,340\,885\,888x^6 - 7801\,708\,756\,272\,263\,419\,344x^7 \\ & + 10\,929\,800\,840\,921\,163\,293\,912x^8 - 45\,123\,693\,732\,451\,251\,946\,951x^9 \\ & + 72\,469\,347\,127\,122\,031\,920\,503x^{10} - 57\,004\,940\,985\,415\,997\,264\,335x^{11} \\ & + 18\,768\,946\,563\,950\,160\,696\,028x^{12} + 7071\,447\,265\,880\,540\,507\,399x^{13} \\ & - 12\,955\,437\,249\,853\,164\,400\,164x^{14} + 8753\,288\,431\,922\,263\,351\,485x^{15} \\ & - 3567\,263\,867\,581\,256\,107\,712x^{16} + 863\,585\,677\,517\,038\,259\,840x^{17} \\ & - 106\,841\,634\,606\,075\,117\,511x^{18} + 3\,388\,641\,803\,935\,920\,834x^{19} \\ & + 366\,870\,695\,477\,680\,992x^{20} + 460\,940\,531\,245\,824x^{21} \\ & - 2210\,964\,576\,190\,464x^{22} + 16\,454\,358\,835\,200x^{23} \end{aligned} \tag{E.6}$$

$$
\begin{aligned}
P_4(x) = {} & 2366\,653\,681\,631\,232\,000 - 121\,147\,207\,913\,255\,731\,200x \\
& + 1204\,116\,891\,510\,820\,044\,800x^2 - 4471\,702\,378\,853\,957\,632\,000x^3 \\
& + 3882\,247\,113\,210\,548\,060\,160x^4 + 19\,050\,981\,728\,126\,452\,531\,200x^5 \\
& - 60\,173\,463\,653\,707\,794\,252\,800x^6 + 61\,647\,224\,332\,324\,323\,946\,560x^7 \\
& - 8902\,468\,613\,107\,108\,100\,784x^8 + 12\,826\,049\,043\,414\,461\,768\,619x^9 \\
& - 130\,168\,614\,999\,998\,943\,881\,548x^{10} + 224\,001\,261\,613\,376\,678\,757\,254x^{11} \\
& - 177\,856\,326\,190\,823\,713\,976\,989x^{12} + 61\,054\,390\,293\,590\,247\,070\,213x^{13} \\
& + 16\,702\,537\,875\,405\,165\,273\,835x^{14} - 34\,504\,858\,377\,234\,313\,267\,137x^{15} \\
& + 23\,145\,136\,640\,956\,770\,024\,901x^{16} - 9263\,503\,411\,249\,994\,637\,068x^{17} \\
& + 2206\,094\,527\,520\,924\,501\,173x^{18} - 270\,104\,745\,702\,122\,623\,849x^{19} \\
& + 8\,842\,661\,536\,198\,397\,940x^{20} + 853\,061\,197\,515\,225\,600x^{21} \\
& - 104\,681\,638\,038\,528x^{22} - 4965\,596\,931\,735\,552x^{23} \\
& + 35\,863\,949\,721\,600x^{24}
\end{aligned}
\tag{E.7}
$$

$$
\begin{aligned}
P_3(x) = {} & -11\,833\,268\,408\,156\,160\,000 + 296\,345\,375\,978\,029\,056\,000x \\
& - 2144\,074\,027\,349\,363\,916\,800x^2 + 6754\,783\,276\,113\,459\,937\,280x^3 \\
& - 8346\,802\,888\,116\,933\,918\,720x^4 - 3858\,851\,854\,352\,328\,468\,480x^5 \\
& + 18\,632\,235\,498\,956\,809\,831\,680x^6 - 5896\,832\,882\,414\,934\,369\,872x^7 \\
& - 20\,092\,581\,953\,702\,558\,202\,578x^8 + 9039\,258\,514\,198\,720\,200\,789x^9 \\
& + 30\,407\,815\,233\,273\,202\,124\,059x^{10} - 44\,070\,504\,829\,347\,073\,747\,333x^{11} \\
& + 24\,353\,838\,992\,694\,268\,159\,678x^{12} - 2634\,376\,094\,447\,814\,881\,673x^{13} \\
& - 5786\,419\,996\,453\,628\,346\,078x^{14} + 5219\,893\,433\,655\,665\,719\,185x^{15} \\
& - 2420\,429\,903\,132\,324\,343\,712x^{16} + 639\,931\,684\,854\,726\,532\,004x^{17} \\
& - 85\,290\,635\,341\,359\,290\,409x^{18} + 3288\,947\,218\,141\,266\,164x^{19} \\
& + 249\,618\,983\,188\,012\,416x^{20} - 1896\,484\,503\,705\,600x^{21} \\
& - 1406\,458\,574\,438\,400x^{22} + 9\,870\,981\,120\,000x^{23}
\end{aligned}
\tag{E.8}
$$

$$
\begin{aligned}
P_2(x) = {} & 36\,978\,963\,775\,488\,000\,000 - 561\,176\,784\,419\,998\,924\,800x \\
& + 3011\,344\,824\,723\,628\,359\,680x^2 - 7448\,695\,810\,925\,636\,485\,120x^3 \\
& + 7522\,952\,469\,866\,279\,936\,000x^4 + 1571\,640\,457\,415\,428\,633\,600x^5 \\
& - 6448\,286\,163\,781\,893\,470\,272x^6 - 5723\,224\,826\,762\,698\,171\,248x^7 \\
& + 17\,523\,982\,043\,162\,827\,496\,339x^8 - 10\,871\,737\,924\,438\,446\,047\,496x^9 \\
& - 1585\,720\,216\,853\,837\,551\,908x^{10} + 4808\,127\,638\,071\,252\,147\,363x^{11} \\
& - 1996\,433\,111\,317\,397\,547\,653x^{12} - 255\,305\,750\,980\,018\,902\,593x^{13} \\
& + 746\,533\,058\,733\,318\,361\,745x^{14} - 451\,215\,245\,481\,551\,846\,087x^{15} \\
& + 139\,082\,951\,228\,390\,470\,494x^{16} - 20\,614\,123\,445\,854\,835\,699x^{17} \\
& + 931\,298\,896\,466\,844\,519x^{18} + 53\,593\,802\,936\,851\,536x^{19} \\
& - 845\,315\,534\,083\,200x^{20} - 297\,344\,753\,664\,000x^{21} + 2006\,484\,480\,000x^{22}
\end{aligned}
\tag{E.9}
$$

$$\begin{aligned}
P_1(x) = &-14\,878\,398\,029\,915\,750\,400 + 132\,065\,832\,050\,507\,120\,640x \\
&- 489\,750\,894\,279\,960\,821\,760x^2 + 843\,938\,312\,965\,142\,528\,000x^3 \\
&- 436\,158\,709\,746\,751\,846\,400x^4 - 504\,050\,378\,774\,640\,931\,456x^5 \\
&+ 362\,705\,915\,662\,042\,967\,696x^6 + 930\,077\,072\,222\,989\,626\,372x^7 \\
&- 1516\,628\,759\,565\,419\,342\,933x^8 + 919\,766\,159\,750\,634\,600\,991x^9 \\
&- 236\,614\,768\,621\,801\,865\,201x^{10} - 4425\,339\,208\,940\,491\,244x^{11} \\
&+ 15\,751\,985\,529\,081\,115\,811x^{12} + 356\,035\,582\,683\,101\,460x^{13} \\
&- 3725\,421\,245\,175\,199\,951x^{14} + 1704\,676\,309\,420\,947\,212x^{15} \\
&- 304\,427\,228\,661\,108\,302x^{16} + 16\,257\,997\,137\,256\,137x^{17} \\
&+ 635\,239\,438\,099\,728x^{18} - 16\,870\,300\,732\,800x^{19} \\
&- 3641\,326\,080\,000x^{20} + 23\,063\,040\,000x^{21}
\end{aligned} \tag{E.10}$$

Appendix F

Let us recall a few elementary facts on factorization of differential operators, without introducing any sophisticated $\mathcal{D}$-module theory. Two differential operators $\mathcal{D}_1$ and $\mathcal{D}_2$ of order N are said to be equivalent (see for instance equation (5) in [21]) if there exist two (intertwining) differential operators R and S of orders at most $N - 1$, such that

$$\mathcal{D}_2 \cdot R = S \cdot \mathcal{D}_1. \tag{F.1}$$

The factorization of differential operators is not in general unique. There can be several factorizations, *even an infinite number* of factorizations, as can be seen in the simple exemple:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} = \left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{a}{ax+b}\right) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{a}{ax+b}\right) \tag{F.2}$$

where factorization (F.2) holds for every constant a and b. This can be seen as a consequence of the relation:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{a}{ax+b}\right) \cdot (ax+b) = (ax+b) \cdot \frac{\mathrm{d}}{\mathrm{d}x} \tag{F.3}$$

which means that the multiplication by $ax + b$ transforms the solutions of the differential operator $\mathrm{d}/\mathrm{d}x$ into a solution of $\mathrm{d}/\mathrm{d}x - a/(ax + b)$. Relation (F.3) is a simple example of equivalence (F.1) of two differential operators. The occurrence of this infinite set of factorizations can be seen to be a consequence of the fact that the various differential operators of (F.2), (F.3) have solutions the ratio of which can be rational functions. In general this is not the case and one will only have a *finite number* of factorizations, these various factorizations being basically the same *up to some permutation of the differential operators*, each operator in the factorization being changed ('by some transmutation' (F.1)) *into an equivalent one* during the permutation process.

This is typically the situation we encounter with the operator $\mathcal{L}_{10}$ associated with the ordinary differential equation of order ten satisfied by $\tilde{\chi}^{(4)}$.

This order ten operator $\mathcal{L}_{10}$ has 36 factorizations with two operators of order four (M_1 and M_2), ten operators of order two ($N_0, N_1, \ldots, N_9$) and 26 operators of order one ($L_0, L_1, \ldots L_{25}$). *All these operators have been normalized in such a way that the coefficient of the highest derivative is normalized to* +1. The operators L_0, L_1, L_2, N_0 and N_8 below are such that $L_0(\mathcal{S}_0) = 0, L_1(\mathcal{S}_1) = 0, L_2(\mathcal{S}_2) = 0, N_0(\mathcal{S}_3) = 0$ and $N_8^*(\mathcal{S}_1^*) = 0$. These 36 factorizations[17] of $\mathcal{L}_{10}$ read

$$\begin{aligned}
\mathcal{L}_{10} &= N_8 \cdot M_2 \cdot L_{25} \cdot L_{12} \cdot L_3 \cdot L_0 \\
&= M_1 \cdot N_9 \cdot L_{25} \cdot L_{12} \cdot L_3 \cdot L_0 \\
&= M_1 \cdot L_{24} \cdot N_4 \cdot L_{12} \cdot L_3 \cdot L_0 \\
&= M_1 \cdot L_{24} \cdot L_{13} \cdot N_6 \cdot L_3 \cdot L_0 \\
&= M_1 \cdot L_{24} \cdot L_{13} \cdot L_{17} \cdot N_3 \cdot L_0 \\
&= M_1 \cdot L_{24} \cdot L_{13} \cdot L_{17} \cdot L_{11} \cdot N_0
\end{aligned} \tag{F.4}$$

$$\begin{aligned}
\mathcal{L}_{10} &= N_8 \cdot M_2 \cdot L_{25} \cdot L_{14} \cdot L_4 \cdot L_0 \\
&= M_1 \cdot N_9 \cdot L_{25} \cdot L_{14} \cdot L_4 \cdot L_0 \\
&= M_1 \cdot L_{24} \cdot N_4 \cdot L_{14} \cdot L_4 \cdot L_0 \\
&= M_1 \cdot L_{24} \cdot L_{15} \cdot N_7 \cdot L_4 \cdot L_0 \\
&= M_1 \cdot L_{24} \cdot L_{15} \cdot L_{16} \cdot N_3 \cdot L_0 \\
&= M_1 \cdot L_{24} \cdot L_{15} \cdot L_{16} \cdot L_{11} \cdot N_0
\end{aligned} \tag{F.5}$$

$$\begin{aligned}
\mathcal{L}_{10} &= N_8 \cdot M_2 \cdot L_{25} \cdot L_{18} \cdot L_5 \cdot L_1 \\
&= M_1 \cdot N_9 \cdot L_{25} \cdot L_{18} \cdot L_5 \cdot L_1 \\
&= M_1 \cdot L_{24} \cdot N_4 \cdot L_{18} \cdot L_5 \cdot L_1 \\
&= M_1 \cdot L_{24} \cdot L_{19} \cdot N_5 \cdot L_5 \cdot L_1 \\
&= M_1 \cdot L_{24} \cdot L_{19} \cdot L_{23} \cdot N_1 \cdot L_1 \\
&= M_1 \cdot L_{24} \cdot L_{19} \cdot L_{23} \cdot L_{10} \cdot N_0
\end{aligned} \tag{F.6}$$

$$\begin{aligned}
\mathcal{L}_{10} &= N_8 \cdot M_2 \cdot L_{25} \cdot L_{14} \cdot L_6 \cdot L_1 \\
&= M_1 \cdot N_9 \cdot L_{25} \cdot L_{14} \cdot L_6 \cdot L_1 \\
&= M_1 \cdot L_{24} \cdot N_4 \cdot L_{14} \cdot L_6 \cdot L_1 \\
&= M_1 \cdot L_{24} \cdot L_{15} \cdot N_7 \cdot L_6 \cdot L_1 \\
&= M_1 \cdot L_{24} \cdot L_{15} \cdot L_{20} \cdot N_1 \cdot L_1 \\
&= M_1 \cdot L_{24} \cdot L_{15} \cdot L_{20} \cdot L_{10} \cdot N_0
\end{aligned} \tag{F.7}$$

$$\begin{aligned}
\mathcal{L}_{10} &= N_8 \cdot M_2 \cdot L_{25} \cdot L_{18} \cdot L_7 \cdot L_2 \\
&= M_1 \cdot N_9 \cdot L_{25} \cdot L_{18} \cdot L_7 \cdot L_2 \\
&= M_1 \cdot L_{24} \cdot N_4 \cdot L_{18} \cdot L_7 \cdot L_2 \\
&= M_1 \cdot L_{24} \cdot L_{19} \cdot N_5 \cdot L_7 \cdot L_2 \\
&= M_1 \cdot L_{24} \cdot L_{19} \cdot L_{21} \cdot N_2 \cdot L_2 \\
&= M_1 \cdot L_{24} \cdot L_{19} \cdot L_{21} \cdot L_9 \cdot N_0
\end{aligned} \tag{F.8}$$

$$\begin{aligned}
\mathcal{L}_{10} &= N_8 \cdot M_2 \cdot L_{25} \cdot L_{12} \cdot L_8 \cdot L_2 \\
&= M_1 \cdot N_9 \cdot L_{25} \cdot L_{12} \cdot L_8 \cdot L_2 \\
&= M_1 \cdot L_{24} \cdot N_4 \cdot L_{12} \cdot L_8 \cdot L_2 \\
&= M_1 \cdot L_{24} \cdot L_{13} \cdot N_6 \cdot L_8 \cdot L_2 \\
&= M_1 \cdot L_{24} \cdot L_{13} \cdot L_{22} \cdot N_2 \cdot L_2 \\
&= M_1 \cdot L_{24} \cdot L_{13} \cdot L_{22} \cdot L_9 \cdot N_0.
\end{aligned} \tag{F.9}$$

[17] The DFactor command [27] of DEtools can give the second factorization in (F.4), namely $\mathcal{L}_{10} = M_1 \cdot N_9 \cdot L_{25} \cdot L_{12} \cdot L_3 \cdot L_0$.

All these 36 factorizations can be seen as the consequence of 'transmutations' like (F.1), where one differential operator becomes an equivalent one, thus yielding some elementary permutation of the operator. For instance, the two differential operators of order four, namely M_1 and M_2, are equivalent, their 'intertwiners' being two order two differential operators N_8 and N_9:

$$N_8 \cdot M_2 = M_1 \cdot N_9. \tag{F.10}$$

Actually, one sees clearly that the two first factorizations in (F.4) are just deduced from one another by (F.10).

Nine order-two differential operators $N_0, \ldots, N_8$ can be seen to be related by various equivalence relations requiring the introduction of 26 differential operators L_i of order one (the intertwiners):

$$\begin{aligned}
&N_1 \cdot L_1 = L_{10} \cdot N_0, && N_2 \cdot L_2 = L_9 \cdot N_0 \\
&N_3 \cdot L_0 = L_{11} \cdot N_0, && N_4 \cdot L_{12} = L_{13} \cdot N_6 \\
&N_4 \cdot L_{14} = L_{15} \cdot N_7, && N_4 \cdot L_{18} = L_{19} \cdot N_5 \\
&N_5 \cdot L_5 = L_{23} \cdot N_1, && N_5 \cdot L_7 = L_{21} \cdot N_2 \\
&N_6 \cdot L_3 = L_{17} \cdot N_3, && N_6 \cdot L_8 = L_{22} \cdot N_2 \\
&N_7 \cdot L_4 = L_{16} \cdot N_3, && N_7 \cdot L_6 = L_{20} \cdot N_1 \\
&N_9 \cdot L_{25} = L_{24} \cdot N_4. &&
\end{aligned} \tag{F.11}$$

The 26 order-one differential operators L_i can also be seen to be related by various equivalence relations. The L_i being order one differential operators, their intertwiners (R and S in (F.1)) should be order zero differential operators, that is some function $f_{i,j}$. As a consequence of the normalization to +1 of the highest order derivative (here $\mathrm{d}/\mathrm{d}x$), the two intertwiners R and S in (F.1) are identical and one thus gets relations like:

$$L_i \cdot f_{i,j} = f_{i,j} \cdot L_j. \tag{F.12}$$

Furthermore one also verifies the following relations between the order one operators L_i:

$$L_3 \cdot L_0 = L_8 \cdot L_2, \qquad L_4 \cdot L_0 = L_6 \cdot L_1 \qquad L_5 \cdot L_1 = L_7 \cdot L_2 \tag{F.13}$$

$$\begin{aligned}
&L_{12} \cdot L_3 = L_{14} \cdot L_4, \qquad L_{13} \cdot L_{17} = L_{15} \cdot L_{16} \qquad L_{18} \cdot L_5 = L_{14} \cdot L_6, \\
&L_{19} \cdot L_{23} = L_{15} \cdot L_{20} \qquad L_{18} \cdot L_7 = L_{12} \cdot L_8, \qquad L_{19} \cdot L_{21} = L_{13} \cdot L_{22}
\end{aligned} \tag{F.14}$$

$$L_{17} \cdot L_{11} = L_{22} \cdot L_9, \qquad L_{16} \cdot L_{11} = L_{20} \cdot L_{10} \qquad L_{23} \cdot L_{10} = L_{21} \cdot L_9. \tag{F.15}$$

It is straightforward to see that a relation like $L_i \cdot L_j = L_k \cdot L_m$ yields relations like:

$$\begin{aligned}
&L_i \cdot (L_j - L_m) = (L_k - L_i) \cdot L_m, \qquad && (L_i - L_k) \cdot L_j = L_k \cdot (L_m - L_j), \\
&f_{i,m} = L_j - L_m = L_k - L_i, && f_{k,j} = L_m - L_j = L_i - L_k
\end{aligned}$$

The number of intertwiners $f_{i,j}$ between the operators of order one in (F.12), is at first sight 325, but due to the above identities only 15 are independent.

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