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A Challenge in Enumerative Combinatorics: The Graph of Contributions* of Professor Fa-Yueh Wu

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We will try to sketch Professor F. Y. Wu's contributions in lattice statistical mechanics, solid state physics, graph theory, enumerative combinatorics and so many other domains of physics and mathematics. We will recall F. Y. Wu's most important and well-known classic results, and we will also sketch his most recent research dedicated to the connections of lattice statistical mechanical models with deep problems in pure mathematics. Since it is hard to provide an exhaustive list of all his contributions, to give some representation of F. Y. Wu's "mental connectivity", we will concentrate on the interrelations between the various results he has obtained in so many different domains of physics and mathematics. Along the way we will also try to understand Wu's motivations and his favorite concepts, tools and ideas.

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I. Introduction

The publish-or-perish period of science could soon be seen as a golden age: our brave new world now celebrates the triumph of Enron's financial and accounting creativity. Sadly science is now also, increasingly, considered from an accountant's viewpoint. In this respect, if one takes this "modern" point of view, Professor F. Y. Wu's contribution[1] is clearly a very good return on investment: he has given more than 270 talks in meetings or conferences, published over 200 papers and monographs in refereed journals, and had many students. He has also published in, or is the editor[2] of, many books [21, 31, 71, 122, 138, 157, 171, 178, 179, 196].

Professor Wu was trained in theoretical condensed matter physics [3, 4, 19, 20, 27, 35, 108], but he is now seen as a mathematical physicist who is a leading expert in mathematical modeling of phase transition phenomena occurring in complex systems. Wu's research includes

[1] Professor F. Y. Wu is presently the Matthews University Distinguished Professor at Northeastern University. He is a fellow of the American Physical Society and a permanent member of the Chinese Physical Society (Taipei). His research has been supported by the National Science Foundation since 1968, a rare accomplishment by itself in an environment of declining research support in the U.S., and he currently serves as the editor of three professional journals: the Physica A, International Journal of Modern Physics B and the Modern Physics Letters B.

[2] For instance, Ref. [180] contains the proceedings of the conference on "Exactly Soluble Models in Statistical Mechanics: Historical Perspectives and Current Status", held at Northeastern University in March 1996 – the first ever international conference to deal exclusively with this topic. The proceedings reflect the broad range of interest in exactly soluble models as well as the diverse fields in physics and mathematics that they connect.

both theoretical studies and practical applications[3]. Among his recent researches he has studied connections of statistical mechanical models with deep problems in pure mathematics. This includes the generation of knot and link invariants from soluble models of statistical mechanics and the study of the long-standing unsolved mathematical problem of multidimensional partitions of integers in number theory using a Potts model approach.

Professor Wu's contributions to lattice statistical mechanics have been mostly in the area of exactly solvable lattice models. While integrable models have continued to occupy a prominent place in his work (such as the exact solution of two- and three-dimensional spin models and interacting dimer systems), his work has ranged over a wide variety of problems including exact lattice statistics in two and three dimensions, graph theory and combinatorics, to mention just a few. His work in many-body theory [3, 4, 7, 8, 15, 22, 28, 36, 66], especially those on liquid helium [2, 3, 6, 25, 26], has also been influential for many years.

F. Y. Wu joined the faculty of Northeastern University to work with Elliott Lieb in 1967, and in 1968 they published a joint paper[4] on the ground state of the Hubbard model [11] which has since become a classic. The Baxter-Wu model [45, 49] is also, clearly, an important milestone in the history of integrable lattice models.

F. Y. Wu has published several very important reviews of lattice statistical mechanics. First, Lieb and Wu wrote a monograph in 1970 on vertex models which became the fundamental reference in the field for decades [31]. Wu's 1982 review on the Potts model is another classic [89]. At more than one hundred citations per year ever since it was published, it is one of the most cited papers in physics[5]. In 1992 F. Y. Wu published yet another extremely well-received review on knot theory and its connection with lattice statistical mechanics [154]. In addition, in 1981, F. Y. Wu and Z. R. Yang published a series of expository papers on critical phenomena written in Chinese [84] - [88]. This review is well-known to Chinese researchers.

I-1. The choice of presentation: a challenge in enumerative combinatorics

An intriguing aspect of lattice statistics is that seemingly totally different problems are sometimes related to each other, and that the solution of one problem can often lead to solving other outstanding unsolved problems. At first sight, most of the work of F. Y. Wu could be said to correspond to *exact results in lattice statistical mechanics*, but because of the relations between seemingly totally different problems it can equivalently be seen, and sometimes be explicitly presented, as exact results in various domains of mathematical physics or mathematics: sometimes exact results in *graph theory*, sometimes in *enumerative combinatorics*, sometimes in *knot theory*, sometimes in *number theory*, etc. Wu's "intellectual walk" goes from vertex models to circle theorems or duality relations, from dimers to Ising models and back, from percolations or animal problems to Potts models, from Potts models to the Whitney-Tutte Polynomials, to polychromatic

[3] He has considered, for instance, the modeling of physical adsorption and applied it to describe processes used in chemical and environmental engineering [148, 175]. He has even published one experimental paper on slow neutron detectors [5].

[4] This paper has become prominent in the theory of high-T_c superconductors. P. W. Anderson even attributed to this paper as "predicting" the existence of quarks in his *Physics Today* (October, 1997) article on the centennial of the discovery of electrons.

[5] There was once a study published in 1984 (E. Garfield, Current Comments **48**, 3 (1984)) on citations in physics for the year of 1982. It reports that in 1982, the year this Potts review was published, it was the fifth most-cited paper among papers published in all of physics.

polynomials or to knot theory, from results, or conjectures, on critical manifolds[6] to Yang-Baxter integrability, perhaps on the way revisiting duality or Lee-Yang zeros, etc., etc. The simple listing of Professor Wu's results and contributions, and the inter-relations between these results and the associated concepts and tools, is by itself a *challenge in enumerative combinatorics*.

Actually it is impossible to describe Wu's contributions *linearly*, in a sequence of sections in a review paper like this, or even with a website-like "*tree* organization" of paragraphs. F. Y. Wu's contributions really correspond to a quite large "graph" of concepts, results, tools and models, with many "intellectual loops". The only possible "linear" and exhaustive description of Wu's contributions is his list of publications.

We have therefore chosen to give his exhaustive list of publications at the end of this paper. No other references are given.

We have chosen to keep the notation F.Y. Wu used in his publications[7], and *not to normalize them*, so that the reader who wants to see more and goes back to the cited publications will immediately be able to recover the equations and notations.

Obviously, we will not try to provide an exhaustive description of Wu's contributions but, rather, to provide some considered well-suited specific "morceaux choisis[8]", *comments* on some of his results, some hints of the kind of concepts he likes to work with, and try to explain why his results are important, fruitful and stimulating for anyone who works in lattice statistical mechanics or in mathematical physics.

II. Even before vertex models: the exact solution of the Hubbard model

Elliott H. Lieb and F. Y. Wu published in 1968 a joint paper on the ground state of the Hubbard model [11] which has since become a classic, and served as a cornerstone in the theory of high-T_c superconductors. An important question there corresponds to the spin-charge decoupling, which is exact and explicit in one-dimensional models: is the spin-charge decoupling a characteristic of one dimension? Is it possible that some "trace" of spin-charge decoupling remains for quantum two-dimensional models which are supposedly related to high-T_c superconductors?

Let us describe briefly the classic Lieb-Wu solution of the Hubbard model. One assumes that the electrons can hop between the Wannier states of neighboring lattice sites and that each site is capable of accommodating two electrons of opposite spins with an interaction energy $U > 0$. The corresponding Hamiltonian reads:

$$H = T \sum_{<ij>} \sum_{\sigma} \, c^{\dagger}_{i\sigma} c_{j\sigma} + U \sum_{i} c^{\dagger}_{i\uparrow} \, c_{i\uparrow} \, c^{\dagger}_{i\downarrow} \, c_{i\downarrow},$$

[6] The critical manifolds deduced or conjectured by F. Y. Wu are mostly algebraic varieties and not simple differentiable or analytical manifolds.

[7] The price paid is, for instance, that the spin edge Boltzmann weights will sometimes be denoted e^{K_1}, e^{K_2}, e^{K_3}, e^{K_4}, or a, b, c, d, or x_1, x_2, x_3, x_4, and the vertex Boltzmann weights $\omega_1, \omega_2, \cdots$ or $a, b, c, d, a', b', c', d'$. This corresponds to the spectrum of notations used in the lattice statistical mechanics literature. These different notations were often introduced when one faced large polynomial expressions and the e^{K_i} or $e^{-\beta \cdot J_i}$ notations for Boltzmann weights would be painful.

[8] I apologize, in advance, for the fact that these "morceaux choisis" are obviously biased by my personal taste for effective birational algebraic geometry in lattice statistical mechanics.

where $c_{i\sigma}^{\dagger}$ and $c_{i\sigma}$ are the creation and annihilation operators for an electron of spin σ in the Wannier state at the i-th lattice site and the first sum is taken over nearest neighbor sites. Denoting $f(x_1, x_2, \cdots, x_M; x_{M+1}, \cdots, x_N)$ the amplitude of the wavefunction for which the down spins are located at sites $x_1, x_2, \cdots, x_M$ and the up spins are located at sites $x_{M+1}, \cdots, x_N$. The eigenvalue equation $H\psi = E\psi$ leads to:

$$\begin{aligned} -\sum_{i=1}^{N}\sum_{s=\pm 1} & f(x_1, x_2, \cdots, x_i + s, \cdots, x_N) \\ & +U\sum_{i<j}\delta(x_i - x_j) f(x_1, x_2, \cdots, x_N) = E\, f(x_1, x_2, \cdots, x_N), \end{aligned} \tag{1}$$

where $f(x_1, x_2, \cdots, x_N)$ is antisymmetric in the first M and the last $N - M$ variables (separately). Let μ_+ (resp. μ_-) denote the chemical potential of adding (resp. removing) one electron. In the half-filled band one has $\mu_+ = U - \mu_-$, and the calculation of μ_- can be done in closed form with the result:

$$\mu_- = 2 - 4\int_0^{\infty} \frac{J_1(\omega)\cdot d\omega}{\omega\cdot(1+\exp(\omega U/2))}, \tag{2}$$

where J_1 is the Bessel function. It can be established from (2) and $\mu_+ = U - \mu_-$ that $\mu_+ > \mu_-$ for $U > 0$. In other words, the ground state for a half-filled band is insulating for any nonzero U, and conducting for $U = 0$. Equivalently, there is no Mott transition for nonzero U, *i.e.*, the ground state is analytic in U on the real axis except at the origin.

III. Vertex models

The distinction between vertex models and spin models is traditional in lattice statistical mechanics, but there are “bridges” between these two sets of lattice models [78]. Roughly speaking one can say that F. Y. Wu first obtained results on vertex models [13, 14] (five-vertex models [9, 10], free-fermion vertex models [50], dimer models seen as vertex models, ...) and then obtained results on spin models (Ising model with second-neighbor Interactions [12], the Baxter-Wu model [45, 49], Potts model, ...), introducing more and more graph theoretical approaches, up to looping the loop with knot theory, which is, in fact, closely related to vertex models and to Potts models! As far as vertex models are concerned, we will first sketch the approach given in his monograph with Lieb (section (III-1)), in a second step we will sketch his free-fermion results (section (III-2-1)) closely followed by his dimer results (section (III-3)), and, then, we will discuss some miscellaneous results he obtained on five-, six- and eight-vertex models (section (III-4)).

III-1. Two-dimensional ferroelectric models

Elliott Lieb and F. Y. Wu wrote a monograph on vertex models in 1970, entitled “Two-dimensional Ferroelectric Models”, which became a fundamental reference in the field for decades [31]. This monograph gives the best introduction to the sixteen-vertex model, which is a fundamental model in lattice statistical mechanics. Unfortunately it is not known well enough, even to many specialists of lattice models, that it contains the most general eight-vertex model, most of the (Yang-Baxter) integrable vertex models (the symmetric eight-vertex model, various free-fermion models, the asymmetric free-fermion model, the asymmetric and symmetric six-vertex model, the five-vertex models, three-coloring of square maps, and others) and also fundamental *non-integrable* models such as, for instance, the Ising model in a magnetic field. In particular the monograph mentions explicitly the weak-graph duality (see section (V) below) on the sixteen-vertex model (see page 457 of [31]):

$$\begin{aligned}
\omega_1^* &= \frac{1}{4} \cdot \sum_{i=1}^{16} \omega_i, \qquad\qquad \omega_2^* = \frac{1}{4} \cdot \Big(\sum_{i=1}^{8} \omega_i - \sum_{i=9}^{16} \omega_i \Big) \\
\omega_3^* &= 14 \cdot \Big(\sum_{i=1}^{4} \omega_i - \sum_{i=5}^{8} \omega_i + (\omega_{10} + \omega_{12} + \omega_{14} + \omega_{16}) \\
&\qquad - (\omega_9 + \omega_{11} + \omega_{13} + \omega_{15}) \omega_i \Big), \cdots
\end{aligned} \tag{3}$$

The 154 pages of this monograph are still, by today's standard, an extremely valuable document for any specialist of lattice models. Beyond the taxonomy of ferro and ferrielectric models (ice model, KDP [9, 18], modified KDP [41], F model [13], modified F model [38, 75, 80], F model with a staggered field, ...), this monograph remains extremely modern and valuable from a technical viewpoint.

Among the exactly soluble models (the bread-and-butter of F. Y. Wu) was one that, for a long time, was a "sleeper", namely, Bethe's 1931 solution of the ground state energy and elementary excitations of the one-dimensional quantum-mechanical spin-$\frac{1}{2}$ Heisenberg model of antiferromagnetism. We will see below a large set of results from the Lieb-Wu monograph on *vertex models*, in particular the six-vertex model.

The monograph gives an extremely lucid exposition of the Bethe ansatz for the six-vertex model. The Bethe ansatz is analyzed and explained in the most general framework (with horizontal and vertical fields) and it is a must-read anyone who wants to work seriously on the coordinate Bethe ansatz. It is certainly much more interesting and deeper than so many subsequent papers that have revisited, at nauseum, the Bethe ansatz of the symmetric six-vertex model, re-styling this simple Bethe ansatz with a conformal resp. quantum group, resp. knot theory, resp. ... framework. The analysis of the conditions for the transfer matrix T of the most general sixteen-vertex model to have a non-trivial "linear operator" (1D quantum Hamiltonian) that commutes[9] with T (pages 367 to 373) are probably one of the first pages any student who wants to study integrable lattice models should read.

The monograph makes crystal clear the fact that the Bethe ansatz is related to the conservation of a certain charge. This can be seen from the fact that most of the analysis (from page 374 to page 444) relies on the use (page 363 equation (81)) of the variable $y = 1 - 2\, n/N$, which in spin language is the average z-component of the spin per vertical bond, namely, $y = < S_z > /N$ for a square lattice of size $N \times M$, where n denotes the number of down arrows and N the number of vertical bonds in a row.

We use the same notation as in Lieb-Wu. In particular, let us introduce the horizontal and vertical fields H and V, respectively. The partition function per site in the thermodynamic limit is:

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[9] Which is the most obvious manifestation of the Yang-Baxter integrability.

$$\lim_{N\to\infty} \frac{1}{N} \cdot \ln(\Lambda) = \max_{-1 \leq y \leq +1} [z(y) + V \cdot y], \tag{4}$$

where Λ denotes the largest eigenvalue of the transfer matrix. The monograph details a large set of situations. Let us consider here the regime

$$\Delta \equiv (\omega_1\omega_2 + \omega_3\omega_4 - \omega_5\omega_6)/2\sqrt{\omega_1\omega_2\omega_3\omega_4} < -1, \tag{5}$$

and introduce the variable:

$$e^{\theta_0} = \frac{1+\eta e^{\lambda}}{e^{\lambda}+\eta}, \qquad 0 \leq \theta_0 \leq \lambda \qquad \text{where:}$$

$$\eta = e^{|K_1|} = \Big(\frac{\omega_1\omega_2}{\omega_3\omega_4}\Big)^{1/2} \qquad \text{or:} \qquad \Big(\frac{\omega_3\omega_4}{\omega_1\omega_2}\Big)^{1/2}.$$

When $\Delta < -1$, $z(y)$ reads:

$$z(y) = -K_2 + \max(0, -K_1) + \frac{1}{4\pi} \cdot \int_{-b}^{+b} R(\alpha) \cdot C(\alpha) \cdot d\alpha,$$

$$\text{where:} \qquad C(\alpha) = ln\Big(\frac{\cosh(2\lambda - \theta_0) - \cos(\alpha)}{\cosh(\theta_0) - \cos(\alpha)}\Big),$$

and the (normalized) density[10] $R(\alpha)$ satisfies the Bethe-ansatz integral equation with the kernel $K(\alpha)$:

$$R(\alpha) = \frac{\sinh(\lambda)}{\cosh(\lambda) - \cos(\alpha)} - \int_{-b}^{+b} K(\alpha-\beta) \cdot R(\beta)\, d\beta \tag{6}$$

$$\text{with:} \qquad 2\pi \cdot K(\alpha - \beta) = \frac{\sinh(2\lambda)}{\cosh(2\lambda) - \cos(\alpha - \beta)}.$$

The integral equation (6) is nothing but the well-known Yang-Yang Bethe ansatz integral equation on the density $\rho(q)$:

$$1 = 2\pi \cdot \rho(p) - \int_{-Q}^{+Q} \frac{d\theta(p,q)}{dp} \rho(q) \cdot dq \quad \text{with:} \quad Q = \frac{\pi \cdot (1-y)}{2}.$$

The range b of the new variable α in the integral relation (6) can be deduced from the definition of the density $R(\alpha)$:

$$\pi \cdot (1-y) = \int_{-b}^{+b} R(\alpha)\, d\alpha = \int_{-Q}^{+Q} \rho(q) \cdot dq.$$

When $y = 0$, the integral attains its maximum range and one can solve (6) by using a Fourier series of a Fourier transform. One thus gets $R(\alpha)$ as a simple dn elliptic function. Not surprisingly one can also calculate all the derivatives of $z(y)$ at $y = 0$. One can thus expand $z(y)$ namely, write $z(y) = z(0) - z'(0) \cdot y + z'''(0) \cdot y^3/6 + \cdots$. To first order in y one obtains $z'(0) = -\Xi(\lambda - \theta_0)$, where the function Ξ is related to the Jacobian elliptic function nd:

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[10] One has $R(\alpha) \cdot d\alpha = 2\pi\rho(p) \cdot dp$.

$$\Xi(\phi) = \ln\Big(\frac{\cosh((\lambda+\phi)/2)}{\cosh((\lambda-\phi)/2)}\Big) - \frac{\phi}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n \cdot e^{-2n\lambda} \cdot \sinh(n\phi)}{n \cdot \cosh(n\lambda)}.$$

The function $\Xi(\phi)$ also satisfies the nice involutive functional relations[11]:

$$\Xi(\phi) = -\Xi(-\phi), \quad \Xi(\lambda+\phi)\Xi(\lambda-\phi), \quad \Xi(4\lambda+\phi) = \Xi(\phi).$$

Let us consider the thermodynamic properties of the model when $H = 0$ and $V \neq 0$. From (4) one sees that the thermodynamic properties depend on the optimal choice of y given by:

$$z'(y) = -V.$$

When lowering the temperature the slope of $z(y)$ corresponding to the transition sticks at $y \simeq 0$, and one thus has (see page 425 of [31]) an antiferroelectric transition occurring at $T_c(V)$ given by:

$$V = \Xi(\lambda - \theta_0). \qquad (7)$$

This gives a beautiful example of a *transcendental critical manifold* which reduces, in some domain of the parameters (low temperatures), to a transcendental equation (7) and *not to an algebraic one*, as one is used to seeing in exactly solvable models. One thus has a *transcendental critical manifold* for a vertex model for which one can actually write down the exact Bethe ansatz (see equation (6)). Writing a closed simple formula for the solution is not possible, but one can certainly find numerical solutions on a computer. Should we say that the model is exactly solvable but not "computable"? We will revisit these questions of the algebraicity of the critical manifold versus integrability in other sections of this paper with other critical manifold conjectures, or results, of F. Y. Wu (see for instance sections (VI-3), (IV-1) below). For those who have a "naive" point of view on the character of critical manifolds[12], example (7) shows that a model having a Bethe ansatz can have a *transcendental* critical manifold.

The Lieb-Wu review provides wonderful pieces of analytical work (analysis in one complex variable, see for instance pages 410-411 and the analysis of the analytic structure of the F model or the temperature Riemann structure for the free energy of the F model). One finds a festival of one complex variable analytical tools (the Maclaurin formula, tools for the evaluations of asymptotic behaviors, path integration, etc.).

Many more results can be found in the monograph (the three-color problem, the hard square model, the F model on the triangular lattice, three coloring of the edges of the hexagonal lattice ...). Let us mention, in particular, the six-vertex model with *site-dependent* weights (which can be considered as the first example of a Z-invariant model). Let us introduce $\omega_j(I, J)$ where $j = 1$,

[11] In agreement with the inversion relations on the model.

[12] With, for instance, a prejudice of algebraicity of the critical manifolds of "solvable" models: all examples known in the literature are polynomial expressions in well-suited variables e^{K_i}. These include, for instance, the critical varieties of the anisotropic Ising, or Potts, models on square, and triangular lattices, or the critical varieties of the Baxter model. For non-integrable models the common wisdom is, probably, that critical manifolds are always analytic, or may be differentiable, and the algebraicity of the critical manifolds is ruled out by the non-integrability. This is also a naive point of view: see (36) in section (VI-1).

$\cdots, 6$, are the six possible Boltzmann factors of the vertex in row I and column J, and let us require that the algebraic invariant Δ be independent of I, J:

$$\Delta = \frac{\omega_1(I,J)\omega_2(I,J) + \omega_3(I,J)\omega_4(I,J) - \omega_5(I,J)\omega_6(I,J)}{2 \cdot (\omega_1(I,J)\omega_2(I,J)\omega_3(I,J)\omega_4(I,J))^{1/2}}. \tag{8}$$

Up to a multiplicative factor $\square_{I,J}$, a (rational) parametrization of these invariance conditions (8) is:

$$\begin{aligned}
&\omega_1(I,J) = (1 - t \cdot p_{I,J}) \cdot \alpha_{I,J}\beta_{I,J}, \\
&\omega_2(I,J) = (1 - t \cdot p_{I,J}) \cdot \frac{1}{\alpha_{I,J}\beta_{I,J}}, \\
&\omega_3(I,J) = (p_{I,J} - t) \cdot \frac{\alpha_{I,J}}{\beta_{I,J}}, \quad \omega_4(I,J) = (p_{I,J} - t) \cdot \frac{\beta_{I,J}}{\alpha_{I,J}}, \\
&\omega_5(I,J) = \Big(\frac{1}{t} - t\Big) \cdot p_{I,J} \cdot \gamma_{I,J}, \quad \omega_6(I,J) = (1 - t^2) \cdot \frac{1}{\gamma_{I,J}}.
\end{aligned} \tag{9}$$

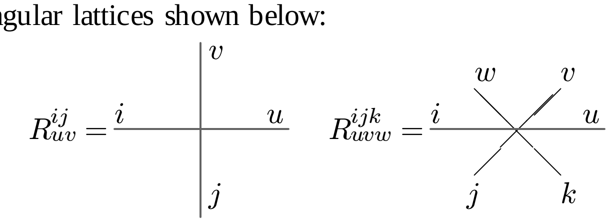
Baxter's Z-invariance condition for integrability requires that the $p_{I,J}$'s are actually products of a (spectral) parameter depending on the row and another parameter depending on the column: $p_{I,J} = \rho_I \cdot \sigma_J$. We will see in section (7) when sketching the correspondence between the standard scalar Potts model and a staggered asymmetric six-vertex model, that these product conditions, $p_{I,J} = \rho_I\sigma_J$, actually correspond in the case of the checkerboard Potts model to criticality, or to the vanishing conditions of a staggering field H_{stag} from a Lee-Yang zeros viewpoint: $|z| = e^{H_{\text{stag}}} = 1$.

Provided that the $p_{I,J} = \rho_I \cdot \sigma_J$ integrability conditions are satisfied, the partition function, with parametrization (9), can be expressed as a multiplicative closed formula:

$$\begin{aligned}
&Z = 2 \prod_{I=1}^{M}\prod_{I=1}^{M} \frac{\sqrt{\omega_5(I,J)\omega_6(I,J)}}{1 - t^2} \cdot F(\rho_I\sigma_J) \cdot F\Big(\frac{1}{\rho_I\sigma_J}\Big), \\
&\qquad \text{where:} \qquad F(z) = \prod_{m=1}^{\infty} \frac{1 - t^{4m-1}z}{1 - t^{4m+1}z}.
\end{aligned} \tag{10}$$

III-2. Vertex models: free fermions

Another classic work of F. Y. Wu is his 1970 paper with C. Fan in which they coined the term the *free-fermion model* [16]. This work was later extended to its checkerboard version during one of Wu's visits to Taiwan [50, 52]. In the following we shall arrange the homogeneous vertex weights in a matrix R, whose size and form vary according to the number of edge states and the coordination number of the lattice. Typical examples we will consider are the 2D square and triangular lattices shown below:



2D square 2D triangular

III-2-1. Free-fermion asymmetric eight-vertex model

C. Fan and F. Y. Wu obtained many free-fermion results [12, 16]. The free energy of the most general free-fermion model on a square lattice evaluated by Fan and Wu reads:

$$f = \frac{1}{16\pi^2}\int_0^{2\pi} d\theta\, d\phi\, \ln\Big(2a + 2b\cos\theta + 2c\cos\phi + 2d\cos(\theta-\phi) + 2\,e\,\cos(\theta+\phi)\Big),$$

where:

$$a = \frac{1}{2}\cdot(\omega_1^2+\omega_2^2+\omega_3^2+\omega_4^2), \qquad b = \omega_1\,\omega_3 - \omega_2\,\omega_4,$$
$$c = \omega_1\,\omega_4 - \omega_2\,\omega_3, \qquad d = \omega_3\,\omega_4 - \omega_7\,\omega_8, \qquad e = \omega_3\,\omega_4 - \omega_5\,\omega_6,$$

provided that the free fermion condition:

$$\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8 \tag{11}$$

is satisfied.

Let us revisit some of their results from an inversion relation viewpoint. Renaming the vertex weights as $a=\omega_1$, $a'=\omega_2$, $b=\omega_3$, $b'=\omega_4$, $c=\omega_5$, $c'=\omega_6$, $d=\omega_7$, $d'=\omega_8$, the matrix R of the eight-vertex model is then:

$$R = \begin{pmatrix} a & 0 & 0 & d' \\ 0 & b & c' & 0 \\ 0 & c & b' & 0 \\ d & 0 & 0 & a' \end{pmatrix}. \tag{12}$$

A matrix of the form (12) can be brought, by a similarity transformation, to a block-diagonal form:

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}, \quad \text{with} \quad R_1 = \begin{pmatrix} a & d' \\ d & a' \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} b & c' \\ c & b' \end{pmatrix}.$$

If one introduces $\delta_1 = aa' - dd'$ and $\delta_2 = bb' - cc'$, the determinants of the two blocks, then the (homogeneous) matrix inverse I (namely $R \to \det(R)\cdot R^{-1}$) reads:

$$\begin{aligned} (a, a', d, d') \quad &\rightarrow \quad (a'\cdot\delta_2,\, a\cdot\delta_2, -d\cdot\delta_2, -d'\cdot\delta_2) \\ (b, b', c, c') \quad &\rightarrow \quad (b'\cdot\delta_1,\, b\cdot\delta_1, -c\cdot\delta_1, -c'\cdot\delta_1). \end{aligned} \tag{13}$$

It is straightforward to see that the free-fermion condition (11) is $\delta_1 = -\delta_2$ which has the effect of *linearizing* the inversion (13) into an involution given by:

$$a \leftrightarrow a', \quad b \leftrightarrow -b', \quad (d,\, d') \rightarrow (-d,\, d'), \quad b \leftrightarrow -b', \quad (c,\, c') \rightarrow (c,\, c')$$

The group generated by the two inversion relations of the model is then realized by permutations of the entries mixing with sign changes, and its orbits are thus *finite*. The *finiteness* condition of the group is a common feature of all free-fermion models.

III-2-2. Free-fermion for the 32-vertex model on a triangular lattice

We next consider the free-fermion conditions of J. E. Sacco and F. Y. Wu [53] for the 32-vertex model on a triangular lattice. Using the same notation as in [53], we have:

$$R = \begin{bmatrix} f_0 & 0 & 0 & f_{23} & 0 & f_{13} & f_{12} & 0 \\ 0 & f_{36} & f_{26} & 0 & f_{16} & 0 & 0 & \bar{f}_{45} \\ 0 & f_{35} & f_{25} & 0 & f_{15} & 0 & 0 & \bar{f}_{46} \\ f_{56} & 0 & 0 & \bar{f}_{14} & 0 & \bar{f}_{24} & \bar{f}_{34} & 0 \\ 0 & f_{34} & f_{24} & 0 & f_{14} & 0 & 0 & \bar{f}_{56} \\ f_{46} & 0 & 0 & \bar{f}_{15} & 0 & \bar{f}_{25} & \bar{f}_{35} & 0 \\ f_{45} & 0 & 0 & \bar{f}_{16} & 0 & \bar{f}_{26} & \bar{f}_{36} & 0 \\ 0 & \bar{f}_{12} & \bar{f}_{13} & 0 & \bar{f}_{23} & 0 & 0 & \bar{f}_0 \end{bmatrix}. \qquad (14)$$

By permuting rows and columns, this matrix can be brought into the block diagonal form:

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}, \qquad \text{with:} \qquad (15)$$

$$R_1 = \begin{bmatrix} f_0 & f_{13} & f_{12} & f_{23} \\ f_{46} & \bar{f}_{25} & \bar{f}_{35} & \bar{f}_{15} \\ f_{45} & \bar{f}_{26} & \bar{f}_{36} & \bar{f}_{16} \\ f_{56} & \bar{f}_{24} & \bar{f}_{34} & \bar{f}_{14} \end{bmatrix}, \quad R_2 = \begin{bmatrix} f_{14} & f_{34} & f_{24} & \bar{f}_{56} \\ f_{16} & f_{36} & f_{26} & \bar{f}_{45} \\ f_{15} & f_{35} & f_{25} & \bar{f}_{46} \\ \bar{f}_{23} & \bar{f}_{12} & \bar{f}_{13} & \bar{f}_0 \end{bmatrix}. \qquad (16)$$

The inverse I, written polynomially (homogeneous matrix inverse), is now a transformation of degree 7. If one introduces the two determinants, $\Delta_1 = \det(R_1)$ and $\Delta_2 = \det(R_2)$, then each term in the expression of $I(R)$ is a product of a degree three minor, taken within a block, times the determinant of the other block. This inverse I clearly singles out one of the three directions of the triangular lattice. These three involutions do not commute and generate a quite large infinite discrete group $\Gamma_{\rm triang}$ (see also section (IV-1) below).

The free-fermion conditions of Sacco and Wu [53] read:

$$\begin{aligned} f_0 f_{ijkl} &= f_{ij} f_{kl} - f_{ik} f_{jl} + f_{il} f_{jk}, \qquad \forall\, i,j,k,l = 1,\ldots,6 \\ f_0 \bar{f}_0 &= f_{12}\bar{f}_{12} - f_{13}\bar{f}_{13} + f_{14}\bar{f}_{14} - f_{15}\bar{f}_{15} + f_{16}\bar{f}_{16}, \end{aligned} \qquad (17)$$

which we denote by $\mathcal{V}$. What is remarkable is that, not only is the rational variety $\mathcal{V}$ globally invariant under $\Gamma_{\rm triang}$, but *again* the realization of this (generically very large infinite discrete group) $\Gamma_{\rm triang}$ on this variety becomes *finite*. This comes about from the degeneration of I into a mixture of sign changes and permutations of the entries, as in the preceding subsection.

Remark. The ordinary *matrix product of three matrices* (14) *solutions* of (17), is *another solution*! In other words, if R_α, R_β, $R_\gamma \in \mathcal{V}$, then $R_\alpha \cdot R_\beta \cdot R_\gamma \in \mathcal{V}$, while $R_\alpha \cdot R_\beta \notin \mathcal{V}$. This was also the case for involutions of (11) in the case of the square lattice, but the mechanism is more subtle here as the conditions (17) imply $\Delta_1 = \Delta_2$.

III-3. Dimers and spanning trees

Before Fan and Wu's free-fermion vertex models, the Onsager solution of the two-dimensional Ising model was clearly the first free-fermion model ever solved. There were also several approaches to the two-dimensional Ising model that did not use the transfer matrix formalism; the most interesting one is perhaps the mapping of the problem onto a dimer-covering problem on a slightly more complicated lattice. The dimer problem was first solved by Temperley-Fisher and Kasteleyn. Kasteleyn found out how to treat the most general planar graph.

The dimer problem has a life of its own and has generated since many followup works, not only in statistical mechanics, but also in combinatorial theory. In this regard, Wu has provided a large number of new results [37, 173, 184, 194, 207, 205], including applications to condensed matter physics, as well as in pure combinatorial analysis. In addition, Wu has obtained new results on the spanning tree problem [198, 200], a problem intimately related to the dimer problem through a bijection due to Temperley. In the following we shall describe some of the contributions in this area.

III-3-1. Revisiting dimers: the honeycomb lattice

The dimer model on the honeycomb lattice was first solved by Kasteleyn, but he never published the solution, except for hinting at the existence of a transition. This deficiency was made up by Wu in a 1968 paper [10] in which he presented details of the analysis for the honeycomb lattice, and applied the results to describe the physics of a modified KDP model.

III-3-2. Revisiting dimers: Interacting dimers in 2 and 3 dimensions

Almost 30 years after the publication of the solution for the dimers on the honeycomb lattice [10], Wu and his co-workers made two important extensions of the earlier Kasteleyn solution. In the first, H. Y. Huang, F. Y. Wu, H. Kunz, and D. Kim [173] considered the case where the dimers have nearest-neighbor interaction. This model turns out to be identical to the most general five-vertex model, a degenerate case of the six-vertex model which requires a special Bethe ansatz analysis. The resulting phase diagram of this five-vertex model is very complicated and the analysis extremely lengthy.

In the second work H. Y. Huang, V. Popkov and F. Y. Wu [177, 184] introduced, and solved, a three-dimensional model consisting of layered honeycomb dimer lattices, as described in the preceding subsection, but with a specific layer-layer interaction. Again, the phase diagram is very complicated. It is noted that this model is the only solvable three-dimensional lattice model with *physical* Boltzmann weights (the Baxter solution of the 3D Zamolodchikov model has negative weights). However, the layered dimer model, while having strictly positive weights, describes dimer configurations in which the dimers are confined in planes. As a consequence the critical behavior is essentially two-dimensional.

III-3-3. Revisiting dimers: a continuous-line model

F. Y. Wu and H. Y. Huang [158] have further used a dimer mapping to solve a continuous-line lattice model in three and higher dimensions. They have also applied it to model a type-II superconductor [160. In three dimensions, the model is a special case of an $O(n)$ model on a finite $L_1 \times L_2 \times L_3$ cubic lattice with periodic boundary conditions with the partition function:

$$Z(n) = \sum_{closed\ polygons} n^l \cdot z^b,$$

where the summation is taken over all closed non-intersecting polygonal configurations, l is the number of polygons, and b is the number of edges of each configuration. They considered the $n = -1$ special case, which they showed to be in one-to-one correspondence with a dimer problem whose partition function can be evaluated as a *Pfaffian*. The result for a finite lattice is:

$$Z(-1) = \prod_{n_1=1}^{L_1} \prod_{n_2=1}^{L_2} \prod_{n_3=1}^{L_3} \left| 1 + \sum_{i=1}^{3} z \cdot e^{2\pi n_i / N_i} \right| .$$

In the thermodynamic limit, this leads to the per-site free energy:

$$f = \frac{1}{(2\pi)^3} \cdot \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^{2\pi} d\theta_3 \, \ln \left| 1 + \sum_{i=1}^{3} z \cdot e^{2\pi n_i / N_i} \right| .$$

The phase diagram is rich and quite non-trivial.

However, it must be said that this exactly soluble three-dimensional $\mathrm{O}(-1)$ model describes line configurations running only in a *preferred direction* and, secondly,[13] the Boltzmann weights can be negative.

III-3-4. Revisiting dimers: nonorientable surfaces

More recently W. T. Lu and F. Y. Wu initiated studies on dimers and Ising models on *nonorientable* surfaces [194, 203, 205]. For dimers on an $M \times N$ net, embedded on non-orientable surfaces, they solved *both* the Möbius strip and the Klein bottle problems for all sizes M and N and obtained the dimer generating function $Z_{M,N}$ as:

$$\frac{Z_{M,N}}{z^{M\,N/2}} = \mathrm{Re}\Big((1-i) \cdot \prod_{m=1}^{(M+1)/2} \prod_{n=1}^{N} \Big(2i(-1)^{M/2+m+1} \sin\Big(\frac{(4\,n-1)\pi}{2N} \Big) + 2X_m \Big) \Big),$$

where Re denotes the real part, z_v and z_h are the dimer weights in the vertical and horizontal directions, respectively, and X_m is given for the Möbius strip and the Klein bottle respectively by:

$$X_m = \frac{z_v}{z_h} \cdot \cos\Big(\frac{m\pi}{M+1} \Big), \qquad X_m = \frac{z_v}{z_h} \cdot \cos\Big(\frac{(2m-1)\pi}{M} \Big).$$

In paper [205] they also obtained an *extension of the Stanley-Propp reciprocity theorem* for dimers[14]. Inspired by this work, there is now much activity in this area. There is also very much current interest in finite-size corrections and conformal field theories on more complicated surfaces (higher genus pretzels, ...).

III-3-5. Dimers on a square lattice with a boundary defect

In a very recent paper [207], fittingly dedicated to the 70th birthday of Michael Fisher, who first solved the dimer problem for the square lattice, W. J. Tzeng and F. Y. Wu obtained the dimer

[13] But we are used to this after R. J. Baxter's solution of the 3D Zamolodchikov model.

[14] A subject matter of pertinent interest to mathematicians.

generating function for the square lattice with one corner (or some other boundary site) of the lattice missing. In this work they made use of a bijection between the dimer and spanning tree configurations due to Temperley (and extended by Wu in his unpublished 1976 lecture notes as well as well as more recently by Kenyon, Propp and Wilson). They also carried out a finite-size analyses which lead to a logarithmic correction term in the large-size expansion for the vacancy problem with free boundary conditions. They found a central charge $c = -2$ for the vacancy problem, to be compared with $c = -1$ when there is no vacancy. This central charge $c = -2$ is in contradiction with the prediction of a naive conformal field theory.

III-3-6. Spanning trees

As mentioned above, the problem of spanning trees in graph theory is intimately related to the dimer problem, and it is not surprising that Wu found his way to spanning trees. In 1977 he published a paper [62] on the counting of spanning trees on two-dimensional lattices using the equivalence with a Potts model. Very recently he refined the tools by using a result in algebraic graph theory, which he and W. J. Tzeng rederived using elementary means. Tzeng and Wu enumerated spanning trees for general d-dimensional lattices as well as non-orientable surfaces [198]. Applying these results to general graphs and regular lattices, R. Shrock and F. Y. Wu [200] published a lengthy paper in which they established new theorems on spanning trees as well as enumerating spanning trees for a large number of regular lattices in the thermodynamic limit.

III-4. Miscellaneous results on vertex models

In this section we describe an arbitrary choice of miscellaneous results obtained by F. Y. Wu on vertex models.

III-4.1. Boundary conditions

The six-vertex model is known to be a boundary condition dependent model. However H. J. Brascamp, H. Kunz and F. Y. Wu [43] established, for the first time, that, at sufficiently low temperatures or sufficiently high fields, the six-vertex models with either periodic or free boundary conditions are equivalent.

III-4-2. The eight-vertex model in a field

A simple result due to F. Y. Wu [105] is that a very general staggered eight-vertex model in the Ising language (as introduced by Kadanoff and Wegner and by F. Y. Wu [78]) with the special Yang-Lee magnetic field $i\pi kT/2$, is equivalent to Baxter's symmetric eight-vertex model and hence is soluble. This result is remarkable, since the general eight-vertex model without this field is not known to be soluble.

III-4-3. The eight-vertex model on the honeycomb lattice

F. Y. Wu has always been keen in providing results for the *honeycomb* lattice [48, 130]. One interesting result is that he has established the exact equivalence of the eight-vertex model on the honeycomb lattice with an Ising model in a nonzero magnetic field [48, 130]. The equivalence also leads to exact analysis of the Blume-Emery-Griffiths model for the honeycomb lattice (for details see section (IV-3) below).

In pursuit of applications of these results, P. Pant and J. H. Barry and Wu [181, 182] obtained exact results for a model of a ternary polymer mixture which is equivalent to an eight-vertex model. A model ternary polymer mixture was considered with bi- and tri-functional monomers and a solvent placed on the sites of a honeycomb lattice. Using the equivalence with an eight-vertex model which further maps the problem into an Ising model in a nonzero magnetic field, an exact analysis of the model was carried out. The phase boundary of the three-phase equilibrium polymerization regime was determined exactly.

Comment: These kinds of results are particularly interesting when one realizes that concepts and structures corresponding to the Yang-Baxter integrability do not exist at first sight. This leads to the natural question: *how to construct a Yang-Baxter relation for the honeycomb lattice*?

III-4-4. Exact critical line of a vertex model in 3 dimensions

Wu [46] has introduced a vertex model in three dimensions with real vertex weights, and determined its exact first-order phase transition line by mapping it to an Ising model in a field. It also exhibits a critical point. This is one of the very few lattice statistical models for which exact results can be deduced in higher-than-two dimensions.

IV. Spin models: Ising models and other models

We will consider in this section F. Y. Wu's results on *spin models*, mostly *Ising* models. Due to its importance the Potts models will be treated separately in section (7).

The distinction between vertex models and spin models (more generally Interaction Round a Face (IRF) models) is an important one in lattice statistical mechanics. However Wu showed in paper [78] an equivalence between an Ising model with a vertex model. Similarly in paper [114] Wu and K. Y. Lin studied the Ising model on the Union Jack lattice, showing it to be a free-fermion model. Many of the free-fermion results on the vertex models in sections (III-2-1) and (III-3) can also be re-styled as free-fermion Ising models.

As far as the Union Jack lattice is concerned Wu has also obtained the spontaneous magnetization of the three-spin Ising model [51]. It is, in fact, obtained in terms of the magnetic and ferroelectric orderings of the eight-vertex model, or, equivalently, the spontaneous magnetization and polarization of the eight-vertex model. It was found that the two sublattices possess different critical exponents.

An important development in the history of lattice models is the analysis of the phase diagram of the Ashkin-Teller model on the square lattice by F. Y. Wu and K. Y. Lin [47]. The Ashkin-Teller model is another example of spin models for which the traditional distinction of lattice statistical mechanics between spin and vertex models is irrelevant. The Ashkin-Teller model can be seen as two Ising models coupled together with four-spin interactions. Performing a dual transformation on one of the two Ising models and interpreting the result as a vertex model, one finds that the Ashkin-Teller model is equivalent to a *staggered* eight-vertex model [29], thus exhibiting two phase transitions.

Wu's analysis of spin models was not restricted to two dimensions. For instance, Barry and Wu have obtained exact results for a four-spin-interaction Ising model on the three-dimensional pyrochlore lattice [128], and Wu has also obtained various results for spin models on the Bethe lattice and Cayley trees [54, 56].

Wu also performed *real-space renormalization* studies for Ising models [129], but, not surprisingly, using some duality ideas, namely, the duality-decimation transformation of T. W.

Burkhardt. Wu had previously applied the duality-decimation transformation in order to solve the two-dimensional Ising model with nearest-neighbor, next-nearest-neighbor and four-spin interactions in a pure imaginary field [105] (see section III-4.2). In paper [129] Burkhardt's method, which combines a bond-moving and duality-decimation transformation, is modified, in order to preserve the free energy in the renormalization transformation.

IV-1. Generalized transmissivities for spin models

We will see below that a large number of F. Y. Wu's work correspond to graph expansions (see section (VII-1)). For spin models with edge interactions this requires the introduction of certain "transmissivity" variables. Thermal transmissivities are introduced when considering high-temperature expansions of an edge-interaction spin model or performing renormalization analyses. They are also the natural variables to use in the decimation of spins in a simple multiplicative way.

Introduce the edge Boltzmann weight $W(K_1, K_2, \cdots K_n; a, b)$, where $K_1, K_2, \cdots K_n$ denote a set of coupling constants describing the model, and a and b are two nearest-neighbor spin states which can take on q values. Let us assume that the decimation procedure yields a Boltzmann weight of the *same* form:

$$\sum_b W(K_1, K_2, \cdots K_n; a, b) \cdot W(K_1', K_2', \cdots K_n'; b, c) = \lambda \cdot W(K_1'', K_2'', \cdots K_n''; a, c). \quad (18)$$

Alternatively, one can build a $q \times q$ Boltzmann matrix $\mathbf{W}$ with entries $\mathbf{W}_{i,j} = W(K_1, K_2, \cdots K_n; i, j)$. In terms of such matrices relation (18) becomes $\mathbf{W} \cdot \mathbf{W}' = \mathbf{W}''$. The decimation procedures, and also the high-temperature expansions in such models, are greatly simplified by introducing a "transmissivity" function t_α, such that the matrix relation $\mathbf{W} \cdot \mathbf{W}' = \mathbf{W}''$ becomes one or more multiplicative relations of the form:

$$t_\alpha(\mathbf{W}) \cdot \mathbf{t}_\alpha(\mathbf{W}') = \mathbf{t}_\alpha(\mathbf{W}''), \qquad \alpha = \mathbf{1}, \cdots, \mathbf{r}.$$

The simplest example is the transmissivity variable for the q state standard scalar Potts model, for which one has $t = (e^K - 1)/(e^K + q - 1)$. This is the natural expansion variable for the high-temperature series of the model (see also the $f_{i,j}$'s in (45) and (46) introduced in section (VII-1) below). For the Ising model this reduces to the $\tanh(K)$ variable. F. Y. Wu *et al*. [147] underlined the fact that two quite different situations must be considered. If the family of Boltzmann matrices $\mathbf{W}$ is a set of *commuting* matrices, then they can be diagonalized simultaneously and the transmissivity variables are nothing but all the possible ratios of eigenvalues of the Boltzmann matrices $\mathbf{W}$. If, alternatively, the Boltzmann matrices $\mathbf{W}$ do not commute, then one must perform a simultaneous block-diagonalization of this family of Boltzmann matrices, and, therefore, some of the t_α's will be block matrices from which one can extract functions ϕ_α satisfying $\phi_\alpha(\mathbf{W}'') = \phi_\alpha(\mathbf{W}) \cdot \phi_\alpha(\mathbf{W}')$. One obvious choice for ϕ_α is the (ratio of) determinants of these blocks. A number of non-trivial non-commuting transmissivities are given in [147].

IV-2. Three-spin interactions: the Baxter-Wu model

Another important work in the history of exact solutions of lattice statistics is the Baxter-Wu model, which is an Ising model on the triangular lattice with *three-spin* interactions. This model was solved exactly by R. J. Baxter and F. Y. Wu in 1973 [45, 49].

The three spins surrounding every triangular face interact with a *three-body* interaction of strength $-J$, so that the Hamiltonian reads:

$$H = -J \cdot \sum \sigma_i \sigma_j \sigma_k. \tag{19}$$

Baxter and Wu found that the per-site partition function Z has a remarkably simple expression:

$$Z = \sqrt{6yt} \qquad \text{with:} \qquad t = \sinh(2\,|J|/kT), \tag{20}$$

and where y is the solution of the algebraic equation:

$$(y-1)^3\,(1+3y)\,(1+t^2)\cdot t = 2(1-t)^4 \cdot y^3. \tag{21}$$

The partition function has a singular part which behaves as $|t-1|^{4/3}$.

Some interesting duality properties of the Baxter-Wu model are very clearly detailed in [45], and used to convert the Baxter-Wu model into a *coloring* problem. This provides a very heuristic example showing that duality is *not* specific to edge-interaction spin models, but can also be introduced with many-body interactions. In the following we briefly describe how the Baxter-Wu model is transformed into a coloring problem.

First we introduce a $\mathbf{Z}_2$-Fourier transform with function $g(\lambda, \mu)$ which enables us to simply write the Kramers-Wannier duality for this three-spin model (λ and μ are Ising spins) as:

$$\begin{aligned} g(\lambda,\mu) &= +1 \quad \text{if:} \quad \lambda = +1, \\ g(\lambda,\mu) &= \mu \quad\;\; \text{if:} \quad \lambda = -1. \end{aligned} \tag{22}$$

Note that this function is symmetric in λ and μ, namely, $g(\lambda, \mu) = +1$ when $\mu = +1$ and $g(\lambda, \mu) = \lambda$ when $\mu = -1$.

Returning to the Baxter-Wu model, each spin σ_i of the triangular lattice belongs to six triangles around vertex i, which form a hexagon with the spin σ_i at the center. Let us now consider the close-packing of such hexagons. The spins σ_i now form a (triangular) sublattice of the initial triangular lattice.

Consider next the spins σ_i, and denote the edge connecting nearest-neighboring spins, σ_k and σ_l, sitting on the hexagon surrounding σ_i, by $< kl >$. Let us introduce Ising edge variables λ_r corresponding to the six edges $< kl >$ of the hexagon: $\lambda_r = \sigma_k \cdot \sigma_l$, $r = 1, \cdots 6$. The local Boltzmann weight of a hexagonal cell around a spin σ_i can be written as:

$$W_{\text{hex}} = \frac{1}{2} \cdot \Big(1 + \prod_{r=1}^{6} \lambda_r\Big) \cdot \exp\Big(K \cdot \sigma_i \cdot \sum_{r=1}^{6} \lambda_r\Big), \tag{23}$$

where the factor $(1 + \prod \lambda_r)$ takes into account the fact that the Ising edge variables λ_r are not independent, but are constrained by the condition $\prod \lambda_r = 1$. As usual this condition, associated with every hexagon, can be written by introducing a dummy variable μ_i also associated with every hexagon:

$$\sum_{\mu_i=\pm 1} \prod_{r=1}^{6} g(\lambda_r, \mu_i) = 1 + \prod_{r=1}^{6} \lambda_r,$$

enabling us to rewrite (23) as:

$$W_{\rm hex} = \sum_{\mu_i=\pm 1} \prod_{r=1}^{6} g(\lambda_r, \mu_i) \cdot \exp\Big(K \cdot \sigma_i \cdot \sum_{r=1}^{6} \lambda_r \Big).$$

The partition function of the Baxter-Wu model is now seen as a summation over all the (initial) spins σ_i and the (dummy) spins μ_i of a triangular sublattice, and the edge Ising spins λ_r. Let us focus on one λ_r. The edge $r =< kl >$ belongs to a hexagon around spin σ_i and a neighboring hexagon around another spin, say, σ_j. The edge Ising spin λ_r thus occurs in the Boltzmann factors with a factor of

$$W_{\lambda_r} = e^{K \cdot (\sigma_i + \sigma_j) \cdot \lambda_r} \cdot g(\lambda_r, \mu_i) \cdot g(\lambda_r, \mu_j).$$

Summing over the edge Ising spin λ_r in the partition function and using relations (22), one thus obtains a factor:

$$\omega_{ij} = \sum_{\lambda_r=\pm 1} W_{\lambda_r} = e^{K(\sigma_i+\sigma_j)} + \mu_i \mu_j \cdot e^{-K(\sigma_i+\sigma_j)} \qquad (24)$$

between two spins σ_i and σ_j on the sublattice. This can be interpreted as the edge weight associated with a coloring problem, and the Baxter-Wu model is transformed into a coloring problem.

IV-3. The Blume-Emery-Griffiths model

The Blume-Emery-Griffiths (BEG) model is a model that F. Y. Wu and his coworkers quite naturally considered [106, 116, 136, 148], since it reduces to an Ising model on the honeycomb lattice on a special manifold [106, 116].

The BEG model is defined by the Hamiltonian:

$$-\beta \cdot H = -J \cdot \sum_{<i,j>} S_i S_j - K \cdot \sum_{<i,j>} S_i^2, S_j^2 - \Delta \sum_i S_i^2 - H \cdot \sum_i S_i, \qquad (25)$$

where the spins are classical spin-1 spins taking on the values $S_i = 0, \pm 1$.In the high-temperature expansion the nearest-neighbor Boltzmann factor assumes the form

$$\exp(J\, S_i\, S_j + K\, S_i^2\, S_j^2) = 1 + (e^K \sinh J) S_i S_j + (e^K \cosh J - 1) S_i^2 S_j^2. \qquad (26)$$

It follows then in the subspace $K = -\ln(\cosh J)$, one has the simple relation

$$\exp(J S_i S_j + K S_i^2 S_j^2) = 1 + S_i\, S_j\, \tanh(J),$$

and the partition function of the BEG model assumes the simpler form:

$$Z_{BEG} = \sum_{S_i=0,\pm 1} \prod_{<i,j>} (1 + S_i S_j \tanh J) \prod_i \exp(-\Delta S_i^2 + H S_i).$$

Expanding the products over neighboring pairs, representing each term by a graph and making use of the identities

$$\sum_{S_i=0,\pm 1} S_i^n \cdot e^{-\Delta S_i^2} = \rho(n) \qquad \text{with:} \qquad \rho(2) = 2e^{-\Delta},$$
$$\rho(0) = 2e^{-\Delta} + 1, \qquad \rho(1) = 0, \qquad (27)$$

one finds that one has eight possible configurations at each vertex of the honeycomb lattice, corresponding to the following Boltzmann weights of an (isotropic) eight-vertex model:

$$a = 1 + 2e^{-\Delta}\cosh(H), \qquad b = 2\sqrt{\tanh(J)}e^{-\Delta}\sinh(H),$$
$$c = 2\tanh(J)e^{-\Delta}\cosh(H), \quad d = 2(\tanh(J))^{3/2}e^{-\Delta}\sinh(H).$$

With these notations one deduces the identity of the partition function of the BEG model (25) with the eight-vertex model on a honeycomb lattice (see also sections (V-1) and (V-3) below):

$$Z_{BEG} = Z_{8v}(a, b, c, d).$$

Performing a weak-graph duality transformation on this eight-vertex model (see section (V) below) associated with the 2×2 (gauge) matrix [48]:

$$g_1 = g_2 = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & y \\ -y & 1 \end{pmatrix},$$

one finds that the partition function of the eight-vertex model $Z_{8v}(a, b, c, d)$ remains invariant under a weak-graph duality transformation [7, 116]:

$$\tilde{a} = (a + 3yb + 3y^2c + y^3d)/(1 + y^2)^{3/2}, \;\cdots\; Z_{8v}(a, b, c, d) = Z_{8v}(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}). \qquad (28)$$

The four parameters a, b, c, d or $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ can be seen, as far as the calculation of the partition function is concerned, as four *homogeneous* parameters. Taking into account an irrelevant overall factor and the irrelevant gauge variable y from the weak-graph symmetry (28), one sees that the partition function of the eight-vertex model $Z_{8v}(a, b, c, d)$ basically depends on two variables instead of four. Not surprisingly, Wu found that $Z_{8v}(a, b, c, d)$ is equivalent to the partition function of an Ising model with nearest-neighbor interactions K_I and a magnetic field L:

$$\begin{aligned} Z_{\rm Ising}(L, K_I) &= \sum_{\sigma} \prod_{<ij>} \exp(K_I\sigma_i\sigma_j) \prod_i \exp(L\sigma_i) \\ &= Z_{8v}(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) \cdot \left(\frac{2\cosh(L)\cosh^{3/2}(K_I)}{\tilde{a}} \right)^N, \end{aligned} \qquad (29)$$

where N is the number of lattice sites. The explicit expressions of K_I and L in terms of the BEG parameters are complicated. But for $H = 0$, one has $L = 0$ and

$$\tanh(K_I) = \frac{2}{2 + e^{\Delta}} \cdot \tanh(J),$$

using which one determines the critical line $K_I = 1/\sqrt{3}$ in the $J > 0$ regime. The spontaneous magnetization of the BEG model for $J > 0$ and the phase boundary of the $J < 0$ BEG model can be similarly determined [116].

The proof of the equivalence of the honeycomb eight-vertex model with an Ising model in a field, as outlined in the above, is quite tedious. However, a more direct derivation has since been given by Wu [130].

The eight-vertex model on the honeycomb lattice can also be seen to be related to a lattice-gas grand partition function $\Xi^{Kag}(z, J, J_3)$ on the *Kagomé lattice* [126] (see also section (VI-3) below), where $-J$ is the nearest-neighbor interaction, $-J_3$ the triplet interactions existing among three sites surrounding a triangular face of the lattice, and z denotes the fugacity. Then one has the equivalence:

$$\Xi^{Kag}(z, J, J_3) = Z_{8v}(a, b, c, d)$$
$$a = 1, \quad b = \sqrt{z}, \quad c = z \cdot e^J, \quad d = z^{3/2} \cdot e^{3J+J_3}. \tag{30}$$

From (29) and (30), F. Y. Wu and X. N. Wu were able to obtain results for the liquid and vapor densities, showing that an observed anomalous critical behavior occurs in the lattice gas only when there are nonzero triplet interactions [126]. This analysis has been extended to a lattice gas on the 3-12 lattice by J. L. Ting, S. C. Lin and F. Y. Wu [140].

Wu's tricks for the honeycomb BEG model are not limited to the weak-graph transformation for the eight-vertex model. Using a syzygy analysis of the invariants under the O(3) transformation L. H. Gwa and F. Y. Wu have obtained an expression for the critical variety of the honeycomb BEG model to an extremely high degree of accuracy [148] (see section (V-4) below).

IV-4. Other spin results: disorder points

Let us finally describe, among many results obtained by F. Y. Wu on spin models, one result concerning disorder points. Disorder solutions are particularly simple solutions corresponding to some "dimensional reduction" of the model, which provide simple exact results for models which are generically quite involved. While this yields severe constraints on the phase diagrams, the series expansion, and the analyticity properties of the model, it does lead to exact solutions of models which are otherwise nonintegrable.

For example, using a decimation approach, Wu [100] has deduced the disorder solution for the triangular Ising model in a nonzero magnetic field. Wu and K. Y. Lin [120] have used a checkerboard Ising lattice to illustrate that there may exist more than one disorder point in a given spin system. Along the same vein, N. C. Chao and Wu [101] have explored the validity of the decimation approach by considering the disorder solutions of a general checkerboard Ising model in a field.

V. Weak-graph dualities and Hilbert's syzygies

In a pioneering paper F. Y. Wu and Y. K. Wang [58] introduced a duality transformation for a general spin model which can have chiral interactions. This is the first time that a chiral spin model was explicitly considered. In terms of R matrices such as (12) these transformations are the tensor product of two similarities:

$$R \quad \longrightarrow \quad g_1 \quad g_2 \cdot R \cdot g_1^{-1} \quad g_2^{-1} \tag{31}$$

where g_1 and g_2 are two $q \times q$ matrices, R is a $q^2 \times q^2$ matrix ($q = 2$ for the sixteen-vertex model). This symmetry group is an $\mathrm{sl}(q) \times \mathrm{sl}(q)$ symmetry group. The high- and low-temperature duality (3) given in section (III-1) for the sixteen-vertex model is a particular case of such transformations, corresponding to g_1 and g_2 being two involutions:

$$g_1 = g_2 = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

This duality relation [58, 131] is now well-known for vertex models. It corresponds to symmetries of the model and can be used, as will be seen in the next two subsections, to find good variables to express the critical manifolds of a lattice model, and, hopefully, to determine their exact expressions when algebraic.

V-1. Hilbert's syzygies, gauge-like dualities and critical manifolds

Let us give some hints as to how the gauge-like dualities enable us to deduce results on critical manifolds or varieties. The main idea is to construct algebraic invariants under these gauge transformations.

Hilbert has shown that all invariants of a linear transformation are algebraic and can be expressed in terms of a set of homogeneous polynomials, the *syzygies*. Considering the sixteen-vertex model, the transformation is O(2) and the fundamental invariants corresponding to the O(2) group have been constructed by J. H. H. Perk, F. Y. Wu and X. N. Wu in [131]. Likewise for 3-state vertex models the transformation is O(3) and the associated invariants have been constructed by L. H. Gwa and F. Y. Wu [146] (see section (V-4) below).

V-2. Hilbert's syzygies and the square lattice Ising model in a magnetic field

With an algebraic prejudice for critical manifolds, it is very tempting to conjecture closed algebraic formula for critical manifolds that will reproduce known exact results in various limits. For instance, closed-form expressions for the critical line of the square lattice antiferromagnetic Ising model in a magnetic field were proposed[15] by Müller-Hartmann and Zittartz. However, it has been shown that the expression is numerically incorrect.

X. N. Wu and F. Y. Wu [135] considered the square lattice antiferromagnetic Ising model in a magnetic field, which can be seen as a subcase of the sixteen-vertex model under the O(2) group. Introducing the variables

$$a = 1, \quad b = \sqrt{v}h, \quad c = v, \quad d = v^{3/2}h, \quad e = v^2$$
$$\text{with:} \qquad v = \tanh(J/kT), \qquad h = \tanh(H/kT),$$

the five fundamental Hilbert invariants of O(2) read:

$$I_1 = a + 2c + e \qquad I_2 = (a - 6c + e)^2 + 16(b - d)^2,$$
$$I_3 = (a - e)^2 + 4(b + d)^2, \qquad I_5 = a^2 d - be^2 - 3(a - e)(b + d)c,$$
$$I_4 = (a - 6c + e) \cdot ((a - e)^2 - 4(b + d)^2) + 4(a - e)(b^2 - d^2),$$

and the critical line proposed by Wu and Wu assumes the form

$$c_1 \cdot I_1^4 \, c_2 \cdot I_1^2 \, I_2 + c_3 \cdot I_1^2 \, I_3 + c_4 \cdot I_2^2 + c_5 \cdot I_3^2 + c_6 \cdot I_2 \, I_3 = 0.$$

[15] This is just one example in a very long list of incorrect algebraic conjectures for critical manifolds that one can find in the literature.

They then determined the c_i's using the various known results, including a constraint dictated by the known zero-field critical point, as well as the results of a finite-size analyses which they carried out. This lead to the values $c_1 = 1$, $c_2 = -0.044\ 338$, $c_3 = 0.362\ 73$, $c_4 = 0.000\ 4938$, $c_5 = 0.042\ 779$, $c_6 = -0.008\ 9149$. The resulting closed form expression for the critical line reproduces all known numerical data to a high degree of accuracy. For instance, the critical line yields for a small magnetic field: $T_c \simeq T_0 \cdot (1 - u \cdot (H/J)^2)$ with $u \simeq 0.038\ 022$. This is compared to the presumably exact value obtained by M. Kauffman: $u \simeq 0.038\ 0123\ 259 \cdots$

V-3. Hilbert's syzygies and the honeycomb lattice Ising model in a magnetic field

Similar analyses have been carried out for the honeycomb lattice by F. Y. Wu, X. N. Wu and H. W. J. Blöte [132]. For the corresponding honeycomb eight-vertex model we have

$$a = 1, \qquad b = \sqrt{v}h, \qquad c = v, \qquad d = v^{3/2}h$$
$$\text{with:} \qquad v = \tanh(J/kT), \qquad h = \tanh(H/kT).$$

Analogous to (32), we introduce the following Hilbert's syzygies:

$$P = a^2 + 3ac + 3bd + d^2, \qquad Q = b^2 - ac + c^2 - bd$$
$$P_2 = 2(a^4 + d^4) - 6(a^2\,c^2 + b^2\,d^2) + 12(a^2\,b^2 + c^2\,d^2) - 5a^2\,d^2$$
$$+27\,b^2\,c^2 + 36(ab + cd)bc + 18\,a\,b\,c\,d.$$

The critical line proposed by Wu, Wu and Blöte now reads [125]:

$$c_1 \cdot P_2 + c_2 \cdot P^2 + c_3 \cdot PQ + c_4 \cdot Q^2 = 0. \tag{32}$$

After an extensive search by mapping with all known exact results, they proposed the numbers: $c_1 = 1$, $c_2 = -(4 + 3\sqrt{3})/6$, $c_3 = -(1 - 9\sqrt{3})/8$, and $c_4 = -3(3 - \sqrt{3})/8$.

The initial slope of this critical frontier for small H is $-\ln(z_c)$ where z_c is the critical fugacity of the nearest-neighbor exclusion gas. Their expression leads to the value $z_c \simeq 7.851\ 780\ 04 \cdots$ which is in very good agreement with the value obtained from finite-size analysis, namely, $z_c \simeq 7.851\ 725\ 175(13)$. The critical line (32) is probably not the exact one but certainly a very accurate approximation.

Comment: Hilbert's invariant theory amounts to considering *linear* gauge-like symmetries of the model and the associated invariants. From the inversion relations one has further an infinite discrete set of birational *non-linear* symmetries, that one can couple with these continuous linear groups. In fact, all the above analyses can be revisited by combining the gauge transformation with the *infinite discrete symmetries* generated by the *inversion relations* of the sixteen- (or simply eight)-vertex models. This would lead us to consider a unique "superinvariant" which is, in fact, the *modular invariant* of the elliptic curves parametrizing the sixteen-vertex model.

V-4. Hilbert's syzygies and the honeycomb BEG model

To apply the syzygy consideration to the BEG model, which is a 3-state spin model, one needs to consider the O(3) gauge transformation and its associated invariants, but the construction of the O(3) invariants is very complicated. However, the day is saved, since there exists a mapping between O(3) and sl(2), and invariants for the latter have been worked out by mathematicians a long time ago. While the mathematics to decipher the old results is involved, L. H. Gwa and F. Y. Wu [146] have succeeded in carrying out such an analysis and deduced that there are 5 independent invariants for the O(3) group. They next applied the analysis to the isotropic BEG model on the honeycomb lattice [148], and found one of the invariants to vanish identically. The remaining 4 invariants are then used to determine the critical variety of the BEG model, as in the case of the O(2) gauge. The resulting closed-form expression for the critical variety agrees extremely well with numbers obtained from a finite-size analysis, which they also carried out.

VI. Critical manifolds and critical varieties

A problem solver like F. Y. Wu first tries to find the exact solution of a problem. He tries to "dig out" problems that can be solved. However since most of the problems one looks at cannot be solved exactly, one then tries to study models for which some exact results can be "salvaged". This could be the critical manifolds, which are submanifolds along which the models are Yang-Baxter integrable. In such cases the critical manifolds are, in fact, critical *varieties*. For other models the critical manifolds are *algebraic* varieties without hidden Yang-Baxter integrability [79, 161].

For two-dimensional lattice models the situation is more specific: one can have some "conformal prejudice" that critical manifolds should be submanifolds where the model has a two-dimensional conformal (infinite) symmetry yielding some integrability in the scaling limit. Therefore, as far as critical manifolds are concerned, it is crucial to understand the inter-relation between 1) algebraicity consequences of Yang-Baxter integrability, 2) conformal integrability consequences of a two dimensional criticality, and 3) self-duality.

Many criticality conditions have been obtained, or simply conjectured, in the literature of lattice models in statistical mechanics [47, 65, 72-74, 79, 80], and all these conjectures were *algebraic* [125]. A straightforward situation corresponds to the case where the model possesses a duality symmetry (see section (V)) for which it is always possible to give a *linear* representation of this duality transformation. One can sometimes find varieties which are globally invariant under this symmetry.

Let us, instead, consider the fixed points of the linear duality transformation, which belong to some algebraic variety (hyperplane). If the algebraic variety separates the phase diagram into two disconnected parts, and *if one assumes that the critical temperature is unique*, one can actually deduce that this algebraic variety is a critical variety [79]. Of course if the algebraic variety is only globally invariant (and not invariant *point-by-point* on the algebraic variety) one cannot draw any conclusion.

In fact, for most of the time one is not in a situation where a simple self-dual argument allows the determination of the critical points. A good example is the duality transformation (31) for the sixteen-vertex model for which self-dual arguments are insufficient. In fact there exists a "super-invariant" in this model after taking into account the gauge (weak-graph) duality symmetries (31) and the inversion relation symmetries[16]. But that is another story.

When the critical varieties are exact, *they are almost always related to some integrability of the model, the algebraicity thus being a consequence of the integrability*. A paradigm is the

[16] Remarkably this consideration even extends the $sl(2)$ $sl(2)$ weak-graph symmetry group to an $sl(2)$ $sl(2)$ $sl(2)$ $sl(2)$ symmetry group.

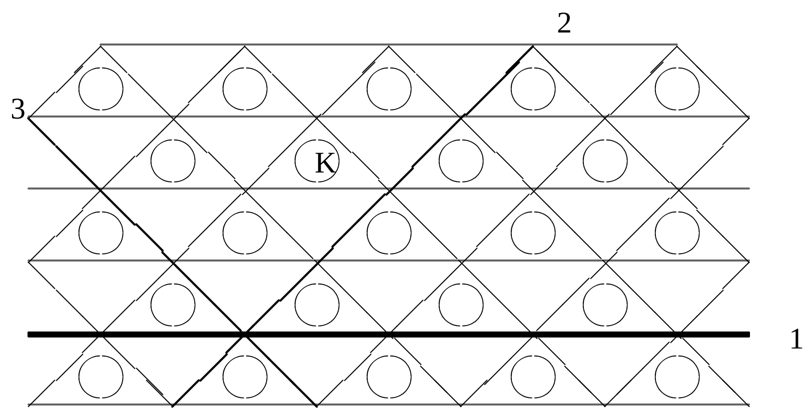
standard scalar Potts model (see section (VII-1)) which is integrable at criticality. However, we will give below, in the case of the two- and three-site interaction Potts model [161], an algebraic variety which is the critical condition [125] but is *unrelated* to any simple Yang-Baxter-like integrability.

Throughout the years F. Y. Wu has obtained numerous results on critical varieties for two- and three-dimensional spin models, and many related conjectures as well. One can only say that the seeking of critical manifolds and critical varieties is a fascinating subject matter by itself for specialists like Wu.

VI-1. Inversion relations, duality and critical varieties

The considerations of *inversion relations* has been shown to be a powerful tool for analyzing the phase diagram of lattice models and, particularly, for obtaining critical algebraic manifolds in the form of algebraic varieties (see (56) in section (VII-2)).

Let us consider the standard scalar q-state Potts model on an anisotropic triangular lattice with nearest-neighbor and three-spin interactions around up-pointing triangles [79, 161] as shown:



The partition function of the models reads:

$$\mathcal{Z} = \sum_{\{\sigma_i\}} \prod_{<i,j>} e^{K_1 \delta_{\sigma_i,\sigma_j}} \prod_{<j,k>} e^{K_2 \delta_{\sigma_j,\sigma_k}} \prod_{<k,l>} e^{K_3 \delta_{\sigma_k,\sigma_l}} \prod_{\Delta} e^{K \delta_{\sigma_i,\sigma_j} \delta_{\sigma_j,\sigma_k}}.$$

Here the summation is taken over all spin configurations, the first three products denote edge Boltzmann weights and the last product is over all up-pointing triangles.

A *duality* transformation exists for this model [79]. We introduce the following notation:

$$x = e^K, \qquad x_i = e^{K_i}, \qquad i = 1, 2, 3$$
$$y = x\, x_1\, x_2\, x_3 - (x_1 + x_2 + x_3) + 2. \tag{33}$$

With the notation (33) the duality

$$D: \begin{cases} x_i \;\; \longrightarrow x_i^* = 1 + q\dfrac{x_i - 1}{y}, \qquad y \longrightarrow y^* = \dfrac{q^2}{y}, \\ x \;\; \longrightarrow x^* = \dfrac{x_1 + x_2 + x_3 - 2 + q^2/y}{x_1\, x_2\, x_3}, \end{cases} \tag{34}$$

and the partition transforms as

$$Z(x_1, x_2, x_3, y) = (y/q)^N \cdot Z(x_1^*, x_2^*, x_3^*, y^*), \tag{35}$$

where N is the number of sites.

On the basis of this duality Baxter *et al.* proposed that the critical points are located on the *algebraic variety*:

$$x\ x_1\ x_2\ x_3 - (x_1 + x_2 + x_3) + 2 - q = 0, \tag{36}$$

which corresponds to the set of fixed points of D. The critical variety (36) is not only globally invariant under (34), it is also *point-by-point* invariant, namely, every point on the variety is invariant. In general when an algebraic variety is such that every point of the variety is invariant under a duality symmetry, it is possible to argue, subject to some continuity and uniqueness arguments, that the variety actually corresponds to the criticality variety. This has been done by Wu and Zia [125] for $q > 4$ in the ferromagnetic region. It is important to note that the critical variety (36) is *not* an algebraic variety on which the model becomes *Yang-Baxter* (*star-triangle*) *integrable*. This is an interesting example of a model where algebraic criticality does not automatically imply Yang-Baxter integrability.

Comment: In suitable variables the duality transformations can be seen as a *linear* transformation. There are two globally invariant hyperplanes under D: $y = +q$ and $y = -q$. The (ferromagnetic) criticality variety (36) corresponds to $y = +q$. The second hyperplane $y = -q$ is not a point-by-point invariant although it is globally self-dual. It is not a locus for critical or transition points.

This illustrates a fundamental question one frequently encounters when trying to analyze a lattice model: is the critical manifold an algebraic variety or a transcendental manifold? It will be seen that a first-order transition manifold exists for this model for $q = 3$, and its algebraic or transcendental status is far from being clear (see [166] and (67) in section (VII-4-3)). The existence of such a very large (nonlinear) group of (birational) symmetries provides drastic constraints on the critical manifold and therefore the phase diagram.

There exist three inversion relations associated with the three directions of the triangular lattice for this model [161]. For instance, the inversion relation which singles out direction 1 (see figure 1) is the (involutive) rational transformation I_1:

$$\begin{aligned} I_1 : (x, x_1, x_2, x_3) \longrightarrow \Big(& \frac{(x\,x_1 - 1)^2 (x_1 + q - 2)}{(x\,x_1^2 + x\,x_1(q-3) - q + 2)(x_1 - 1)}, \\ & 2 - q - x_1 + \frac{x_1(x-1)}{x_1\,x - 1}, \frac{x_1 - 1}{x_3(x\,x_1 - 1)}, \frac{x_1 - 1}{x_2(x\,x_1 - 1)} \Big). \end{aligned} \tag{37}$$

These three inversion relations generate a group of symmetries which is naturally represented in terms of birational transformations in a four dimensional space. This infinite discrete group of *birational* symmetries is generically a *very large one* (as large as a free group). The algebraic variety (36) is remarkable from an algebraic geometry viewpoint: it is invariant under this very large group generated by three involutions (37).

In this framework of a very large group of symmetries of the model, an amazing situation arises: the one for which q, the number of states of the Potts model, corresponds to *Tutte-Beraha* *numbers* $q = 2 + 2\cos(2\pi/N)$ where N is an integer. For these selected numbers of q, the group of birational transformations is generated by generators of finite order: it is seen as a Coxeter group generated by generators and relations between the generators. The elements of the group can be seen as the words one can build from an alphabet of three letters A, B and C with the constraints $A^{N+1} = A$, $B^{N+1} = B$, $C^{N+1} = C$. Since the generators A, B and C do not commute (nor does any power of A, B and C) the number of words of length L still grows exponentially with L (hyperbolic group). Among these values of q, two Tutte-Beraha numbers play a special role: $q = 1$ and $q = 3$. For these two values the hyperbolic Coxeter group degenerates[17] into a group isomorphic to $\mathbf{Z} \times \mathbf{Z}$.

For the standard scalar nearest-neighbor Potts model the Tutte-Beraha numbers correspond to the values of q for which the critical exponents of the model are *rational* (see (53) in section (VII-1)).

VI-2. The exact critical frontier of the Potts model on the 3-12 lattice

F. Y. Wu *et al.* considered a general 3-12 lattice with two *and three-site* interaction on the triangular cells [155]. This model has eleven coupling constants and includes the Kagomé lattice as a special case.

In a special parameter subspace of the model, condition (38) below, an exact critical frontier for this Potts model on a general 3-12 lattice Potts model was determined. The Kagomé lattice limit is unfortunately not compatible with the required condition (38).

The condition under which they obtained the exact critical frontier reads:

$$\begin{aligned} & x^2\, x_1^2\, x_2^2\, x_3^2 - x\, x_1\, x_2\, x_3 \cdot (x_1\, x_2 + x_2\, x_3 + x_1\, x_3 - 1) \\ & \quad +(x_1 + x_2 + x_3 + q - 4) \cdot (x_1 x_2 + x_2 x_3 + x_1 x_3 + 3 - q) \\ & \quad -q\, x_1\, x_2\, x_3 - (x_1^2 + x_2^2 + x_3^2) + q^2 - 6\, q + 10 = 0. \end{aligned} \tag{38}$$

This is nothing but the condition which corresponds to the star-triangle relation of the Potts model.

Comment: One can show that condition (38) *is actually invariant under the inversion relation* (37) of the previous section (VI-1), and therefore, since (38) is symmetric under the permutations of K_1, K_2 and K_3, under the three inversions generating the very large group of birational transformations previously mentioned in section (VI-1). More generally, introducing D_1, D_2 and D_3:

$$\begin{aligned} & D_1 = x_1 + x_2 + x_3 - x\, x_1\, x_2\, x_3 + q - 2, \quad D_3 = x\, x_1\, x_2\, x_3 - x_1\, x_2\, x_3, \\ & D_2 = x_1 + x_2 + x_3 + x\, x_1\, x_2\, x_3 - 1 - (x_1\, x_2 + x_2\, x_3 + x_1\, x_3), \end{aligned}$$

one can show that the algebraic expression

$$\mathbf{I}_1(x_1, x_2, x_3, x) = \frac{D_1 \cdot D_2}{D_1 D_2 - q \cdot D_3} \tag{39}$$

is invariant under the three inversion relations and the large group of birational transformation they

[17] Up to semi-direct products by finite groups.

generate, the (star-triangle) condition (38) corresponding to $\mathbf{I}_1(x_1, x_2, x_3, x) = \infty$, namely[18] $D_1\, D_2 - q\, D_3 = 0$.

When $x = 1$, or $q = 1$ or 3, there are additional invariants of the three inversions (37). For instance, for $x = 1$, one can build an invariant from a covariant we give below (see (57)). For $q = 3$, introducing

$$D_5 = x_1 x_2 x_3 \cdot (x_1^2 x_2^2 x_3^2 x^2 - x_2^2 x x_1^2 - x_3^2 x x_1^2 - x_2^2 x_3^2 x + x_1^2 + x_2^2 + x_3^2 - 1),$$

one finds that the expression:

$$\mathbf{I}_2(x_1, x_2, x_3, x) = \frac{D_1^3 \cdot D_2}{3^5 \cdot D_5},$$

is invariant under the three inversions (37). One can try to find the manifold corresponding to the first order transition (see (VII-4-3) below) in the form $F(\mathbf{I}_1, \mathbf{I}_2) = 0$. It still remains an open question whether this variety is algebraic or transcendental. The $x = 1$ limit corresponds to $\mathbf{I}_1 = +1$. The condition $\mathbf{I}_2(x_1, x_2, x_3, x) = 1$ yields $x_1 = x_2 = x_3 = 0.215\ 816$ (to be compared with $0.226\ 681$ from (57) in section (VII-2) below), still different from 0.204 (see (66) in section (VII-4-3) below), which is believed to be the location of the first-order transition point.

VI-3. The embarrassing Kagomé critical manifold

At the end of the 80's there was a surge of interest in the Kagomé lattice coming from the theoretical study of high-T_c or strongly interacting fermions in two dimensions (the 2D Hubbard model, resonating valence bond (RVB), ground state of the Heisenberg model). The two-dimensional Gutzwiller product RVB ansatz strategy promoted by P. W. Anderson for describing strongly interacting fermions seemed to fail for regular lattices (square, triangular, ...). Thus, because of its ground state entropy and other specific properties, the Kagomé lattice seemed to be the "last chance" for the RVB approach.

Since one can obtain a critical frontier (38) for the general 3-12 lattice model, and since the 3-12 model includes the Kagomé lattice as a special case, it is tempting to try to obtain the critical frontier for the Potts Kagomé lattice.

The Kagomé Potts critical point was first conjectured by Wu [74] as

$$\begin{aligned} &y^6 - 6y^4 + 2(2-q)\cdot y^3 + 3(3-2\,q)\cdot y^2 - 6(q-1)\cdot(q-2)\,y \\ &\quad -(q-2)\,(q^2 - 4\,q + 2) = 0, \end{aligned} \tag{40}$$

which gives, for $q = 2$, the correct critical point $y^4 - 6y^2 - 3 = 0$ and for $q = 0$ gives (also correctly) $y = 1$. Furthermore, for large q, y behaves like $\sqrt{q}$, as it should. However in the percolation limit $q \to 1$, it gives a percolation threshold p_c for the Kagomé lattice of $p_c = 0.524\ 43\cdots$, which compares to the best numerical estimate[19] obtained by R. M. Ziff and P. N. Suding, namely $p_c = 0.524\ 405\ 3\cdots$, with uncertainty in the last quoted digit. Wu's conjecture is thus wrong,

[18] For $x = 1$ (no three-spin interaction, $D_3 = 0$), condition (38) factorizes and one recovers the ferromagnetic critical condition (36) of the q-state Potts model on an anisotropic triangular lattice.

[19] R. M. Ziff and P. N. Suding, Determination of the bond percolation threshold for the Kagomé lattice, J. Phys. *A 30*, 5351 (1997) and cond-mat/9707110.

but by less than $5 \cdot 10^{-5}$. Some very long high-temperature series of I. Jensen, A. J. Guttmann and I. G.Enting on the q-state Potts model on the Kagomé lattice further confirm that the conjecture is wrong for arbitrary values of q. Nevertheless the Wu conjecture remains an extraordinary approximation.

It is a bit surprising that no exact result on integrability (along some algebraic subvariety) or exact expression for the critical variety is known for the standard scalar Potts model on the *Kagomé lattice*, as generally one expects that the integrability on one lattice, say the square lattice, implies integrability for most of the other Euclidian lattices. This is certainly not the case for the Kagomé lattice.

VII. Potts models

The Potts model encompasses a very large number of problems in statistical physics and lattice statistics. The Potts model, which is a generalization of the two-component Ising model to q components for arbitrary q, has been the subject matter of intense interest in many fields ranging from condensed matter to high-energy physics. It is also related to coloring problems in graph theory.

However, exact results for the Potts model have proven to be extremely elusive. Rigorous results are limited, and include essentially only a closed-form evaluation of its free energy for $q = 2$, the Ising model, and critical properties for the square, triangular and honeycomb lattices [70]. Much less is known about its correlation functions.

VII-1. Wu's review of the Potts model

F. Y. Wu's 1982 review of the Potts model is very well-known [89] (see also [98]). It is an exhaustive expository review of most of the results known about the Potts model up to 1981, a time when interest in the model began to mount. It has remained extremely valuable for anyone wishing to work on the standard scalar Potts model. In particular, it explains the $q \to 1$ limit of the percolation problem (see also [64]), the $q \to 1/2$ limit of the dilute spin glass problem, and the $q \to 0$ limit of the resistor network problem; the equivalences with the Whitney-Tutte polynomial [89] (see section (7.7) and also [57])) and many other related models are also detailed. For instance, the Blume-Capel and the Blume-Emery-Griffiths model (see (25) in (IV-3)) can also be seen as a Potts models. More generally, it is shown that any system of classical q-state spins, the Potts model included, can be formulated as a spin $(q-1)/2$ system.

However, Wu's review was not written in time to include discussions of the *inversion functional relations*. For the two- and three-dimensional anisotropic q-state Potts models, the partition functions satisfies, respectively, the functional relations:

$$Z(e^{K_1}, e^{K_2}) \cdot Z(2 - q - e^{K_1}, e^{-K_2}) = (e^{K_1} - 1) \cdot (1 - q - e^{K_1}), \tag{41}$$

$$Z_{cubic}(e^{K_1}, e^{K_2}, e^{K_3}) \cdot Z_{cubic}(2 - q - e^{K_1}, e^{-K_2}, e^{K_3}) = (e^{K_1} - 1) \cdot (1 - q - e^{K_1}). \tag{42}$$

There are also permutation symmetries like, in 3 dimensions, $Z_{cubic}(e^{K_1}, e^{K_2}, e^{K_3}) = Z_{cubic}(e^{K_3}, e^{K_1}, e^{K_2}) = Z_{cubic}(e^{K_3}, e^{K_2}, e^{K_1})$. Combining these relations one generates an infinite set of discrete symmetries which yield a canonical rational parametrization of the Potts model at and beyond[20] $T = T_c$, and shows clearly the role played by the Tutte-Beraha numbers. These infinite sets of discrete symmetries impose very severe constraints on the critical manifolds and the integrability (see sections (6), (VI-1)). An inversion relation study has subsequently been carried out by F. Y. Wu *et al.* [161].

Graph theory plays a central role in Wu's work on the Potts model. The Potts partition function can be written as [89]

$$Z \equiv Z_G(q,K) = \sum_{G' \subseteq G} (e^K - 1)^b q^n, \tag{43}$$

where $K = J/kT$, the summation is taken over all subgraphs $G' \subseteq G$, and b and n are, respectively, the number of edges and clusters, including isolated vertices, of G'. The duality relation of the Potts model can be obtained from a graph-theoretical viewpoint by using the Euler relation $c + N = b + n$, where c is the number of independent circuits in the subgraph G', and N is the total number of vertices in G. This leads to the duality relation

$$Z_G(q,K) = v^{|E|} \cdot q^{1-N_D} \cdot Z_D(q,K^*),$$

where D is the graph dual to G, and the dual variable K^* is given by:

$$(e^K - 1) \cdot (e^{K^*} - 1) = q. \tag{44}$$

The generalization of the duality to multisite interactions is also given in the review.

A consequence of (43) is that one finds the following connection with the *chromatic polynomial* $P_G(q)$ on G by taking the antiferromagnetic zero-temper-ature limit $K \to -\infty$:

$$Z_G(q, K = -\infty) = P_G(q).$$

The (high- and low-temperature) series expansions are described from a graph-theoretical viewpoint. For instance, the high-temperature expansions are written in the (Domb) form:

$$Z_G(q,K) = \sum_{\sigma_i=0}^{q-1} \prod_{<ij>} \frac{q+v}{q} \cdot (1 + f_{ij}), \quad f_{ij} = \frac{v}{q+v} \cdot (q\delta(\sigma_i,\sigma_j) - 1). \tag{45}$$

The introduction of these f_{ij} variables comes from the fact that

$$\sum_{\sigma_j=0}^{q-1} f_{ij} = 0, \tag{46}$$

and, consequently, all subgraphs with vertices of degree 1 give rise to zero contributions. The number of subgraphs that occur in the expansion is therefore greatly reduced.

The location of the critical points of the anisotropic Potts model on a square, triangular and honeycomb lattice were given in terms of the variables $x_r = (e^{K_r} - 1)/\sqrt{q}$ (see also (61) below).

[20] At $T = T_c$, one recovers the well-known rational parametrization of the model (occurring in (52) see below).

These expressions are invariably the various special cases of Wu's conjecture [89] for the critical point of the more general checkerboard lattice, namely,

$$\sqrt{q} + x_1 + x_2 + x_3 + x_4 = x_1\, x_2\, x_3 + x_1\, x_2\, x_4 + x_1\, x_3\, x_4 + x_2\, x_3\, x_4 + \sqrt{q} \cdot x_1\, x_2\, x_3\, x_4. \qquad (47)$$

Using the notation $a = e^{K_1},\ b = e^{K_2},\ c = e^{K_3},\ d = e^{K_4}$, this critical algebraic variety (47) reads:

$$-(q-1)(q-3) + (a+b+c+d)(2-q) - (ab+ac+bc+ad+bd+cd) + abcd = 0. \qquad (48)$$

The critical point of a mixed ferromagnetic-antiferromagnetic square Potts model considered by Kinzel, Selke and Wu [82]:

$$(e^{K_1} - 1) \cdot (e^{K_2} + 1) = -q, \qquad (49)$$

was also given. While this expression coincides with the critical point for $q = 2$, it is incompatible with the inversion relations (41) for general q, and hence is not a critical variety. This is because the infinite discrete group of symmetries generated from the inversion relations of the square Potts model transforms (49) into an *infinite set* of other algebraic varieties, and hence cannot be critical.

Generally, critical manifolds need to be (globally) invariant under this infinite set of transformations (discrete symmetries). Actually, for the anisotropic square Potts model, for instance, one can show that, when q is not a Tutte-Beraha number[21], the *only algebraic varieties compatible with the inversion relation symmetries* (41) are given by the well-known ferromagnetic condition:

$$(e^{K_1} - 1) \cdot (e^{K_2} - 1) = q, \qquad (50)$$

and the antiferromagnetic condition obtained by R. J. Baxter:

$$(e^{K_1} + 1) \cdot (e^{K_2} + 1) = 4 - q, \qquad (51)$$

(for which the model is exactly soluble). Note that these two varieties can be deduced from the conjecture (47) by taking $x_1 = x_3$ and $x_2 = x_4$. In this limit, the critical condition (47) factors into conditions (50) and (51). In fact, it has since been shown that the critical condition (47) corresponds to an integrability condition of the checkerboard Potts model.

At *criticality* the Potts model is *exactly solvable*. Let us give the example of the square lattice. The free energy of the isotropic Potts model at the *ferromagnetic* critical point $T = T_c$ reads:

$$f(q, T_c) = \frac{1}{2} + \theta + 2 \cdot \sum_{n=1}^{\infty} \frac{e^{-n\theta} \cdot \tanh(n\theta)}{n}, \quad \text{for:} \quad q > 4$$

[21] When q is a Tutte-Beraha number the generically infinite discrete group Γ generated by the inversion relations on the model becomes a finite group, and many algebraic varieties can be invariant under such a finite group: a simple way to build such algebraic Γ-invariants amounts to performing summations over the group Γ.

$$f(q, T_c) = \ln(2) + 4 \cdot \ln\left(\frac{\Gamma(1/4)}{2\Gamma(3/4)}\right), \qquad \text{for:} \qquad q = 4 \tag{52}$$

$$f(q, T_c) = \frac{1}{2} + \int_{-\infty}^{+\infty} \frac{dx}{x} \tanh(\mu x) \frac{\sinh((\pi - \mu)x)}{\sinh(\pi x)}, \quad \text{for:} \quad q < 4,$$

where the variables θ and μ correspond to the rational parametrization of the model at $T = T_c$, namely $\cosh(\theta) = \sqrt{q}/2$ or $\cos(\mu) = \sqrt{q}/2$.

The exact critical exponents of the standard scalar Potts model are also given in this review. These critical exponents, which are *rational* when q is the Tutte-Beraha numbers, are:

$$\alpha = \alpha' = \frac{2}{3} \cdot \frac{1 - 2u}{1 - u}, \quad \beta = \frac{1 + u}{12}, \quad \gamma = \gamma' = \frac{7 - 4u + u^2}{6 \cdot (1 - u)},$$

$$\delta = \frac{(3 - u) \cdot (5 - u)}{1 - u^2}, \quad \nu = \nu' = \frac{2 - u}{3 \cdot (1 - u)}, \qquad \eta = \frac{1 - u^2}{2 \cdot (2 - u)},$$

where the parameter u is related to q by:

$$2 \cos\left(\frac{\pi u}{2}\right) = \sqrt{q}, \qquad \text{or:} \qquad 2 + 2\cos(\pi u) = q. \tag{53}$$

These results played a key role in the emergence of the conformal theory.

VII-2. Comments on the checkerboard Potts model

The Wu conjecture (47) for the criticality condition of the q-state checkerboard has since been confirmed from an inversion relation analysis. To discuss the inversion relation we introduce variables u, v, w, z, t defined by:

$$\frac{e^{K_1} - 1}{e^{K_1} + q - 1} = t \cdot \frac{u - t}{1 - t^3 u}, \quad \frac{e^{K_2} - 1}{e^{K_2} + q - 1} = t \cdot \frac{v - t}{1 - t^3 v}, \cdots \tag{54}$$

$$\text{or:} \quad e^{K_1} = a = \frac{u - t^3}{t \cdot (1 - ut)}, \quad e^{K_2} = b = \frac{v - t^3}{t \cdot (1 - vt)}, \cdots$$

$$\text{with:} \qquad t + t^{-1} = \sqrt{q}.$$

In these variables the criticality condition (47) reads:

$$u\, v\, w\, z = 1. \tag{55}$$

Using the inversion trick[22], the partition function of the checkerboard model at *criticality* can be written in a multiplicative form

[22] With the well-suited variables (54) the inversion trick naturally yields exact formulae as products over an infinite discrete group, in this case Eulerian products (56).

$$Z(u,v,w,z)_{[uvwz=1]} = \frac{q}{t^2}\cdot\frac{F(u)\cdot F(1/u)}{1-tu}\cdot\frac{F(v)\cdot F(1/v)}{1-tv}$$
$$\cdot\frac{F(w)\cdot F(1/w)}{1-tw}\cdot\frac{F(z)\cdot F(1/z)}{1-tz} \tag{56}$$

where: $$F(u) = \prod_{n=1}^{\infty}\frac{1-t^{4\,n-1}u}{1-t^{4\,n+1}u}.$$

This formula is reminiscent of (10) for the six-vertex model (see section (III-1)), as it should since it also follows from the Potts and six-vertex model correspondence (see section (VII-3) below).

Note that, in the anisotropic case $w = u$ and $z = v$, the critical condition (55) factorizes into $uv = +1$ and $uv = -1$ which are, respectively, the aforementioned ferromagnetic and antiferromagnetic criticalities (and integrability).

The checkerboard model reduces to a honeycomb model, and hence a triangular model by taking the dual, if one of the four interactions vanishes. Therefore it is useful to re-examine the triangular lattice limit of the checkerboard variety. P. Martin *et al.* have established two varieties for the triangular Potts model: a ferromagnetic variety which is precisely (36) with $x = 1$, and an antiferromagnetic variety:

$$(q-2)^2 - 2 + (a+b+c+abc)(q-2) + 2(ac+ab+bc) = 0. \tag{57}$$

These two algebraic varieties can be written, respectively, as $u\,v\,w = +t$ and $u\,v\,w = -t$, which can be deduced by taking, respectively, $d = 1$ and $d = -(q-2)/2$ in (48).

For $q = 3$ the antiferromagnetic algebraic variety yields an algebraic critical point very close to the first-order transition point for the three-state Potts model [91, 143].

Along the well-known ferromagnetic variety (44), which for the isotropic model reads $(e^K - 1)\cdot(e^{K^*} - 1) = q$, it is of interest to point out the existence of some “hidden” duality $K \to K^\dagger$, such that the antiferromagnetic condition (51) reads $K_2 = K_1^\dagger$. This hidden duality is the involution:

$$e^{K^\dagger} = -\frac{e^K + q - 3}{e^K + 1}. \tag{58}$$

Note that the dualities (44) and (58) commute, and their product gives the involution:

$$e^K \quad\longrightarrow\quad e^{K^{*\dagger}} = -\frac{(q-2)\cdot e^K + 2}{.}2e^K + (q-2) \tag{59}$$

The ferromagnetic critical variety of the anisotropic triangular Potts model (36) now transforms into the anti-ferromagnetic critical variety (57) under (59). One also finds that the ferromagnetic critical variety of the anisotropic square-lattice Potts model (50), and its anti-ferromagnetic critical variety (51), are *both* invariant under transformations (44), (58) and thus (59). More generally, for the checkerboard model, one finds that (48) is invariant under both (44), (58), and thus (59). These results can be very simply seen in the variables u, v, w, z. For example, (59) is simply $u \to -u$. Involutions like the “hidden duality” (58), or like (59), do not yield simple functional equations on the partition function like the Kramers-Wannier duality does (see (44) and (44)).

They are not symmetries of the model, but, rather, "symmetries of the second kind": symmetries of the symmetries.

VII-3. Equivalence of the q-state Potts model with a six-vertex model

Recall that the partition function of the standard scalar q-state Potts model can readily be written as a Whitney-Tutte polynomial. In a further step, Temperley and Lieb used operator methods to show that, for the square lattice, the Whitney-Tutte polynomial is, in turn, equivalent to a staggered ice-type vertex model.

In a classic paper [57] by R. J. Baxter, S. B. Kelland and F. Y. Wu, this equivalence is *rederived* from a graphical approach (see (72) in section (VII-7)), which is easier than the algebraic method of Temperley and Lieb and is applicable to an *arbitrary* planar graph. In particular, the equivalence was extended to triangular or honeycomb Potts models and a staggered six-vertex model on the Kagomé lattice.

The equivalences are as follows (page 404 in [57]). For the square, triangular and honeycomb lattices, the equivalent ice-type vertex model has the Boltzmann weights:

$$(\omega_1, \omega_2, \cdots, \omega_6) = (1, 1, x_r, x_r, A_r, B_r), \tag{60}$$

where (A_r, B_r) are, respectively, for the square, triangular and honeycomb lattices:

$$\left(\frac{1}{s} + x_r \cdot s, \frac{x_r}{s} + s\right), \quad \left(\frac{1}{t} + x_r \cdot t^2, \frac{x_r}{t^2} + t\right), \quad \left(\frac{1}{t^2} + x_r \cdot t, \frac{x_r}{t} + t^2\right),$$

with:

$$s = e^{\theta}, \qquad t = e^{\theta/3}, \quad 2\cosh(\theta) = \sqrt{q}, \qquad x_r = \frac{e^{K_r} - 1}{\sqrt{q}}. \tag{61}$$

These mappings play an important role in the Lee-Yang theorem to be discussed latter (see section (VII-5-1)).

VII-4. Miscellaneous Potts model phase diagrams

VII-4-1. Potts model with competing interactions

Kinzel, Selke and Wu [82] have studied a square lattice Potts model with the competing interactions alluded to earlier in section (VII-1). A similar model in 3 dimensions with next-nearest-neighbor competing interactions has been studied by J. R. Banavar and Wu using mean-field theory and Monte-Carlo simulations [97]. A rich phase diagram was found, and they established positively that the behavior of the 4-state three dimensional Potts model is mean-field-like.

VII-4-2. First-order transition in the antiferromagnetic Potts model

F. Y. Wu *et al.* have analyzed specifically the three-state triangular Potts model and considered the (tricky) tricritical behavior of this model [166].

Recalling the results of section (VI-1), and in particular the special role played by $q = 3$ for the two- and three-site interaction Potts model (33), Monte-Carlo calculations of the $q = 3$ isotropic limit of the model have been performed [144]. These studies confirmed the existence of an antiferromagnetic critical point (in addition to the well-known ferromagnetic one), probably corresponding to a first-order transition occurring near the variety $x = 1$ isotropic region.

For the triangular Potts model with two-spin interactions, Monte-Carlo calculations [166] confirm the localization $a \simeq 0.204 \pm .003$ for a *first-order* transition point. This transition point is confirmed to be different from *the algebraic antiferromagnetic point localized* at $a = 0.22665$ derived below (see (67) below).

The antiferromagnetic transition point $a \simeq 0.204 \pm .003$ can be interpreted as belonging to some singular manifold in the parameter space of the model with two- and three-spin interactions. This singular manifold corresponds to a first-order transition frontier. Recalling discussions in section (VI-1), the question of the algebraic or transcendental status of this first-order transition frontier still remains open.

VII-4-3. Chiral Potts models

F. Y. Wu *et al.* have analyzed a particular two-dimensional chiral Potts model, namely, the 3-state chiral Potts, in order to understand the relation between the (higher genus) integrability and criticality conditions [143]. On the checkerboard model the higher genus integrability of the 3-state chiral Potts model is restricted to the following algebraic variety:

$$\begin{aligned} &3 \cdot (Q_1 P_2 P_3 P_4 + Q_2 P_1 P_3 P_4 + Q_3 P_1 P_2 P_4 + Q_4 P_1 P_2 P_3) \\ &\quad -(P_1 Q_2 Q_3 Q_4 + P_2 Q_1 Q_3 Q_4 + P_3 Q_1 Q_2 Q_4 + P_4 Q_1 Q_2 Q_3) = 0, \end{aligned} \tag{62}$$

where:

$$\begin{aligned} &P_i = f_i - 3\,h_i, \qquad Q_i = f_i - 2\,g_i + 3\,h_i, \qquad h_i = a_i^2\, b_i^2\, c_i^2 \\ &f_i = a_i\, b_i\, c_i \cdot (a_i^3 + b_i^3 + c_i^3), \qquad g_i = a_i^3\, b_i^3 + b_i^3\, c_i^3 + c_i^3\, a_i^3, \end{aligned} \tag{63}$$

where a_i, b_i, c_i denote the three possible values of the four edge Boltzmann weights $w_i(\sigma_k - \sigma_l)$ of the checkerboard lattice:

$$a_i = w_i(0), \quad b_i = w_i(1) = w_i(-2), \quad c_i = w_i(2) = w_i(-1), \quad i = 1,\, 2,\, 3,\, 4.$$

Recalling the conformal theory prejudice (criticality in two dimensions versus integrability), one can also wonder if an integrability condition like (62) could correspond to a critical subvariety of the phase diagram. Let us consider the standard scalar limit of this model. The higher genus integrability condition (62) reduces to the critical condition (47) or (48) of the standard scalar Potts model on the checkerboard lattice for $q = 3$:

$$abcd - (ab + ad + ac + bc + bd + cd) - (a + b + c + d) = 0, \tag{64}$$

together with another algebraic variety:

$$\begin{aligned} &abcd + 2 \cdot (acd + bcd + abd + abc) + ab + bd + cd + ac + ad + bc \\ &\quad -(a + b + c + d) - 2 = 0. \end{aligned} \tag{65}$$

With the variables (54) taken for $q = 3$, the critical condition (64) reads $u\,v\,w\,z = +1$, and the algebraic condition (65) reads: $uvwz = -1$. Considering the similarity of (65), namely

$uvwz = -1$, with (64), namely $uvwz = +1$, it is tempting to imagine that (65) could also be in some domain of the parameter space a, b, c, d, a critical variety.

In the isotropic triangular and standard scalar limit the higher genus integrability condition (62) factorizes into two conditions. One is the ferromagnetic critical condition of the standard scalar Potts model (see also (36)) $2 - q - a - b - c + a\,b\,c = 0$ (or $u\,v\,w = t$) with $q = 3$, the other one is:

$$(q-2)^2 - 2 + (a+b+c+abc)(q-2) + 2(ac+ab+bc) = 0, \qquad \text{or:} \quad u\,v\,w = -t, \tag{66}$$

with $q = 3$. In the isotropic limit and for $q = 3$, these two algebraic varieties give, respectively

$$a^3 - 3a - 1 = 0 \qquad \text{and} \qquad a^3 + 6a^2 + 3a - 1 = 0, \tag{67}$$

yielding the ferromagnetic critical point, a = 1.8793 and *an antiferromagnetic* transition point at a = 0.22665. This antiferromagnetic point must be compared with the antiferromagnetic critical and *first-order* transition point $a \simeq 0.204 \pm 0.003$ obtained from series analysis by I. G. Enting and F. Y. Wu [91].

VII-5. Zeros of partition functions of Potts models

In 1952 Yang and Lee introduced the concept of considering the zeros of the grand partition function of statistical mechanical systems, a consideration that has since opened new avenues to the study of phase transitions. While Yang and Lee considered the zeros in the complex fugacity plane, or equivalently the complex magnetic field plane in the case of spin systems, Fisher in 1964 called attention to the relevance of the zeros of the canonical partition function in the complex temperature plane. Generally speaking, there exist several different kinds of exact results on lattice models in statistical mechanics. Ideally, one would like to obtain the exact, closed-form, expressions of thermodynamic quantities such as the per-site free energy, the surface tension, spontaneous magnetization, and correlation functions. A knowledge of these exact expressions leads to a complete description of the system including the phase boundary (critical frontier) and the location of the zeros of the partition function. However, exact evaluations of physical quantities are not always possible. In such cases one can sometimes determine the critical frontier from properties such as the duality and the inversion relations, or analyze the analyticity properties of the free energy by locating the zeros of the partition function. But other than in the case of some special one-dimensional model, exact results on the zeros have been confined mostly to the Ising model.

VII-5-1. Fugacity variable for checkerboard Potts model, staggering field, Lee-Yang theorem, duality

Let us consider a q-state Potts model on a square, or triangular, lattice, with no magnetic field. Since one does not have a magnetic field, one does not expect a Lee-Yang theorem to exist. Actually *this is not true*: there is a "hidden" field for the standard scalar q-state Potts model!

A. Hinterman, H. Kunz and F. Y. Wu [70] combined the equivalence of the Potts model with a staggered six-vertex model, together with the Lee-Yang circle theorem due to Suzuki and Fisher (see also [30]), to deduce the critical variety of the Potts model. In this consideration a fugacity variable z is associated with the staggering field that occurs in the aforementioned correspondence. In terms of the variables (A_r, B_r) given in (VII-3), the fugacity variable z reads, respectively, for the square and triangular lattices:

$$z^4 = \frac{A_1 A_2}{B_1 B_2}, \qquad\qquad z^6 = \frac{A_1 A_2 A_3}{B_1 B_2 B_3}. \tag{68}$$

A condition on the Suzuki-Fisher extension of the Lee-Yang theorem and for real temperatures requires that we must have $q > 4$.

All these results can be generalized to the q-state Potts model on the checkerboard lattice. Not surprisingly, (68) is generalized into:

$$z^8 = \frac{A_1 A_2 A_3 A_4}{B_1 B_2 B_3 B_4}.$$

Note that the duality (44) of the Potts model *has a very simple representation in terms of these fugacity variables* $z : z \to 1/z$.

The zeros of the partition function of the checkerboard model will later be seen to lie on $|z| = 1$, as they should from the Lee-Yang theorem which applies when $q > 4$. Remarkably, it was also seen later by other authors, that the $|z| = 1$ condition can be extended to $q < 4$, and that one then recovers the well-known Fisher's circles for the Ising model!!

The fugacity variable z is a fundamental variable. It corresponds to a crucial combination of variables encapsulating the action of the infinite discrete group generated by the inversion relations. The criticality conditions corresponding to $z = 1$ and $z = -1$ seem also to play some role (see (65) for $q = 3$ in section (VII-4-3)), but not a critical or transition point role. In terms of the variables u, v, w, z of section (VII-4-3), the fugacity variable z is simply the product $u\,v\,w\,z$.

VII-5-2. Zeros for the square lattice: A graph-theoretical viewpoint

Following F. Y. Wu's approach, let us consider the q-state Potts model on the square lattice from a graph-theoretical viewpoint with the partition function (43). Introducing the variable

$$x = (e^K - 1)/\sqrt{q}, \tag{69}$$

the partition function (43) can be written as a polynomial in x

$$Z \equiv P_G(q, x) = \sum_{b=0}^{E} c_b(q) x^b, \quad \text{where} \qquad c_b(q) = q^{b/2} \sum_{G' \subseteq G} q^n,$$

where the second summation is taken over all $G' \subseteq G$ for a fixed b. Then, the duality relation (44) can be rewritten as a duality relation for the polynomial P_G [58]:

$$P_G(q, x) = q^{N-1-E/2}\, x^E \cdot P_D(q, x^{-1}). \tag{70}$$

In the case of the square lattice for which D is identical to G in the thermodynamic limit regardless of boundary conditions, (70) implies that the system is critical at $x_c = 1$. For finite self-dual lattices relation (70) gives an example of a self-dual polynomial[23] [197].

To describe the density of zeros on the Lee-Yang circle, we introduce an angle θ associated with the location of the zero on the unit circle. For small θ we have $g(\theta) = a|\theta|^{1-\alpha(q)}$, for $q \cdot\ 4$, and $g(\theta) = \epsilon(q)$, when $q > 4$. This leads to the specific heat singularity $|t|^{-\alpha(q)}$, for $q \cdot\ 4$, and a jump discontinuity of $\epsilon(q)$ in U for $q \cdot\ 4$. This is the known critical behavior of the Potts model [89].

The zeros of the partition function of the q-state Potts model on the square lattice have been evaluated numerically [170]. On the basis of these numerical results, it was conjectured [170] that, for both finite self-dual lattices and for lattices with free or periodic boundary conditions in the thermodynamic limit, the zeros in the $\mathrm{Re}(x) > 0$ region of the complex x plane *are located on the unit circle* $|x| = 1$.

VII-5-3. Zeros for two- and three-spin interactions on triangular lattice

We now return to the q-state Potts model on the triangular lattice with two- and three-site interactions in alternate triangular faces [79] (section VI-1)). The partition function is:

$$Z(x, x_1, x_2, x_3) = \sum_G W(G), \qquad \text{where:}$$
$$W(G) = \prod_\Delta (1 + v\delta_{abc})(1 + v_1\delta_{bc})(1 + v_2\delta_{ca})(1 + v_3\delta_{ab}),$$
$$\text{and} \qquad v = e^K - 1, \qquad v_i = e^{K_i} - 1,$$

and the product is taken over all up-pointing triangles. It is convenient to represent terms in the expansion of the partition function by graphs G in which the up-pointing triangular faces are either occupied by a solid triangle with a fugacity v or unoccupied.

We next evaluate the weight $W(G)$ associated with the graph G. It is clear that each solid triangle contributes a factor v to $W(G)$, and each bond a factor v_i. In addition, by including the associated bond factors, each solid triangle contributes an additional factor $(1 + v_1)(1 + v_2)(1 + v_3) = e^{K_1+K_2+K_3}$. Consider next the q dependence of $W(G)$. For the graph representing N isolated points, we have simply $W(G) = q^N$. For other graphs, each triangle reduces the factor q^N by q^2, and each bond by q. But whenever the triangles and bonds close up to form a circuit[24], this restores a factor q, due to the overlapping of one lattice site summation. Thus we have:

$$Z = q^N \sum_G \left[\frac{v}{q^2} e^{K_1+K_2+K_3}\right]^m \left[\frac{v_1}{q}\right]^{b_1} \left[\frac{v_2}{q}\right]^{b_2} \left[\frac{v_3}{q}\right]^{b_3} q^c, \tag{71}$$

where the summation is over the 2^N graphs G, m is the number of solid triangles, b_i is the number of bonds with weight v_i, and $c(G)$ is the number of independent circuits in G. Expression (71) generates the high-temperature expansion of the partition function.

In the case of pure three-site interactions, (71) reduces to

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[23] A polynomial $P(x)$ in x is self-dual if it is proportional to $P(1/x)$. Self-dual polynomials occur naturally in lattice models in statistical physics and in restricted partitions of an integer in number theory (see section (VIII-3) below).

[24] Here, we use the term *circuit* in the topological sense that solid triangles can be regarded as stars having three branches, each of which can be connected to other triangles and bonds.

$$Z = q^N \sum_G (w/q)^{m(G)} q^{c(G)}, \quad \text{where} \quad w = (e^K - 1)/q.$$

For pure three-site interactions, the partition function is self-dual and the critical variety assumes the simple form $w = 1$. On the basis of a reciprocal symmetry and numerical results, Wu *et al.* [170] conjectured that zeros for the three-site Potts model (in up-pointing triangles) lie in the thermodynamic limit on the *unit circle* $|w| = 1$, as well as a line segment on the real negative axis.

VII-5-4. Density of Fisher zeros for the Ising model

One trademark of F. Y. Wu's research is that he often looks at old problems and finds new life or new solution that others have not previously seen. A good example of a new look at an old problem is the density of Fisher zeros for the Ising model, which is the $q = 2$ Potts model.

In 1964 Fisher pointed out that in the thermodynamic limit, the zeros of the Ising partition function for a square lattice lie on two circles, now known as the Fisher circles, in the complex $\tanh K$ plane, where K is the nearest-neighbor interaction. However, Fisher had not made the argument rigorous and, furthermore, no one had bothered to look into the distribution of the zeros on the circles, except at small angles which dictates the Ising critical behavior.

Both of these two deficiencies have been rectified by W. T. Lu and F. Y. Wu. First, by considering the zeros of the Fisher zeros for the Ising model on finite self-dual lattices, Lu and Wu [190] established rigorously that, indeed, the Fisher zeros approach two circles in the thermodynamic limit. In a subsequent paper published in 2000 [202], they deduced the close-form expression for the density of the Fisher zeros for many regular two-dimensional lattices, thus completing the story of the Fisher zeros some 25 years after it was first proposed!

VII-6. Duality relation for Potts correlation functions

VII-6-1. Correlation dualities

Duality considerations are not often applied to correlation functions [133], but F. Y. Wu initiated a new method for generating duality relations for correlation functions of the Potts model on planar graphs. In a pioneering paper [183], he obtained duality relations for 2- and 3-point correlation functions, for spins residing on the boundary of a lattice. The consideration was soon extended to n-point correlations [187, 189] and to the case where the spins reside on 2 or more faces in the interior of the lattice [206]. A graph-theoretical formulation of the results in terms of rooted Tutte polynomials (see section (VII-6-2) below) was also given [186, 187]. In addition, C. King and Wu [206] showed that, generally, it is linear combinations of correlation functions, not the individual correlations, that are related by dualities.

VII-6-2. Correlation functions as rooted Tutte Polynomials

As previously mentioned, the Potts partition function is also the Whitney-Tutte polynomial, or in short, the Tutte polynomial, considered in graph theory. In one further step, F. Y. Wu, C. King and W. L. Lu formulated the Potts correlation function as a rooted Tutte polynomial [193].

In graph theory a vertex is rooted if it is colored with a prescribed (fixed) color and a graph is rooted if it contains a rooted vertex. If one interprets the color of a site (vertex) as spin states, then the Potts correlation functions for which the spin states of given sites are fixed can naturally be formulated as rooted-Tutte polynomials. This is the basis of their graph-theoretical formulation of the Potts correlation function, from which duality relations of the Potts correlations become transparent and can be analyzed [193].

VII-7. Potts model and graph theory

F. Y. Wu has written several review papers dedicated to the analysis of the Potts model from a graph theoretical viewpoint [57, 71, 117, 122]. The most important one is the classic paper [57] written in collaboration with Baxter and Kelland alluded to above, which gives the graphical construction of the equivalence of the partition function of the Potts model with an ice-type model [57]. This derivation simplifies the algebraic method of Temperley and Lieb, which is based on the Temperley-Lieb algebra[25]:

$$U_{i,i+1}^2 = \sqrt{q} \cdot U_{i,i+1},$$
$$U_{i,i-1} \cdot U_{i,i+1} \cdot U_{i,i-1} = U_{i,i-1}, \qquad U_{i,i+1} \cdot U_{i,i-1} \cdot U_{i,i+1} = U_{i,i+1},$$
$$U_{i,i+1} \cdot U_{j,j+1} = U_{j,j+1} \cdot U_{i,i+1} \qquad \text{if} \qquad |i-j| > 3,$$

and applies to an arbitrary planar graph. This then opens the door for analyzing the triangular and honeycomb Potts models.

Another important graphical analysis of a Potts model is the joint work with J. H. H. Perk [103, 104] on the non-intersecting string (NIS) model of Stroganov and Schultz, the close-packed loop model. The NIS model formulated by Perk and Wu in [103] turns out to be nothing but the bracket polynomial introduced by L. H. Kauffman in his state-model formulation of knot invariants. This fact offers a most natural approach to knot invariants from a statistical mechanical viewpoint [154].

VIII. Other miscellaneous topics

F. Y. Wu has worked on a diverse array of topics in mathematics and mathematical physics. In this section we present a random choice of topics that are not included above.

VIII-1. Topics in graph theory

F. Y. Wu is fond of graphs and has made many contributions to graph theory. A fine example is the aforementioned introduction of the rooted Tutte polynomial. Even when Wu does not obtain new results, he tries to provide simpler derivations, or find new consequences of known results. A good example is his work on random graphs [92].

Random graphs is a topic well-known to graph theorists after the work by Erdös and Renyi, who introduced the problem in 1960. In the simplest formulation each pair of points of a set of N can be connected (by a bond, say) with a probability α/N, where α is a constant. One then asks questions such as what is the mean cluster size and the probability $P(\alpha)$ that the set becomes fragmented, namely not connected, etc.

Using a Potts model formulation, Wu [93] reproduced the Erdös-Renyi result in just a few steps, showing that a transition occurs at $\alpha_c = 1$ and computed the critical exponents as well as the mean cluster size at criticality. This work has drawn considerable attention from graph

[25] A matrix representation of the $U_{i,i\pm1}$ is, for instance, the $q^n \times q^n$ matrices with entries $q^{-1/2}\prod_{j=1;j\neq i}^n \delta(\sigma_j, \sigma_j')$ and $q^{1/2}\delta(\sigma_i,\sigma_{i+1})\prod_{j=1}^n \delta(\sigma_j,\sigma_j')$.

theorists. Wu has also applied the result to evaluate the reliability probability of a communication network [92].

As another example of Wu's work in graph theory, one can mention his paper on the Temperley-Nagle identity for graph embeddings [69], where he provides a simple derivation of the Temperley-Nagle identity:

$$\sum_G x^v y^l = (1+x)^N \sum_L (y-1)^l \Big(\frac{x}{1+x}\Big)^v,$$

where G denotes section graphs of the original graph of v vertices and l lines, and L is a line set of G containing l lines covering v vertices. Using a weak-graph expansion he also deduced a sum-rule relation connecting the lattice constants of weak and strong embeddings.

VIII-2. The vicious neighbor problem

Consider N points randomly distributed in a bounded d-dimensional space. At a given instance, each point destroys his nearest neighbor (vicious neighbors) with a probability p. What is the probability $P_N(p)$ that a given point will survive in the $N \longrightarrow \infty$ limit?

The $d=2$, $p=1$ version of this problem was first posed by the Brandeis mathematician R. Abilock in American Mathematical Monthly in 1967, and remained unsolved for almost two decades. In 1986 the *Omni* magazine posted a prize for its correct solution, and R. Tao and Wu claimed the prize by publishing the solution for general d and p in 1987 [111].

The idea of their solution is very simple. In d dimensions a given point can be killed by at most a finite number n_d of other points. In two dimensions, for example, the number is $n_2 = 5$ (theoretically a point can also be killed by 6 other points, but the phase space for that to happen has a zero measure). Therefore, one computes the volumes of the phase space for a point to be killed by 1, 2, $\cdots$, n_d neighbors, and uses the inclusion-exclusion principle to write the probability in question as an alternate series in p, whose highest power is n_d. However, the evaluation of the volumes of the phase space is tedious, requiring special techniques.

For $d=1$ the result is quite simple and one has $P_\infty(p) = 1 - p + p^2/2$. For $d=2$ the result is:

$$P_\infty(p) = 1 - p + 0.316\ 3335\, p^2 - 0.032\ 9390\, p^3 + 0.000\ 6575\, p^4 - 0.000\ 0010\, p^5, \quad (72)$$

where the coefficient of each term is evaluated from integrals which can, in principle, be computed to any numerical accuracy. The coefficient of the last term, for example, is obtained by combining two 8-fold integrations.

For $p=1$, (72) yields $P_\infty(1) = 0.284\ 051...$, a solution which claimed the *Omni* prize. Tao and Wu also carried out Monte Carlo simulations to obtain the solution for $d = 3, 4, 5$.

VIII-3. Counting partitions: from Potts to three-dimensional enumeration and beyond

F. Y. Wu *et al.* [172] considered a directed lattice animal problem on the d-dimensional hypercubic lattice, and established its equivalence first with the infinite-range Potts model and, in a second step, with the enumeration of $(d-1)$-dimensional restricted partitions of an integer. The directed compact lattice animal problem was solved exactly in two and three dimensions, using known results in number theory. They found that the number of lattice animals of n sites grows as:

$$\exp(c \cdot n^{(d-1)/d}).$$

Furthermore, the infinite-state Potts model solution leads to a conjectured limiting form for the generating function of restricted partitions for $d > 3$, which is a long-standing unsolved problem in number theory.

Let us denote by $A_n(L_1, L_2, \cdots L_d)$ the number of n-site animals that can grow on an $L_1 \times L_2 \times \cdots \times L_d$ lattice. F. Y. Wu *et al.* showed that A_n is precisely the number of $(d-1)$-dimensional restricted partitions of the integer n into non-negative parts to units of a hypercube of size $L_1 \times L_2 \times \cdots \times L_{d-1}$, with the size of each part being at most L_d.

Define the generating function

$$G(L_1, L_2, L_3, \cdots L_d; t) = 1 + \sum_{n=1}^{L_1 \cdot L_2 \cdot L_3 \cdots L_d} A_n(L_1, L_2, \cdots L_d) \cdot t^n,$$

which is of interest in number theory. F. Y. Wu *et al.* showed that G is precisely the partition function of a Potts model on the d-dimensional lattice in the infinite-state limit, provided one identifies t with x^d where $x = (e^K - 1)/q^{1/d}$. This then connects the Potts model with the theory of partitions in number theory.

For $d = 2$, the generating function corresponding to the square lattice (L_1, L_2) reads[26]:

$$G(L_1, L_2; t) = \frac{(t)_{L_1+L_2}}{(t)_{L_1} \cdot (t)_{L_2}}, \tag{73}$$

where $(t)_p = \prod_{q=1}^{p} (1 - t^q)$. $G(L_1, L_2)$ is a polynomial in t, also known as the Gaussian polynomial or the "q-coefficient". For $d = 3$, the generating function reads:

$$G(L_1, L_2, L_3; t) = \frac{[t]_{L_1+L_2+L_3} \cdot [t]_{L_1} \cdot [t]_{L_2} \cdot [t]_{L_3}}{[t]_{L_1+L_2} \cdot [t]_{L_2+L_3} \cdot [t]_{L_3+L_1}}, \tag{74}$$

where $[t]_L$ denotes:

$$[t]_L = \prod_{p=1}^{L-1} (t)_p, \quad L > 1, \qquad (t)_p = \prod_{q=1}^{p} (1 - t^q). \tag{75}$$

For $d = 4$, $G(L_1, L_2, L_3, L_4; t)$ is the generating function of restricted solid partitions of a positive integer into parts on a $L_1 \times L_2 \times L_3$ cubic lattice, with each part being no greater than L_4. The evaluation of a closed-form expression for G in this case has remained an unsolved problem for almost a century.

The expression which straightforwardly generalizes (73) and (74) would be:

$$G_{\mathrm{straight}}(L_1, L_2, L_3, L_4; t) = \frac{N_G(t)}{D_G(t)}, \qquad \text{with:} \tag{76}$$

[26] Let us recall the classic analysis due to Rademacher, which yields the celebrated Hardy-Ramanujan asymptotic result: $A_n \simeq 1/(4n\sqrt{3}) \cdot \exp(\pi\sqrt{2n/3})$. One gets the following asymptotic behavior for $A_n(L_1, L_2)$ when $L_1 = L_2$: $A_n(L, L) \simeq (\sqrt{3}/(2\pi n)) \cdot 2^{2\sqrt{2n}}$.

$$N_G(t) = \{t\}_{L_1+L_2+L_3+L_4} \cdot \{t\}_{L_1+L_2} \cdot \{t\}_{L_1+L_3} \cdot \{t\}_{L_1+L_4} \cdot \{t\}_{L_2+L_3} \cdot \{t\}_{L_2+L_4} \cdot \{t\}_{L_3+L_4},$$

$$D_G(t) = \{t\}_{L_1+L_2+L_3} \cdot \{t\}_{L_2+L_3+L_4} \cdot \{t\}_{L_1+L_3+L_4} \cdot \{t\}_{L_1+L_2+L_4} \cdot \{t\}_{L_1} \cdot \{t\}_{L_2} \cdot \{t\}_{L_3} \cdot \{t\}_{L_4},$$

where:

$$\{t\}_L = \prod_{p=1}^{L-1} [t]_p \quad L > 2. \tag{77}$$

But the explicit expression of $G(2, 2, 2, L_4; t)$, obtained by Major P. A. MacMahon in 1916 is

$$G(2, 2, 2, L_4; t) = G_{\rm straight}(2, 2, 2, L_4; t) + C(2, 2, 2, L_4; t), \tag{78}$$

where

$$G(2, 2, 2, L_4; t) = \sum_{i=0}^{4} g_i \cdot \frac{(t)_{L_4+8-i}}{(t)_8 \cdot (t)_{L_4-i}}, \qquad \text{with:}$$

$$g_0 = 1, \qquad g_1 = 2t^2 \cdot (1 + t + t^2 + t^3 + t^4) + t^4,$$

$$g_2 = t^5 \cdot (1 + 3t + 4t^2 + 8t^3 + 4t^4 + 3t^5 + t^6)$$

$$g_3 = 2t^{10} \cdot (1 + t + t^2 + t^3 + t^4) + t^{12}, \qquad g_4 = t^{16},$$

and

$$C(2,\, 2,\, 2,\, P;\, t) = -\Big(\frac{t^6 \cdot (t+1)^2 \cdot (t^4 - 2\,t^3 + t^2 - 2\,t + 1)}{t^2 + t + 1}\Big) \cdot \Big(\frac{(t)_{P+6}}{(t)_8 (t)_{P-2}}\Big).$$

H. Y. Huang and F. Y. Wu [179] decided to look into the zeros of the generating function $G(2, 2, 2, L_4; t)$ for various increasing values of L_4. They found that the zeros are not exactly on the unit circle, but seem to converge to the unit circle as L_4 increases. This indicates that a multiplicative correction $C_{\rm mult}(2, 2, 2, L_4; t) = G(L_1, L_2, L_3, L_4; t)/G_{\rm straight}(L_1, L_2, L_3, L_4; t)$, would not have any simple Eulerian product form as in (76) and (77):

$$C_{\rm mult}(2, 2, 2, L_4; t) = \prod_{n=1} (1 - t^n)^{\alpha_n}, \tag{79}$$

where α_n are positive integers, since these product forms (79) would necessarily yield zeros on the unit circle. H. Y. Huang and F. Y. Wu conjectured however, on the basis on their numerical results, that the zeros tend to be on the unit circle in the limit, when any one of L_1, L_2, L_3, $L_4 \to \infty$.

VIII-3-1. Directed percolation and random walk problems

F. Y. Wu and H. E. Stanley [90] have considered a directed percolation problem on square and triangular lattices in which the occupation probability is unity along one spatial direction.

They formulated the problem as a random walk, and evaluated in closed-form the percolation probability, or the arriving probability of a walker. To this date this solution stands as the only exactly solved model of directed percolation.

In another random walk problem, Wu and H. Kunz [192] considered *restricted random walks* on graphs, which keep track of the number of immediate reversal steps, by using a transfer matrix formulation. A closed-form expression was obtained for the number of n-step walks with r immediate reversals for any graph. In the case of graphs of a uniform valence, they established a probabilistic meaning of the formulation, and deduced explicit expressions for the generating function in terms for the eigenvalues of the adjacency matrix.

IX. Knot theory

The connection between knot theory and statistical mechanics was probably first discovered by Jones. His derivation of the V. Jones polynomial reflects the resemblance to the von Neumann algebra when he uses with the Lieb-Temperley algebra occurring in the Potts model (see section (VII-7)). This direct connection came to light when L. Kauffman produced a simple derivation of the Jones polynomial using the very diagrammatic formulation of the non-intersecting string (NIS) model of J. H. H. Perk and F. Y. Wu [103, 104]. Soon thereafter Jones worked out a derivation of the Homfly polynomial using a vertex-model approach. The connection between knot theory and lattice statistical mechanics was further extended by Jones to include spin and IRF models.

F. Y. Wu has written several papers on the connection between knot theory and statistical mechanics [150, 151, 154], including a comprehensive review [150]. In hindsight, knot invariants arose naturally in statistical mechanics even before the connection with solvable models was discovered. In their joint paper [103], for example, J. H. H. Perk and F. Y. Wu described a version of an NIS model which is precisely the bracket polynomial of L. Kauffman. Similarly, the q-color NIS model studied by J.H.H. Perk and C. Schultz is a $q(q-1)$ vertex model which generates the Homfly polynomial. Here we briefly describe the latter connection.

The q-color NIS model has vertex weights ($\delta_{abcd} = 1$ if $a = b = c = d$ and zero otherwise):

$$w(a,b,c,d) = (W(u) - S(u) - T(u)) \cdot \delta_{abcd} + S(u) \cdot \delta_{ab}\delta_{cd} + T(u) \cdot \delta_{ac}\delta_{bd}, \qquad (80)$$

$$\text{where:} \qquad W(u) = \sinh(u) = \sinh(\eta + u), \quad S(u) = \sinh(u),$$

$$T(u) = \sinh(\eta - u), \qquad q = e^{\eta} + e^{-\eta},$$

and the *Homfly polynomial* is a *two variable* knot invariant polynomial, discovered after Jones' work, by Freyd *et al.* The Homfly polynomial knot invariant has since been re-derived and analyzed by Jones using the Hecke algebra of the braid group. It can also be constructed from the Perk-Schultz NIS model. Actually the partition function $Z(q, e^{\eta})$ of the NIS model is a knot invariant related to the Homfly polynomial $P(t, z)$:

$$P(t,z) = \frac{\sinh(\eta)}{\sinh(q\eta)} \cdot Z(q, e^{\eta}).$$

In the infinite rapidity limit this model leads to the Jones polynomial. The Boltzmann weight (80) of the non-intersecting string model becomes:

$$w(a,b,c,d) = -e^{\pm 2\eta}\delta_{a,b}\delta_{c,d} + e^{\pm\eta}\delta_{b,d} \quad \text{with:} \quad q = e^{\eta} + e^{-\eta}. \qquad (81)$$

The Jones polynomial $V(t)$ is then obtained from the Homfly polynomial $P(t,z)$ by taking $z = \sqrt{t} - 1/\sqrt{t}$.

F. Y. Wu discussed many knot invariants in his review [154]: the Alexander-Conway polynomial, the Jones polynomial, the Homfly polynomial, the Kauffman polynomial and the Akutsu-Wadati polynomial, etc. The Alexander-Conway polynomial can be obtained from the Homfly polynomial by setting $t = 1$ in the Homfly polynomial $P(t,z)$. The Akutsu-Wadati polynomial is an example of a *new* knot invariant derived from exactly solvable models in statistical mechanics.

As our final example of F. Y. Wu's versatility, he and P. Pant and C. King [162] have obtained a new knot invariant using the exactly solvable chiral Potts model and a generalized Gaussian summation identity. Starting from a general formulation of link invariants using edge-interaction spin models, they establish the uniqueness of the invariant for self-dual models. They applied the formulation to the self-dual chiral Potts model, and obtain a *link invariant* in the form of a lattice sum defined by a matrix associated with the link diagram. A generalized Gaussian summation identity was then used to carry out this lattice sum, enabling them to cast the invariant into a tractable form. The resulting expression for the link invariant was characterized by the roots of unity and does not appear to belong to the usual quantum group family of invariants.

Finally, Pant and Wu [185] have derived a link invariant associated with the Izergin-Korepin model.

X. Conclusion

It would not be fair to summarize F. Y. Wu's contributions by a quick conclusion such as: he wrote several important monographs on vertex models, on the Potts model and on knot theory, obtained many important results, in particular the Lieb-Wu solution of the Hubbard model, the Fan-Wu free-fermion vertex model, the solution of the Baxter-Wu model, and many other results on dimers or free-fermion models, 3D dimers, d-dimensional free-fermion models, Potts models, Ising and vertex models, using a large set of tools including analytic calculations, expansions, series analysis, Monte-Carlo, ..., with a particular emphasis on graph-theoretical methods.

Most of the work of F. Y. Wu could be said to correspond to exact results in lattice statistical mechanics, or in mathematics, with particular emphasis on *graph theory* and enumerative combinatorics. We have tried to give here some hints as to the space of F. Y. Wu's very large "graph" of concepts, results, tools, models, with many "intellectual loops". We have not tried to provide an exhaustive description of F. Y. Wu's contributions but, rather, only to provide a few *comments* on some of his results, emphasizing the fruitful cross-fertilizations between the various domains of mathematical physics and mathematics, and also to show the motivation and relevance of these results, tools, concepts and methods.

Beyond a post-modern accountant's evaluation and from a research viewpoint, one must say that the important and numerous results F. Y. Wu has obtained are not due to publish-or-perish productivity pressure, but, on the contrary, are the natural consequence of the pleasure of a scientist who loves to play with concepts and mathematical objects (dimers, graphs with particular boundary conditions, dualities, Potts models, series expansions with transmissivities, ...) and who has a strong desire to reach ambitious goals, such as obtaining new results in three dimensions, new results for non-critical Potts models, or even for the Ising model in a magnetic field.

A scientist does not become as productive as F. Y. Wu in response to external pressure but, on the contrary, only by forces being in harmony with his deep personal motivations. This is the only way to be as efficient and productive as F. Y. Wu and, as the famous French mathematician Jean Dieudonné once wrote, to work efficiently, “pour l’honneur de l’esprit humain”.

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References

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[1] F. Y. Wu, On the discussions of ‘free waveforms’ (in Chinese), Naval Tech. **2**, 9-11 (1955).

[2] F. Y. Wu, Ground state of liquid He^4, in *Proceedings of the Midwest Conference on Theoretical Physics*, 182-196 (1961).

[3] F. Y. Wu and E. Feenberg, Ground state of liquid helium (Mass 4), Phys. Rev. **122**, 739-742 (1961).

[4] F. Y. Wu and E. Feenberg, Theory of Fermi liquids, Phys. Rev. **128**, 943-955 (1962).

[5] F. Y. Wu, T. S. Kuo and K. H. Sun, Four slow neutron converters, Nucleonics **20**, 61-62 (1962).

[6] F. Y. Wu, The ground state of liquid He^4, Prog. Theoret. Phys. (Kyoto) **28**, 568-569 (1962).

[7] F. Y. Wu, Cluster development in an N-body problem, J. Math. Phys. **4**, 1438-1443 (1963).

[8] F. Y. Wu, H. T. Tan and E. Feenberg, Necessary conditions on radial distributions functions, J. Math. Phys. **8**, 864-869 (1967).

[9] F. Y. Wu, Exactly soluble model of ferroelectric phase transition in two dimensions, Phys. Rev. Lett. **18**, 605-607 (1967).

[10] F. Y. Wu, Remarks on the modified KDP model, Phys. Rev. **168**, 539-543 (1968).

[11] E. H. Lieb and F. Y. Wu, Absence of Mott transition in an exact solution of the short-range one-band model in one dimension, Phys. Rev. Lett. **20**, 1445-1448 (1968).

[12] C. Fan and F. Y. Wu, Ising model with next-neighbor interactions: Some exact results and an approximate solution, Phys. Rev. **179**, 560-570 (1969).

[13] F. Y. Wu, Critical behavior of two-dimensional hydrogen-bonded antiferroelectrics, Phys. Rev. Lett. **22**, 1174-1176 (1969).

[14] F. Y. Wu, Exact solution of a model of an antiferroelectric transition, Phys. Rev. **183**, 604-607 (1969).

[15] F. Y. Wu and M. K. Chien, Convolution approximation for the n-particle distribution function, J. Math. Phys. **11**, 1912-1916 (1970).

[16] C. Fan and F. Y. Wu, General Lattice Model of Phase Transitions, Phys. Rev. **B2**, 723-733 (1970).

[17] M. F. Chiang and F. Y. Wu, One-dimensional binary alloy with Ising interactions, Phys. Lett. A **31**, 189-191 (1970).

[18] F. Y. Wu, Critical behavior of hydrogen-bonded ferroelectrics, Phys. Rev. Lett. **24**, 1476-1478 (1970).

[19] F. Y. Wu, C. W. Woo and H. W. Lai, Communications on a microscopic theory of helium submonolayers, J. Low Temp. Phys. **3**, 331-333 (1970).

[20] F. Y. Wu, H. W. Lai and C. W. Woo, Theory of helium submonolayers: Formulation of the problem and solution in the zero coverage limit, J. Low Temp. Phys. **3**, 463-490 (1970).

[21] F. Y. Wu, *Introduction to Crystal Physics* (in Chinese), translation of a popular science book of Doubleday science study series (Chung Hwa Book Co., Taipei 1970).

[22] F. Y. Wu, Density correlations in a many particle system, Phys. Lett. A **34**, 446-447 (1971).

[23] F. Y. Wu and D. M. Kaplan, On the eigenvalues of orbital angular momentum, Chin. J. Phys. **9**, 31-40 (1971).

[24] F. Y. Wu, Modified KDP Model in a staggered field, Phys. Rev. B **3**, 3895-3900 (1971).

[25] F. Y. Wu, H. T. Tan and C. W. Woo, Sum rules for a binary solution and effective interactions between He^3 quasi-particles in superfluid He^4, J. Low Temp. Phys. **5**, 261-265 (1971).

[26] F. Y. Wu, H. W. Lai and C. W. Woo, Theory of helium submonolayers II: Bandwidths and correlation effects, J. Low Temp. Phys. **5**, 499-518 (1971).

[27] F. Y. Wu and C. W. Woo, Physically adsorbed monolayers, Chin. J. Phys. **9**, 68-91 (1971).

[28] F. Y. Wu, Multiple density correlations in a many particle system, J. Math. Phys. **12**, 1923-1929 (1971).

[29] F. Y. Wu, Ising model with four-spin interactions, Phys. Rev. B **4**, 2312-2314 (1971).

[30] K. S. Chang and S. Y. Wang and F. Y. Wu, Circle theorem for the ice-rule ferroelectric models, Phys. Rev. A **4**, 2324-2327 (1971).

[31] E. H. Lieb and F. Y. Wu, Two-dimensional ferroelectric models, in *Phase Transitions and Critical Phenomena*, Vol. 1, Eds. C. Domb and M. S. Green (Academic Press, London, 1972) pp. 331-490.

[32] F. Y. Wu, Exact results on a general lattice statistical model, Solid State Commun. **10**, 115-117 (1972).

[33] F. Y. Wu, Solution of an Ising model with two- and four-spin interactions, Phys. Lett. A **38**, 77-78 (1972).

[34] F. Y. Wu, Phase transition in a sixteen-vertex lattice model, Phys. Rev. B **6**, 1810-1813 (1972).

[35] G. Keiser and F. Y. Wu, Electron gas at metallic densities, Phys. Rev. A **6**, 2369-1377 (1972).

[36] F. Y. Wu, Matrix elements of a many-boson Hamiltonian in a representation of correlated basis functions, J. Low Temp. Phys. **9**, 177-187 (1972).

[37] F. Y. Wu, Statistics of parallel dimers on an $m \times n$ lattice, Phys. Lett. A **43**, 21-22 (1973).

[38] H. J. Brascamp, H. Kunz and F. Y. Wu, Critical exponents of the modified F model, J. Phys. C **6**, L164-L166 (1973).

[39] K. S. Chang, S. Y. Wang and F. Y. Wu, Statistical mechanics of finite ice-rule ferroelectric models, Chin. J. Phys. **11**, 41-48 (1973).

[40] F. Y. Wu, Phase diagram of a decorated Ising system, Phys. Rev. B **8**, 4219-4222 (1973).

[41] F. Y. Wu, Modified KDP model on the Kagomé lattice, Prog. Theoret. Phys. (Kyoto) **49**, 2156-2157 (1973).

[42] G. Keiser and F. Y. Wu, Correlation energy of an electron gas in the quantum strong-field limit, Phys. Rev. A **8**, 2094-2098 (1973).

[43] H. J. Brascamp, H. Kunz and F. Y. Wu, Some rigorous results for the vertex model in statistical mechanics, J. Math. Phys. **14**, 1927-1932 (1973).

[44] F. Y. Wu, Phase transition in an Ising model with many-spin interactions, Phys. Lett. A **46**, 7-8 (1973).

[45] R. J. Baxter and F. Y. Wu, Exact solution of an Ising model with three-spin interactions on a triangular lattice, Phys. Rev. Lett. **31**, 1294-1297 (1973).

[46] F. Y. Wu, Phase transition in a vertex model in three dimensions, Phys. Rev. Lett. **32**, 460-463 (1974).
[47] F. Y. Wu and K. Y. Lin, Two phase transitions in the Ashkin-Teller model, J. Phys. C **7**, L181-L184 (1974).
[48] F. Y. Wu, Eight-vertex model on the honeycomb lattice, J. Math. Phys. **6**, 687-691 (1974).
[49] R. J. Baxter and F. Y. Wu, Ising model on a triangular lattice with three-spin interactions: I. The eigenvalue equations, Aust. J. Phys. **27**, 357-367 (1974).
[50] F. Y. Wu and K. Y. Lin, Staggered ice-rule model - The Pfaffian solution, Phys. Rev. B **12**, 419-428 (1975).
[51] F. Y. Wu, Spontaneous magnetization of the three-spin Ising model on the Union Jack lattice, J. Phys. C **8**, 2262-2266 (1975).
[52] C. S. Hsue, K. Y. Lin and F. Y. Wu, Staggered eight-vertex model, Phys. Rev. B **12**, 429-437 (1975).
[53] J. E. Sacco and F. Y. Wu, Thirty-two vertex model on a triangular lattice, J. Phys. A **8**, 1780-1787 (1975).
[54] Y. K. Wang and F. Y. Wu, Multi-component spin model on the Bethe lattice, IUPAP Conference on Statistical Physics (Akademiai Kiado, 1975), p. 142.
[55] M. L. C. Leung, B. Y. Tong and F. Y. Wu, Thermal denaturation and renaturation of DNA molecules, Phys. Lett. A **54**, 361-362 (1975).
[56] Y. K. Wang and F. Y. Wu, Multi-component spin model on the Cayley tree, J. Phys. A **9**, 593-604 (1976).
[57] R. J. Baxter, S. B. Kelland and F. Y. Wu, Equivalence of the Potts model or Whitney polynomial with the ice-type model: A new derivation, J. Phys. A **9**, 397-406 (1976).
[58] F. Y. Wu and Y. K. Wang, Duality transformation in a many component spin model, J. Math. Phys. **17**, 439-440 (1976).
[59] F. Y. Wu, Two phase transitions in triplet Ising models, J. Phys. C **10**, L23-L27 (1977).
[60] F. Y. Wu, Ashkin-Teller model as a vertex problem, J. Math. Phys. **18**, 611-613 (1977).
[61] K. G. Chen, H. H. Chen, C. S. Hsue and F. Y. Wu, Planar classical Heisenberg model with biquadratic interactions, Physica A **87**, 629-632 (1977).
[62] F. Y. Wu, Number of spanning trees on a lattice, J. Phys. A **10**, L113-L115 (1977).
[63] H. Kunz and F. Y. Wu, Site percolation as a Potts model, J. Phys. C **11**, L1-L4 (1978).
[64] F. Y. Wu, Percolation and the Potts model, J. Stat. Phys. **18**, 115-123 (1978).
[65] F. Y. Wu, Some exact results for lattice models in two dimensions, Ann. Israel Phys. Soc. **2**, 370-376 (1978).
[66] V. T. Rajan, C.-W. Woo and F. Y. Wu, Multiple density correlation for inhomogeneous systems, J. Math. Phys. **19**, 892-897 (1978).
[67] F. Y. Wu, Absence of phase transitions in tree-like percolation in two dimensions, Phys. Rev. B **18**, 516-517 (1978).
[68] F. Y. Wu, Phase diagram of a spin-one Ising system, Chin. J. Phys. **16**, 153-156 (1978).
[69] F. Y. Wu, On the Temperley-Nagle identity for graph embeddings, J. Phys. A **11**, L243 (1978).
[70] A. Hinterman, H. Kunz and F. Y. Wu, Exact results for the Potts model in two dimensions, J. Stat. Phys. **19**, 623-632 (1978).
[71] F. Y. Wu, Graph Theory in Statistical Physics, in *Studies in Foundations of Combinatorics*, Adv. in Math.: Supp. V.1, Ed. G-C. Rota, 151-166, (Academic Press, New York 1978).
[72] K. Y. Lin and F. Y. Wu, Phase diagram of the antiferromagnetic triangular Ising model with anisotropic interactions, Z. Phys. B **33**, 181-185 (1979).
[73] F. Y. Wu, Phase diagram of a five-state spin model, J. Phys. C **12**, L317-L320 (1979).

[74] F. Y. Wu, Critical point of planar Potts models, J. Phys. C **12**, 645-649 (1979).

[75] X. Sun and F. Y. Wu, Critical polarization of the modified F model, J. Phys. C **12**, L637-L641 (1979).

[76] F. Y. Wu and K. Y. Lin, On the triangular Potts model with two- and three-site interactions, J. Phys. A **14**, 629-636 (1980).

[77] F. Y. Wu, Exact results for a dilute Potts model, J. Stat. Phys. **23**, 773-782 (1980).

[78] F. Y. Wu, On the equivalence of the Ising model with a vertex model, J. Phys. A **13**, L303-L305 (1980).

[79] R. K. P. Zia and F. Y. Wu, Critical point of the triangular Potts model with two- and three-site interactions, J. Phys. A **14**, 721-727 (1981).

[80] X. Sun and F. Y. Wu, The critical isotherm of the modified F model, Physica A **106**, 292-300 (1981).

[81] F. Y. Wu, Dilute Potts model, duality and site-bond percolation, J. Phys. A **14**, L39-L44 (1981).

[82] W. Kinzel, W. Selke and F. Y. Wu, A Potts model with infinitely degenerate ground state, J. Phys. A **14**, L399-L404 (1981).

[83] S. Sarbach and F. Y. Wu, Exact results on the random Potts model, Z. Phys. B **44**, 309-316 (1981).

[84] F. Y. Wu and Z. R. Yang, Critical phenomena and phase transitions I: Introduction, Prog. in Physics (in Chinese) **1**, 100-124 (1981).

[85] F. Y. Wu and Z. R. Yang, Critical phenomena and phase transitions II: The Ising Model, Prog. in Physics (in Chinese) **1**, 314-364 (1981).

[86] F. Y. Wu and Z. R. Yang, Critical phenomena and phase transitions III: The vertex model, Prog. in Physics (in Chinese) **1**, 487-510 (1981).

[87] F. Y. Wu and Z. R. Yang, Critical phenomena and phase transitions IV: Percolation, Prog. in Physics (in Chinese) **1**, 511-525 (1981).

[88] F. Y. Wu and Z. R. Yang, Critical phenomena and phase transitions V: Model systems, Prog. in Physics (in Chinese) **1**, 525-541 (1981).

[89] F. Y. Wu, The Potts Model, Rev. Mod. Physics **54**, 235-268 (1982).

[90] F. Y. Wu and H. E. Stanley, Domany-Kinzel model of directed percolation: Formulation as a random-walk problem and some exact results, Phys. Rev. Lett. **48**, 775-777 (1982).

[91] I. G. Enting and F. Y. Wu, Triangular lattice Potts model, J. Stat. Phys. **28**, 351-378 (1982).

[92] F. Y. Wu, Random graphs and network communication, J. Phys. A **15**, L395-L398 (1982).

[93] F. Y. Wu, An infinite-range bond percolation, J. Appl. Phys. **53**, 7977 (1982).

[94] F. Y. Wu and H. E. Stanley, Universality in Potts models with two- and three-site interactions, Phys. Rev. B **26**, 6326-6329 (1982).

[95] F. Y. Wu and Z. R. Yang, The Slater model of $K(H_{1-x}D_x)_2PO_4$ in two dimensions, J. Phys. C **16**, L125-L129 (1983).

[96] F. Y. Wu and H. E. Stanley, Polychromatic Potts model: A new lattice-statistical problem and some exact results, J. Phys. A **16**, L751-755 (1983).

[97] J. R. Banavar and F. Y. Wu, Antiferromagnetic Potts model with competing interactions, Phys. Rev. B **29**, 1511-1513 (1984).

[98] F. Y. Wu, Potts model of ferromagnetism, J. Appl. Phys. **55**, 2421-2425 (1984).

[99] D. H. Lee, R. G. Caflish, J. D. Joannopoulos, and F. Y. Wu, Antiferromagnetic classical XY-model: A mean-field analysis, Phys. Rev. B **29**, 2680-2684 (1984).

[100] F. Y. Wu, Exact solution of a triangular Ising model in a nonzero magnetic field, J. Stat. Phys. **40**, 613-620 (1985).

[101] N. C. Chao and F. Y. Wu, Disorder solution of a General checkerboard Ising model in a field and validity of the decimation approach, J. Phys. A **18**, L603-L607 (1985).

[102] Z. R. Yang and F. Y. Wu, Exact solution of a dilute-bond Potts model on the decorated square lattice, Acta Physica Sinica (in Chinese), **34** 484-492 (1985).

[103] J. H. H. Perk and F. Y. Wu, Non-intersecting string model and graphical approach: Equivalence with a Potts model, J. Stat. Phys. **42**, 727-742 (1986).

[104] J. H. H. Perk and F. Y. Wu, Graphical approach to the non-intersecting string model: Star-triangle equation, inversion relation and exact solution, Physica A **138**, 100-124 (1986).

[105] F. Y. Wu, Two-dimensional Ising model with crossing and four-spin interactions and a magnetic field $i\pi kT/2$, J. Stat. Phys. **44**, 455-463 (1986).

[106] F. Y. Wu, On the Horiguchi's Solution of the Blume-Emery-Griffiths Model, Phys. Lett. A **116**, 245-247 (1986).

[107] F. Y. Wu, Thermodynamics of particle in a box, Ann. Report Inst. Phys. Academia Sinica **16**, 25-30 (1986).

[108] H. K. Sim, R. Tao, and F. Y. Wu, Ground-state energy of charged quantum liquids in two dimensions, Phys. Rev. B **34**, 7123-7128 (1986).

[109] W. Selke and F. Y. Wu, Potts models with competing interactions, J. Phys. A **20**, 703-711 (1987).

[110] Y. Chow and F. Y. Wu, Residual entropy and validity of the third law of thermodynamics in discrete spin systems, Phys. Rev. B **36**, 285-288 (1987).

[111] R. Tao and F. Y. Wu, The Vicious Neighbor Problem, J. Phys. A **20**, L299-L306 (1987).

[112] G. O. Zimmerman, A. K. Ibrahim, and F. Y. Wu, The effect of defects on a two-dimensional dipolar system on a honeycomb lattice, J. Appl. Phys. **61**, 4416-4418 (1987).

[113] F. Y. Wu, Book Review: Statistical mechanics of periodic frustrated Ising systems (R. Liebmann), J. Stat. Phys. **48**, 953 (1987).

[114] F. Y. Wu and K. Y. Lin, Ising model on the Union Jack lattice as a free-fermion model, J. Phys. A **20**, 5737-5740 (1987).

[115] G. O. Zimmerman, A. K. Ibrahim, and F. Y. Wu, A planar classical dipolar system on a honeycomb lattice, Phys. Rev. B **37**, 2059-2065 (1988).

[116] X. N. Wu and F. Y. Wu, The Blume-Emery-Griffiths Model on the honeycomb lattice, J. Stat. Phys. **50**, 41-55 (1988).

[117] F. Y. Wu, Potts model and graph theory, J. Stat. Phys. **52**, 99-112 (1988).

[118] K. Y. Lin and F. Y. Wu, Magnetization of the Ising model on the generalized checkerboard lattice, J. Stat. Phys. **52**, 669-677 (1988).

[119] K. Y. Lin and F. Y. Wu, Ising models in the magnetic field $i\pi/2$, Int. J. Mod. Phys. B **2**, 471-481 (1988).

[120] F. Y. Wu and K. Y. Lin, Spin model exhibiting multiple disorder points, Phys. Lett. A **130**, 335-337 (1988).

[121] H. Kunz and F. Y. Wu, Exact results for an $O(n)$ model in two dimensions, J. Phys. A **21**, L1141-L1144 (1988).

[122] F. Y. Wu, Potts model and graph theory, in *Progress in Statistical Mechanics*, Ed. C. K. Hu, (World Scientific, Singapore 1988).

[123] K. Y. Lin and F. Y. Wu, Three-spin correlation of the free-fermion model, J. Phys. A **22** 1121-1130 (1989).

[124] X. N. Wu and F. Y. Wu, Duality properties of a general vertex model, J. Phys. A **22**, L55-L60 (1989).

[125] F. Y. Wu, X. N. Wu and H. W. J. Blöte, Critical frontier of the antiferromagnetic Ising model in a nonzero magnetic field: The honeycomb lattice, Phys. Rev. Lett. **62**, 2273-2276 (1989).

[126] F. Y. Wu and X. N. Wu, Exact coexistence-curve Diameter of a lattice gas with pair and triplet interactions, Phys. Rev. Lett. **63**, 465 (1989).

[127] X. N. Wu and F. Y. Wu, Exact results for lattice models with pair and triplet interactions, J. Phys. A **22**, L1031-L1035 (1989).

[128] J. H. Barry and F. Y. Wu, Exact solutions for a four-spin-interaction Ising model on the three-dimensional pyrochlore lattice, Int. J. Mod. Phys. B **8**, 1247-1275 (1989).

[129] F. Lee, H. H. Chen and F. Y. Wu, Exact duality-decimation transformation and real-space renormalization for the Ising model on a square lattice, Phys. Rev. B **40**, 4871-4876 (1989).

[130] F. Y. Wu, Eight-vertex model and Ising model in a nonzero magnetic field on the honeycomb lattice, J. Phys. A **23**, 375-378 (1990).

[131] J. H. H. Perk, F. Y. Wu and X. N. Wu, Algebraic invariants of the symmetric gauge transformation, J. Phys. A **23**, L131-L135 (1990).

[132] H. W. J. Blöte, F. Y. Wu and X. N. Wu, Critical point of the honeycomb antiferromagnetic Ising model in a nonzero magnetic field: Finite-size analysis, Int. J. Mod. Phys. B **4**, 619-630 (1990).

[133] A. C. N. de Magalhães, J. W. Essam and F. Y. Wu, Duality relation for Potts multispin correlation functions, J. Phys. A **23**, 2651-2669 (1990).

[134] K. Y. Lin and F. Y. Wu, General eight-vertex model on the honeycomb lattice, Mod. Phys. Lett. B **4**, 311-316 (1990).

[135] X. N. Wu and F. Y. Wu, Critical line of the square-lattice Ising model, Phys. Lett. A **144**, 123-126 (1990).

[136] F. Y. Wu, Comment on 'Rigorous Solution of a Two-Dimensional Blume-Emery-Griffiths Model', Phys. Rev. Lett. **64**, 700 (1990).

[137] F. Y. Wu, Yang-Lee edge in the high-temperature limit, Chin. J. Phys. **28**, 335-338 (1990).

[138] F. Y. Wu, Ising Model, in *Encyclopedia of Physics*, Eds. R. G. Lerner and G. L. Trigg (Addison-Wesley, New York 1990).

[139] K. Y. Lin and F. Y. Wu, Unidirectional-convex polygons on the honeycomb lattice, J. Phys. A **23**, 5003-5010 (1990).

[140] J. L. Ting, S. C. Lin and F. Y. Wu, Exact analysis of a lattice gas on the $3-12$ lattice with two- and three-site interactions, J. Stat. Phys. **62**, 35-43 (1991).

[141] D. L. Strout, D. A. Huckaby and F. Y. Wu, An exactly solvable model ternary solution with three-body interactions, Physica A **173**, 60-71 (1991).

[142] F. Y. Wu, Rigorous results on the Ising model on the triangular lattice with two- and three-spin interactions, Phys. Lett. A **153**, 73-75 (1991).

[143] J. C. Anglès d'Auriac, J.-M. Maillard, and F. Y. Wu, Three-state chiral Potts models in two dimensions: Integrability and symmetry, Physica A **177**, 114-122 (1991).

[144] J. C. Anglès d'Auriac, J.-M. Maillard, and F. Y. Wu, Three-state chiral Potts model on the triangular lattice: A Monte Carlo study, Physica A **179**, 496-506 (1991).

[145] K. Y. Lin and F. Y. Wu, Rigorous results on the anisotropic Ising model, Int. J. Mod. Phys. B **5**, 2125-2132 (1991).

[146] L. H. Gwa and F. Y. Wu, The $O(3)$ gauge transformations and three-state vertex models, J. Phys. A **24**, L503-L507 (1991).

[147] J.-M. Maillard, F. Y. Wu and C. K. Hu, Thermal transmissivity in discrete spin systems: Formulation and applications, J. Phys. A **25**, 2521-2531, (1992).

[148] L. H. Gwa and F. Y. Wu, Critical surface of the Blume-Emery-Griffiths model on the honeycomb lattice, Phys. Rev. B **43**, 13755-13777 (1991).

[149] C. K. Lee, A. S. T. Chiang and F. Y. Wu, Lattice model for the adsorption of benzene in silicalite I, J. Am. Int. Chem. Eng. **38**, 128-135 (1992).

[150] F. Y. Wu, Knot theory and statistical mechanics, in *AIP Conf. Proc.*, No. 248, Ed. C. K. Hu, 3-11 (1992).

[151] F. Y. Wu, Jones polynomial as a Potts model partition function, J. Knot Theory and Its Ramifications **1**, 47-57 (1992).
[152] F. Y. Wu, Transformation properties of a spin-1 system in statistical physics, Chin. J. Phys. **30**, 157-163 (1992).
[153] F. Y. Wu, Jones polynomial and the Potts model, Helv. Phys. Acta **65**, 469-470 (1992).
[154] F. Y. Wu, Knot theory and statistical mechanics, Rev. Mod. Phys. **64**, 1099-1131 (1992).
[155] J.-M. Maillard, G. Rollet and F. Y. Wu, An exact critical frontier for the Potts model on the $3-12$ lattice, J. Phys. A **26**, L495-L499 (1993).
[156] F. Y. Wu, The Yang-Baxter equation in knot theory, Int. J. Mod. Phys. B **7**, 3737-3750 (1993).
[157] F. Y. Wu, Knot Invariants and Statistical Mechanics: A Physicist's Perspective, in *Braid Group, Knot Theory and Statistical Mechanics*, Eds. M. L. Ge and C. N. Yang (World Scientific, Singapore 1993).
[158] F. Y. Wu and H. Y. Huang, Exact solution of a vertex model in d dimensions, Lett. Math. Phys. **29**, 205-213 (1993).
[159] J. C. Anglès d'Auriac, J.-M. Maillard, G. Rollet, and F. Y. Wu, Zeros of the partition function of the triangular Potts model: Conjectured distribution, Physica A **206**, 441-453 (1994).
[160] H. Y. Huang, and F. Y. Wu, Exact solution of a model of type-II superconductors, Physica A **205**, 31-40 (1994).
[161] J.-M. Maillard, G. Rollet, and F. Y. Wu, Inversion relations and symmetry groups for Potts models on the triangular lattice, J. Phys. A **27**, 3373-3379 (1994).
[162] F. Y. Wu, P. Pant, and C. King, A new knot invariant from the chiral Potts model, Phys. Rev. Lett. **72**, 3937-9340 (1994).
[163] C. K. Hu, J. A. Chen, and F. Y. Wu, Histogram Monte-Carlo renormalization group approach to the Potts model on the Kagomé lattice, Mod. Phys. Lett. B **8**, 455-459 (1994).
[164] F. Y. Wu, P. Pant and C. King, Knot invariant and the chiral Potts model, J. Stat. Phys. **78**, 1253-1276 (1995).
[165] F. Y. Wu and H. Y. Huang, Soluble free-fermion model in d dimensions, Phys. Rev. E **51**, 889-895 (1995).
[166] H. Meyer, J. C. Anglès d'Auriac, J.-M. Maillard, G. Rollet, and F. Y. Wu, Tricritical behavior of the three-state triangular Potts model, Phys. Lett. A **201**, 252-256 (1995).
[167] F. Y. Wu, Statistical mechanics and the knot theory, AIP Conf. Proc. No. **342**, Ed. A. Zepeda, 750-756 (1995).
[168] G. Rollet and F. Y. Wu, Gauge transformation and vertex models: A polynomial formulation, Int. J. Mod. Phys. B **9**, 3209-3217 (1995).
[169] C. K. Hu, C. N. Chen, and F. Y. Wu, Histogram Monte-Carlo renormalization group: Applications to the site percolation, J. Stat. Phys. **82**, 1199-1206 (1996).
[170] C. N. Chen, C. K. Hu, and F. Y. Wu, Partition function zeros of the square lattice Potts model, Phys. Rev. Lett. **76**, 169-172 (1996).
[171] *Statistical Models, Yang-Baxter Equation and Related Topics and Symmetry, Statistical Models and Applications*, Eds. M. L. Ge and F. Y. Wu, Advances in Statistical Mechanics, Vol. 11 (World Scientific, Singapore 1996).
[172] F. Y. Wu *et al*., Directed compact lattice animals, restricted partitions of numbers, and the infinite-state Potts model, Phys. Rev. Lett. **76**, 173-176 (1996).
[173] H. Y. Huang, F. Y. Wu, H. Kunz, and D. Kim, Interacting dimers on the honeycomb lattice: An exact solution of the five-vertex model, Physica A **228**, 1-32 (1996).
[174] F. Y. Wu, Soluble free-fermion models in d dimensions, in *Statistical Models, Yang-Baxter Equation and Related Topics, and Symmetry, Statistical Mechanical Models and Applications*, Eds. M. L. Ge and F. Y. Wu, 334-339, (World Scientific, Singapore 1996).

[175] A. S. T. Chiang, C. K. Lee, and F. Y. Wu, Theory of adsorbed solutions: analysis of one-dimensional systems, J. Am. Chem. Eng. **42**, 2155-2161 (1996).

[176] F. Y. Wu, The infinite-state Potts model and restricted multidimensional partitions of an integer, J. Comp. and Math. Comp. Modeling **26**, 269-274 (1997).

[177] H. Y. Huang, V. Popkov, and F. Y. Wu, Exact solution of a three-dimensional dimer system, Phys. Rev. Lett. **78**, 409-413 (1997).

[178] C. King and F. Y. Wu, Star-star relation for the Potts model, Int. J. Mod. Phys. B **11**, 51-56 (1997).

[179] H. Y. Huang and F. Y. Wu, The infinite-state Potts model and solid partitions of an integer, Int. J. Mod. Phys. B **11**, 121-126 (1997).

[180] *Exactly Soluble Models in Statistical Mechanics, Historical Perspectives and Current Status*, Eds. C. King and F. Y. Wu, Advances in Statistical Mechanics, Vol. 13 (World Scientific, Singapore 1997).

[181] P. Pant, F. Y. Wu, and J. H. Barry, Exact results for a model ternary polymer mixture, Int. J. Mod. Phys. B **11**, 169-174 (1997).

[182] J. H. Barry, P. Pant, and F. Y. Wu, A lattice-statistical model for ternary polymer mixtures: Exact phase diagrams, Physica A **238**, 149-162 (1997).

[183] F. Y. Wu, Duality relations for Potts correlation functions, Phys. Lett. A **228**, 43-47 (1997).

[184] V. Popkov, D. Kim, H. Y. Huang, and F. Y. Wu, Lattice statistics in three dimensions: Exact solution of layered dimer and layered domain wall models, Phys. Rev. E **56**, 3999-4008 (1997).

[185] P. Pant and F. Y. Wu, Link invariant of the Izergin-Korepin model, J. Phys. A **30**, 7775-7782 (1997)

[186] F. Y. Wu and H. Y. Huang, New sum rule identities and duality relation for the Potts n-point correlation function, Phys. Rev. Lett. **79**, 4954-4957 (1997).

[187] F. Y. Wu and W. T. Lu, Correlation duality relation of the (N_α, N_β) model, Chin. J. Phys. **35**, 768-773 (1997).

[188] F. Y. Wu, The exact solution of a class of three-dimensional lattice-statistical models, in *Proceedings of the 7th Asia Pacific Physics Conference*, Ed. H. Chen, 20-28 (Science Press, Beijing 1998).

[189] W. T. Lu and F. Y. Wu, On the duality relation for correlation functions of the Potts model, J. Phys. A **31**, 2823-2836 (1998).

[190] W. T. Lu and F. Y. Wu, Partition function zeros of a self-dual Ising model, Physica A **258**, 157-170 (1998).

[191] C. A. Chen, C. K. Hu and F. Y. Wu, Critical point of the Kagomé Potts model, J. Phys. A **31**, 7855-7864 (1998).

[192] F. Y. Wu and H. Kunz, Restricted random walks on a graph, Ann. Combinatorics **3**, 475-481 (1999).

[193] F. Y. Wu, C. King and W. T. Lu, On the rooted Tutte polynomial, Ann. Inst. Fourier **49**, 101-112 (1999).

[194] W. T. Lu and F. Y. Wu, Dimer statistics on a Möbius strip and the Klein bottle, Phys. Lett. A **259**, 108-114 (1999).

[195] F. Y. Wu, Exactly solvable models in statistical mechanics, Notices of the AMS, 1353-1354 (1999).

[196] F. Y. Wu, Two solvable lattice models in three dimensions, in *Statistical Mechanics in the Eve of the 21st century*, Eds. M. T. Batchelor and T. Wille, 336-350 (World Scientific, Singapore 1999).

[197] F. Y. Wu, Self-dual polynomials in statistical physics and number theory, in *Yang-Baxter Systems, Nonlinear Models and Their Applications*, Eds. B. K. Chung, Q-Han Park and C. Rim, 77-82 (World Scientific, Singapore 2000).

[198] W. J. Tzeng and F. Y. Wu, Spanning trees on hypercubic lattices and non-orientable surfaces, Appl. Math. Lett. **13**, 19-25 (2000).

[199] N. Sh. Izamailian, C. K. Hu and F. Y. Wu, The 6-vertex model of hydrogen-bonded crystals with bond defects, J. Phys. A **33**, 2185-2193 (2000).

[200] R. Shrock and F. Y. Wu, Spanning trees on graphs and lattices in d dimensions, J. Phys. A **33**, 3881-3902 (2000).

[201] W. Guo, H. W. J. Blöte and F. Y. Wu, Phase transitions in the $n > 2$ honeycomb $O(n)$ model, Phys. Rev. Lett. **85**, 3874-3877 (2000).

[202] W. T. Lu and F. Y. Wu, Density of the Fisher zeros for the Ising model J. Stat. Phys. **102**, 953-970 (2000).

[203] W. T. Lu and F. Y. Wu, Ising model for non-orientable surfaces, Phys. Rev. E **63**, 026107 (2001).

[204] F. Y. Wu and J. Wang, Zeros of the Jones polynomial, Physica A **296**, 483-494 (2001).

[205] W. T. Lu and F. Y. Wu, Closed-packed dimers on non-orientable surfaces, Phys. Lett. A **293**, 235-246 (2002), cond-mat/0110035.

[206] C. King and F. Y. Wu, New correlation relations for the planar Potts model, J. Stat. Phys. to appear (2002).

[207] W. J. Tzeng and F. Y. Wu, Dimers on a simple-quartic net with a vacancy, J. Stat. Phys. to appear (2002), cond-mat/0203149.

[208] F. Y. Wu, Dimers and spanning trees: Some recent results, Int. J. Mod. Phys. B, to appear (2002).