

LETTER TO THE EDITOR

An exact critical frontier for the Potts model on the 3-12 lattice

J M Maillard†, G Rollet† and F Y Wu‡

† Laboratoire de Physique Théorique et Hautes Energies, Tour 16, 1^{er} étage, 4 place Jussieu, 75252 Paris Cedex, France

‡ Department of Physics, Northeastern University, Boston, Massachusetts 02115, USA

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Abstract. An exact critical frontier for the Potts model on the 3-12 lattice, which includes the Kagomé lattice as a special case, is determined in a special parameter space of the two- and three-site interactions. The determination is made possible by the use of a star-triangle transformation which converts the lattice into one whose exact critical point is known.

The determination of the exact critical frontier for the Potts model [1] for general two-dimensional lattices has remained very much an open problem. Unlike the Ising model for which the exact critical point can be worked out for any two-dimensional lattice, the critical point for the Potts model has been determined only for the square, triangular, and honeycomb lattices [2, 3]. In particular, the exact critical point for the Kagomé Potts lattice has proven to be exceptionally elusive to analyse. In view of the considerable recent interest in investigating higher spin systems on the Kagomé lattice in the context of high-temperature superconductivity (see for example [4]), it is of some interest to revisit the Kagomé Potts model. Here we report on some exact results in this connection.

We begin by considering the Potts model on the more general 3-12 lattice. The 3-12 lattice, shown in figure 1 with (reduced) two-site and three-site interactions K_i, K'_i, J_i and M, M' , respectively, reduces to the Kagomé lattice by taking the $J_i = \infty$ limit. We shall determine its exact critical frontier in a special manifold of the parameter space.

The first step of our consideration is a transformation which converts the 3-12 lattice into a triangular one. This is accomplished by introducing the star-triangle transformation shown in figure 2. Specifically, we require the up-pointing triangle consisting of two-site interactions K_1, K_2, K_3 and the three-site interaction M to be replaced by the two-site L_1, L_2, L_3 interactions forming a 'star' in the form of a 'Y'. While generally this transformation cannot be carried through for $q \geq 3$, where q is the number of states, a little algebra shows that the transformation does hold in a special parameter manifold [5]. To obtain an explicit expression of this manifold, we write out the transformation which reads

$$e^{M+K_1+K_2+K_3} = \frac{e^{L_1+L_2+L_3} + q - 1}{e^{L_1} + e^{L_2} + e^{L_3} + q - 3} \tag{1}$$

$$e^{K_i} = \frac{e^{L_j+L_k} + e^{L_i} + q - 2}{e^{L_i} + e^{L_j} + e^{L_k} + q - 3} \quad i \neq j \neq k.$$

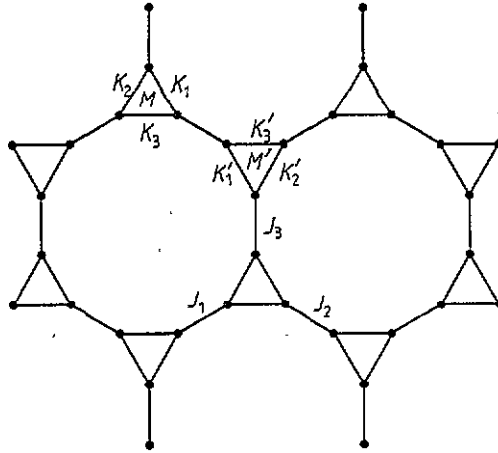


Figure 1. The 3-12 lattice.

Then, an expression of the special manifold in the $\{M, K_1, K_2, K_3\}$ space is obtained from (1) by eliminating L_1, L_2, L_3 . This leads to the equation

$$\begin{aligned}
 e^{2(M+K_1+K_2+K_3)} - e^{M+K_1+K_2+K_3} (e^{K_1+K_2} + e^{K_2+K_3} + e^{K_3+K_1} - 1) \\
 + (e^{K_1} + e^{K_2} + e^{K_3} + q - 4)(e^{K_1+K_2} + e^{K_2+K_3} + e^{K_3+K_1} - q + 3) \\
 - qe^{K_1+K_2+K_3} - (e^{2K_1} + e^{2K_2} + e^{2K_3}) + q^2 - 6q + 10 = 0.
 \end{aligned} \quad (2)$$

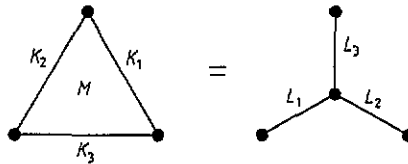


Figure 2. The star-triangle relation.

Carry out this star-triangle transformation for all up-pointing triangles. The 3-12 lattice is reduced into a triangular one shown in figure 3, where each shaded down-pointing triangle possesses a structure as shown in figure 4. The Boltzmann factor $F(\sigma_1, \sigma_2, \sigma_3)$ of the shaded triangle whose three terminal spins are in states $\sigma_1, \sigma_2, \sigma_3 = 1, 2, \dots, q$ can be readily worked out. After some algebra and disregarding an overall constant which does not concern us, we arrive at the expression

$$F(\sigma_1, \sigma_2, \sigma_3) = A + B_1\delta_{23} + B_2\delta_{31} + B_3\delta_{12} + C\delta_{123} \quad (3)$$

where $\delta_{ij} = \delta(\sigma_i, \sigma_j)$, $\delta_{123} = \delta(\sigma_1, \sigma_2)\delta(\sigma_2, \sigma_3)$ and

$$\begin{aligned}
 A = (q + v_1 + v_2 + v_3)[q^2 + q(w_1 + w_2 + w_3) + h] + v_1v_2v_3 \\
 + v_1v_2(q + w_1 + w_2) + v_2v_3(q + w_2 + w_3) + v_3v_1(q + w_3 + w_1) \\
 B_1 = v_2v_3[h + (q + v_1)w_1]
 \end{aligned} \quad (4)$$

$$C = v_1v_2v_3h$$

and similar expressions for B_2 and B_3 . Here

$$\begin{aligned}
 h &= e^{M'+K'_1+K'_2+K'_3} - e^{K'_1} - e^{K'_2} - e^{K'_3} + 2 \\
 v_i &= (e^{J_i} - 1)(e^{L_i} - 1)/(e^{J_i} + e^{L_i} + q - 2) \\
 w_i &= e^{K_i} - 1.
 \end{aligned}
 \tag{5}$$

Then, the partition function of the 3-12 Potts model becomes

$$Z_{3-12} = \sum \prod_{\nabla} F(\sigma_1, \sigma_2, \sigma_3)
 \tag{6}$$

where the product is taken over all down-pointing triangles in figure 3.

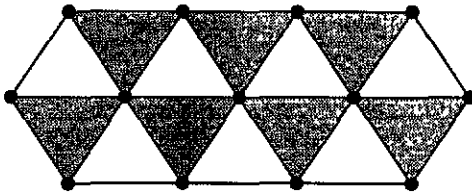


Figure 3. The triangular lattice.

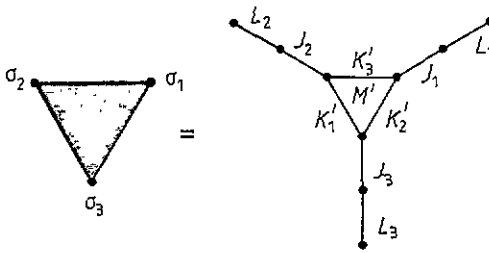


Figure 4. The internal structure of a shaded down-pointing triangle.

Now, the Potts model with the partition function (6) is self-dual [6, 7]. It has been established [8] that in the regime†

$$F(\sigma, \sigma, \sigma) \geq \{F(\sigma, \sigma, \sigma'), F(\sigma, \sigma', \sigma), F(\sigma', \sigma, \sigma), F(\sigma, \sigma', \sigma'')\}
 \tag{7}$$

the critical frontier is located at the self-dual point

$$F(\sigma, \sigma, \sigma) - F(\sigma, \sigma, \sigma') - F(\sigma, \sigma', \sigma) - F(\sigma', \sigma, \sigma) = q - 2
 \tag{8}$$

where $\sigma \neq \sigma'$ in (8). Substituting from (3) we find that (7) and (8) become, respectively

$$B_1 + B_2 + B_3 + C > 0 \quad B_i + B_j + C > 0 \quad i \neq j
 \tag{9}$$

$$qA = C.
 \tag{10}$$

† Assuming that, in this ferromagnetic regime, one has a unique critical point in some variable such as the one denoted by y in [6-8].

Thus, we have located an exact critical frontier for the Potts model on the 3-12 lattice. That is, in the regime (1) and (9), the critical frontier is (10).

It is informative to examine the critical frontier (10) in some special instances.

Isotropic case. For the isotropic lattice with $K_i = K$, $K'_i = K'$, $J_i = J$ and hence $L_i = L$, we can solve e^L from the second line of (1), which now reads

$$(e^L - 1)^2 = (e^K - 1)[3(e^L - 1) + q]. \quad (11)$$

The substitution of this expression of e^L into (10) now yields an explicit expression of the critical frontier in terms of the Kagomé parameters K , K' , M , M' , J . Here, M is given by (2) or, equivalently

$$e^{2M+6K} + (1 - 3e^{2K})e^{M+3K} + q - 2 = (q - 9)e^{3K} + 3(5 - q)e^{2K} + 3(q - 3)e^K. \quad (12)$$

Indeed, we have verified that in this case the critical condition (10) does possess solutions in the physical ferromagnetic regime.

Up-down symmetry. In the case of the lattice with up-down symmetry $K'_i = K_i$, $M' = M$ one can alternately perform the star-triangle transformation to both the up and down triangles of the 3-12 lattice. This results in a honeycomb lattice whose edges are sequences of two L_i and one J_i interactions and whose critical condition is known. The sequence of two L_i and one J_i interactions can be replaced by a single equivalent interaction $K_i^* = K_i^*(K_i, J_i)$ given by [9]

$$\frac{e^{K_i^*} - 1}{e^{K_i^*} + q - 1} = \left(\frac{e^{L_i} - 1}{e^{L_i} + q - 1} \right)^2 \left(\frac{e^{J_i} - 1}{e^{J_i} + q - 1} \right) \quad (13)$$

or, equivalently

$$e^{K_i^*} - 1 = \frac{(e^{L_i} - 1)^2}{2e^{L_i} + q - 2 + (e^{L_i} + q - 1)^2 / (e^{J_i} - 1)}. \quad (14)$$

where $e^{L_i} - 1$ is given in terms of K_i as in (11). Thus, from the known critical point for the honeycomb lattice [2, 3], one obtains an alternative expression for the critical frontier in the $\{K, J\}$ space of

$$t_1 t_2 t_3 = q(t_1 + t_2 + t_3) + q^2 \quad (15)$$

where $t_i = e^{K_i^*} - 1$. For isotropic interactions $K_i^* = K^*$, this reduces to

$$(e^{K^*} - 1)^3 = 3q(e^{K^*} - 1) + q^2. \quad (16)$$

Kagomé lattice. The Kagomé lattice is recovered by setting $J_i = \infty$. Generally, the critical frontier (10) does not intersect the manifold $M = 0$ for the Kagomé lattice. This can be seen from the fact that, using $M = 0$, one obtains from (1) the relations

$$v_1 v_2 v_3 = q(v_1 + v_2 + v_3) + q^2 \quad (17)$$

$$v_i \equiv e^{L_i} - 1 = q / (e^{K_i} - 1) \quad i = 1, 2, 3. \quad (18)$$

In the case of $M' = M = 0$ it is clear that (17) is incompatible with the critical frontier (15), as both equations are of the same form but in terms of different variables t_i and v_i . Indeed, it is straightforward to verify for general M' that, by first using (17) to eliminate $v_1 v_2 v_3$ and then using (18) to eliminate v_i (inside the square bracket in (19)), one obtains from (4) the identity

$$qA - C = v_1 v_2 v_3 \left[\sum_{i \neq j} e^{K_i + K'_j} + (q-2) \sum_i (e^{K_i} + e^{K'_i}) + (q-2)(q-3) \right]. \quad (19)$$

This expression cannot vanish for integral $q \geq 2$. Hence (10) has no solution for $M = 0$. In addition, since M' does not appear in (19), this implies that (10) has no solution whenever one of the three-spin interactions M, M' vanishes.

Finally, for the isotropic Kagomé lattice with up-down symmetry $K' = K, M' = M$, the critical frontier is (16) with

$$e^{K^*} - 1 = (e^L - 1)/(2e^L + q - 2). \quad (20)$$

Here, e^L is a function of e^K given by (11), and e^M is related to e^K through (12).

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