Physics Letters B 314 (1993) 79–88
North-Holland

PHYSICS LETTERS B

Towards three-dimensional Bethe ansatz

M. Bellon, S. Boukraa, J.-M. Maillard and C.-M. Viallet

Laboratoire de Physique Théorique et des Hautes Energies, Unité associée au CNRS (URA 280), Université de Paris 6–Paris 7, Tour 16, 1er étage, boîte 126, 4 Place Jussieu, F-75252 Paris Cedex 05, France

Received 18 January 1993
Editor: R. Gatto

We introduce a "pre-Bethe-Ansatz" system of equations for three dimensional vertex models. We bring to the light various algebraic curves of high genus and discuss some situations where these curves simplify. As a result we describe remarkable subvarieties of the space of parameters.

1. Introduction

The purpose of this paper is to sketch how ideas introduced in the study of the sixteen-vertex model in [1] can be generalized to higher lattice dimensions. We think that the ideas developed here are relevant tools for the analysis of lattice models in three or more dimensions, a widely unexplored area.

In this paper we introduce the simplest three-dimensional generalization of the results of [1]. We describe a specific model which naturally generalizes the Baxter model. Finally, we show how this general construction points to a number of algebraic varieties of interest.

2. Towards three-dimensional Bethe ansatz

We denote by $w(i,j,k,l,m,n)$ the Boltzmann weight of a given three-dimensional vertex. We shall only consider the simplest case where each of the spins i, j, k, l, m and n can take only two values. The vertex weights may be arranged in an 8×8 matrix of entries

$$R^{ijk}_{lmn} = w(i,j,k,l,m,n) . \tag{1}$$

The natural generalization of the "pre-Bethe Ansatz" equation of [1] is

$$R\,(u \otimes v \otimes w) = \mu \cdot u' \otimes v' \otimes w' . \tag{2}$$

Let us introduce the notation

$$u = \begin{pmatrix} 1 \\ p \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ q \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ r \end{pmatrix}, \quad u' = \begin{pmatrix} 1 \\ p' \end{pmatrix}, \quad v' = \begin{pmatrix} 1 \\ q' \end{pmatrix}, \quad w' = \begin{pmatrix} 1 \\ r' \end{pmatrix} . \tag{3}$$

In the following subsections, we shall recall the symmetries of a three dimensional vertex model, as described in [2].

2.1. The group of inversions Γ_{3D}

As in [2], we first introduce the involution I changing R to its matrix inverse (we let appear an overall factor λ since the entry of R are taken projectively):

$$\sum_{\alpha_1,\alpha_2,\alpha_3} (IR)^{i_1 i_2 i_3}_{\alpha_1\alpha_2\alpha_3} \cdot R^{\alpha_1\alpha_2\alpha_3}_{j_1 j_2 j_3} = \lambda\, \delta^{i_1}_{j_1}\, \delta^{i_2}_{j_2}\, \delta^{i_3}_{j_3} . \tag{4}$$

Multiplying both sides of (2) by IR, we get an equation of the same form as (2) with u and u', v and v' and w and w' exchanged and R replaced by IR.

In [2], we also introduced three partial transposition t_1, t_2 and t_3. t_1 is defined by

$$(t_1 R)^{i_1 i_2 i_3}_{j_1 j_2 j_3} = R^{j_1 i_2 i_3}_{i_1 j_2 j_3} . \tag{5}$$

The definitions of t_2 and t_3 are similar.

The four involutions I and t_i $(i = 1, 2, 3)$ generate an infinite discrete group Γ_{3D} [2]. The so-called inversion relations of the statistical mechanics model can be simply expressed with these building blocks. They are

$$I, \quad J = t_1 I t_2 t_3, \quad K = t_2 I t_3 t_1, \quad L = t_3 I t_1 t_2 . \tag{6}$$

Considering the parameter space as a projective space (the entries of the R-matrix are homogeneous parameters), the elements of the group Γ_{3D} have a *non-linear* representation in terms of *birational transformations*. This group of symmetry of the parameter space of the model is very large. The number of elements of length l grows exponentially with l. It is actually a *hyperbolic* Coxeter group [3]. The symmetry group of the Yang–Baxter equations in two dimensions is a mere affine Coxeter group [3,4,2].

The group Γ_{3D} has been shown in [2] to enter the description of the group of automorphisms of the tetrahedron equations (generalization of the Yang–Baxter equations in three dimensions). We shall use this symmetry group beyond integrability, that is to say for models which do not have to verify the tetrahedron equations.

2.2. Weak-graph duality for 3D models: the gauge group G

A "gauge" group $G = sl_2 \times sl_2 \times sl_2$ acts *linearly* on the matrix R by similarity transformations (the weak-graph transformations, see [5] for details). If $g = g_1 \times g_2 \times g_3$, we define

$$g(R) = g_1\, g_2\, g_3 \cdot R \cdot g_1^{-1} g_2^{-1} g_3^{-1} . \tag{7}$$

Each of the g_i's acts on the corresponding vector space and g_1 for example is a short hand notation for $g_1 \otimes \mathbb{1} \otimes \mathbb{1}$. The action of G and Γ_{3D} do not commute. However, G and I do commute, and the commutation relation between the t_i's and G gives a rather simple semi-direct product structure to the combined group:

$$t_1 g = g^{t_1} t_1 , \tag{8}$$

with

$$g^{t_1} = {}^t g_1^{-1} \times g_2 \times g_3 , \tag{9}$$

and similar relations for t_2 and t_3. In particular, Γ_{3D} sends orbits of G onto orbits of G. The compatibility of these two groups is described in [1] in a two-dimensional case, the sixteen-vertex model.

The effect of such a transformation on the pre-Bethe-Ansatz equation (2) is simple: g_1 acts naturally on u and u', g_2 on v and v' and g_3 on w and w'.

3. A three-dimensional model

The most general vertex models on a cubic lattice has a large number of parameters (sixty-four). We therefore impose some relations on the Boltzmann weights of the three-dimensional vertex. We require that these relations are invariant under the inverse I [6,7] and the three partial transpositions t_1, t_2 and t_3 (eq. (5)). They will thus be invariant under the group Γ_{3D}. We are particularly interested in generalizations of the Baxter symmetric eight-vertex model, and define here a specific three-dimensional model, denoted in the sequel $\mathcal{B}_{3D}$. It is possible to "project" down a three-dimensional model onto a bidimensional one by just taking the trace of the matrix R on one of the spaces 1, 2, or 3: take for example space 3.

$$\widehat{R}_{kl}^{ij} = \sum_{\alpha_3} R_{k,l,\alpha_3}^{i,j,\alpha_3} . \tag{10}$$

The constraints verified by $\mathcal{B}_{3D}$ are such that the three possible projections are symmetric Baxter models.

We define $\mathcal{B}_{3D}$ by imposing the following restrictions on the entries [2]:

$$R_{j_1 j_2 j_3}^{i_1 i_2 i_3} = R_{-j_1,-j_2,-j_3}^{-i_1,-i_2,-i_3} , \tag{11}$$

$$R_{j_1 j_2 j_3}^{i_1 i_2 i_3} = 0 \quad \text{if } i_1 i_2 i_3 j_1 j_2 j_3 = -1 . \tag{12}$$

These constraints imply that the 8×8 matrix R is the direct product of two times the same 4×4 submatrix [8]. It is further possible to impose that this 4×4 matrix is symmetric, since such a symmetry is preserved by the partial transpositions t_1, t_2, t_3 [2,8], that is,

$$R_{j_1 j_2 j_3}^{i_1 i_2 i_3} = R_{i_1 i_2 i_3}^{j_1 j_2 j_3} . \tag{13}$$

We shall use the following notations for the entries of this 4×4 submatrix:

$$\begin{pmatrix} a & d_1 & d_2 & d_3 \\ d_1 & b_1 & c_3 & c_2 \\ d_2 & c_3 & b_2 & c_1 \\ d_3 & c_2 & c_1 & b_3 \end{pmatrix} . \tag{14}$$

The four rows and columns of this matrix correspond to the states $(+,+,+)$, $(+,-,-)$, $(-,+,-)$ and $(-,-,+)$ of the triplets (i_1, i_2, i_3) or (j_1, j_2, j_3). The matrix R can be completed by spin reversal, according to (11). Note that t_1 (respectively t_2, t_3) simply exchanges c_2 with d_2 and c_3 with d_3 (respectively circular permutations). I acts as the inversion of this 4×4 matrix.

It is quite remarkable that there exist four quantities which are covariant by all the four generating involutions I, t_1, t_2, t_3, and therefore the whole group Γ_{3D}. Let us introduce

$$p_3 = ab_3 + b_1 b_2 - c_3^2 - d_3^2 , \quad q_3 = c_1 d_1 - c_2 d_2 , \tag{15}$$

and the polynomials obtained by permutations of 1, 2 and 3. They form a five-dimensional space of polynomials. Any ratio of these polynomials is invariant under all the four generating involutions I, t_1, t_2, t_3. CP_9 is thus foliated by five dimensional algebraic varieties invariant under the *whole group* Γ_{3D}. We can also express it by saying that the polynomials (15) define a map from the parameter space CP_9 to CP_4 invariant under Γ_{3D}.

If we consider a subgroup Γ_2 generated by *only* two involutions, say I and L (6) or equivalently I and t_3, one gets *three more* independent covariant polynomials leading to algebraic surfaces (see fig. 1). They read

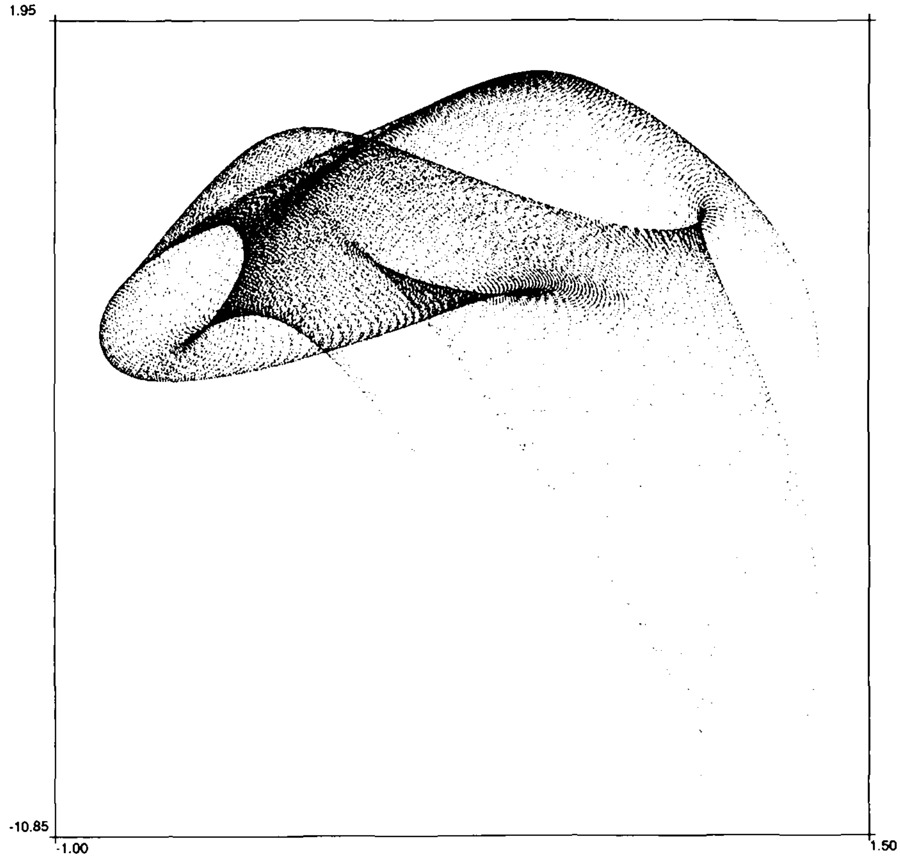

Fig. 1.

$$r_3 = ab_3 \cdot b_1 b_2 - c_3^2 + d_3^2, \quad s_3 = (a + b_3)c_3 - d_1 d_2 - c_1 c_2, \quad t_3 = (b_1 + b_2)d_3 - d_1 c_2 - c_1 d_2. \tag{16}$$

From the projection (10) we get a Baxter model. If we denote by a_B, b_B, c_B, d_B the non-zero entries of the R-matrix of this model, we have

$$a_B = a + b_3, \quad b_B = b_1 + b_2, \quad c_B = 2c_3, \quad d_B = 2d_3. \tag{17}$$

4. Study of the three-dimensional "pre-Bethe" equations

4.1. A first attempt

In the study of eq. (2), we can start by eliminating the variables p and p'. We obtain the following system of five equations for the remaining variables q, q', r and r':

$$0 = c_3d_3 - b_2b_3\,q^2 - c_1^2\,r^2 + (d_2d_3 - b_2c_1 + c_2c_3 - b_3c_1)\,rq + c_2d_2\,q^2r^2 - ab_1\,q'^2 + (ab_3 - c_3^2 + b_1b_2 - d_3^2)\,qq'$$
$$+ (ac_1 - c_2c_3 + b_1c_1 - d_2d_3)\,rq' + (b_2d_1 - c_3d_2 - c_2d_3 + b_3d_1)\,rq^2q' + 2(c_1d_1 - c_2d_2)\,r^2qq'$$
$$- (ad_1 + b_1d_1 - c_2d_3 - c_3d_2)\,rqq'^2 + c_3d_3\,q^2q'^2 + c_2d_2\,r^2q'^2 - d_1^2\,q^2r^2q'^2\,. \tag{18}$$

$$0 = (c_1c_3 - b_2c_2)\,q + (b_3c_3 - c_1c_2)\,r + (c_1d_2 - b_2d_3)\,q^2r + (b_3d_2 - c_1d_3)\,qr^2 + ac_2\,q' - ac_3\,r' - c_1d_3\,q^2q' - b_3d_2\,r^2q'$$
$$+ b_2d_3\,q^2r' + c_1d_2\,r^2r' + d_1d_3\,q^2r^2q' - d_1d_2\,q^2r^2r' + (ad_3 + c_2d_1 - b_3d_3 - c_1d_2)\,qrq'$$
$$- (ad_2 + c_3d_1 - b_2d_2 - c_1d_3)\,qrr'\,, \tag{19}$$

$$0 = c_3d_2 - b_2c_1\,(q^2 + r^2) + (c_3^2 - b_2^2 + d_2^2 - c_1^2)\,rq + (ab_2 - d_2^2)\,rq' + (ac_1 - d_2d_3)\,qq' + (b_1b_2 - c_3^2)\,qr'$$
$$- ab_1\,q'r' + c_3d_2\,q^2r^2 + (b_2d_1 - c_3d_2)\,r^2qq' + (c_1d_1 - c_3d_3)\,rq^2q' + (c_1d_1 - c_2d_2)\,r^2qr'$$
$$- d_1^2\,q^2r^2q'r' + (c_2d_3 + c_3d_2 - ad_1 - b_1d_1)\,rqr'q' + (b_1c_1 - c_2c_3)\,rr' + c_2d_2\,r^2q'r' + (b_2d_1 - c_3d_2)\,r'rq^2$$
$$+ c_3d_3\,q^2q'r'\,, \tag{20}$$

$$0 = c_3d_1 - b_2c_2\,q^2 - c_1c_3\,r^2 + (d_1d_2 - c_1c_2 - b_2c_3 + b_1c_3)\,rq + b_1d_2\,q^2r^2 + (ac_2 - d_1d_3)\,qq'$$
$$+ (ac_3 - d_1d_2)\,rq' - ac_3\,q'r' + (c_3d_1 - b_1d_2)\,r^2qq' + (c_2d_1 - b_1d_3)\,rq^2q' + c_1d_2\,r^2q'r' + b_2d_3\,q^2q'r'$$
$$- (ad_2 + c_3d_1 - c_1d_3 - b_2d_2)\,rqr'q' - d_1d_2\,q^2r^2q'r'\,, \tag{21}$$

$$0 = (c_3d_2 - b_2d_1)\,q + (c_3d_3 - c_1d_1)\,r - (ab_2 - d_2^2)\,q^2r - (ac_1 - d_2d_3)\,qr^2 + ad_1\,q' + (a^2 + d_1^2 - d_3^2 - d_2^2)\,rqq'$$
$$- d_2d_3\,(q^2 + r^2)q' + ad_1\,q^2r^2q' + (b_1b_2 - c_3^2)\,qq'r' + (b_1c_1 - c_2c_3)\,rr'q' - (ad_1 + b_1d_1 - c_2d_3 - c_3d_2)\,rqr'q'^2$$
$$+ c_2d_2\,r^2q'^2r' - ab_1\,q'^2r' + (c_1d_1 - c_2d_2)\,r^2qr'q' + (b_2d_1 - c_3d_2)\,rq^2r'q' - d_1^2\,q^2r^2q'^2r' + c_3d_3\,q^2q'^2r'\,. \tag{22}$$

Two similar system of equations can be obtained by the elimination of the pair of variables q and q' or r and r'. The equations are of degree two in each of the variables, that is an overall maximum degree of eight. In fact only one is of degree seven, three are of degree six and one is of degree five.

The only apparent property of this system is the invariance by changing the sign of each of the variables q, q', r and r', which is linked to the zeroes of the R matrix for $ijklmn = -1$.

The spin reversal symmetry of the R matrix and the change of R into its inverse IR have no visible consequences. This is due to the particular choice made in the elimination of p and p'. The equations we have written are just five out of a system of thirty-six equations with a number of relations among them. On this full system, the symmetries should be more manifest.

4.2. *Necessary conditions for $\mathcal{B}_{3D}$*

Since this direct attempt to find a full solution to (2) leads to such a confuse result, we shall study some necessary conditions. We replace the tensor product $u \otimes v$ and $u' \otimes v'$ by general vectors in the tensor product of space 1 and 2 U and U'. (2) becomes

$$R\,(U \otimes w) = \mu U' \otimes w'\,. \tag{23}$$

This can be written as

$$\begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix} \begin{pmatrix} U \\ rU \end{pmatrix} = \mu \begin{pmatrix} U' \\ r'U' \end{pmatrix}, \tag{24}$$

with $R_1, \ldots, R_4$ the 4×4 blocs of R.

Eliminating U and U' gives the *necessary* condition

$$\det(R_1 r' + R_2 rr' - R_3 - R_4 r) = 0. \tag{25}$$

For the model introduced in section 3 (eqs. (11), (12), (13)), eq. (25) reads in terms of r and r'

$$A_3 \cdot (r^4 r'^4 + 1) + B_3 \cdot (r^4 r'^2 + r^2 r'^4 + r^2 + r'^2) + C_3 \cdot (r^4 + r'^4) + D_3 \cdot (r^3 r'^3 + rr') + E_3 \cdot (r^3 r' + rr'^3) + F_3 \cdot r^2 r'^2 = 0. \tag{26}$$

Here $A_3, \ldots, F_3$ are polynomial expressions of degree four in the homogeneous entries of the R-matrix (14) ($a, \ldots, d_3$). The simplest expressions are

$$A_3 = (c_1 d_1 - c_2 d_2)^2 = q_3^2, \quad C_3 = (b_1 b_2 - c_3^2)(ab_3 - d_3^2) = \tfrac{1}{4}(p_3 + r_3)(p_3 - r_3).$$

In fact, all the coefficients $A_3, \ldots, F_3$ can be expressed as quadratic expressions in the polynomials invariant by the subgroup Γ_2 of Γ_{3D} listed in (15) and (16). This shows that eq. (26) is invariant by this infinite group. Moreover, they verify the relation

$$4A_3 \cdot (F_3 - 2A_3 + 2C_3 + 2E_3) = (D_3 + 2B_3)^2. \tag{27}$$

Relation (27) is actually the condition for relation (26) to become, when $r = r'$, the square of

$$(c_2 d_2 - c_1 d_1)(r^4 + 1) + r^2(ab_2 + b_1 b_3 - c_2^2 - d_2^2 - ab_1 - b_2 b_3 + c_1^2 + d_1^2). \tag{28}$$

We recognize q_3 in the coefficient of $r^4 + 1$ and $p_2 - p_1$ (15) in the coefficient of r^2. These coefficients are thus Γ_{3D}-covariant polynomials.

Of course two similar eliminations can be performed on (2) yielding constraints between p and p' (respectively q and q').

To take into account the symmetry of (26) by the exchanges $r \leftrightarrow r'$, $r \leftrightarrow 1/r$ and $r' \leftrightarrow 1/r'$, one may introduce the variables $X = rr' + 1/rr'$ and $Y = r/r' + r'/r$. Eq. (26) then becomes a *conic*:

$$A_3 X^2 + B_3 XY + C_3 Y^2 + D_3 X + E_3 Y + \widetilde{F}_3 = 0, \tag{29}$$

with $\widetilde{F}_3 = F_3 - 2A_3 - 2C_3$.

An invariant $\mathcal{I}_{\text{proj}}$ is naturally associated to the conic (29), it is the determinant of the 3×3 matrix M [9]:

$$M = \begin{pmatrix} A_3 & \frac{1}{2}B_3 & \frac{1}{2}D_3 \\ \frac{1}{2}B_3 & C_3 & \frac{1}{2}E_3 \\ \frac{1}{2}D_3 & \frac{1}{2}E_3 & \widetilde{F}_3 \end{pmatrix}. \tag{30}$$

The value of this invariant is (taking into account the relations between the entries of M): $\mathcal{I}_{\text{proj}} = -\mathcal{I}^2/4A_3$, with

$$\mathcal{I} = 2B_3^2 + D_3 B_3 - 2E_3 A_3 - 8A_3 C_3. \tag{31}$$

$\mathcal{I}_{\text{proj}} = 0$ is a projectively invariant condition for the conic, meaning that it is the union of two lines. Remark that this does not imply the existence of a rational uniformization of (26). A similar phenomenon happens in the Baxter model, for which the corresponding equation becomes linear in X and Y, but this does not provide the elliptic parametrization.

In order to obtain a parametrization of (26), we look at it as a polynomial of degree four in r, the coefficients being polynomials in r':

$$\alpha(r')r^4 + 4\beta(r')r^3 + 6\gamma(r')r^2 + 4\beta'(r')r + \alpha'(r') = 0. \tag{32}$$

Its discriminant reads [9,10]

$$\Delta(r') = g_2(r')^3 - 27\,g_3(r')^2, \tag{33}$$

with

$$g_2(r') = \alpha(r')\alpha'(r') - 4\beta(r')\beta'(r') + 3\gamma(r')^2, \tag{34}$$

and

$$g_3(r') = \alpha(r')\gamma(r')\alpha'(r') + 2\beta(r')\gamma(r')\beta'(r') - \alpha(r')\beta'(r')^2 - \beta(r')^2\alpha'(r') - \gamma(r')^3. \tag{35}$$

For a general vertex model in three dimensions, eq. (25) leads to an equation like (32) where $\alpha(r')$, $\beta(r')$, $\gamma(r')$, $\beta'(r')$, $\alpha'(r')$ are polynomials of degree four in r'. The polynomials $g_2(r')$, $g_3(r')$, and $\Delta(r')$ are polynomials of degree 8, 12 and 24 respectively. For the general vertex model the hyperelliptic curve $y^2 = \Delta(r')$ is a genus eleven curve.

However, one verifies easily that, for the model B_{3D}, $\Delta(r')$ is a polynomial of degree twelve in r'^2. Moreover the polynomial $\Delta(r')/r'^{12}$ is symmetric under the inversion $r' \leftrightarrow 1/r'$. Hence, introducing the variable $s' = r'^2 + r'^{-2}$, $\Delta(r')/r'^{12}$ becomes a *degree six* polynomial in s', denoted $P_6(s')$.

If $P_6(s')$ were a generic polynomial of degree six, the hyperelliptic curve $y^2 = P_6(s')$ would be a genus two curve, meaning that, as far as parametrization is concerned, one is obliged to deal with theta functions of two variables (the Jacobian associated to the genus two curve [11]) or automorphic functions (see [12] and page 455 of [13]). One can envisage a handy parametrization when the hyperelliptic curves degenerate into elliptic ones, that is, when two roots of $P_6(s')$ coincide, or equivalently when the discriminant of $P_6(s')$ vanishes [#1].

It is important to note that the model of section 3, *corresponds to such a situation* where $P_6(s')$ can be written as

$$P_6(s') = (s' - s_0)^2 \cdot P_4(s'), \tag{36}$$

where $P_4(s')$ is a polynomial of the fourth degree in s', containing 289 monomials of degree eight in the coefficients A_3, ..., E_3. Noticeably, s_0 is a quite simple expression,

$$s_0 = -\frac{D_3 + 2\,B_3}{2\,A_3}, \tag{37}$$

which reads in terms of the entries a, ..., d_3 of R:

$$s_0 = \frac{\left(c_1^2 - c_2^2 + d_1^2 - d_2^2\right) - (a - b_3)\,(b_1 - b_2)}{c_1\,d_1 - c_2\,d_2}. \tag{38}$$

Expression (38) is invariant under the group Γ_{3D} (see eqs. (15)). In eq. (26), index 3, and the equations similar to (26) relating p and p', or q and q', lead to equations like (38), where 1, 2 and 3 are permuted. It would be interesting to look for conditions on the entries of the R such that these three elliptic curves identify.

[#1] One should note that this is just an auxiliary parametrization and not a uniformization of eq. (32).

Clearly, a particular variety plays a special role: the subvariety in the space of models where the three elliptic curves reduce to rational ones. This algebraic variety is a good candidate for being a set of critical points (or disorder points) for $\mathcal{B}_{3D}$ though it is only a codimension-three subvariety of the codimension-one critical manifold we are looking for.

To sum up, the three-dimensional vertex model $\mathcal{B}_{3D}$ yields a generalization of the intertwining quadratic Frobenius relations in the form of an intertwining of three different elliptic curves by an R-matrix living on an algebraic variety of dimension five given by the intersection of five quadrics (eq. (15)).

4.3. Further analysis

The polynomial $P_4(s')$ appearing in eq. (36) is worth analyzing. We see that g_2 and $\Delta = g_2^3 - 27g_3^2$ factorize:

$$g_2 = A_3^3 \cdot g_2^{(1)} \cdot g_2^{(2)}, \quad g_3 = A_3^4 \cdot g_3^{(1)}, \quad \Delta = A_3^8 \cdot \Delta_1\Delta_2\Delta_3\Delta_4\Delta_5^3, \tag{39}$$

with

$$g_2^{(1)} = 3\,E_3^2A_3 + 8\,C_3E_3A_3 + 16\,A_3^2C_3 - D_3^2C_3 - 16\,C_3^2A_3 - E_3B_3D_3 - 2\,B_3^2E_3 - 4\,B_3^2A_3 + 4\,C_3B_3^2,$$

$$\Delta_1 = 2\,E_3A_3 - D_3B_3, \quad \Delta_2 = 2\,B_3^2 + D_3B_3 - 2\,E_3A_3 - 8\,A_3C_3,$$

$$\Delta_3 = 4\,B_3^2A_3 + 16\,C_3^2A_3 - 4\,C_3B_3^2 - D_3^2C_3 + E_3^2A_3 + 8\,C_3E_3A_3 - 16\,A_3^2C_3 - 4\,C_3B_3D_3 + 8\,C_3D_3A_3 - 4\,B_3E_3A_3,$$

$$\Delta_4 = 4\,B_3^2A_3 + 16\,C_3^2A_3 - 4\,C_3B_3^2 - D_3^2C_3 + E_3^2A_3 + 8\,C_3E_3A_3 - 16\,A_3^2C_3 - 4\,C_3B_3D_3 - 8\,C_3D_3A_3 + 4\,B_3E_3A_3.$$

The expressions $g_2^{(2)}$ and Δ_5 are polynomials of degree ten in the variables A_3, B_3, C_3, D_3 and E_3 (for instance, $g_2^{(2)}$ contains 147 monomials). $g_3^{(1)}$ is a polynomial of degree 20 in the same variables. Their explicit expressions involve too many terms to be reproduced here.

Expressing the coefficients $A_3, \ldots, E_3$ in terms of the entries of the R-matrix, one discovers further factorizations

$$g_2^{(1)} = (c_1d_1 - c_2d_2) \cdot G_2^{(1)}, \quad \Delta_2 = (c_1d_1 - c_2d_2) \cdot \delta_2,$$

$$\Delta_3 = (c_1d_1 - c_2d_2)^2 \cdot \delta_3, \quad \Delta_4 = (c_1d_1 - c_2d_2)^2 \cdot \delta_4, \tag{40}$$

where $G_2^{(1)}$ is the sum of 1570 monomials of degree ten, δ_2 is the sum of 104 monomials of degree six, δ_3 and δ_4 are the sum of 780 monomials of degree eight, and Δ_1 is the sum of 256 monomials of degree eight in the entries of the three-dimensional R-matrix, i.e. $a, \ldots, d_3$.

Let us note that $\Delta_2 = \mathcal{I}$ (see eq. (31)), and is thus related to the projective invariant $\mathcal{I}_{\rm proj}$ of the conic (29).

4.4. Subcases of $\mathcal{B}_{3D}$

The three-dimensional model $\mathcal{B}_{3D}$ was built in such a way that it "projects" down to the two dimensional Baxter model, as defined in section 3. It is natural to consider the conditions on $9\mathcal{B}_{3D}$ obtained by writing that the three projections lie on the critical or disorder varieties of the Baxter model.

For example, writing the three disorder conditions $a_B + d_B = b_B + c_B$ [14] of the Baxter model for the three projections ($i = 1, 2, 3$), yields a codimension-three subvariety of the three-dimensional model parametrized as follows:

$$a = b_1 + b_2 + b_3 - 2z, \quad c_1 = b_1 + d_1 - z, \quad c_2 = b_2 + d_2 - z, \quad c_3 = b_3 + d_3 - z. \tag{41}$$

On the subvariety (41), the discriminant of $P_4(s')$ vanishes, $\Delta_2 = \Delta_4 = 0$, the conic (29) degenerates since $\mathcal{I}_{\text{proj}} = \Delta_2 = 0$ and even more remarkably, $P_4(s')$ gets proportional to $(s' - 2)^4$.

Similarly, the three criticality conditions $a_B = b_B + c_B + d_B$ [14] yield a codimension three subvariety:

$$2d_1 = a + b_1 - b_2 - b_3 - 2c_1, \quad 2d_2 = a + b_2 - b_3 - b_1 - 2c_2, \quad 2d_3 = a + b_3 - b_1 - b_2 - 2c_3. \tag{42}$$

On the subvariety (42), the discriminant of $P_4(s')$ vanishes, $\Delta_2 = \Delta_3 = 0$, and now $P_4(s')$ gets proportional to $(s' + 2)^4$. This last codimension-three subvariety is particularly interesting, since it is $\Gamma_{3\text{D}}$ invariant.

4.5. *A solvable case*

Another interesting model is obtained by setting $d_1 = d_2 = d_3 = 0$. The projections yield *six-vertex models* [15]. The biquartic equation (26) becomes a homogeneous equation of degree 4 and the solution is the union of four lines $r' = \lambda r$. Remarkably, for this model the left-hand side of (26) factorizes for $r' = -r$:

$$r^4(ab_2 + b_1b_3 - c_2^2 + ab_1 + b_2b_3 - c_1^2 - 2ac_3 - 2b_3c_3 + 2c_1c_2)(ab_2 + b_1b_3 - c_2^2 + ab_1 + b_2b_3 - c_1^2 + 2ac_3 + 2b_3c_3 - 2c_1c_2). \tag{43}$$

What is more interesting is that we can get the conditions for the existence of solutions to (2) in this case. In (2), there are two equations which fix uniquely some scale factors:

$$a = \mu, \quad a\,pqr = \mu\,p'q'r'. \tag{44}$$

Using these informations, the six others components of (2) fall into three pairs of equations like

$$ap' = b_1 p + c_3 q + c_2 r, \quad a\frac{1}{p'} = b_1\frac{1}{p} + c_3\frac{1}{q} + c_2\frac{1}{r}. \tag{45}$$

In writing these equations, we discarded the trivial solution $p = q = r = 0$ which always exists in this case. Multiplying pairwise these equations, we obtain three linear equations for the variables $X_p = q/r + r/q$, $X_q = r/p + p/r$ and $X_r = p/q + q/p$. This system of equation can be easily solved. These three variables are not independent since they depend only on the ratios of p, q and r. They satisfy the relation

$$X_pX_qX_r - (X_p^2 + X_q^2 + X_r^2) + 4 = 0. \tag{46}$$

Rewritten in the homogeneous variables a, b_1, b_2, b_3, c_1, c_2, c_3, eq. (46) is a necessary condition for equation (2) to have non-trivial solutions. This is however not the end of the story since the relation $pqr = p'q'r'$ yields another condition on R once we have solved for r/p and q/p. Note that the normalization of the variables p, q, r and p', q' and r' remains free.

A complete analysis therefore yields the existence of non-trivial solutions to (2) when R is on some codimension-two subvariety in the parameter space.

5. Conclusion

We have shown how to associate algebraic curves with three-dimensional vertex models. We have described a specific model for which the analysis of these curves is handable. We have introduced a generalization of the quadratic Frobenius relations (associated to elliptic functions). It corresponds to new intertwining relations of products of more than two algebraic curves by R-matrices living on algebraic varieties which are no longer curves. In the example detailed in this paper, one has an intertwining of three curves by an R-matrix living on a higher-dimensional algebraic variety. We think that these equations are a key ingredient for the construction of the generalization of the Bethe Ansatz in higher dimensions, the quest of solutions of the tetrahedron equations and more generally any exact calculation (inversion trick [16], quest of critical manifolds [17]) performed on higher dimensional models.

Acknowledgement

We thank J. Avan and M. Talon for many discussions and encouragements.

References

[1] M.P. Bellon, J-M. Maillard and C-M. Viallet, Phys. Lett. B 281 (1992) 315.
[2] M.P. Bellon, J-M. Maillard and C-M. Viallet, Phys. Lett. B 260 (1991) 87.
[3] J.E. Humphreys, Reflection Groups and Coxeter Groups (Cambridge U.P., Cambridge, 1990).
[4] M.P. Bellon, J-M. Maillard and C-M. Viallet, Phys. Lett. A 157 (1991) 343.
[5] A. Gaaf and J. Hijmans, Phys. A 80 (1975) 149.
[6] M.P. Bellon, J-M. Maillard and C-M. Viallet, Phys. Lett. A 159 (1991) 221.
[7] M.P. Bellon, J-M. Maillard and C-M. Viallet, Matrix Patterns for Integrability, in preparation.
[8] M.P. Bellon, J-M. Maillard and C-M. Viallet, Phys. Rev. Lett. 67 (1991) 1373.
[9] J. Dixmier, Gaz. Mat. (Société Mathématique de France) 43 (1990) 39.
[10] Ch. Hermite, Cambridge and Dublin Math. J. (1854); Œuvres, t.1 296–349.
[11] D. Mumford, Tata Lectures on Theta II (Birkhauser, Basel, 1984).
[12] I.R. Shafarevich, Basic algebraic geometry, Springer study (Springer, Berlin, 1977) p. 216.
[13] E.T. Whittaker and G.N. Watson, A course of modern analysis (Cambridge U.P., Cambridge, 1965).
[14] R.J. Baxter, Exactly solved models in statistical mechanics (Academic Press, London, 1981).
[15] E.H. Lieb and F.Y. Wu, Two dimensional ferroelectric models, in: Phase Transitions and Critical Phenomena, eds. C. Domb and M.S. Green, Vol. 1 (Academic Press, New York, 1972) pp. 331–490.
[16] R.J. Baxter, in: Proc. 1980 Enschede Summer School: Fundamental problems in statistical mechanics V (North-Holland, Amsterdam, 1981).
[17] F.Y. Wu, X.N. Wu and H.W.J. Blöte, Phys. Rev. Lett. 62 (1989) 2773.