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Rational Symmetries of Differential Equations and Renormalization

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Abstract

We give some examples infinite order rational transformations that leave a linear differential equation covariants. These examples can be seen as a non-trivial but still simple illustration of an exact representation of the renormalization group.

Key-words: Renormalization group, infinite order rational symmetries of ODE's, Gauss hypergeometric functions, elliptic functions.

ПРЕПРИНТЫ

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1 Introduction

The purpose of this paper is to provide simple (but *non trivial*) examples of *exact* renormalization transformations that are not degenerate like the previous transformations on one-dimensional model. In several papers [1, 2] for Yang-Baxter integrable models with a canonical genus one parametrization [3, 4] (elliptic functions of modulus k), we underlined that the *exact* generators of the renormalization group must necessarily identify with the various isogenies which amounts to multiplying or dividing τ, the ratio of the two periods of the elliptic curves, by an integer. The simplest example is the *Landen transformation* [2] which corresponds to multiplying (*or dividing* because of the modular group symmetry $\tau \leftrightarrow 1/\tau$), the ratio of the two periods:

$$k \quad \longrightarrow \quad k_L = \frac{2\sqrt{k}}{1+k}, \qquad\qquad \tau \longleftrightarrow 2\,\tau. \tag{1}$$

The other transformations[1] correspond to $\tau \leftrightarrow N \cdot \tau$, for various integers N. In the natural variables of the model (e^K, $\tanh(K)$, $k = \sinh(2\,K)$, not transcendental variables like τ), these transformations are *algebraic* transformations corresponding in fact to the *fundamental modular curves*. For instance (1) corresponds to the *genus zero fundamental modular curve*

$$\begin{aligned} j^2 \cdot j'^2 - (j+j') \cdot (j^2 + 1487 \cdot j\,j' + j'^2) & \\ +3 \cdot 15^3 \cdot (16\,j^2 - 4027\,j\,j' + 16\,j'^2) & \\ -12 \cdot 30^6 \cdot (j+j') + 8 \cdot 30^9 &= 0, \end{aligned} \tag{2}$$

or:

$$\begin{aligned} 5^9\,v^3\,u^3 - 12 \cdot 5^6\,u^2\,v^2 \cdot (u+v) + 375\,u\,v \cdot (16\,u^2 + 16\,v^2 - 4027\,v\,u) & \\ -64\,(v+u) \cdot (v^2 + 1487\,v\,u + u^2) + 2^{12} \cdot 3^3 \cdot u\,v = 0, & \end{aligned} \tag{3}$$

which relates the two Hauptmoduls $u = 12^3/j(k)$, $v = 12^3/j(k_L)$:

$$\begin{aligned} j(k) &= 256 \cdot \frac{(1-k^2+k^4)^3}{k^4 \cdot (1-k^2)^2}, \\ j(k_L) &= 16 \cdot \frac{(1+14\,k^2+k^4)^3}{(1-k^2)^4 \cdot k^2}. \end{aligned}$$

One verifies easily that (2) is verified with $j = j(k)$ and $j' = j(k_L)$.

[1] See for instance (2.18) in [5].

The selected values of k, the modulus of elliptic functions, $k = 0, 1, \infty$ are actually *fixed points of the Landen transformations.* For the Ising (resp. Baxter) model these selected values of k correspond to the three selected subcases of the model ($T = \infty$, $T = 0$ and the critical temperature $T = T_c$), for which the elliptic parametrization of the model degenerates into a rational parametrization [2]. We have the same property for all the other algebraic modular curves corresponding to $\tau \leftrightarrow N\cdot\tau$. This is certainly the main property most physicists expect for an exact representation of a *generator of the renormalization group.* Modular transformations are, in fact, the only transformations to be compatible with all the other symmetries of the Ising (resp. Baxter) model like for instance, the gauge transformations, some extended $sl(2) \times sl(2) \times sl(2) \times sl(2)$ symmetry [6], etc. It has also been underlined in [1, 2] that seeing (1) as a transformation on *complex variables* (instead of real variables) provides two other complex fixed points which actually correspond to *complex multiplication* for the elliptic curve, and are, actually, fundamental new singularities discovered on the $\chi^{(3)}$ linear ODE [7, 8, 9]. In general, this underlines the deep relation between the renormalization group and the theory of elliptic curves in a deep sense, namely *isogenies of elliptic curves, Hauptmoduls*[2], *modular curves and modular forms.*

Note that an algebraic transformation like (1) or (3) cannot be obtained from any *local* Migdal-Kadanoff transformation which naturally yields *rational* transformations: an exact renormalization group transformation like (1) can only be deduced from *non-local* decimations. The emergence of modular transformations as representations of exact generators of the renormalization group explains, in a quite subtle way, the difficult problem of how renormalization group transformations can be compatible with *reversibility*[3] (iteration forward and backwards). An algebraic modular transformation (3) corresponds to $\tau \rightarrow 2\tau$ *and* $\tau \rightarrow \tau/2$ *in the same time, as a consequence of the modular group symmetry* $\tau \leftrightarrow 1/\tau$.

A simple rational parametrization[4] of the genus zero modular curve (3) reads:

$$u \, = \, 1728 \, \frac{z}{(z+16)^3}, \qquad v \, = \, 1728 \, \frac{z^2}{(z+256)^3} \, = \, u \left(\frac{2^{12}}{z} \right). \tag{4}$$

[2]It should be recalled that the mirror symmetry found with Calabi-Yau manifolds [10, 11, 12, 13, 14] can be seen as higher order generalizations of Hauptmoduls. We thus have already generalizations of this identification of the renormalization and modular structure when one is not restricted to elliptic curves anymore.

[3]The fact that the renormalization group must be reversible has apparently been totally forgotten by most of the authors who just see a semi-group corresponding to forward iterations converging to the critical points (resp. manifolds).

[4]Corresponding to Atkin-Lehner polynomials and Weber's functions.

Note that the previously mentioned reversibility is also associated with the fact that the modular curve (3) is invariant by $u \leftrightarrow v$, and, within the previous rational parametrization (4), with the fact that permuting u and v corresponds to the Atkin-Lehner involution $z \leftrightarrow 2^{12}/z$.

For many Yang-Baxter integrable models of lattice statistical mechanics the physical quantities (partition function per site, correlation functions, ...) are solutions of selected linear differential equations. For instance the partition function per site of the square (resp. triangular, etc.) Ising model is an integral of an elliptic integral of the third kind. It would be too complicated to show the precise covariance of these physical quantities with respect to (algebraic) modular transformations like (3). Instead, let us give, here, an illustration of the non-trivial action of the renormalization group on some elliptic function that actually occurs in the 2-D Ising model: a weight-one modular form. This modular form actually, and remarkably, emerged [16] in a second order linear differential operator factor denoted Z_2 occurring [7] for $\chi^{(3)}$, and that the reader can think as a physical quantity solution of a particular linear ODE replacing the too complicated integral of an elliptic integral of the third kind. Let us consider the second order linear differential operator (D_z denotes d/dz):

$$\alpha = \quad D_z^2 \; + \frac{(z^2 + 56\,z + 1024)}{z \cdot (z+16)\,(z+64)} \cdot D_z \; - \; \frac{240}{z \cdot (z+16)^2\,(z+64)},$$

which has the (modular form) solution:

$$\begin{aligned} &{}_2F_1\left([1/12, 5/12], [1]; 1728\, \frac{z}{(z+16)^3}\right) \qquad (5)\\ &\quad = \; 2 \cdot \left(\frac{z+256}{z+16}\right)^{-1/4} \cdot \; {}_2F_1\left([1/12, 5/12], [1]; 1728\; \frac{z^2}{(z+256)^3}\right). \end{aligned}$$

Do note that the two pull-backs in the arguments of the *same* hypergeometric function are *actually related by the modular curve relation* (3) (see (4)). The covariance (5) is thus the very expression of a modular form property with respect to a modular transformation ($\tau \leftrightarrow 2\,\tau$) corresponding to the modular transformation (3).

The hypergeometric function at the rhs of (5) is solution of the second order linear differential operator

$$\beta = \; D_z^2 \; + \frac{z^2 + 416\,z + 16384}{(z+256)\;(z+64)\,z} \cdot D_z \; - \; \frac{60}{(z+64)\,(z+256)^2},$$

which is the transformed of operator α by the Atkin-Lehner duality $z \leftrightarrow$

$2^{12}/z$, and, also, a conjugation of α:

$$\beta \; = \; \left(\frac{z+16}{z+256}\right)^{-1/4} \cdot \alpha \cdot \left(\frac{z+16}{z+256}\right)^{1/4}. \qquad (6)$$

Along this line we can also recall that the (modular form) function[5]:

$$F(j) \; = \; j^{-1/12} \cdot {}_2F_1\left([1/12, 5/12], [1]; \frac{12^3}{j}\right), \qquad (7)$$

verifies:

$$F\left(\frac{(z+16)^3}{z}\right) \; = \; 2 \cdot z^{-1/12} \cdot F\left(\frac{(z+256)^3}{z^2}\right). \qquad (8)$$

A relation like (5) is a straight generalization of the covariance we had in the one-dimensional model $Z(t) \, = \, C(t) \cdot Z(t^2)$, which basically amounts to seeing the partition function per site as some "automorphic function" with respect to the renormalization group, the simple renormalization group transformation $t \, \rightarrow \, t^2$ being replaced by the algebraic modular transformation (3) corresponding to $\tau \, \leftrightarrow \, 2\,\tau$ (that is the Landen transformation (1)).

The purpose of this paper is to present another elliptic hypergeometric function and other much simpler (Gauss hypergeometric) second order linear differential operators covariant by infinite order rational transformations.

2 Infinite number of rational symmetries on a Gauss hypergeometric ODE

Keeping in mind modular form expressions like (5), let us recall a particular Gauss hypergeometric function introduced by R. Vidunas in [17]:

$$\begin{aligned} {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{4}\right], \left[\frac{5}{4}\right]; \, z\right) \; &= \; \frac{1}{4} \cdot z^{-1/4} \cdot \int_0^z t^{-3/4} \, (1-t)^{-1/2} dt \\ &= \; (1-z)^{-1/2} \cdot \; {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{4}\right], \left[\frac{5}{4}\right]; \, \frac{-4\,z}{(1-z)^2}\right). \qquad (9) \end{aligned}$$

This hypergeometric function corresponds to the integral of a holomorphic form on a *genus-one* curve $P(y, \, t) \; = \; 0$:

$$\frac{dt}{y}, \qquad \text{with:} \qquad y^4 \; - \; t^3 \cdot (1-t)^2 \; = \; 0. \qquad (10)$$

[5] Where j is typically the j-function [5].

Note that the function

$$\mathcal{F}(z) \;=\; z^{1/4} \cdot \; {}_2F_1\left(\left[\frac{1}{2}, \frac{1}{4}\right], \left[\frac{5}{4}\right]; \, z\right), \tag{11}$$

which is exactly an integral of an algebraic function, has an extremely simple covariance property with respect to the *infinite order rational* transformation $z \rightarrow -4\,z/(1-z)^2$:

$$\mathcal{F}\left(\frac{-4\,z}{(1-z)^2}\right) \;=\; (-4)^{1/4} \cdot \mathcal{F}(z). \tag{12}$$

The occurrence of this specific infinite order transformation is reminiscent of Kummer's quadratic relation

$$\begin{aligned} & {}_2F_1\,([a,\, b], [1+a-b];\, z) \;= \qquad\qquad\qquad (13) \\ & \quad = \; (1-z)^{-a} \cdot \; {}_2F_1\left(\left[\frac{a}{2}, \frac{1+a}{2} - b\right], [1+a-b]; \, -\frac{4\,z}{(1-z)^2}\right), \end{aligned}$$

but it is crucial to note that, relation (12) does not relate two different functions, but is an "automorphy" relation on the *same function.*

It is clear from the previous paragraph that we want to see such functions as 'ideal' examples of physical functions covariant by an exact (here, rational) generator of the renormalization group. The function (11) is actually solution of the second order linear differential operator:

$$\begin{aligned} \Omega \;&=\; D_z^2 \;+\frac{1}{4}\,\frac{3\,-5\,z}{z\cdot(1-z)}\cdot D_z \;=\; \omega_1 \cdot D_z, \qquad \text{with:} \qquad (14) \\ \omega_1 \;&=\; D_z \;+\frac{1}{4}\,\frac{3\,-5\,z}{z\cdot(1-z)} \;=\; D_z \,+\, \frac{1}{4}\cdot\frac{d\ln(z^3\,(1-z)^2)}{dz}. \end{aligned}$$

From the previous expression of ω_1 involving a log-derivative of a rational function it is obvious that this second order linear differential operator has two solutions, the constant function and an integral of an algebraic function. Since these two solutions behave very simply under the infinite order rational transformation $z \rightarrow -4\,z/(1-z)^2$, it is totally and utterly natural to see how the linear differential operator Ω transforms under the rational change of variable $z \rightarrow R(z) = -4\,z/(1-z)^2$ (which amounts to seeing how the two order-one operators ω_1 and D_z transform). It is a straightforward calculation to see that introducing the cofactor $C(z)$ which is the inverse of the derivative of $R(z)$

$$C(z) \;=\; -\frac{1}{4}\cdot\frac{(1-z)^3}{1+z}, \qquad\qquad \frac{1}{C(z)} \;=\; \frac{dR(z)}{dz}, \tag{15}$$

D_z and ω_1 respectively transform under the rational change of variable $z \rightarrow R(z) \, = \, -4\,z/(1-z)^2$ as:

$$D_z \, \longrightarrow \, C(z) \cdot D_z, \qquad \omega_1 \, \longrightarrow \, (\omega_1)^{(R)} \, = \, C(z)^2 \cdot \omega_1 \cdot \frac{1}{C(z)}, \tag{16}$$

$$\text{yielding:} \qquad \qquad \Omega \, \longrightarrow \, C(z)^2 \cdot \Omega. \tag{17}$$

Since $z \rightarrow -4\,z/(1-z)^2$ is of infinite order, the second order linear differential operator (14) has *an infinite number of rational symmetries* (isogenies):

$$\begin{aligned} z \quad \longrightarrow \quad \frac{-4\,z}{(1-z)^2} \quad \longrightarrow \quad 16 \cdot \frac{(1-z)^2 \cdot z}{(1+z)^4} \quad \longrightarrow \\ \longrightarrow \quad -64 \cdot \frac{(1-z)^2\,(1+z)^4\,z}{(1-6\,z\,+z^2)^4} \quad \longrightarrow \quad \cdots \end{aligned} \tag{18}$$

More generally, let us consider a rational transformation $z \rightarrow R(z)$, the corresponding cofactor $C(z) \, = \, 1/R'(z)$, and the order-one operator $\omega_1 \, = \, D_z \, + \, A(z)$. We have the identity:

$$C(z) \cdot D_z \cdot \frac{1}{C(z)} \quad = \quad D_z \, - \, \frac{d\ln(C(z))}{dz}. \tag{19}$$

The change of variable $z \rightarrow R(z)$ on ω_1 reads:

$$D_z \, + \, A(z) \quad \longrightarrow \quad C(z) \cdot D_z \, + \, A(R(z)) \, = \, C(z) \cdot (D_z \, + \, B(z)) \, .$$

We want to impose that this rhs expression can be written (see (16)) as:

$$C(z)^2 \cdot (D_z \, + \, A(z)) \cdot \frac{1}{C(z)},$$

which, because of (19), occurs if

$$B(z) \, = \, A(z) \, - \, \frac{d\ln(C(z))}{dz},$$

yielding a "Rota-Baxter-like" [18, 19] functional equation on $A(z)$ and $R(z)$:

$$\left(\frac{dR(z)}{dz}\right)^2 \cdot A(R(z)) \, = \, \frac{dR(z)}{dz} \cdot A(z) \, + \frac{d^2R(z)}{dz^2}. \tag{20}$$

3 Symmetries of Ω, solutions the "Rota-Baxter-like" functional equation.

A first natural generalization amounts to keeping the remarkable factorization (14) which will, in fact, reduce the covariance of a second order operator to the covariance of a first order operator[6]. Such a situation occurs for Gauss hypergeometric functions ${}_2F_1([a,\, b], [1+a];\, z)$ solution of the $(a,\, b)$-symmetric linear differential operator

$$z \cdot (1-z) \cdot D_z^2 \; + (c - (a+b+1) \cdot z) \cdot D_z \; - a \cdot b, \tag{21}$$

as soon as $c \, = \, 1+a$. For instance

$$\mathcal{F}(z) \, = \, z^a \cdot {}_2F_1([a,\, b],\, [1\, +a];\, z)\, , \tag{22}$$

is an integral of a simple algebraic function and is solution with the constant function of the second order operator:

$$\begin{aligned} \Omega \, &= \, \Big(D_z \, + \frac{(a-b-1)\, z \, + 1 - a}{z \cdot (1-z)} \Big) \cdot D_z \\ &= \, \Big(D_z \, + \frac{d \ln((1-z)^b \cdot z^{1-a})}{dz} \Big) \cdot D_z, \end{aligned} \tag{23}$$

yielding a new $A(z)$:

$$A(z) \, = \, \frac{(1\, -a) \, + (a-b-1)\, z}{(1-z) \cdot z} \, = \, \frac{1-a}{z} \, - \, \frac{b}{1\, - z}. \tag{24}$$

The adjoint of (23) has the simple solution $z^{1-a} \cdot (1-z)^b$:

$$F(z) \, = \, z \cdot (1-z)^b \cdot {}_2F_1([a,\, b],\, [1\, +a];\, z)\, . \tag{25}$$

Due to the $(a,\, b)$-symmetry of (21) we have a similar result for $c \, = \, 1+b$. The function $\mathcal{F}(z) \, = \, z^b \cdot {}_2F_1([a,\, b],\, [1\, +b];\, z)$ is solution of (23) where a and b have been permuted:

$$\Big(D_z \, + \frac{(b-a-1)\, z \, + 1 - b}{z \cdot (1-z)} \Big) \cdot D_z \tag{26}$$

yielding another $A(z)$:

$$A(z) \, = \, \frac{(1\, -b) \, + (b-a-1)\, z}{(1-z) \cdot z}, \tag{27}$$

[6]Thus avoiding the full complexity (and subtleties) of the covariance of ODE's by algebraic transformations like modular transformations (3).

The adjoint of (26) has the solution $(1-z)^a \cdot z^{1-b}$ together with the hypergeometric function:

$$F(z) \; = \; z \cdot (1-z)^a \cdot {}_2F_1\left([a,b],[1+b];\, z\right). \qquad (28)$$

We are seeking for (Gauss hypergeometric) second order differential equations with an *infinite number* of (hopefully rational, if not algebraic) symmetries: this is another way to say that we are not looking for generic Gauss hypergeometric differential equations, but Gauss hypergeometric differential equations *related to elliptic curves*, and thus having an infinite set of such isogenies. We are necessarily in the framework where the two parameters a and b of the Gauss hypergeometric are *rational numbers* in order to have *integral of algebraic functions* (yielding globally nilpotent [16] second order differential operators). Let us denote by D the common denominator of the two rational numbers $a = N_a/D$ and $b = N_b/D$, the function (22) is associated to a period of the algebraic curve:

$$y^D \; = \; (1-t)^{N_b} \cdot t^{D-N_a}. \qquad (29)$$

We just need to restrict to triplets of integers $(N_a,\, N_b,\, D)$ such that the previous curve is an elliptic curve.

Let us now analyse all the symmetries of the linear differential operator $\Omega = (D_z + A(z)) \cdot D_z$ by analyzing all the solutions of (20) for a given $A(z)$ (i.e. N_a, N_b and D). Let us seek for general solutions that are *analytic at* $z = 0$:

$$R(z) \; = \; a_1 \cdot z + a_2 \cdot z^2 + a_3 \cdot z^3 \; + \; \cdots \qquad (30)$$

The straightforward calculation gives, order by order from (20), the successive coefficients a_n in (30) as polynomial expressions (with rational coefficients) of the first coefficient a_1.

It is occur that we have here a series depending on one parameter a_1 we will denote it $R_{a_1}(z)$. This is a quite remarkable series depending on one parameter[5].

Furthermore, one can also verify a quite essential property we expect for a representation of the renormalization group namely that two $R_{a_1}(z)$ for different values of a_1 commute, the result corresponding to the product of these two a_1:

$$R_{a_1}\Big(R_{b_1}(z)\Big) \; = \; R_{b_1}\Big(R_{a_1}(z)\Big) \; = \; R_{a_1 \cdot b_1}(z). \qquad (31)$$

[5]For $A(z)$ given we get a one-parameter family of $R(z)$ solution of (20). Conversely, for $R(z)$ given one can ask if there are several $A(z)$ such that (20) is verified.

The neutral element must necessarily correspond to $a_1 = 1$ which is actually the identity transformation $R_1(z) = z$. We have an "absorbing" element corresponding to $a_1 = 0$, namely $R_0(z) = 0$. Performing the inverse of $R_{a_1}(z)$ (with respect to the composition of functions) amounts to changing a_1 into its inverse $1/a_1$.

4 Rational symmetries corresponding to the hypergeometric function ${}_2F_1\left([\frac{1}{2},\frac{1}{4}],[\frac{5}{4}];\, z\right)$.

Let us recall first the symmetries of the linear differential operator Ω for $A(z) = (3-5z)/z/(1-z)/4$ corresponding to the hypergeometric function ${}_2F_1\left([\frac{1}{2},\frac{1}{4}],[\frac{5}{4}];\, z\right)$ (see Section 2). It is a straightforward calculation to get, order by order from (20), the successive coefficients a_n in (30) as polynomial expressions (with rational coefficients) of the first coefficient a_1 with

$$
\begin{aligned}
a_2 &= -\frac{2}{5}\cdot a_1\cdot(a_1-1), \qquad a_3 = \frac{1}{75}\cdot a_1\cdot(a_1-1)\cdot(7\,a_1-17),\\
a_4 &= -\frac{2}{4875}\cdot a_1\cdot(a_1-1)\cdot(41\,a_1^2-232\,a_1+366), \qquad \cdots\\
a_n &= -\frac{n}{5}\cdot a_1\cdot(a_1-1)\cdot\frac{P_n(a_1)}{P_n(-4)},
\end{aligned}
\tag{32}
$$

where $P_n(a_1)$ is a polynomial with integer coefficients of degree $n-2$.

It is occur that this one-parameter family of "functions" reduces to rational functions for an infinite number of selected values a_1 namely $a_1 = (m_1 + i\cdot m_2)^4$ where m_1 and m_2 are two integers .

The more simple expression for $R_{a_1}(z)$ is obtained for $a_1 = (1+i)^4 = -4$

$$
R_{-4}(z) = \frac{-4z}{(z-1)^2}. \tag{33}
$$

For real values $a_1 = 2^4$, $a_1 = 3^4$ and $a_1 = 4^4$ one have

$$
\begin{aligned}
R_{2^4}(z) &= \frac{16z(-1+z)^2}{(1+z)^4}\\
R_{3^4}(z) &= \frac{z(z^2+6z-3)^4}{(3z^2-6z-1)^4}\\
R_{4^4}(z) &= \frac{256z(-1+z)^2(z+1)^4(z^2-6z+1)^4}{z^4+20z^3-26z^2+20z+1)^4}
\end{aligned}
\tag{34}
$$

It is easy to verify that $R_{2^4}(R_{2^4}(z)) = R_{4^4}(z)$. Let us give also two other examples, namely for $a_1 = (2+2i)^4 = -4 \cdot 2^4$ and $a_1 = (3+3i)^4 = -4 \cdot 3^4$

$$
\begin{aligned}
R_{(2+2i)^4}(z) &= \frac{-64z(-1+z)^2(z+1)^4}{(z^2-6z+1)^4} \\
R_{(3+3i)^4}(z) &= \frac{-4z(3z^2-6z-1)^4(z^2+6z-3)^4}{(-1+z)^2(z^4-28z^3+6z^2-28z+1)^4}
\end{aligned}
\qquad (35)
$$

For the first complex values of a_1 one gets

$$
\begin{aligned}
R_{(1+2i)^4}(z) &= \frac{(7-24i)z(-z+1+2i)^4}{(5z-1+2i)^4} \\
R_{(1+3i)^4}(z) &= \frac{-16z(-5z+1+2i)^4(z-1+2i)^4}{(1-3i)^4(-1+z)^2(-z^2-2(1-4i)z-1)^4} \\
R_{(2+3i)^4}(z) &= \frac{(119-120i)z(-z^3+(11+10i)z^2-(7+4i)z-3+2i)^4}{(13z^3+(13+26i)z^2-(13+52i)z+3+2i)^4}
\end{aligned}
\qquad (36)
$$

We are thus starting to build an infinite number of (elementary) *commuting* rational transformations, *any* composition of these (infinite number of) rational transformations giving rational transformations satisfying (20) and preserving the linear differential operator Ω. This set of rational transformations is a pretty large set.

5 Rational symmetries corresponding to the hypergeometric function ${}_2F_1\left([\frac{3}{4}, \frac{1}{4}], [\frac{5}{4}];\ z\right)$.

The hypergeometric function ${}_2F_1\left([\frac{3}{4}, \frac{1}{4}], [\frac{5}{4}];\ z\right)$ corresponds to a genus one curve

$$y^4 \, - \, t \cdot (1-t) \, = \, 0. \qquad (37)$$

In this case the one-parameter family of $R_{a_1}(z)$ reduces to rational functions also for an infinite number of selected values a_1 namely $a_1 = (m_1 + i \cdot m_2)^4$ where m_1 and m_2 are two integers . The rational expressions for $R_{a_1}(z)$ are more complicated than in previous section.

For integer values of a_1 we have

$$\begin{aligned}
R_{-4}(z) &= \frac{4z(-1+z)}{(-1+2z)^2}\\
R_{2^4}(z) &= \frac{-16z(-1+z)}{(-1-4z+4z^2)^2}\\
R_{3^4}(z) &= \frac{z(4z^2-3)^4}{(-1-24z+72z^2-96z^3+48z^4)^2}\\
R_{4^4}(z) &= \frac{256z(-1+z)(-1+2z)^4(-1-4z+4z^2)^4}{(1+80z-496z^2+2112z^3-4000z^4+2816z^5+256z^6-1024z^7+256z^8)^2}
\end{aligned} \tag{38}$$

For the first complex value of $a_1 = (1+2i)^4$ one gets

$$R_{(1+2i)^4}(z) \frac{(3+4i)z(2z-2+i)^4}{(-20z^2+20z-1+2i)^2} \tag{39}$$

One can reasonably conjecture that the fourth power of *any* complex number $m_1 + i \cdot m_2$ will provide a new example of $R_{a_1}(z)$ being a rational function.

6 Rational symmetries corresponding to the hypergeometric function ${}_2F_1\left([\frac{2}{3}, \frac{1}{3}], [\frac{4}{3}];\ z\right)$.

The hypergeometric function ${}_2F_1\left([\frac{3}{4}, \frac{1}{4}], [\frac{5}{4}];\ z\right)$ corresponds to a genus one curve

$$y^3 \ - \ t \cdot (1-t) \ = \ 0. \tag{40}$$

In this case the one-parameter family of $R_{a_1}(z)$ reduces to rational functions for an infinite number of selected values a_1 namely $a_1 = (m_1 s_1 + m_2 s_2 + m_3 s_3)^3$ where $m_1,\ m_2$ and m_3 are integers and $s_1 + 1$ and $s = s_{2,3} = 1/2 \pm i\sqrt{3}/2$ are three roots of $s_i^3 = 1$. Let us note that $(s_1 + s_2)^3 = -1$.

For integer values of a_1 one gets the following rational expressions for $R_{a_1}(z)$

$$
\begin{aligned}
R_{-1}(z) &= \frac{z}{-1+z} \\
R_{2^3}(z) &= \frac{z}{-1+z}\left(\frac{z-2}{z+1}\right)^3 \\
R_{-2^3}(z) &= -z\left(\frac{z-2}{-1+2z}\right)^3 \\
R_{3^3}(z) &= -27z(z-1)\left(\frac{z^2-z+1}{z^3-6z^2+3z+1}\right)^3 \\
R_{-3^3}(z) &= -27z(z-1)\left(\frac{z^2-z+1}{z^3+3z^2-6z+1}\right)^3
\end{aligned}
\tag{41}
$$

For complex values of a_1 one gets the following rational expressions for $R_{a_1}(z)$

$$
\begin{aligned}
R_{(1+2s)^3} &= \frac{3z(1+2s)(z-1)}{(z+s)^3} \\
R_{(1-s)^3} &= \frac{3\sqrt{3}iz(-1+z)}{(z+s)^3} \\
R_{-(2+3s)^3} &= z\left(\frac{(z^2+(3s+2)z-(3s+2))}{1+(3s+2)z-(3s+2)z^2}\right)^3
\end{aligned}
\tag{42}
$$

7 Two examples of rational symmetries corresponding to the hypergeometric function of the $c = 1+b$ type with $b = 1/6$

Let us give two examples corresponding to a genus one curve, with ${}_2F_1\left([\frac{2}{3},\frac{1}{6}],[\frac{7}{6}];\, z\right)$ and ${}_2F_1\left([\frac{1}{2},\frac{1}{6}],[\frac{7}{6}];\, z\right)$. One expect to obtain the rational expressions for $R_{a_1}(z)$ for $a_1 = (m_1s_1 + m_2s_2 + m_3s_3)^6$ where m_1, m_2 and m_3 are integers and s_1+1 and $s = s_{2,3} = 1/2 \pm i\sqrt{3}/2$ are three roots of $s_i^3 = 1$. We have the following results for the first integer values of a_1

$$
\begin{aligned}
R_{(1+2s_2)^6} &= \frac{-27(-1+z)^2z}{(9z-1)^2} \\
R_{2^6}(z) &= \frac{64z}{(-1-18z+27z^2)^2} \\
R_{3^6}(z) &= \frac{729z(-1+z)^2(3z+1)^6}{(9z-1)^2(243z^3-405z^2+225z+1)^2}
\end{aligned}
\tag{43}
$$

for ${}_2F_1\left([\frac{2}{3},\frac{1}{6}],[\frac{7}{6}];\,z\right)$ and

$$\begin{aligned}
R_{(1+2s_2)^6} &= \frac{27z}{(4z-1)^3} \\
R_{2^6}(z) &= \frac{64z}{(-1+z)^3(1+8z)^3} \\
R_{3^6}(z) &= \frac{-729z(4z-1)^6}{(64z^3-48z^2-96z-1)^3}
\end{aligned} \tag{44}$$

for ${}_2F_1\left([\frac{1}{2},\frac{1}{6}],[\frac{7}{6}];\,z\right)$.

8 Two examples of algebraic symmetries corresponding to the hypergeometric function of the $c = 1+b$ type with $b = 1/6$

We have found two examples of algebraic symmetries which corresponds to the *genus two* curves. The first one is associated with the hypergeometric function ${}_2F_1\left([\frac{5}{6},\frac{1}{6}],[\frac{7}{6}];\,z\right)$. One gets

$$\begin{aligned}
R_{(1+2s_2)^6} &= \frac{1}{2} - \frac{30z-96z^2+1+64z^3}{2(1+16z^2-16z)^{3/2}} \\
R_{2^6}(z) &= \frac{1}{2} + \frac{-1+80z+48z^2-256z^3+128z^4}{2(-32z^2+32z+1)^{3/2}} \\
R_{3^6}(z) &= \frac{1}{2} + \frac{(2z-1)(32z^2-32z-1)P_1(z)}{2\,P_2(z)^{(3/2)}}
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
P_1(z) &= 4096z^6-12288z^5+13056z^4-5632z^3+1680z^2-912z+1 \\
P_2(z) &= 4096z^6-12288z^5+13056z^4-5632z^3+384z^2+384z+1
\end{aligned}$$

And the second one corresponds to hypergeometric function ${}_2F_1\left([\frac{1}{3},\frac{1}{6}],[\frac{7}{6}];\,z\right)$ for witch we have

$$R_{(1+2s_2)^6} = -\frac{1}{54}\cdot\frac{P(z)+(z+1)(z^2-34z+1)(z^2+14z+1)^{(3/2)}}{z(1-z)^4} \tag{46}$$

with

$$P(z) = -1+12z-807z^2-2504z^3-807z^4+12z^5-z^6$$

9 Conclusion

We have shown that several selected Gauss hypergeometric linear differential operators associated to elliptic curves and factorised into order one differential operators actually present an *infinite number of rational symmetries* that actually identify with the *isogenies of the associated elliptic curves* that are perfect illustrations of *exact representations of the renormalization group*. We actually displayed all these calculations, results and structures because they are perfect examples of exact renormalization transformations. For more realistic models (corresponding to Yang-Baxter models with elliptic parametrizations), the previous calculations and structures become more involved and subtle, the previous rational transformations being replaced by algebraic transformations corresponding to *modular curves*.

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