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The Ising model and special geometries

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Abstract
We show that the globally nilpotent G-operators corresponding to the factors of the linear differential operators annihilating the multifold integrals $\chi^{(n)}$ of the magnetic susceptibility of the Ising model ($n \leqslant 6$) are homomorphic to their adjoint. This property of being self-adjoint up to operator homomorphisms is equivalent to the feature of their symmetric squares, or their exterior squares, having rational solutions. The differential Galois groups are in the special orthogonal, or symplectic, groups. This self-adjoint (up to operator equivalence) property means that the factor operators that we already know to be derived from geometry are special globally nilpotent operators: they correspond to 'special geometries'. Beyond the small order factor operators (occurring in the linear differential operators associated with $\chi^{(5)}$ and $\chi^{(6)}$), and, in particular, those associated with modular forms, we focus on the quite large order-12 and order-23 operators. We show that the order-12 operator has an exterior square which annihilates a rational solution. Then, its differential Galois group is in the symplectic group $Sp(12, \mathbb{C})$. The order-23 operator is shown to factorize into an order-2 operator and an order-21 operator. The symmetric square of this order-21 operator has a rational solution. Its differential Galois group is, thus, in the orthogonal group $SO(21, \mathbb{C})$.

Keywords: susceptibility of the Ising model, differential Galois groups, special geometries, exterior square of differential operators, symmetric square of differential operators, self-adjoint operators, homomorphisms of differential operators
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1. Introduction

In previous papers [1–4] some calculations have been performed on the magnetic susceptibility of the square Ising model, and on the n-particle contributions $\tilde{\chi}^{(n)}$ of the susceptibility defined by multifold integrals. In particular, the linear differential operator for $\tilde{\chi}^{(5)}$ was analyzed [2, 3], and similar calculations were carried out [4] for $\tilde{\chi}^{(6)}$.

The small order factors occurring in the factorization of the linear differential operators for $\tilde{\chi}^{(5)}$ and $\tilde{\chi}^{(6)}$ are found to be associated with elliptic functions, or are found to have a *modular form interpretation* [5, 6]. An order-4 operator, emerging for $\chi^{(6)}$, is found to be associated with a Calabi–Yau ODE (and more precisely, associated with a ${}_4F_3$ hypergeometric function with an *algebraic pullback*). Two linear differential operators of large orders (12 and 23) are too involved and have not been analyzed.

While the occurrence of linear differential operators *associated with elliptic curves* for square Ising correlation functions, or *modular forms*, can be expected [7], it is far from being clear what kinds of linear differential operators should emerge in quite involved *highly composite* objects, like the n-particle components $\chi^{(n)}$ of the susceptibility of the square Ising model. We only have the expectation, inherited from the Yang–Baxter integrability of the Ising model, that these differential operators should be 'special' and could possibly be associated with elliptic curves[5]. The result [6] for the order-4 operator of $\chi^{(6)}$, which is a Calabi–Yau ODE, clearly shows that one moves away from the elliptic curve framework, and that the Ising model *does not restrict to the theory of elliptic curves* [9] (and their associated elliptic functions and *modular forms*).

The integrand of the multifold integrals of the Ising model is *algebraic* in the variables of integration and in the other remaining variables. As a consequence, these multifold integrals can be interpreted as *'periods' of algebraic varieties* and should verify *globally nilpotent* [5] linear differential equations[6], i.e. they are [10–12] '*derived from geometry*'. In a recent paper [13], we showed that the multifold integrals of the Ising model *actually correspond to diagonals of rational functions*. This remarkable property does explain, maybe not the modularity property [6] of these n-fold integrals, but at least the *integrality* (i.e. the globally bounded character) property of the corresponding series [13].

Inside this 'geometry' framework [14], the multifold integrals of the Ising model seem to be even more 'selected'. This justifies exploring[7] these 'special geometries'.

Actually, in a previous paper [16, 17], and with a learn-by-example approach, we displayed a set of enumerative combinatorics examples corresponding to miscellaneous *lattice Green functions* [18–24], as well as Calabi–Yau examples, together with order-7 operators [25, 26] associated with differential Galois groups which are exceptional groups. For the *irreducible* operators of these examples, two differential algebra properties occur simultaneously [16]. On the one hand, these operators are *homomorphic to their adjoint*, and, on the other hand, their symmetric, or exterior, squares have a *rational solution* [16]. These properties are equivalent, and correspond to special differential Galois groups. The differential Galois groups are not the $SL(N, \mathbb{C})$, or extensions of $SL(N, \mathbb{C})$, groups that one could expect generically, but *selected* $SO(N, \mathbb{C})$, $Sp(N, \mathbb{C})$, $G_2, \ldots$ differential Galois groups [27].

[5] Corresponding to the canonical parameterization of the Ising model [8] in terms of elliptic functions.

[6] These linear differential operators factorize into irreducible operators that are also necessarily globally nilpotent [5].

[7] Not to be confused, *at first sight*, with the 'special geometry' of *extended supersymmetries* that appears in string theory with moduli spaces of Calabi–Yau 3-folds [15]. However, the questions that we address in this paper, and the ones in [15], do correspond to variations of Hodge structures. One cannot exclude the possibility that these two concepts of 'special geometry' could be related.

An irreducible linear differential operator L_q, of order q, has, generically, a symmetric square ($Sym^2(L_q)$) of order $N_s = q\,(q+1)/2$ and an exterior square ($Ext^2(L_q)$) of order $N_e = q\,(q-1)/2$. If the Wronskian of L_q is rational and $Sym^2(L_q)$ annihilates a rational solution, or is of order $N_s - 1$, the group is in the *orthogonal group* $SO(q, \mathbb{C})$ that admits an invariant quadratic form. If the Wronskian of L_q is rational and $Ext^2(L_q)$ has a rational solution, or is of order $N_e - 1$, the group is in the *symplectic group* $Sp(q, \mathbb{C})$ which admits an invariant alternating form, and the order q is necessarily *even*.

We are going to use these tools on the globally nilpotent operators of the n-particle (multifold-integral) contributions of the magnetic susceptibility of the Ising model, and show that these operators are not only 'derived from geometry', but also actually correspond to 'special geometries'.

The paper is organized as follows. In section 2, we recall the factorizations of the linear differential operators corresponding to the linear differential equations for $\tilde{\chi}^{(5)}$ and $\tilde{\chi}^{(6)}$. In section 3, we show (and recall) that all of the 'small order' factor operators are *associated with elliptic functions*, have a modular form interpretation, or are Calabi–Yau ODEs. These factor operators are, therefore, homomorphic to their adjoints. Sections 4 and 5 are devoted to the 'large order' linear differential operators which occur for $\tilde{\chi}^{(5)}$ and $\tilde{\chi}^{(6)}$. Seeking homomorphisms of these 'large' factors with their corresponding adjoints is beyond the scope of our current computer resources. Instead, keeping in mind the results of [16], we look for, and produce, the rational solutions of their exterior and symmetric squares. Section 6 displays a set of (quite technical) remarks on the subtleties of these massive calculations. Section 7 contains the conclusion.

2. Recalled material

2.1. The linear differential equation for $\tilde{\chi}^{(5)}$

With series of 10 000 terms (modulo a prime), we have obtained [1] the Fuchsian differential equation annihilating $\tilde{\chi}^{(5)}$, which is of order-33. Subsequently [2], it was shown that the linear combination of the five-, three- and one-particle contributions to the magnetic susceptibility,

$$\Phi^{(5)} = \tilde{\chi}^{(5)} - \tfrac{1}{2}\tilde{\chi}^{(3)} + \tfrac{1}{120}\tilde{\chi}^{(1)}, \tag{1}$$

is annihilated by an order-29 linear ODE. The corresponding linear differential operator, L_{29}, factorizes as

$$L_{29} = L_5 \cdot L_{12}^{(\mathrm{left})} \cdot \tilde{L}_1 \cdot L_{11}, \tag{2}$$

with

$$L_{11} = (Z_2 \cdot N_1) \oplus V_2 \oplus \left(F_3 \cdot F_2 \cdot L_1^s\right). \tag{3}$$

The linear differential equations have been obtained in primes, and we have obtained in exact arithmetic some factors occurring in the factorization. All of the factors have been reconstructed and are known *in exact arithmetic*, except for $L_{12}^{(\mathrm{left})}$, L_5 and $\tilde{L}_1$, which are known *only modulo some primes*. The linear differential operator L_5 is irreducible. Its analytical solution (at 0), denoted by $sol(L_5)$ has been written [1] as a homogeneous polynomial of (homogeneous) degree 4 of the complete elliptic integrals K and E.

Then, considering the inhomogeneous equation

$$L_{24}(\Phi^{(5)}) = sol(L_5), \tag{4}$$

where

$$L_{24} = L_{12}^{(\mathrm{left})} \cdot \tilde{L}_1 \cdot L_{11}, \tag{5}$$

we have reconstructed [3], in exact arithmetic, L_{24} and L_5, and have shown that $L_{12}^{(\rm left)}$ is *irreducible* [3].

2.2. The linear differential equation for $\tilde{\chi}^{(6)}$

The order-52 linear differential equation for $\tilde{\chi}^{(6)}$ and the factorization of the corresponding linear differential operator have been given in [4]. It was shown that the linear combination of the six-, four- and two-particle contributions to the magnetic susceptibility,

$$\Phi^{(6)} = \tilde{\chi}^{(6)} - \tfrac{2}{3}\tilde{\chi}^{(4)} + \tfrac{2}{45}\tilde{\chi}^{(2)}, \qquad (6)$$

is annihilated by an order-46 linear ODE. The corresponding linear differential operator, L_{46}, factorizes as

$$L_{46} = L_6 \cdot L_{23} \cdot L_{17}, \qquad (7)$$

with

$$L_{17} = L_4^{(4)} \oplus \left(D_x - \frac{1}{x}\right) \oplus L_3 \oplus \ (L_4 \cdot \tilde{L}_3 \cdot L_2),$$
$$L_4^{(4)} = L_{1,3} \cdot (L_{1,2} \oplus L_{1,1} \oplus D_x), \qquad (8)$$

where D_x denotes the derivative with respect to x.

The linear differential equation has been obtained in primes, and we have obtained, in exact arithmetic, some factor operators occurring in the factorization. All the factors are known *in exact arithmetic* except for L_{23} and L_6, which are known *only modulo some primes*. While L_6 is irreducible, since its analytical solution (at 0) has been written as a polynomial expression of homogeneous degree 5 in the complete elliptic integrals K and E, we had not reached [4] any conclusion on whether the operator L_{23} is reducible or not. Performing the factorization based on the combination method presented in section 4 of [2] requires prohibitive computational times.

2.3. Summary

We have plenty[8] of linear differential operators occurring as factor operators in the linear differential equations for $\tilde{\chi}^{(5)}$ and $\tilde{\chi}^{(6)}$. Beyond the order-1 differential operators, L_1, N_1 (given in [28]), L_1^s, and $\tilde{L}_1$ (given in [2, 3]), and the fully factorizable order-4 operator $L_4^{(4)}$ (given in [4] with its solution in [29]), there are linear differential operators of higher orders, Z_2, V_2, F_2, F_3, L_5 and $L_{12}^{(\rm left)}$ for $\tilde{\chi}^{(5)}$, and L_2, L_3, $\tilde{L}_3$, L_4, L_6 and L_{23} for $\tilde{\chi}^{(6)}$.

The next section deals with these small order operators (up to order-6) where we show (and/or recall) for each one that it is either equivalent to a symmetric power of L_E, the differential operator corresponding to the complete elliptic integral E, or has a symmetric (or exterior) square which annihilates a rational solution. Next, it is shown that each operator L_q is *homomorphic to its adjoint* [16],

$$L_q \cdot R_n = \mathrm{adjoint}(R_n) \cdot \mathrm{adjoint}(L_q), \qquad (9)$$

and we focus on the order of R_n, and on the coefficient in front of the higher derivative. Note that, for irreducible operators, there is another equivalence relation,

$$\mathrm{adjoint}(L_q) \cdot S_p = \mathrm{adjoint}(S_p) \cdot L_q, \qquad (10)$$

which sends the solutions of L_q into the solutions of the adjoint. In the sequel, we will consider the relation (9).

[8] Typically these problems can be rephrased in terms of the variation of mixed Hodge structures. To some extent, this explains why the minimal order operators annihilating the $\tilde{\chi}^{(n)}$ factorize in a *quite large number of factor operators*.

Notation. The linear differential operators of $\tilde{\chi}^{(5)}$ and $\tilde{\chi}^{(6)}$ have large orders and factorize in many factors. With a few exceptions, all the factor operators carry large degree polynomials whose roots are *apparent singularities*. In the sequel we adopt the notation $A_n(L_q)$ for these apparent polynomials of a linear differential operator L_q, n denoting the degree of the 'apparent' polynomial.

3. The rational solution for Sym^2 or Ext^2 versus homomorphism with the adjoint

3.1. The rational solution for Sym^2 or Ext^2

The order-2 operator V_2 (given in [2]) is equivalent to the second-order operator associated with $\tilde{\chi}^{(2)}$ (or equivalently with L_E, the linear differential operator corresponding to the complete elliptic integral of the second kind E). Similarly, the order-2 operator L_2 (given in [4]) is equivalent to the second-order operator associated with $\tilde{\chi}^{(2)}$. This is also the case for the order-3 operator L_3 (given in [4]) which is equivalent to the symmetric square of the second-order operator associated with $\tilde{\chi}^{(2)}$. The equivalence occurs also for the order-5 operator L_5, occurring [2] in $\tilde{\chi}^{(5)}$, which is the *symmetric fourth power* of L_E, and the order-6 operator L_6, occurring [4] in $\tilde{\chi}^{(6)}$, which is the *symmetric fifth power* of L_E. There are also the order-3 operator Y_3 (given in [30]), and the order-4 operator [5, 29] M_2, which are symmetric second and third powers, respectively, of L_E.

For all the linear differential operators which are symmetric powers of L_E, their solutions are given as polynomials of homogeneous degree in the complete elliptic integrals. For those which are equivalent to L_E (or to a symmetric power of L_E), the solutions can be written as homogeneous polynomials in the complete elliptic integrals and their derivatives [2, 4].

Other operators are equivalent to hypergeometric functions *up to rational pullbacks*. The order-2 operator Z_2 (given in [31]), occurring (also) in the factorization of the linear differential operator [28] associated with $\tilde{\chi}^{(3)}$, is seen to correspond to a *modular form of weight 1* [5]. The order-2 operator F_2 (given in [2]) corresponds to a *modular form*: its solutions can be written in terms of Gauss hypergeometric functions with pullbacks [6].

Now there are linear differential operators of order $\geqslant 3$ which are equivalent to symmetric powers of hypergeometric functions with a pullback.

3.1.1. The symmetric square of F_3. The symmetric square of the order-3 operator F_3 is an order-6 linear differential operator which is a direct sum of an order-5 differential operator and an order-1 differential operator. The rational solution of the symmetric square of F_3, denoted as $S_R(Sym^2(F_3))$, reads

$$S_R(Sym^2(F_3)) = \frac{P_{34}(x)}{D_{26}(x) \cdot A_7(F_2)^2}, \tag{11}$$

with

$$D_{26}(x) = x^2 \cdot (x-1)^2\,(1+2x)^2\,(4x-1)^9\,(1+4x)^7\,(4x^2+3x+1)^2,$$

where $P_{34}(x)$ is a polynomial of degree 34 given in appendix A, and where the degree 7 apparent polynomial for the order-2 operator F_2 appearing in (3) reads

$$A_7(F_2) = 1 + x - 24x^2 - 145x^3 - 192x^4 + 96x^5 + 128x^7. \tag{12}$$

The irreducible linear differential operator F_3 has a symmetric square of order-6 that annihilates a rational function. The differential Galois group of the operator F_3 is in the orthogonal group $SO(3, \mathbb{C})$, or equivalently the group $PSL(2, \mathbb{C})$. The differential operator F_3

is equivalent (up to a multiplicative function) to the symmetric square of an order-2 differential operator[9]. The independent solutions of this order-2 operator (call it O_2) are

$$x^{-1/3} \cdot \; {}_2F_1\Big(\big[\tfrac{1}{6}, \tfrac{1}{6}\big], \big[\tfrac{1}{2}\big]; \, P_1\Big), \qquad x^{1/3} \cdot \sqrt{P_1} \cdot {}_2F_1\Big(\big[\tfrac{2}{3}, \tfrac{2}{3}\big], \big[\tfrac{3}{2}\big]; \, P_1\Big), \qquad (13)$$

with

$$P_1(x) \, = \, \frac{1}{108} \, \frac{(1-4x^2)\,(1+32x^2)^2}{x^2} \, = \, \frac{1}{108} \, \frac{(1-16x^2)^3}{x^2} \, + \, 1. \qquad (14)$$

The solutions of F_3 can be written in terms of the solutions (13) of this order-2 operator O_2 (the order-2 intertwiner R_2 is given in appendix B):

$$sol(F_3) = (1-4x)^{-9/2}\,(1+4x)^{-7/2} \cdot R_2(sol(Sym^2(O_2))). \qquad (15)$$

Anticipating some comments in section 6, we give the rational solution of the *symmetric square of the adjoint* of[10] F_3:

$$S_R(Sym^2(\mathrm{adjoint}(F_3))) \, = \, \frac{N_{33}(x) \cdot P_{53}(x)}{A_{37}(F_3)^2}, \qquad (16)$$

with

$$N_{33}(x) = x^5 \cdot (1-x)^2(1-2x)^2\,(1+2x)^4\,(1-4x)^{10}(1+4x)^6\,(1+3x+4x^2)^2. \qquad (17)$$

The degree 37 apparent polynomial $A_{37}(F_3)$ and the degree 53 polynomial $P_{53}(x)$ are given, respectively, in (A.1) and (A.3).

3.1.2. The symmetric square of $\tilde{L}_3$.

Similarly, the order-3 differential operator $\tilde{L}_3$ has a symmetric square differential operator of order-6 which is a direct sum of an order-5 linear differential operator and an order-1 linear differential operator, the latter annihilating the rational solution $S_R(Sym^2(\tilde{L}_3))$:

$$S_R(Sym^2(\tilde{L}_3)) = \frac{2\,-42\,x\,+225\,x^2\,-260\,x^3\,-4352\,x^4\,+49\,152\,x^5}{(1\,-16\,x)^7}. \qquad (18)$$

The linear differential operator $\tilde{L}_3$ is equivalent (up to a multiplicative function) to the symmetric square of an order-2 differential operator. The independent solutions of this order-2 operator (call it O_2) are

$${}_2F_1\Big(\big[\tfrac{1}{8}, \tfrac{3}{8}\big], \big[\tfrac{1}{2}\big]; \, P_1\Big), \qquad \sqrt{P_1} \cdot \; {}_2F_1\Big(\big[\tfrac{5}{8}, \tfrac{7}{8}\big], \big[\tfrac{3}{2}\big]; \, P_1\Big), \qquad (19)$$

with

$$P_1(x) \, = \, \frac{(1-12\,x)^2}{(1-16x)(1-4x)^2} \, = \, \frac{256x^3}{(1-16x)(1-4x)^2} + 1. \qquad (20)$$

The solutions of $\tilde{L}_3$ can be written in terms of the solutions (19) of the order-2 operator (the order-2 intertwiner R_2 is given in appendix B):

$$sol(\tilde{L}_3) = \frac{(1-16\,x)^{9/2}\,(1-4\,x)^{3/2}}{x^2 \cdot (1024\,x^3 - 1232\,x^2 + 160\,x - 5)} \cdot R_2(sol(Sym^2(O_2))). \qquad (21)$$

The rational solution of the *symmetric square of the adjoint* of $\tilde{L}_3$ (taken in monic form) reads

$$S_R(Sym^2(\mathrm{adjoint}(\tilde{L}_3))) = \frac{x^4 \cdot (1-16x)^6 \cdot P_{10}(x)}{(1-4x)^3 \cdot A_4(\tilde{L}_3)^2}, \qquad (22)$$

where the degree 4 apparent polynomial of $\tilde{L}_3$, $A_4(\tilde{L}_3)$, reads

$$A_4(\tilde{L}_3) = 4352x^4 + 3607x^3 - 1678x^2 + 252x - 8. \qquad (23)$$

The polynomial P_{10} is given in (A.4).

[9] See [32] for the reduction of order-3 ODEs to order-2 ODEs.
[10] Here F_3 is taken to be monic: its head coefficient of D_x^3 is normalized to 1.

3.1.3. The exterior square of L_4. The last of the 'small order' linear differential operators is the order-4 operator L_4 occurring in the linear differential operator of $\tilde{\chi}^{(6)}$. The linear differential operator L_4 has been analyzed in [6] and was shown to be equivalent to a *Calabi–Yau equation* with a ${}_4F_3$ hypergeometric function with an *algebraic pullback*. This algebraic pullback is simply related to the modulus of the elliptic functions parameterizing the Ising model. The solution of this order-4 operator L_4 is sketched in section B.3.

The exterior square of L_4 is of order-6. It is a *direct sum* of an order-5 differential operator and an order-1 differential operator. The order-1 operator annihilates the rational solution, $S_R(Ext^2(L_4))$, that we recall [6] here:

$$S_R(Ext^2(L_4)) = \frac{P_{17}(x)}{x^9 \cdot (1-16x)^{13}(1-4x)^2 \cdot A_4(\tilde{L}_3)}, \tag{24}$$

where $A_4(\tilde{L}_3)$ is given in (23). The degree 17 polynomial P_{17} is given in (A.6).

The rational solution of the exterior square of the adjoint of L_4 reads

$$S_R(Ext^2(\mathrm{adjoint}(L_4))) = \frac{x^{11} \cdot (1-16x)^{14}(1-4x)^2(1-8x) \cdot P_{17}(x)}{A_{26}(L_4)}. \tag{25}$$

The degree 26 apparent polynomial $A_{26}(L_4)$ is given in (A.5).

3.2. The homomorphism with the adjoint

All the previous linear differential operators (V_2, L_2, L_3, L_5, L_6) which are homomorphic to L_E, or homomorphic to the symmetric square of L_E, are naturally homomorphic with their adjoints. This is straightforward consequence of the homomorphism of L_E with its adjoint.

The more subtle linear differential operators (Z_2, F_2, F_3, $\tilde{L}_3$), which have been shown [6] to be associated with *modular forms*, and more precisely ${}_2F_1$ hypergeometric functions up to rational (or algebraic) pullbacks, are also homomorphic to their adjoint. For instance, Z_2 is conjugated to its adjoint

$$Z_2 \cdot W_Z(x) = W_Z(x) \cdot \mathrm{adjoint}(Z_2), \qquad \text{with:}$$
$$W_Z(x) = \frac{(1+2x)\cdot(1-x)\cdot(96x^4+104x^3-18x^2-3x+1)}{(1+4x)^2\cdot(1-4x)^5\cdot(1+3x+4x^2)\cdot x}, \tag{26}$$

where $W_Z(x)$ is the Wronskian of Z_2.

Similarly, F_2 is homomorphic to its adjoint

$$F_2 \cdot W_F(x) = W_F(x) \cdot \mathrm{adjoint}(F_2), \qquad \text{with:}$$
$$W_F(x) = \frac{x \cdot A_7(F_2)}{(1+3x+4x^2)^2\cdot(1+4x)^3\cdot(1-4x)^6}, \tag{27}$$

where, again, $W_F(x)$ is the Wronskian of F_2, and where $A_7(F_2)$ is the apparent polynomial of F_2 given in (12).

3.3. The homomorphism with the adjoint for F_3

The order-3 linear differential operator F_3 is homomorphic to its adjoint, with a large order-2 intertwiner

$$F_3 \cdot R_2 = \mathrm{adjoint}(R_2) \cdot \mathrm{adjoint}(F_3), \qquad \text{with}$$
$$R_2 = a_2(x)\cdot D_x^2 + a_1(x)\cdot D_x + a_0(x), \tag{28}$$

where $a_2(x)$ is the *rational solution* of $Sym^2(F_3)$ given in (11), and where

$$a_1(x) = \frac{P_{78}(x)}{\rho_1(x)\cdot A_7(F_2)^2\cdot A_{37}(F_3)}, \qquad a_0(x) = \frac{P_{122}(x)}{\rho_1(x)\cdot\rho_0(x)\cdot A_7(F_2)^2\cdot A_{37}(F_3)^2},$$

with

$$\rho_0(x) = x \cdot (x-1)(16x^2-1)(4x^2-1)(4x^2+3x+1),$$
$$\rho_1(x) = x^3 \cdot (x-1)^3 (2x-1)(4x-1)^{10}(1+4x)^8(1+2x)^3(4x^2+3x+1)^3,$$

and where $A_7(F_2)$ is given in (12) and $A_{37}(F_3)$ is given in (A.1). The various polynomials $P_j(x)$ are of degree j.

3.4. The homomorphism with the adjoint for $\tilde{L}_3$

The order-3 linear differential operator $\tilde{L}_3$ is also homomorphic to its adjoint

$$\tilde{L}_3 \cdot R_2 = \mathrm{adjoint}(R_2) \cdot \mathrm{adjoint}(\tilde{L}_3), \qquad \text{with}$$
$$R_2 = a_2(x) \cdot D_x^2 + \frac{P_{11}(x)}{(16x-1)^7 \cdot \rho(x) \cdot A_4(\tilde{L}_3)} \cdot D_x + \frac{P_{17}(x)}{(16x-1)^7 \cdot \rho(x)^2 \cdot A_4(\tilde{L}_3)^2}, \tag{29}$$

where

$$\rho(x) = x \cdot (4x-1)(16x-1), \tag{30}$$

and where $a_2(x)$ is the *rational solution* of $Sym^2(\tilde{L}_3)$ given in (18).

3.5. The homomorphism with the adjoint for L_4

Now, let us consider the last 'small order' linear differential operator occurring in $\tilde{\chi}^{(6)}$, namely the order-4 operator L_4. It is also homomorphic with its adjoint with an *order-2* intertwiner

$$L_4 \cdot R_2 = \mathrm{adjoint}(R_2) \cdot \mathrm{adjoint}(L_4), \qquad \text{with} \tag{31}$$

$$R_2 = a_2(x) \cdot D_x^2 + \frac{P_{46}(x)}{\rho_1(x) \cdot A_4(\tilde{L}_3) \cdot A_{26}(L_4)} \cdot D_x + \frac{P_{75}(x)}{\rho_1(x) \cdot \rho_0(x) \cdot A_4(\tilde{L}_3) \cdot A_{26}(L_4)^2}, \tag{32}$$

where

$$\rho_0(x) = x \cdot (4x-1)(8x-1)(16x-1), \tag{33}$$

$$\rho_1(x) = x^{10} \cdot (4x-1)^3 (8x-1)(16x-1)^{14}, \tag{34}$$

where $A_4(\tilde{L}_3)$ is the degree 4 apparent polynomial of $\tilde{L}_3$ given in (23), $A_{26}(L_4)$ of degree 26 is the apparent polynomial of L_4 given in (A.5), and $a_2(x)$ is the *rational solution* of $Ext^2(L_4)$ given in (24). The polynomials $P_j(x)$ are of degree j.

Note that R_2, the order-2 intertwiner (31), is almost self-adjoint: it is such that

$$Y_2^{(L)} = R_2 \cdot \frac{1}{r(x)} = \alpha_2(x) \cdot D_x^2 + \frac{d\alpha_2(x)}{dx} \cdot D_x + \alpha_0(x), \tag{35}$$

is an order-2 *self-adjoint* operator, where $r(x)$ is the rational function

$$r(x) = 1080 \cdot \frac{P_{26} \cdot A_4(\tilde{L}_3)}{P_{17}^2 \cdot (1-8x)(1-16x) \cdot x^2}. \tag{36}$$

Since $Y_2^{(L)}$ is self-adjoint, the Wronskian of $Y_2^{(L)}$ is also equal to $1/\alpha_2(x)$, the inverse of the head coefficient of $Y_2^{(L)}$. The Wronskian of $Y_2^{(L)}$ is equal to the rational function $r(x)^2 \cdot S_R(Ext^2(\mathrm{adjoint}(L_4)))$, where $S_R(Ext^2(\mathrm{adjoint}(L_4))$ is (25), the rational solution of the adjoint of L_4 (written in monic form).

The other homomorphisms of L_4, with its adjoint, correspond to the intertwining relation

$$\mathrm{adjoint}(L_2) \cdot L_4 = \mathrm{adjoint}(L_4) \cdot L_2, \tag{37}$$

where, again, the order-2 operator L_2 is almost self-adjoint: it is such that

$$Y_2^{(R)} = r(x) \cdot L_2 = \beta_2(x) \cdot D_x^2 + \frac{\mathrm{d}\beta_2(x)}{\mathrm{d}x} \cdot D_x + \beta_0(x) \tag{38}$$

is *self-adjoint*, where $r(x)$ is the *same* rational function (36). The Wronskian of $Y_2^{(R)}$ is nothing but $S_R(Ext^2(L_4))$ given by (24). Since $Y_2^{(R)}$ is self-adjoint, the Wronskian of $Y_2^{(R)}$ is also equal to $1/\beta_2(x)$, the inverse of the head coefficient of $Y_2^{(R)}$.

Remark. Recalling the miscellaneous decompositions $L_n \cdot L_m$ +Cst (up to an overall function), obtained for the lattice Green operators displayed in [16, 17], and since the order-4 operator L_4 is homomorphic to its adjoint with order-2 intertwiners (see (31), (37)), it is tempting to find such an $L_n \cdot L_m$ + Cst decomposition for L_4. One easily deduces [16, 17] the following decomposition for L_4, written in monic form:

$$L_4 = r(x) \cdot \left(Y_2^{(L)} \cdot Y_2^{(R)} + 1\right), \tag{39}$$

where $Y_2^{(L)}$ and $Y_2^{(R)}$ are the two (quite large) *self-adjoint operators* (35) and (38), and where $r(x)$ is the previous rational function (36).

Since L_4 in (39) is monic, one has the following relation between the head coefficients of the two self-adjoint operators $Y_2^{(L)}$ and $Y_2^{(R)}$:

$$r(x) \cdot \alpha_2(x) \cdot \beta_2(x) = 1 \quad \text{or:} \quad \frac{1}{\beta_2(x)} = r(x) \cdot \alpha_2(x) = a_2(x), \tag{40}$$

where $a_2(x)$ is the head coefficient of R_2 in (31).

One easily verifies that $S_R(Ext^2(L_4)) = 1/\beta_2(x)$ (see (24)) is a solution of the exterior square of the order-2 operator $Y_2^{(R)}$. The rational function $1/\alpha_2(x)$ is a solution of the exterior square of the order-2 operator $Y_2^{(L)}$, which we denote as $S_R(Y_2^{(L)})$, and is simply related to $S_R(Ext^2(\mathrm{adjoint}(L_4)))$ (see (25)): it is nothing but $r(x)^2 \cdot S_R(Ext^2(\mathrm{adjoint}(L_4)))$. Actually the exterior square of the adjoint monic order-4 operator $\mathrm{adjoint}(L_4) = (Y_2^{(R)} \cdot Y_2^{(L)} + 1) \cdot r(x)$ has the same rational solution as the exterior square of $Y_2^{(L)} \cdot r(x)$, namely $1/r(x)^2 \cdot S_R(Y_2^{(L)})$.

These two results are a simple consequence of the fact that the rational solution of the exterior square of a decomposition like (39), is, in general, the rational solution of the rightmost order-2 operator in the decomposition (see [16, 17]).

3.6. Comments

In all these previous examples of 'small order' linear differential operators occurring in the $\tilde{\chi}^{(n)}$, we have, as we showed for other examples in [16], the simultaneous occurrence of two properties: the *homomorphism of the irreducible operator with its adjoint* and the *occurrence of a rational solution in the symmetric square, or exterior square*, of the differential operator. The expressions for the intertwiners given above clearly show this link. Each time, the linear differential operator has a symmetric square (or exterior square) annihilating a rational solution (see (11), (18), (24)); it is precisely, this rational solution that appears in the coefficient of the higher derivative of the intertwiners (see (28), (29), (31)). For order-2 linear differential operators, the rational solution of the exterior square is just the Wronskian.

For these small order ($\leqslant$4) examples, one even sees that, for odd (resp. even) order operators, it is the rational solution of the symmetric (resp. exterior) square which builds the intertwiner.

However, beyond these examples, for an order-4 linear differential operator which is homomorphic to its adjoint with an odd order intertwiner, it is the symmetric square which is involved. Appendix C shows the situation with *generic* linear differential operators of order-3 and order-4. The intertwiners of the homomorphism have rational coefficients only when the symmetric (or exterior) square has rational solutions. When there is a homomorphism with the adjoint, for order-3 linear differential operators, the intertwiner can be of order-2, or a function, and it is the symmetric square which annihilates the rational solution (see (C.4)). For order-4 operators, if the intertwiner is of order-2, or a function, it is the *exterior square* which annihilates a rational solution (see (C.14)), while for order-1 and order-3 intertwiners, the rational solution is annihilated by the *symmetric square* (see (C.22)).

Note that we have used the results in appendix C to build the homomorphisms (28) and (31) of F_3 and L_4, which are the largest of our small order differential operators.

We have finished with the small order factors occurring in the linear differential operators of $\tilde{\chi}^{(5)}$ and $\tilde{\chi}^{(6)}$. The differential Galois groups of the order-3 operators are in the *orthogonal group* $SO(3, \mathbb{C})$. The differential Galois group of the order-4 operator L_4 is in the *symplectic group* $Sp(4, \mathbb{C})$. Appendix D gives examples of invariant forms in both cases.

All the factors share many properties (global nilpotence, global boundedness, symplectic or orthogonal differential Galois groups) and all have been solved (in closed forms) in terms of elliptic integrals and modular forms. We should note, however, that the solutions of the factors, which are not rightmost (in the factorization), are not solutions of the differential operators of the $\chi^{(n)}$. For instance, we have the analytical solutions at the origin of both Z_2 and N_1 (see (3)), but we do not have the second analytical solution at the origin[11] of $Z_2 \cdot N_1$, which is a component in $\tilde{\chi}^{(3)}$ and $\tilde{\chi}^{(5)}$. Similarly, we do not have, in closed forms, the second and third solutions of $F_3 \cdot F_2 \cdot L_1^s$ given in (3), while the solutions of F_3 and F_2 are known as hypergeometric functions with pullback. More generally, knowing the solutions of the factors of the minimal order operators annihilating the $\chi^{(n)}$ does not yield automatically a knowledge of the solutions (in closed forms) of these operators (except, of course, if these factors are all in direct sum, which is not the case). To achieve a complete understanding of the solutions of the minimal order operators annihilating the $\chi^{(n)}$, some additional work remains to be done.

We turn now to the 'large order' linear differential operators $L_{12}^{(\mathrm{left})}$ and L_{23}. Here, and as a consequence of the large size of these operators, the approach for finding the intertwiners between $L_{12}^{(\mathrm{left})}$ (and L_{23}) with their corresponding adjoints is hopeless with our current computational resources. In the following two sections, keeping in mind our results on the 'small order' operators, we will claim that $L_{12}^{(\mathrm{left})}$ will be homomorphic to its adjoint if we find a rational solution annihilated by the exterior square (or by the symmetric square). Similarly, L_{23} will be homomorphic to its adjoint if a rational solution of the symmetric square is found.

4. On the linear differential operator $L_{12}^{(\mathrm{left})}$ in $\tilde{\chi}^{(5)}$

To see whether the exterior square of $L_{12}^{(\mathrm{left})}$ has a rational solution it is simpler to start from the definition of the exterior square. The formal solutions (at 0) of $L_{12}^{(\mathrm{left})}$ are obtained (modulo a prime) and are either analytic or logarithmic with a maximum cubic power $\ln(x)^3$. The general solution (at 0) of $L_{12}^{(\mathrm{left})}$ is written as

$$F_3(x) \cdot \ln(x)^3 + F_2(x) \cdot \ln(x)^2 + F_1(x) \cdot \ln(x) + F_0(x). \tag{41}$$

Some 120 starting terms are needed to generate the series $F_3(x)$ with a homogeneous recurrence and the other series $F_j(x)$ with inhomogeneous recurrences. The 12 solutions S_p are taken as

[11] The second analytical solution of $Z_2 \cdot N_1$ is an integral over the solutions of Z_2.

in [3]. We form the linear combination

$$\sum_{k,p} d_{k,p} \cdot \left(S_k \frac{dS_p}{dx} - S_p \frac{dS_k}{dx} \right), \qquad k \neq p = 1, \ldots, 12, \tag{42}$$

which is a general solution of the[12] order-66 exterior square $Ext^2(L_{12}^{(\mathrm{left})})$. Demanding that this combination should not contain logarithms fixes some of the coefficients $d_{k,p}$.

For a rational solution of $Ext^2(L_{12}^{(\mathrm{left})})$ to exist, the form, which is, now, analytic at $x = 0$,

$$D(x) \cdot \sum_{k,p} d_{k,p} \cdot \left(S_k \frac{dS_p}{dx} - S_p \frac{dS_k}{dx} \right), \tag{43}$$

should be a polynomial, and with $D(x)$ the polynomial whose roots are the regular singularities of $L_{12}^{(\mathrm{left})}$. Each regular singularity in $D(x)$ is taken with the power n_j (n_j being twice the maximum local exponent of that singularity in $L_{12}^{(\mathrm{left})}$).

Our series S_j are of length 600 and the coefficients depend on some remaining $d_{k,p}$. By canceling the coefficients of the higher terms in (43), all the coefficients down to a given term are automatically zero, and we obtain a polynomial. The rational solution of the exterior square $Ext^2(L_{12}^{(\mathrm{left})})$ thus reads

$$S_R\big(Ext^2\big(L_{12}^{(\mathrm{left})}\big)\big) = \frac{P_{312}(x)}{A_{131}(\tilde{L}_1 \cdot L_{11}) \cdot D_{211}(x)}, \tag{44}$$

with

$$\begin{aligned} D_{211}(x) &= x^{18} \cdot (2x-1)^2 (x-1)^{12} (x+1)^2 (2x+1)^{13} (4x+1)^{22} (4x-1)^{24} \\ &(4x^2-2x-1)^2 (4x^2+3x+1)^{14} (x^2-3x+1)^2 (8x^2+4x+1)^8 \\ &(4x^3-3x^2-x+1)^6 (4x^3-5x^2+7x-1)^8 (4x^4+15x^3+20x^2+8x+1)^6, \end{aligned}$$

where $P_{312}(x)$ is a polynomial of degree 312, and where $A_{131}(\tilde{L}_1 \cdot L_{11})$ is the apparent polynomial of the product $\tilde{L}_1 \cdot L_{11}$.

The linear differential operator $L_{12}^{(\mathrm{left})}$ is *irreducible* [3]. Its exterior square annihilates a rational function. Its differential Galois group is in the *symplectic group* $Sp(12, \mathbb{C})$. Finding the rational solution of the *exterior square of the adjoint* of $L_{12}^{(\mathrm{left})}$ is, for the moment, beyond our computer facilities. Recalling, for the order-q lattice Green operators displayed in [16, 17], the decompositions of the type $L_n \cdot L_m + \mathrm{Cst}$ (up to an overall function), where L_m and L_n are self-adjoint operators, and where m and n are two integers of the same parity, it is tempting to imagine, for $L_{12}^{(\mathrm{left})}$, a decomposition of the form $L_{2n} \cdot L_{2m} + \mathrm{Cst}$, where the exterior square of L_{2m} would have (44) as a rational solution. This would imply the existence of a rational solution for the *exterior square of the adjoint* of $L_{12}^{(\mathrm{left})}$, that is identified with the rational solution of the exterior square of L_{2n}.

5. On the linear differential operator L_{23} in $\tilde{\chi}^{(6)}$

To fit our scheme wherein the linear differential operators, occurring in the Ising model, correspond to a 'special geometry', the linear differential operator L_{23} should have a rational solution for its symmetric square, which is equivalent to saying that L_{23} is homomorphic to its adjoint.

[12] By canceling all of the coefficients in front of each $x^n \ln(x)^p$ in (42), we find that all the $d_{k,p}$ are zero; this means that $Ext^2(L_{12}^{(\mathrm{left})})$ is of order-66. Similarly, if we carry out the same calculations for adjoint$(L_{12}^{(\mathrm{left})})$ and find that all the $d_{k,p}$ in the corresponding combination are zero, this will mean that $Ext^2(\mathrm{adjoint}(L_{12}^{(\mathrm{left})}))$ has the order 66. This calculation for adjoint$(L_{12}^{(\mathrm{left})})$ has not been carried out.

To see whether the symmetric square[13] of L_{23} has a rational solution, the general solution of $Sym^2(L_{23})$ is built from the formal solutions (modulo a prime) of L_{23} as

$$\sum_{k,p} f_{k,p} \cdot S_k S_p, \qquad\qquad k \geqslant p = 1, \ldots, 23, \tag{45}$$

which should contain neither logarithms nor x^a (a half-integer), thus fixing some of the coefficients $f_{k,p}$.

For a rational solution of $Sym^2(L_{23})$ to exist, the form (analytic at $x = 0$)

$$D(x) \cdot \sum_{k,p} f_{k,p} \cdot S_k S_p, \qquad\qquad k \geqslant p = 1, \ldots, 23, \tag{46}$$

should be a polynomial, where the denominator $D(x)$ reads

$$D(x) = x^{n_1} \cdot (1-16x)^{n_2} (1-4x)^{n_3} (1-9x)^{n_4} (1-25x)^{n_5} (1-x)^{n_6} \\ \times (1-10x+29x^2)^{n_7} (1-x+16x^2)^{n_8},$$

the order of magnitude of the exponents n_j being obtained from the local exponents of the singularities.

With very long series, we have found no rational solution for $Sym^2(L_{23})$. If we trust that our series are long enough, we face one of two situations. Either L_{23} is irreducible and does not follow the general scheme of being a special geometry, contrary to all of the other operators (obtained right now) in the Ising model, or the linear differential operator L_{23} is reducible. In the latter situation (i.e. where L_{23} is *reducible*), the right factor in L_{23} should be of even order: let us fix this order as 2. The linear differential operator L_{23} is assumed to have the factorization

$$L_{23} = L_{21} \cdot \tilde{L}_2. \tag{47}$$

In this case, and assuming that the factorization (47) is unique, $Sym^2(L_{23})$ does not need to have a rational solution, but its exterior square *should have* an order-1 right factor, since (see remark 7 below)

$$Ext^2(L_{23}) = O_{252} \cdot Ext^2(\tilde{L}_2). \tag{48}$$

The next step is, then, to see whether it is the exterior square of L_{23} which has a rational solution. The general solution of $Ext^2(L_{23})$ is written as

$$\sum_{k,p} d_{k,p} \cdot \left(S_k \frac{\mathrm{d}S_p}{\mathrm{d}x} - S_p \frac{\mathrm{d}S_k}{\mathrm{d}x} \right), \qquad\qquad k \neq p = 1, \ldots, 23, \tag{49}$$

and should not contain logarithms or x^a (a half-integer), fixing some of the coefficients $d_{k,p}$.

For a rational solution of $Ext^2(L_{23})$ to exist, the form (analytic at $x = 0$)

$$D(x) \cdot \sum_{k,p} d_{k,p} \cdot \left(S_k \frac{\mathrm{d}S_p}{\mathrm{d}x} - S_p \frac{\mathrm{d}S_k}{\mathrm{d}x} \right) \tag{50}$$

should be a polynomial. We did indeed find a rational solution for $Ext^2(L_{23})$ which has the form

$$\frac{A_{79}(L_{17}) \cdot P_{93}(x)}{x^4 \cdot (1-16x)^6}, \tag{51}$$

where P_{93} is a degree 93 polynomial, and where $A_{79}(L_{17})$ is the degree 79 apparent polynomial of L_{17}.

[13] By canceling all the coefficients in front of each $x^n \ln(x)^p$ and x^a, a half-integer, in (45), we find that all the $f_{k,p}$ are zero, which means that $Sym^2(L_{23})$ is of order-276.

The existence of a rational solution means that an invariant alternating form is preserved, and this implies that the order is even. Since this is not the case, the differential operator L_{23} has a right factor of even order. To establish the factorization (47), we should show that the rational solution (51) is for the exterior square of an *order-2* linear differential operator.

5.1. The factorization $L_{23} = L_{21} \cdot \tilde{L}_2$

The rational solution of $Ext^2(\tilde{L}_2)$ does not determine the linear differential operator $\tilde{L}_2$ whatever its order is. We show now how we have obtained the linear differential operator $\tilde{L}_2$.

Once the combinations of the $d_{k,p}$ (in (49)) in front of the logarithms and x^a (a half-integer) have been canceled, the rational solution is looked for. This fixes all the $d_{k,p}$. We collect (49) over $\mathrm{d}S_j/\mathrm{d}x$, i.e. we consider the linear combination of series in front of $\mathrm{d}S_j/\mathrm{d}x$. Here S_j is the solution with the maximum power of $\ln(x)$, which is $\ln(x)^4$ for L_{23}. We get no combination. There is also no combination when we collect over $\mathrm{d}S_j/\mathrm{d}x$, with S_j the solution with $\ln(x)^3$ and $\ln(x)^2$ terms. When S_j is the solution with a $\ln(x)$ term, we obtain six identical combinations, i.e. series. If there is a right factor in L_{23}, this right factor will be of order $\geqslant 2$. We then search for the linear ODE that annihilates the combination found, and find that its order is, in fact, *2*, this is $\tilde{L}_2$.

We have then shown that L_{23} actually has the factorization (47). Acting with $\tilde{L}_2$ on the solution of L_{23} gives a series which is used to obtain L_{21}.

The linear ODE corresponding to $\tilde{L}_2$ has the (analytic at $x = 0$) solution

$$\frac{(1-16x)\cdot P_{90}(x)\cdot K(x) \;\; + P_{91}(x)\cdot E(x)}{x^{13}\cdot (1-16x)^{15}\,(1-4x)^2\,(1-8x)\cdot A_{79}(L_{17})}, \tag{52}$$

where P_{90} and P_{91} are polynomials of degree 90 and degree 91, and where $K(x)$ and $E(x)$ are the complete elliptic integrals of the first and second kinds: $K(x) = {}_2F_1([1/2, 1/2], [1], 16x)$; $E(x) = {}_2F_1([1/2, -1/2], [1], 16x)$. The linear differential operator $\tilde{L}_2$ is equivalent to L_E, the differential operator of the elliptic integral $E(x)$. The solution (52) is for $\tilde{L}_2$ appearing in the factorization $\tilde{L}_2 \cdot L_{17}$, where all the linear differential operators are *monic*, and of minimal orders.

The rational solution of $Ext^2(\tilde{L}_2)$ reads

$$\frac{P_{93}(x)}{x^{24}\cdot (1-16x)^{26}\,(1-4x)^4\,(1-8x)^2\cdot A_{79}(L_{17})}, \tag{53}$$

which is, as it should be, the rational solution (51), divided by the square of the polynomial in front of the higher derivative of L_{17}. Having obtained the linear differential operator $\tilde{L}_2$, we can see that the roots of the polynomial $P_{93}(x)$ are *apparent singularities* of $\tilde{L}_2$: $A_{93}(\tilde{L}_2) = P_{93}(x)$.

We turn now to the linear differential operator L_{21}, where the same calculations (as for L_{23}) are performed on its *symmetric square*. We find that $Sym^2(L_{21})$ has a rational solution which reads

$$\begin{aligned} S_R(Sym^2(L_{21})) &= \frac{P_{714}(x)}{D_{529}(x)}, \\ D_{529}(x) &= x^{13}\cdot (1-16x)^{56}\,(1-4x)^{63}\,(1-9x)^{47}\,(1-25x)^{63}\,(1-x)^{47}, \\ &\quad \times (1-10x+29x^2)^{57}\,(1-x+16x^2)^{63}, \end{aligned} \tag{54}$$

where P_{714} is a polynomial of degree 714.

With the assumption that L_{21} is irreducible (see remark 8, below), the differential Galois group of L_{21} is seen to be included in the orthogonal group $SO(21, \mathbb{C})$.

6. Remarks

In this section we give some technical remarks on the computations displayed in section 4 and section 5. The way in which the operator $\tilde{L}_2$ has been obtained is applied to show that L_{21} is irreducible. By simple arguments based on the number of logarithmic solutions of maximum degree occurring in L_{21}, we exclude the possibility that L_{21} can be a symmetric power of an operator of smaller order.

Remark 1. From the factorization (5) we obtained $L_{12}^{(\mathrm{left})}$ by right division of L_{24} by the order-12 operator $\tilde{L}_1 \cdot L_{11}$ in its non-monic form, because this is more tractable. This means that the linear differential operator $L_{12}^{(\mathrm{left})}$ that we are using is in fact $L_{12}^{(\mathrm{left})} \cdot P_{12}^{(\mathrm{right})}$, where $P_{12}^{(\mathrm{right})}$ is the polynomial in front of the derivative D_x^{12} of $\tilde{L}_1 \cdot L_{11}$. To obtain and give the rational solution (44), we have corrected by dividing by the square of $P_{12}^{(\mathrm{right})}$. The rational solution (44) is the solution that we would have obtained if we had used the 'exterior power' command then the 'ratsols' command of DEtools in Maple, on the differential operator $L_{12}^{(\mathrm{left})}$ in the factorization $L_{12}^{(\mathrm{left})} \cdot \tilde{L}_1 \cdot L_{11}$ with $\tilde{L}_1 \cdot L_{11}$ in monic form.

Remark 2. Another remark is that we can safely use (and we did) an ODE of non-minimal order (in this case order 37) for $L_{12}^{(\mathrm{left})}$. The minimal order $L_{12}^{(\mathrm{left})}$ has at the higher derivative, besides the regular singularities, the polynomials $A_{131}(\tilde{L}_1 \cdot L_{11})^{11} \cdot A_{828}(L_{12}^{(\mathrm{left})})$, where $A_{828}(L_{12}^{(\mathrm{left})})$ is the degree 828 apparent polynomial of $L_{12}^{(\mathrm{left})}$. The whole coefficient is of degree 2317, to be compared with the degree of the coefficient in front of the higher derivative D_x^{37} of the non-minimal ODE which is 160. At the formal solutions generation step, there are only 12 solutions that appear. This is because, at the point $x = 0$, all the extra and spurious solutions correspond to critical exponents *that are not, in general, rational numbers*. In the modulo prime calculations, these exponents appear as roots of polynomials of degree 2 and higher. When the spurious exponent appears as an integer, we should redo the calculations with another prime, or change to another non-minimal order equation.

Remark 3. One notes that the rational solution given in (54) has not been corrected (as we did for (44), (53); see remark 1) by dividing by the square of the coefficient of the higher derivative of $\tilde{L}_2 \cdot L_{17}$, which reads

$$x^{12} \cdot (1-8x)(1-4x)^2(1-16x)^{11} \cdot A_{79}(L_{17})^2 \cdot A_{93}(\tilde{L}_2). \tag{55}$$

This type of correction is done when, in a given factorization, we deal with non-monic factors, which is more tractable for large operators. In the case of the rational solution (54), this corresponds to L_{21} annihilating the series obtained by acting with a *non-minimal* order-23 $\tilde{L}_2$ on the solution of L_{23}. To correct (54), as we did for (44) and (53), we should have obtained L_{21} by using the $\tilde{L}_2$ in *minimal order*. In this case, the length of the series for encoding L_{21} is very high. However, the occurrence of a rational solution to the symmetric square of L_{21} can be seen whether we use a minimal order ODE or a non-minimal order ODE[14] for $\tilde{L}_2$. Appendix E shows the results on $Sym^2(F_3)$ in the factorization $F_3 \cdot F_2$, which occurs in the linear differential operator of $\tilde{\chi}^{(5)}$ in both ways, i.e. the rational solution of $Sym^2(F_3)$, where F_3 is obtained from the factorization $F_3 \cdot F_2$ with F_2 of minimal order and with F_2 of non-minimal order.

Remark 4. The rational solution of the symmetric (or exterior) square of the operator L_q may carry some or all the regular singularities of L_q. In the expressions for the rational solutions given in (11), (24), (44) and (53), one notes that some *apparent polynomials* occur. These polynomials are apparent for the operator which is *at the right* of L_q. For the operator L_q,

[14] See appendix E, and [1], for a deeper understanding of the non-minimal order representation of an operator.

these polynomials are poles (see Appendix B.1 in [2]). For the rational solution given in (18) there is no such apparent polynomial because the factor L_2, at the right of $\tilde{L}_3$, has no apparent singularities. Note that the apparent polynomial of the operator itself appears in the denominator of the rational solution of the symmetric (or exterior) square when we deal with the adjoint of the operator. If we call ρ_j the local exponents at a regular (or apparent) singularity of L_q, at the same point the local exponents of adjoint(L_q) are $-\rho_j + q - 1$ (and $-\rho_j - q + 1$ for the singularity at infinity). An apparent singularity has the local exponents $0, 1, \ldots, q-2, q$, and for the adjoint one gets automatically a pole.

Remark 5. For all our operators L_q we have obtained a rational solution for the symmetric (or exterior) square having the maximum order $N_s = q(q+1)/2$ (or $N_e = q(q-1)/2$). For the irreducible operator L_q with differential Galois group in $SO(q, \mathbb{C})$ (resp. $Sp(q, \mathbb{C})$), it may happen that $Sym^2(L_q)$ (resp. $Ext^2(L_q)$) does not have the generic order, but has the order $N_s - 1$ (resp. $N_e - 1$). This means that there is no rational solution, but there is, instead, a relation between the solutions of L_q (or the solutions of L_q and their first derivative). Whether this drop in the order of $Sym^2(L_q)$ (resp. $Ext^2(L_q)$) will also be seen for $Sym^2(\text{adjoint}(L_q))$ (resp. $Ext^2(\text{adjoint}(L_q))$) depends on the intertwiner occurring in the homomorphism (9) (details will be given elsewhere).

Recall that we have not obtained a definitive conclusion on whether the exterior square (resp. symmetric square) of the adjoint of $L_{12}^{(\text{left})}$ (resp. L_{21}) has a drop in its order or annihilates a rational solution.

Remark 6. We have succeeded in factorizing the linear differential operator L_{23} via the rational solution of $Ext^2(L_{23})$. This, then, completes the factorization method that we put forward in section 4 of [2]. Recall that, in this method, we produce the general (analytic at 0) solution of L_{23} which begins as

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots \tag{56}$$

with the higher coefficients depending on the a_j ($j = 0, \ldots, 5$). We let the coefficients vary in the range $[1, p_r]$, p_r being the prime, until a differential equation of order less than 23 is found. If this happens, there is a right factor to L_{23}. This computation should have required the maximum time of $2\, p_r^5\, t_0$, as is necessary to produce $\tilde{L}_2$, if t_0 denotes the time needed to obtain L_{23}. The $d_{k,p}$ coefficients that we mentioned in the paragraph before (52) are precisely the actual values of the a_j in (56) for which the series (56) will be solution of an order-2 ODE.

Remark 7. Note that (47) and (48) are obvious properties of the exterior power, which states that if $L_q = L_{q-n} \cdot L_n$, then the exterior power $Ext^n(L_q)$ will have the order-1 right factor

$$Ext^n(L_n) = D_x - \frac{\mathrm{d}}{\mathrm{d}x} \ln(W(x)) \tag{57}$$

where $W(x)$ is the Wronskian of L_n. For our purposes, it happens that the suspected right factor (i.e. $\tilde{L}_2$) is of *order 2*, and we are dealing with the *second* exterior power. If our suspected right factor in L_{23} were not of order 2, we would still use

$$Ext^2(L_{23-n} \cdot L_n) = O_{253-n(n-1)/2} \cdot Ext^2(L_n) \tag{58}$$

which is a general identity and we would expect the rational solution of $Ext^2(L_{23})$ to come from $Ext^2(L_n)$ and use the recipe (see the paragraph before (52)) to obtain L_n.

Remark 8. The general (analytic at 0) solution of L_{21} begins as

$$b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + \cdots \tag{59}$$

with the higher coefficients depending on the b_j ($j = 1, \ldots, 5$). We have then four coefficients to vary, which is very time-consuming. The way in which we have factorized L_{23} can be repeated for L_{21}. Here also, once the coefficients in $\sum f_{k,p} S_k S_p$ have been fixed to encode the rational solution (54), we collect over the series S_j, which is in $\ln(x)^4$. We obtain two series (L_{21} has two series in $\ln(x)^4$; see remark 9). If there is a right factor to L_{21}, it should be of order $\geqslant 5$. If it exists, its solution is a combination of the two series. In this way, we have reduced the ODE search from varying four coefficients to varying one coefficient. The computation time is still high, but the calculation can be done in parallel on many subintervals of $[1, p_r]$. We find that for *any* combination the result is an ODE of order-21. This means that the order-21 differential operator L_{21} is *irreducible*.

Remark 9. A last remark on the irreducibility of the large order linear differential operators $L_{12}^{(\mathrm{left})}$ and L_{21} is worth making. The operator $L_{12}^{(\mathrm{left})}$ has been proved to be irreducible in [3]. We also showed, in [3], that *it is not* a symmetric power, or a symmetric product, of *smaller order operators* (see section 3.1 of [3]). We address the same issue for L_{21}, which, even if it is irreducible, can be built well from factors of lower order, as a symmetric power. The nth symmetric power of the generic order-q operator L_q is $(q+n-1)!/(n!(q-1)!)$. For operators L_{21} and $Sym^n(L_q)$ to be equivalent, where for the doublet (q, n) there are only the possibilities $(2, 20)$, $(3, 5)$ and $(6, 2)$, their singular behaviors at any singular point should match. The linear differential operator L_{21} has the same structure of solutions as the operator L_{23} (see section 4.3 of [4]), except for one analytical solution (at the origin) and one logarithmic solution which are solutions of the right factor $\tilde{L}_2$. The local structure of the formal solutions (around the origin) of L_{21} can be grouped as the following. There are two sets of five solutions, behaving as $\ln(x)^k$, $k = 0, \ldots, 4$, for each set. There are three sets of three solutions behaving as $\ln(x)^k$, $k = 0, \ldots, 2$, for each set. Finally, there are two non-logarithmic solutions behaving as $x^{-11/2}$ and $x^{-13/2}$. For the doublets (q, n), there is no possibility of obtaining *two solutions* with $\ln(x)^4$. The linear differential operator L_{21} *is not a symmetric power of an operator of smaller order*.

7. Conclusion

This work gives a final completion of previous studies on the *factors* of the linear differential operators associated with the n-particle contributions to the magnetic susceptibility of the Ising model (up to $n = 6$).

We have shown that the globally nilpotent G-operators corresponding to the small order ($\leqslant 6$) factor operators of the linear differential operators annihilating the multifold integrals $\chi^{(n)}$, associated with the n-particle contributions of the magnetic susceptibility of the Ising model ($n \leqslant 6$), are *homomorphic to their adjoint*. This 'duality' property of being self-adjoint up to operator homomorphisms is equivalent to the feature that their symmetric (or exterior) squares have rational solutions [16]. These operators are in selected differential Galois groups like $SO(q, \mathbb{C})$ and $Sp(q, \mathbb{C})$. This self-adjoint (up to operator equivalence) property means that the factor operators, which we already know to be derived from geometry, are 'special' globally nilpotent operators: they correspond to 'special geometries'.

Two large order operators occur in the factorization of the linear differential operators associated with $\chi^{(5)}$ and $\chi^{(6)}$. The order-12 operator L_{12}^{left} has an *exterior square* that annihilates a rational solution, and the order-21 operator L_{21} has a *symmetric square* which annihilates a rational solution. The different differential Galois groups are respectively in the symplectic group $Sp(12, \mathbb{C})$ and the orthogonal group $SO(21, \mathbb{C})$.

The two properties (homomorphism with the adjoint and occurrence of a rational solution for the symmetric, or exterior, square), should be verified for these large order operators $L_{12}^{\rm left}$ and L_{21}. Unfortunately, seeking for a homomorphism between these operators, and the corresponding adjoint, is well beyond the abilities of the present computer facilities. One may just imagine that this task could be achievable with dedicated programs, computing the homomorphism[15] modulo primes, with the knowledge that the coefficient of the higher derivative of the intertwiner is[16] the rational solution that we obtained (see (44) and (54)). In view of the results of appendix C, for many examples, and for the examples on the five-dimensional (and six-dimensional) face-centered cubic lattice Green function [21, 22] (namely G_6^{5Dfcc} and G_8^{6Dfcc} in [16]; see the intertwiners in equations (40) and (60) in [16]), one may concentrate on intertwiners with *even* orders. It is thus challenging to obtain the intertwiners occurring in the homomorphisms of $L_{12}^{\rm left}$ and L_{21} with their corresponding adjoint, and see whether a 'decomposition' (see equation (B.3) in [16]) in terms of the intertwiners occurs, the 'decomposition' being probably more complex.

Without waiting for this 'consolidation', we may conclude the intriguing selected character of the globally nilpotent operators annihilating the multifold integrals of the Ising model, which are all *diagonals of rational functions* [13], and we may conjecture that *all* the factors occurring in the differential operators for $\chi^{(n)}$ (for any n) *correspond to selected differential Galois groups*.

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Appendix A. Miscellaneous polynomials occurring in the main formulas of the paper

A.1. The polynomials $A_{37}(F_3)$, P_{34} and P_{53} for F_3

The apparent polynomial $A_{37}(F_3)$ of the (monic) order-3 operator F_3, arises in the expression for the rational solution (16) of the symmetric square of the adjoint of (the monic order-3 operator) F_3, as well as in (29). This degree 27 polynomial reads

$$\begin{aligned}
A_{37}(F_3) = \; & 5629\,499\,534\,213\,120\,x^{37} \; + 5348\,024\,557\,502\,464\,x^{36} \\
& -62\,874\,472\,922\,742\,784\,x^{35} + 339\,080\,589\,913\,096\,192\,x^{34} \\
& +132\,348\,214\,635\,397\,120\,x^{33} \; + 354\,600\,746\,294\,968\,320\,x^{32} \\
& +1383\,732\,497\,338\,073\,088\,x^{31} \; - 269\,118\,080\,922\,157\,056\,x^{30} \\
& -1021\,414\,905\,992\,970\,240\,x^{29} \; + 401\,943\,021\,895\,024\,640\,x^{28} \\
& +378\,516\,473\,892\,569\,088\,x^{27} \; - 379\,126\,125\,978\,189\,824\,x^{26} \\
& -181\,955\,521\,970\,962\,432\,x^{25} \; + 182\,991\,453\,503\,356\,928\,x^{24} \\
& +119\,809\,766\,351\,437\,824\,x^{23} \; - 34\,528\,714\,733\,649\,920\,x^{22}
\end{aligned}$$

[15] For such 'massive' formal calculations, switching to the linear differential systems associated with these operators is probably one way to calculate these homomorphisms. One first obtains the rational solutions of the symmetric or exterior squares of these differential systems. The intertwiners are, then, deduced from these rational solutions.
[16] See, for instance, relations (C.4) and (C.14).

$$
\begin{aligned}
&-46\,719\,523\,456\,286\,720\,x^{21} - 1865\,897\,472\,688\,128\,x^{20}\\
&+9861\,412\,040\,736\,768\,x^{19} + 1690\,374\,175\,916\,032\,x^{18}\\
&-1285\,664\,678\,690\,816\,x^{17} - 304\,716\,171\,767\,808\,x^{16}\\
&+112\,170\,181\,177\,344\,x^{15} + 30517814178816\,x^{14} - 7815\,766\,123\,264\,x^{13}\\
&-2274\,047\,571\,904\,x^{12} + 456\,062\,896\,896\,x^{11} + 150\,282\,885\,872\,x^{10}\\
&-10\,690\,267\,808\,x^{9} - 6048\,942\,832\,x^{8} - 486\,602\,112\,x^{7} + 33\,772\,908\,x^{6}\\
&+25\,075\,632\,x^{5} + 4670\,454\,x^{4} + 13440\,x^{3} - 69\,066\,x^{2} - 5169\,x - 63.
\end{aligned}
\tag{A.1}
$$

A degree 34 polynomial P_{34} arises in the expression (11) for the rational solution of the symmetric square of the order-3 operator F_3. This polynomial P_{34} reads

$$
\begin{aligned}
P_{34}(x) = {}& 17\,592\,186\,044\,416\,x^{34} - 8796\,093\,022\,208\,x^{33} + 204\,509\,162\,766\,336\,x^{32}\\
&-240\,793\,046\,482\,944\,x^{31} + 347\,033\,357\,516\,800\,x^{30}\\
&-356\,447\,925\,829\,632\,x^{29} + 307\,648\,507\,412\,480\,x^{28}\\
&+1547\,605\,565\,767\,680\,x^{27} - 1478\,894\,410\,530\,816\,x^{26}\\
&-3440\,457\,380\,003\,840\,x^{25} + 451\,333\,349\,965\,824\,x^{24}\\
&+3747\,745\,613\,479\,936\,x^{23} + 2236\,072\,096\,432\,128\,x^{22}\\
&-31\,693\,519\,978\,496\,x^{21} - 472\,806\,540\,705\,792\,x^{20}\\
&-202\,845\,119\,840\,256\,x^{19} - 55\,945\,141\,010\,432\,x^{18} - 6522\,043\,670\,528\,x^{17}\\
&+8027\,346\,038\,784\,x^{16} + 5016\,548\,481\,024\,x^{15} + 1158\,549\,638\,912\,x^{14}\\
&-15\,663\,757\,696\,x^{13} - 149\,163\,564\,992\,x^{12} - 59\,735\,088\,608\,x^{11}\\
&-2074\,333\,552\,x^{10} + 4173\,311\,968\,x^{9} + 738\,617\,492\,x^{8} - 85\,245\,032\,x^{7}\\
&-26\,786\,428\,x^{6} + 581\,796\,x^{5} + 383\,308\,x^{4} - 20\,652\,x^{3}\\
&-4867\,x^{2} + 338\,x + 49.
\end{aligned}
\tag{A.2}
$$

A degree 53 polynomial P_{53} arises in the expression (16) for the rational solution of the symmetric square of the adjoint of the (monic) order-3 operator F_3. This polynomial P_{53} reads

$$
\begin{aligned}
P_{53}(x) = {}& 5902\,958\,103\,587\,056\,517\,120\,x^{53} + 4722\,366\,482\,869\,645\,213\,696\,x^{52}\\
&+ 135\,675\,802\,662\,133\,752\,135\,680\,x^{51} + 36\,533\,776\,637\,981\,766\,975\,488\,x^{50}\\
&- 60\,743\,975\,313\,220\,946\,624\,512\,x^{49} + 3954\,166\,813\,899\,570\,825\,658\,368\,x^{48}\\
&+ 96\,486\,199\,280\,696\,075\,223\,040\,x^{47} - 869\,025\,933\,168\,471\,881\,809\,920\,x^{46}\\
&+ 17\,891\,408\,360\,681\,834\,540\,957\,696\,x^{45} + 19\,134\,090\,460\,943\,456\,531\,382\,272\,x^{44}\\
&- 6556\,910\,656\,212\,804\,697\,063\,424\,x^{43} - 18\,888\,872\,271\,338\,563\,015\,016\,448\,x^{42}\\
&- 9641\,687\,070\,213\,801\,940\,877\,312\,x^{41} - 856\,236\,460\,709\,396\,327\,956\,480\,x^{40}\\
&+ 1442\,025\,047\,697\,796\,450\,222\,080\,x^{39} + 3004\,016\,932\,710\,650\,818\,330\,624\,x^{38}\\
&+ 2353\,865\,809\,090\,149\,001\,199\,616\,x^{37} - 846\,024\,305\,296\,182\,175\,858\,688\,x^{36}\\
&- 1509\,872\,584\,625\,178\,282\,033\,152\,x^{35} + 296\,963\,035\,049\,372\,304\,801\,792\,x^{34}\\
&+ 832\,265\,748\,859\,080\,390\,213\,632\,x^{33} + 79\,111\,183\,944\,514\,552\,201\,216\,x^{32}\\
&- 245\,083\,727\,451\,855\,922\,397\,184\,x^{31} - 73\,512\,832\,264\,242\,582\,257\,664\,x^{30}
\end{aligned}
$$

$$+ 31\,367\,300\,451\,777\,147\,568\,128\,x^{29} + 12\,977\,767\,646\,670\,109\,016\,064\,x^{28}$$
$$- 3680\,779\,152\,078\,761\,099\,264\,x^{27} - 1138\,134\,172\,734\,191\,566\,848\,x^{26}$$
$$+ 939\,259\,567\,872\,233\,308\,160\,x^{25} + 292\,487\,910\,921\,964\,093\,440\,x^{24}$$
$$- 129\,084\,866\,249\,262\,874\,624\,x^{23} - 72\,153\,802\,925\,319\,249\,920\,x^{22}$$
$$+ 464\,226\,011\,870\,542\,848\,x^{21} + 8000\,870\,954\,669\,244\,416\,x^{20}$$
$$+ 1689\,242\,686\,839\,294\,720\,x^{19} - 289\,875\,180\,323\,084\,800\,x^{18}$$
$$- 171\,427\,111\,790\,469\,312\,x^{17} - 17\,240\,190\,260\,449\,408\,x^{16}$$
$$+ 4666\,400\,462\,438\,480\,x^{15} + 1816\,703\,798\,900\,448\,x^{14}$$
$$+ 258\,529\,109\,814\,976\,x^{13} - 29\,106\,463\,737\,504\,x^{12}$$
$$- 20\,951\,763\,420\,448\,x^{11} - 2341\,127\,444\,328\,x^{10} + 460\,438\,019\,724\,x^{9}$$
$$+ 115\,534\,150\,804\,x^{8} - 15\,491\,040\,x^{7} - 1792\,901\,976\,x^{6} - 94\,207\,344\,x^{5}$$
$$+ 6320\,658\,x^{4} - 571\,740\,x^{3} - 192\,705x^{2} - 10\,869x - 147. \qquad \text{(A.3)}$$

A.2. The polynomial P_{10} for $\tilde{L}_3$

A degree 10 polynomial P_{10} arises in the expression for the rational solution (22) of the symmetric square of the adjoint of the (monic) order-3 operator $\tilde{L}_3$. This polynomial P_{10} reads

$$P_{10}(x) = 19\,394\,461\,696\,x^{10} - 17\,411\,604\,480\,x^{9} + 6106\,742\,784\,x^{8} - 1095\,237\,312\,x^{7}$$
$$+158\,668\,656\,x^{6} - 36\,766\,920\,x^{5} + 7627\,535\,x^{4} - 900\,594\,x^{3}$$
$$+57\,342\,x^{2} - 1856\,x + 24. \qquad \text{(A.4)}$$

A.3. The polynomials $A_{26}(L_4)$ and P_{17} for L_4

The apparent polynomial $A_{26}(L_4)$ of the order-4 operator L_4 arises in the expression for the rational solution (25) of the exterior square of the adjoint of (the monic order-4 operator) L_4, as well as in the order-2 intertwiner (31). This degree 26 polynomial reads

$$A_{26}(L_4) = 521\,686\,412\,421\,099\,571\,093\,753\,036\,800\,x^{26}$$
$$- 724\,445\,324\,775\,545\,659\,452\,335\,063\,040\,x^{25}$$
$$+ 45\,081\,769\,872\,830\,521\,912\,080\,728\,064\,x^{24}$$
$$+ 616\,797\,192\,523\,902\,897\,669\,611\,192\,320\,x^{23}$$
$$- 636\,026\,962\,079\,787\,427\,490\,890\,252\,288\,x^{22}$$
$$+ 359\,505\,820\,412\,663\,945\,726\,355\,570\,688\,x^{21}$$
$$- 142\,807\,225\,508\,285\,034\,141\,616\,963\,584\,x^{20}$$
$$+ 43\,345\,424\,617\,004\,971\,574\,289\,235\,968\,x^{19}$$
$$- 10\,332\,892\,566\,359\,614\,848\,157\,876\,224\,x^{18}$$
$$+ 1953\,967\,934\,450\,852\,091\,348\,254\,720\,x^{17}$$
$$- 296\,338\,746\,597\,146\,803\,591\,135\,232\,x^{16}$$
$$+ 36\,761\,552\,740\,911\,534\,545\,901\,568\,x^{15}$$
$$- 3850\,023\,960\,384\,577\,768\,909\,952\,x^{14}$$
$$+ 354\,446\,803\,792\,968\,575\,565\,792\,x^{13} - 29\,645\,475\,671\,183\,771\,992\,224\,x^{12}$$

$$
\begin{aligned}
&+ 2252\,938\,824\,290\,334\,087\,840\,x^{11} - 148\,605\,250\,583\,921\,845\,896\,x^{10}\\
&+ 7727\,889\,974\,481\,947\,660\,x^{9} - 251\,264\,549\,473\,230\,968\,x^{8}\\
&- 1305\,110\,830\,870\,633\,x^{7} + 766\,418\,384\,173\,454\,x^{6}\\
&- 52\,582\,954\,690\,298\,x^{5} + 2099\,285\,510\,560\,x^{4}\\
&- 54\,037\,012\,120x^{3} + 873\,083\,400x^{2} - 7854\,000x + 2800. \qquad \text{(A.5)}
\end{aligned}
$$

A degree 17 polynomial P_{17} arises in the expression for the homomorphism (29) of $\tilde{L}_3$ with its adjoint, as well as in the rational solution (24) of the exterior square of L_4 and the rational solution (25) of the exterior square of the adjoint of (the monic order-4 operator) L_4. This polynomial P_{17} reads

$$
\begin{aligned}
P_{17}(x) = {}& 140\,082\,179\,425\,173\,504\,x^{17} - 496\,507\,256\,028\,790\,784\,x^{16}\\
&+705\,909\,942\,330\,064\,896\,x^{15} - 440\,315\,308\,230\,574\,080\,x^{14}\\
&+141\,123\,001\,405\,931\,520\,x^{13} - 25\,595\,376\,023\,494\,656\,x^{12}\\
&+4059\,589\,860\,750\,336\,x^{11} - 1133\,589\,089\,074\,624\,x^{10}\\
&+350\,453\,101\,085\,400\,x^{9} - 74\,115\,473\,257\,440\,x^{8}\\
&+10\,126\,459\,925\,120\,x^{7} - 904\,049\,598\,675\,x^{6} + 52\,738\,591\,890\,x^{5}\\
&-1959\,091\,320\,x^{4} + 43\,407\,720x^{3} - 502\,593x^{2} + 2548x - 12. \qquad \text{(A.6)}
\end{aligned}
$$

Appendix B. Some linear differential operators appearing in section 3

B.1. Linear differential operators for F_3

The order-2 differential operator R_2 occurring in the solution of F_3 given in (15) reads

$$
\begin{aligned}
R_2 = {}& -\frac{x \cdot (1-4x^2)(1-16x^2) \cdot P_{15}(x)}{D_5(x) \cdot A_7(F_2)} \cdot D_x^2\\
&-\frac{P_{20}(x)}{D_5(x) \cdot A_7(F_2)} \cdot D_x + \frac{8x \cdot P_{22}(x)}{D_5(x) \cdot A_7(F_2)},
\end{aligned}
$$

where $A_7(F_2)$ is the apparent polynomial of F_2 given in (12), and where

$$
\begin{aligned}
D_5(x) = {}& x \cdot (1-x)\,(1+2x)\,(4x^2+3x+1),\\
P_{15}(x) = {}& 2 + 9x - 99x^2 - 873x^3 - 1865x^4 + 12\,140x^5 + 83\,412x^6\\
&+238\,912x^7 + 375\,008x^8 - 1397\,504x^9 - 9548\,288x^{10} - 17\,188\,864x^{11}\\
&-7581\,696x^{12} + 2260\,992x^{13} - 7471\,104x^{14} - 786\,432x^{15},\\
P_{22}(x) = {}& 2 - 5x - 233x^2 + 43x^3 + 7343x^4 + 14\,408x^5 + 28\,660x^6\\
&+68224x^7 - 2196\,448x^8 - 13\,292\,608x^9 - 21\,440\,000x^{10} + 94\,341\,632x^{11}\\
&+562\,065\,408x^{12} + 700\,620\,800x^{13} - 1591\,803\,904x^{14} - 4451\,794\,944x^{15}\\
&-2017\,984\,512x^{16} + 467\,664\,896x^{17} - 2013\,265\,920x^{18} - 268\,435\,456x^{19}\\
&+67\,108\,864x^{20},\\
P_{18}(x) = {}& 4 + 13x + 87x^2 + 1472x^3 - 1950x^4 - 34\,896x^5 + 53\,220x^6\\
&+630\,696x^7 + 536\,416x^8 - 4436\,416x^9 - 21\,416\,192x^{10} - 32\,954\,368x^{11}\\
&+80\,510\,976x^{12} + 304\,701\,440x^{13} + 227\,115\,008x^{14} + 5636\,096x^{15}\\
&+151\,519\,232x^{16} + 29\,360\,128x^{17} - 16\,777\,216x^{18}.
\end{aligned}
$$

B.2. Linear differential operators for $\tilde{L}_3$

The order-2 linear differential operator R_2 occurring in the solution of $\tilde{L}_3$ given in (21) is

$$R_2 = \frac{x^2 \cdot Q_3(x) \cdot P_4(x)}{(1-4x)(1-16x)^7} \cdot D_x^2 + \frac{x \cdot Q_3(x) P_6(x)}{(1-4x)^2 (1-16x)^8} \cdot D_x + \frac{216x^3 \cdot (1-12x) \cdot Q_3(x) \cdot P_3(x)}{(1-4x)^3 (1-16x)^9}, \tag{B.1}$$

with

$$Q_3 = 5 - 160x + 1232x^2 - 1024x^3,$$
$$P_4 = 3 - 68x + 976x^2 - 2624x^3 - 61\,440x^4,$$
$$P_6 = 3 - 152x + 3528x^2 - 42\,384x^3 + 89\,024x^4 + 1966\,080x^5 - 5898\,240x^6,$$
$$P_3 = 3 - 92x + 1792x^2 + 4096x^3.$$

B.3. The linear differential operator L_4

The linear differential operator L_4 has been analyzed in [6] and was shown to be equivalent to a *Calabi–Yau equation* with the solution

$$_4F_3\left(\left[\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right], [1, 1, 1]; z\right), \tag{B.2}$$

where the argument z is an *algebraic pullback*, in the variable $w = s/2/(1+s^2)$ (with $s = \sinh(2K)$, where $K = J/kT$ is the Ising model coupling constant):

$$z = \left(\frac{1 + (1 - 16 \cdot w^2)^{1/2}}{1 - (1 - 16 \cdot w^2)^{1/2}}\right)^4 = s^8. \tag{B.3}$$

Note that the variable w deals equally well with the high and low regimes of temperature. One has another pullback which is $1/z = 1/s^8$.

B.3.1. The direct sum structure associated with L_4. When written in the variable $x = w^2$, the $_4F_3$ hypergeometric function *with either of the two pullbacks*, for instance the series with *integer coefficients*

$$_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1, 1, 1];\ \frac{1}{z}\right) = 1 + 16x^4 + 512x^5 + 11264x^6 + \cdots, \tag{B.4}$$

is annihilated by an *order-8* linear operator $L_8 = H_4^{(1)} \oplus H_4^{(2)}$ which is a *direct sum* of *two order-4* linear differential operators $H_4^{(1)}$ and $H_4^{(2)}$:

$$\begin{aligned} H_4^{(1)} = {} & x^3 \cdot (1-16x)^2 (1-8x)^4 \cdot D_x^4 \\ & +2x^2 \cdot (1-8x)^3 (1-16x)(512x^2 - 96x + 3) \cdot D_x^3 \\ & +x \cdot (1-8x)^2 (23\,3472x^4 - 83\,968x^3 + 10\,880x^2 - 512x + 7) \cdot D_x^2 \\ & -(1-8x)(589\,824x^5 - 266\,240x^4 + 40\,960x^3 - 3968x^2 + 144x - 1) \cdot D_x \\ & -256x \cdot (1 - 16x - 128x^2), \end{aligned}$$

$$\begin{aligned} H_4^{(2)} = {} & x^3 \cdot (1-16x)^4 (1-8x) \cdot D_x^4 \\ & +2x^2 \cdot (1-16x)^3 (640x^2 - 96x + 3) \cdot D_x^3 \\ & -x \cdot (1-16x)^2 (50\,688x^3 - 8768x^2 + 456x - 7) \cdot D_x^2 \\ & +(1-16x)(466\,944x^4 - 90\,112x^3 + 5952x^2 - 144x + 1) \cdot D_x \\ & +256x \cdot (1-8x)(192x^2 - 16x + 1). \end{aligned}$$

Each one corresponds to a Calabi–Yau ODE [6].

Note that these two order-4 operators are simply conjugated:

$$\frac{\sqrt{1-16\,x}}{1-8\,x} \cdot H_4^{(1)} = H_4^{(2)} \cdot \frac{\sqrt{1-16\,x}}{1-8\,x}. \tag{B.5}$$

The solution of $H_4^{(2)}$ analytic at $x = 0$ is the series with *integer coefficients*:

$$sol(H_4^{(2)}) = \frac{1}{4} \cdot \frac{\sqrt{1-16x}}{x \cdot z^{1/4}} \cdot {}_4F_3\Big(\Big[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big], [1, 1, 1];\ \frac{1}{z}\Big)$$
$$= 1 - 16\,x^2 - 256\,x^3 - 3568\,x^4 - 48\,640\,x^5 - 664\,832\,x^6 + \cdots. \tag{B.6}$$

The solution of $H_4^{(1)}$ analytic at $x = 0$ is the series with *integer coefficients*:

$$sol(H_4^{(1)}) = 1 + 16\,x^2 + 256\,x^3 + 3600\,x^4 + 49\,664\,x^5 + 687\,360\,x^6$$
$$+9596\,928\,x^7 + 135\,300\,368\,x^8 + 1925\,268\,480\,x^9 + \cdots. \tag{B.7}$$

The simple ${}_4F_3$ hypergeometric function (B.4) is actually equal to the half-sum $(sol(H_4^{(1)}) + sol(H_4^{(2)}))/2$, of the two solutions (B.6) and (B.7) of $H_4^{(2)}$ and $H_4^{(1)}$.

B3.2. The solution of the linear differential operators for L_4. The order-4 operator $H_4^{(2)}$ is *homomorphic to the order-4 operator* L_4, emerging as a factor operator for $\tilde{\chi}^{(6)}$:

$$S_3 \cdot H_4^{(2)} = L_4 \cdot R_3, \tag{B.8}$$

where S_3 and R_3 are two order-3 intertwiners. One immediately deduces the solution of L_4 given in terms of the intertwiner R_3 in (B.8) acting on the solution of the order-4 operator $H_4^{(2)}$:

$$sol(L_4) = R_3\big(sol\big(H_4^{(2)}\big)\big), \tag{B.9}$$

where the order-3 linear differential operator R_3 reads

$$36\,x^5 \cdot (1-4\,x)\,(1-16\,x)^{11} \cdot A_4(\tilde{L}_3) \cdot R_3$$
$$= -x^2 \cdot (1-8\,x)\,(1-16\,x)^6 \cdot Q_3 \cdot D_x^3 - x(1-16\,x)^5 \cdot Q_2 \cdot D_x^2$$
$$-(1-16\,x)^4 \cdot Q_1 \cdot D_x - 128\,x \cdot (1-16\,x)^3 \cdot Q_0, \tag{B.10}$$

$$Q_3 = 20 - 2270\,x + 106\,086\,x^2 - 2675\,757\,x^3 + 40\,471\,555\,x^4 - 389\,549\,218\,x^5$$
$$+2566\,958\,582\,x^6 - 13\,288\,554\,644\,x^7 + 53\,910\,201\,600\,x^8$$
$$-95\,886\,464\,512\,x^9 - 40\,752\,267\,264\,x^{10}$$
$$+93\,413\,441\,536\,x^{11} + 82\,141\,249\,536\,x^{12},$$

$$Q_2 = 60 - 8730\,x + 551\,602\,x^2 - 19\,952\,295\,x^3 + 459\,567\,769\,x^4 - 7113\,445\,902\,x^5$$
$$+76\,621\,809\,730\,x^6 - 596\,173\,812\,436\,x^7 + 3524\,748\,623\,424\,x^8$$
$$-16\,119\,878\,544\,384\,x^9 + 49\,591\,145\,041\,920\,x^{10} - 62\,942\,370\,168\,832\,x^{11}$$
$$-43\,186\,282\,037\,248\,x^{12} + 62\,103\,616\,487\,424\,x^{13} + 63\,084\,479\,643\,648\,x^{14},$$

$$Q_1 = 20 - 3710\,x + 303\,254\,x^2 - 14\,374\,525\,x^3 + 439\,222\,171\,x^4 - 9126\,353\,218\,x^5$$
$$+133\,114\,097\,446\,x^6 - 1396\,508\,587\,356\,x^7 + 10\,831\,258\,373\,280\,x^8$$
$$-63\,997\,739\,175\,680\,x^9 + 285\,429\,913\,462\,784\,x^{10} - 832\,850\,214\,682\,624\,x^{11}$$
$$+969\,294\,168\,981\,504\,x^{12} + 842\,807\,128\,358\,912\,x^{13}$$
$$-1089\,827\,550\,265\,344\,x^{14} - 1135\,520\,633\,585\,664\,x^{15},$$

$$Q_0 = 20 - 2590\,x + 145\,574\,x^2 - 4725\,757\,x^3 + 99\,952\,043\,x^4 - 1473\,719\,054\,x^5$$
$$+15\,848\,325\,886\,x^6 - 128\,583\,477\,160\,x^7 + 795\,236\,207\,808\,x^8$$
$$-3570\,673\,925\,376\,x^9 + 9940\,600\,639\,488\,x^{10} - 10\,105\,313\,820\,672\,x^{11}$$
$$-13\,061\,917\,245\,440\,x^{12} + 14\,868\,774\,125\,568\,x^{13}$$
$$+15\,771\,119\,910\,912\,x^{14},$$

the apparent polynomial $A_4(\tilde{L}_3)$ being given in (23).

Appendix C. Homomorphism with the adjoint for order-3 and order-4 operators

We show here, starting with *generic* (and irreducible) operators, the link between the homomorphism with the adjoint and the occurrence of a rational solution of the symmetric (or exterior) square of the differential operator for operators of order-3 and order-4.

For a linear differential operator of order q, the order of the intertwiner in an equivalence relation may reach the order $q - 1$. Section C.1 considers order-3 generic operator with intertwiners of order-2, 1 and 0 (i.e. a function). Section C.2 deals with order-4 generic operators with order-2 and order-0 intertwiners, and section C.3 is for the case of order-3 and order-1 intertwiners.

C.1. The order-3 linear differential operator

With the generic order-3 differential operator L_3,

$$L_3 = D_x^3 \;\; + p_2(x) \cdot D_x^2 \;\; + p_1(x) \cdot D_x \;\; + p_0(x), \tag{C.1}$$

and the order-2 differential operator

$$R_2 = a_2(x) \cdot D_x^2 + a_1(x) \cdot D_x + a_0(x), \tag{C.2}$$

one demands that the relation

$$L_3 \cdot R_2 = \mathrm{adjoint}(R_2) \cdot \mathrm{adjoint}(L_3), \tag{C.3}$$

be fulfilled, which means that L_3 is homomorphic to its adjoint.

Zeroing the expressions in front of each derivative D_x^j in (C.3) gives a set of equations which are solved as

$$a_2(x) = sol(Sym^2(L_3)) \tag{C.4}$$

$$a_1(x) = -p_2(x) \cdot a_2(x) \;\; - \frac{1}{2}\,\frac{\mathrm{d}a_2(x)}{\mathrm{d}x}, \tag{C.5}$$

and

$$a_0(x) = N_5 \cdot a_2(x). \tag{C.6}$$

The order-5 differential operator N_5 is such that

$$C_1^{(3)} \cdot N_5 = 9\,D_x^5 \; + 30\,p_2(x) \cdot D_x^4 \, + Q_3 \cdot D_x^3 \, + Q_2 \cdot D_x^2 \, + Q_1 \cdot D_x \, + \, Q_0, \tag{C.7}$$

where

$$Q_3 = 25\,p_2(x)^2 \;\; + 45\,p_1(x) \;\; + 15\,\frac{\mathrm{d}p_2(x)}{\mathrm{d}x},$$

$$Q_2 = 75\,p_1(x)\,p_2(x) \;\; + 45\,p_0(x) \;\; + 45\,\frac{\mathrm{d}p_1(x)}{\mathrm{d}x},$$

$$Q_1 = 36\,p_1(x)^2 \;\; - 4\,p_2(x)^4 \;\; + 42\,p_2(x)\,p_0(x) \;\; + 22\,p_2(x)^2\,p_1(x)$$

$$+9\,\frac{d^2 p_1(x)}{dx^2}\;+63\,\frac{dp_0(x)}{dx}\;-18\,\frac{dp_2(x)}{dx}\,p_2(x)^2\;+48\,p_2(x)\,\frac{d}{dx}p_1(x)$$
$$-3\,p_1(x)\,\frac{dp_2(x)}{dx}\;-9\,p_2(x)\,\frac{d^2 p_2(x)}{dx^2},$$

$$Q_0 = 36\,p_1(x)^2 p_2(x)\;-36\,p_1(x)\,p_0(x)\;-8\,p_1(x)\,p_2(x)^3\;+8\,p_2(x)^2 p_0(x)$$
$$-18\,\frac{d^2 p_2(x)}{dx^2}\,p_1(x)\;+18\,\frac{d^2 p_2(x)}{dx^2}\,\frac{dp_2(x)}{dx}\;+36\left(\frac{d}{dx}\,p_2(x)\right)^2 p_2(x)$$
$$+102\,\frac{dp_2(x)}{dx}\,p_0(x)\;-54\,\frac{dp_2(x)}{dx}\,\frac{dp_1(x)}{dx}\;+8\,\frac{dp_2(x)}{dx}\,p_2(x)^3\;+18\,\frac{d^2 p_0(x)}{dx^2}$$
$$+42\,p_2(x)\,\frac{dp_0(x)}{dx}\;+54\,p_1(x)\,\frac{dp_1(x)}{dx}\;-72\,\frac{dp_2(x)}{dx}\,p_1(x)\,p_2(x),$$

and

$$C_1^{(3)} = 36\,p_1(x)\,p_2(x)\;-108\,p_0(x)\;-8\,p_2(x)^3\;-36\,p_2(x)\,\frac{dp_2(x)}{dx}$$
$$+54\,\frac{dp_1(x)}{dx}\;-18\,\frac{d^2 p_2(x)}{dx^2}.$$

The last expression, when $C_1^{(3)} = 0$, is the condition for L_3 to have an order-5 symmetric square instead of an order-6 one.

From the expression (C.4), one sees that if $Sym^2(L_3)$ has a *rational solution*, one may take $a_2(x)$ *as this solution*, and automatically the expressions for $a_1(x)$ and $a_0(x)$ will be rational. If $Sym^2(L_3)$ has no rational solution (assume that $Sym^2(L_3)$ is irreducible), one may still take for $a_2(x)$ *any solution* of $Sym^2(L_3)$ and with the corresponding $a_1(x)$, $a_0(x)$, the relation (C.3) will be verified. But now, the intertwiner R_2 is *no longer over the rationals*.

Note that one may use (C.4) and (C.6) with the derivative, to reduce (C.6) to the following inhomogeneous differential equation:

$$6\,\frac{da_0(x)}{dx}\;+4\,p_2(x)\cdot a_0(x) \;=\; E_3\cdot a_2(x) \tag{C.8}$$

where the order-3 differential operator E_3 is

$$E_3 = D_x^3\;+3\,p_2(x)\cdot D_x^2+\left(4\,p_1(x)\;+2\,p_2(x)^2-3\,\frac{dp_2x)}{dx}\right)\cdot D_x$$
$$+4\,p_1(x)\,p_2(x)\;-4\,p_0(x)\;-6\,\frac{d^2 p_2(x)}{dx^2}\;+6\,\frac{dp_1(x)}{dx}\;-4\,p_2(x)\,\frac{dp_2(x)}{dx}.$$

The way to obtain $a_0(x)$, via (C.6), is more tractable, since this amounts to taking derivatives of the rational solution $a_2(x)$. The route via (C.8) calls for an integration, and re-injection in (C.3), to fix the constants of integration.

Instead of an intertwiner R_2 of order-2, let us consider the situation with an order-1 intertwiner, $a_1(x)\cdot D_x\;+a_0(x)$. In this case, one obtains

$$a_0(x) = -p_2(x)\cdot a_1(x)\;-\frac{da_1(x)}{dx}\quad \text{and:}\quad a_1(x) = sol(Ext^2(L_3)). \tag{C.9}$$

Recall that (with $W_L(x)$ the Wronskian of L_3)

$$Ext^2(L_3)\cdot W_L(x) = W_L(x)\cdot \mathrm{adjoint}(L_3). \tag{C.10}$$

Since L_3 is irreducible, $a_1(x)$ cannot be rational. Therefore, there is no homomorphism between L_3 and its adjoint with an order-1 intertwiner over the rationals. For an order-0 intertwiner, i.e. a function $a_0(x)$, one obtains

$$a_0(x) = W_L(x)^{2/3}\qquad \text{and}\qquad C_1^{(3)} = 0, \tag{C.11}$$

where $W_L(x)$ is the Wronskian of L_3. The condition $C_1^{(3)} = 0$ makes the symmetric square of L_3 of order-5, and L_3 is the symmetric square of an order-2 differential operator.

C.2. The order-4 linear differential operator

What we have done for the generic order-3 linear differential operator can be repeated for a generic order-4 operator L_4.

For the order-4 differential operator,

$$L_4 = D_x^4 \;+ p_3(x) \cdot D_x^3 \;+ p_2(x) \cdot D_x^2 \;+ p_1(x) \cdot D_x \;+ p_0(x), \tag{C.12}$$

and an order-2 operator as in (C.2), the relation

$$L_4 \cdot R_2 = \mathrm{adjoint}(R_2) \cdot \mathrm{adjoint}(L_4), \tag{C.13}$$

is solved to give

$$a_2(x) = sol(Ext^2(L_4)), \qquad a_1(x) = -p_3(x) \cdot a_2(x) - \frac{\mathrm{d}a_2(x)}{\mathrm{d}x}, \tag{C.14}$$

and

$$a_0(x) = N_5 \cdot a_2(x). \tag{C.15}$$

The order-5 differential operator N_5 is such that

$$C_1^{(4)} \cdot N_5 = 4\,D_x^5 \,+ 10\,p_3(x) \cdot D_x^4 \,+ Q_3 \cdot D_x^3 \,+ Q_2 \cdot D_x^2 \;+ Q_1 \cdot D_x \,+ Q_0, \tag{C.16}$$

where

$$Q_3 = 7\,(p_3(x))^2 \;+ 8\,p_2(x) \;+ 8\,\frac{\mathrm{d}p_3(x)}{\mathrm{d}x},$$

$$Q_2 = 14\,p_2(x)\,p_3(x) \;- 4\,p_1(x) \,+ 16\,\frac{\mathrm{d}p_2(x)}{\mathrm{d}x},$$

$$\begin{aligned} Q_1 = {} & 4\,p_2(x)^2 \;- 16\,p_0(x) \;\;+ 5\,p_3(x)^2\,p_2(x) \;- 2\,p_3(x)\,p_1(x) \;- p_3(x)^4 \\ & +8\,\frac{\mathrm{d}p_1(x)}{\mathrm{d}x} \;+ 4\,\frac{\mathrm{d}^2 p_2(x)}{\mathrm{d}x^2} \;- 6\,p_3(x)^2 \frac{\mathrm{d}p_3(x)}{\mathrm{d}x} \;+ 14\,p_3(x)\,\frac{\mathrm{d}p_2(x)}{\mathrm{d}x} \\ & -4\,p_3(x) \frac{\mathrm{d}^2 p_3(x)}{\mathrm{d}x^2}, \end{aligned}$$

$$\begin{aligned} Q_0 = {} & 4\,p_3(x)\,p_2(x)^2 \;- p_2(x)\,p_3(x)^3 \;+ p_3(x)^2\,p_1(x) \;- 8\,p_0(x) p_3(x) \\ & -4\,p_2(x)\,p_1(x) \;+ 4\,\frac{\mathrm{d}p_3(x)}{\mathrm{d}x}\frac{\mathrm{d}^2 p_3(x)}{\mathrm{d}x^2} \;- 4\,p_2(x)\,\frac{\mathrm{d}^2 p_3(x)}{\mathrm{d}x^2} \;+ 4\,\frac{\mathrm{d}^2 p_1(x)}{\mathrm{d}x^2} \\ & +6\,\left(\frac{\mathrm{d}p_3(x)}{\mathrm{d}x}\right)^2 p_3(x) \;+ p_3(x)^3\,\frac{\mathrm{d}p_3(x)}{\mathrm{d}x} \;- 10\,p_3(x)\,p_2(x)\,\frac{\mathrm{d}p_3(x)}{\mathrm{d}x} \\ & +8\,\frac{\mathrm{d}p_3(x)}{\mathrm{d}x}\,p_1(x) \;- 8\,\frac{\mathrm{d}p_3(x)}{\mathrm{d}x}\frac{\mathrm{d}p_2(x)}{\mathrm{d}x} \;+ 8\,p_2(x) \frac{\mathrm{d}p_2(x)}{\mathrm{d}x} \\ & +6\,p_3(x)\,\frac{\mathrm{d}p_1(x)}{\mathrm{d}x} \;- 8\,\frac{\mathrm{d}p_0(x)}{\mathrm{d}x}, \end{aligned}$$

and

$$C_1^{(4)} = 4\,p_2(x)\;p_3(x) \;- 8\,p_1(x) \;- p_3x)^3 \;+ 8\,\frac{\mathrm{d}p_2(x)}{\mathrm{d}x} \;- 4\,\frac{\mathrm{d}^2 p_3(x)}{\mathrm{d}x^2} - 6 p_3(x) \frac{\mathrm{d}p_3(x)}{\mathrm{d}x}. \tag{C.17}$$

Here also, $C_1^{(4)} = 0$ is the so-called [16] 'Calabi–Yau condition'. When verified, $Ext^2(L_4)$ is of order-5, instead of being of order-6.

All that has been said for L_3 (on the rationality of the coefficients) holds. In particular, (C.15) can be reduced to the inhomogeneous linear differential equation

$$2\,\frac{\mathrm{d}a_0(x)}{\mathrm{d}x} \;+ p_3(x) \cdot a_0(x) = E_3 \cdot a_2(x) \tag{C.18}$$

with

$$E_3 = D_x^3 \; + 2\, p_3(x) \cdot D_x^2 \; + \; (p_3(x)^2 \; + p_2(x)) \cdot D_x$$
$$+ p_2(x)\, p_3(x) \; - p_1(x) \; + 2 \frac{dp_2(x)}{dx} \; - 2\, \frac{d^2 p_3(x)}{dx^2} \; - p_3(x)\, \frac{dp_3(x)}{dx}.$$

Like for L_3, one may consider an order-0 intertwiner $a_0(x)$ for R_2. In this case, one obtains

$$a_0(x) = W_{L_4}(x)^{1/2} \qquad \text{and} \qquad C_1^{(4)} = 0. \tag{C.19}$$

$W_{L_4}(x)$ is the Wronskian of L_4 and $C_1^{(4)} = 0$ is the Calabi–Yau condition [16] given in (C.17); if it is fulfilled, then L_4 has an order-5 exterior square instead of an order-6 one.

C.3. The order-4 differential operator and $SO(4, \mathbb{C})$

We consider the equivalence relation

$$L_4 \cdot R_3 = \mathrm{adjoint}(R_3) \cdot \mathrm{adjoint}(L_4), \tag{C.20}$$

with R_3 of order-3:

$$R_3 = a_3(x) \cdot D_x^3 \; + a_2(x) \cdot D_x^2 \; + a_1(x) \cdot D_x \; + a_0(x). \tag{C.21}$$

One obtains

$$a_3(x) = sol(Sym^2(L_4)), \qquad a_2(x) = -p_3(x) \cdot a_3(x) \; - \frac{1}{2}\, \frac{da_3(x)}{dx}. \tag{C.22}$$

The coefficient $a_1(x)$ is a solution of the order-3 inhomogeneous differential equation[17]

$$N_3 \cdot a_1(x) = N_5 \cdot a_3(x), \tag{C.23}$$

where the order-3 operator N_3 is

$$N_3 = 5\, D_x^3 \; + \frac{15}{2}\, p_3(x) \cdot D_x^2 \; + \left(\frac{9}{2}\, \frac{dp_3(x)}{dx} \; + 3\, p_3(x)^2 \; + 2\, p_2(x) \right) \cdot D_x$$
$$+ \left(3\, \frac{dp_2(x)}{dx} \; - 2\, p_1(x) \; + 2\, p_2(x)\, p_3(x) \right), \tag{C.24}$$

and the order-5 operator N_5 reads

$$N_5 = D_x^5 \; + \frac{15}{4}\, p_3(x) \cdot D_x^4 \; + \left(\frac{17}{4}\, p_3(x)^2 - \frac{7}{4}\, \frac{dp_3(x)}{dx} \; + \frac{9}{2}\, p_2(x) \right) \cdot D_x^3$$
$$+ Q_2 \cdot D_x^2 + Q_1 \cdot D_x + Q_0, \tag{C.25}$$

with

$$Q_2 = \frac{3}{2}\, p_3(x)^3 \; + \frac{17}{2}\, p_2(x)\, p_3(x) \; - \frac{5}{2}\, p_1(x) \; - \frac{45}{2}\, \frac{d^2 p_3(x)}{dx^2}$$
$$- \frac{21}{4} p_3(x) \frac{dp_3(x)}{dx} + 15 \frac{dp_2(x)}{dx}, \tag{C.26}$$

$$Q_1 = 2\, p_2(x)^2 \; - 2\, p_1(x)\, p_3(x) \; + 4\, p_2(x)\, p_3(x)^2 \; - 4\, p_0(x) \; - \frac{27}{4}\, (\frac{dp_3(x)}{dx})^2$$
$$- \frac{105}{4}\, p_3(x)\, \frac{d^2 p_3(x)}{dx^2} \; - \frac{9}{2}\, p_3(x)^2\, \frac{dp_3(x)}{dx} \; + \frac{3}{2}\, p_2(x)\, \frac{dp_3(x)}{dx}$$
$$+ \frac{33}{2}\, p_3(x)\, \frac{dp_2(x)}{dx} \; + 15\, \frac{d^2 p_2(x)}{dx^2} \; - \frac{55}{2}\, \frac{d^3 p_3(x)}{dx^3} \; - \frac{3}{2}\, \frac{dp_1(x)}{dx}, \tag{C.27}$$

[17] Note that $a_1(x)$ is also given by $a_1(x) = N_9 \cdot a_3(x)$, where N_9 is an order-9 linear differential operator. Once $a_3(x)$ is rational, $a_1(x)$ will be rational.

$$Q_0 = 2\,p_3(x)\,(p_2(x))^2\,-2\,p_0(x)\,p_3(x)\,\,-2\,p_1(x)\,p_2(x)\,\,-10\,\frac{d^4p_3(x)}{dx^4}$$
$$-3\,\frac{dp_0(x)}{dx}\,\,-15\,p_3(x)\,\frac{d^3p_3(x)}{dx^3}\,\,-\frac{3}{2}\,\frac{dp_2(x)}{dx}\,\frac{dp_3(x)}{dx}\,\,+5\,p_2(x)\frac{d}{dx}p_2(x)$$
$$-9\,\frac{dp_3(x)}{dx}\frac{d^2p_3(x)}{dx^2}\,\,+4\,p_1(x)\,\frac{dp_3(x)}{dx}\,\,-4\,p_2(x)\,\frac{d^2p_3(x)}{dx^2}\,\,+5\,\frac{d^3p_2(x)}{dx^3}$$
$$+\frac{15}{2}\,p_3(x)\,\frac{d^2p_2(x)}{dx^2}\,\,-6\,p_3(x)^2\,\frac{d^2p_3(x)}{dx^2}\,\,-4\,p_2(x)\,p_3(x)\,\frac{dp_3(x)}{dx}$$
$$+3\,\frac{dp_2(x)}{dx}\,\,p_3(x)^2. \qquad (C.28)$$

The coefficient $a_0(x)$ is given by

$$a_0(x) = -p_3(x)\cdot a_1(x) - \frac{3}{2}\frac{da_1(x)}{dx} + \frac{1}{4}E_3\cdot a_3(x), \qquad (C.29)$$

where the order-3 operator E_3 reads

$$E_3 = D_x^3\,\,+3\,p_3(x)\cdot\,D_x^2\,\,+\left(4\,p_2(x)\,\,-7\,\frac{dp_3(x)}{dx}\,\,+2\,p_3(x)^2\right)\cdot\,D_x$$
$$+4p_2(x)p_3(x) - 4p_1(x) - 16\frac{d^2p_3(x)}{dx^2} + 10\frac{dp_2(x)}{dx} - 8p_3(x)\frac{dp_3(x)}{dx}. \qquad (C.30)$$

For the equivalence between the differential operator L_4 and its adjoint with an order-3 intertwiner *over the rationals* to exist, it is the *symmetric square* of L_4 that should annihilate a rational solution.

We consider, now, the case of an order-1 intertwiner $R_1 =\; a_1(x)\cdot\,D_x\,+a_0(x)$ in

$$L_4\cdot\,R_1 = \text{adjoint}(R_1)\cdot\,\text{adjoint}(L_4). \qquad (C.31)$$

One obtains for $a_0(x)$

$$a_0(x) = -p_3(x)\cdot\,a_1(x)\,\,-\frac{3}{2}\,\frac{da_1(x)}{dx}, \qquad (C.32)$$

and

$$a_1(x) = sol(E_2), \qquad (C.33)$$

where the order-2 linear differential operator E_2 reads

$$E_2 = C_1^{(4)}\cdot D_x^2 + \tfrac{1}{105}Q_1\cdot D_x + \tfrac{2}{105}Q_0, \qquad (C.34)$$

$$Q_1 = 212\,\,p_2(x)\,p_3(x)^2\,\,-66\,p_3(x)^4\,\,-440\,p_1(x)\,p_3(x)\,\,+144\,p_2(x)^2\,-1600\,p_0(x),$$
$$+54\,(\frac{dp_3(x)}{dx})^2\,\,-292\,p_2(x)\frac{dp_3(x)}{dx}\,\,-60\,\frac{d^3p_3(x)}{dx^3}\,\,-40\,\frac{d^2p_2(x)}{dx^2}$$
$$+520\,\frac{dp_1(x)}{dx}\,\,+580\,\frac{dp_2(x)}{dx}\,p_3(x)\,\,-423\,p_3(x)^2\,\frac{dp_3(x)}{dx}$$
$$-390\,p_3(x)\,\frac{d^2p_3(x)}{dx^2},$$

$$Q_0 = -22\,p_2(x)\,p_3(x)^3\,\,+22\,p_1(x)\,p_3(x)^2\,\,-72\,p_1(x)\,p_2(x)\,\,+72\,p_2(x)^2\,p_3(x)$$
$$-400\,p_0(x)\,p_3(x)\,\,-13\,\frac{dp_2(x)}{dx}\,\,p_3(x)^2\,\,+18\,\frac{dp_2(x)}{dx}\,\,\frac{dp_3(x)}{dx}\,-20\,\frac{d^3p_2(x)}{dx^3}$$
$$-50\,\frac{d^2p_2(x)}{dx^2}\,p_3(x)\,\,+180\,\frac{dp_1(x)}{dx}\,p_3(x)\,\,-52\,p_1(x)\,\frac{dp_3(x)}{dx}$$

$$-128\, p_3(x)\, p_2(x)\, \frac{dp_3(x)}{dx} \;-200\, \frac{dp_0(x)}{dx} \;+80\, \frac{d^2p_1(x)}{dx^2} \;-80\, p_2(x)\, \frac{d^2p_3(x)}{dx^2}$$
$$+108\, \frac{dp_2(x)}{dx}\, p_2(x).$$

Here, our analysis is a little bit incomplete. For all the examples found so far, this situation corresponds to a symmetric square of order-9 instead one of order-10. We do not know how to prove that (C.34) should have a rational solution in this case. One may just imagine that this is provable if we use the symmetric Calabi–Yau condition (the corresponding condition for order-4 operators that makes the symmetric square of order-9 instead of being of order-10) to factorize, or reduce to order 1, the operator (C.34). Unfortunately, this condition for an order-4 operator is very large, and nonlinear, in the coefficients of L_4 and their derivatives: it is a sum of 3548 monomials and has degree 12 in the coefficients (and their derivatives) of L_4.

Appendix D. Quadratic (or alternating) invariant forms

The differential Galois group of the order-3 linear differential operator F_3 occurring in $\tilde{\chi}^{(5)}$ is in the orthogonal group $SO(3, \mathbb{C})$. We denote by X_0 any of its formal solutions at $x = 0$ and by X_1 (resp. X_2) the first (resp. second) derivative. The first integral of F_3 reads

$$Q(X_0, X_1, X_2) = \mathrm{Cst}, \tag{D.1}$$

with

$$\begin{aligned}
Q(X_0, X_1, X_2) = & \frac{x \cdot (1+4x)^2 (1-4x)^6 \cdot P_{101}(x)}{A_7(F_2)^4 \cdot A_{37}(F_3)^2} \cdot X_0^2 \\
& + \frac{x^3 \cdot (1+4x)^4 (1-4x)^8 (1+2x)^2 \cdot P_{81}(x)}{A_7(F_2)^2 \cdot A_{37}(F_3)^2} \cdot X_1^2 \\
& + \frac{x^5 \cdot (1+4x)^6\, (1-4x)^{10}\, (1-x)^2\, (1+2x)^4\, (1-2x)^2\, (1+3x+4x^2)^2 \cdot P_{53}(x)}{A_{37}(F_3)^2} \cdot X_2^2 \\
& - \frac{x^4 \cdot (1+4x)^5\, (1-4x)^9\, (1-x)\, (1+2x)^3\, (1-2x)\, (1+3x+4x^2) \cdot P_{67}(x)}{A_7(F_2) \cdot A_{37}(F_3)^2} \cdot X_1 X_2 \\
& + \frac{x^3 \cdot (1+4x)^4\, (1-4x)^8\, (1+2x)^2\, (1-x)\, (1-2x)\, (1+3x+4x^2) \cdot P_{77}(x)}{A_7(F_2)^2 \cdot A_{37}(F_3)^2} \cdot X_0 X_2 \\
& - \frac{x^2 \cdot (1+4x)^3\, (1-4x)^7\, (1+2x) \cdot P_{91}(x)}{A_7(F_2)^3 \cdot A_{37}(F_3)^2} \cdot X_0 X_1 .
\end{aligned} \tag{D.2}$$

The $P_j(x)$ are polynomials of degree j. The numerical value of Cst depends on the solution X_0 considered.

The differential Galois group of the order-4 linear differential operator L_4 occurring in $\tilde{\chi}^{(6)}$ is in the symplectic group $Sp(4, \mathbb{C})$. We denote by X_0 any of its formal solutions at $x = 0$ and by X_j, the jth derivative up to $j = 3$. Y_0 is another solution with its derivatives Y_j, $j = 1, \ldots, 3$. We define

$$w_{i,j} = X_i \cdot Y_j - X_j \cdot Y_i, \qquad i = 0, \ldots, 3, \qquad j = 0, \ldots, 3, \qquad j > i. \tag{D.3}$$

The first integral of the order-4 differential operator L_4 reads

$$Q(X_0, X_1, X_2, X_3, Y_0, Y_1, Y_2, Y_3) = \mathrm{Cst}, \tag{D.4}$$

where

$$Q = \frac{x^7\, (1-16x)^{10} \cdot P_{36}}{A_{26}(L_4) A_4(\tilde{L}_3)^3} \cdot w_{0,1} \;+ \frac{x^8 \cdot (1-16x)^{11}\, (1-8x) \cdot P_{34}}{A_{26}(L_4) A_4(\tilde{L}_3)^3} \cdot w_{0,2}$$

$$+\frac{x^9 \cdot (1-16x)^{12}\,(1-4x)(1-8x)\cdot P_{28}}{A_{26}(L_4)\cdot A_4(\tilde{L}_3)^2}\cdot w_{0,3}$$
$$+\frac{x^9 \cdot (1-16x)^{12}\,(1-8x)\cdot P_{29}}{A_{26}(L_4)\cdot A_4(\tilde{L}_3)^2}\cdot w_{1,2}$$
$$+\frac{x^{10} \cdot (1-16x)^{13}\,(1-4x)\,(1-8x)\cdot P_{23}}{A_{26}(L_4)\cdot A_4(\tilde{L}_3)}\cdot w_{1,3}$$
$$+\frac{x^{11}\,(1-16x)^{14}\,(1-4x)^2\,(1-8x)\cdot P_{17}}{A_{26}(L_4)}\cdot w_{2,3}.$$

The $P_j(x)$ are polynomials of degree j. The numerical value of Cst depends on the solutions X_0 and Y_0 considered.

Appendix E. Minimal order versus non-minimal order: $F_3 \cdot F_2$

For a given series $S(x)$, the solution of a Fuchsian operator of order q, one can consider the family of *Fuchsian* linear differential operators of order $Q > q$ annihilating this series, the degree D of the polynomial coefficients being taken as small as possible [1]. The Fuchsian operator of minimal order annihilating this series is unique, and right divides all the operators of the previous family. Let us denote by N the minimum number of coefficients needed to find the linear ODE in this family within the constraint that the order Q and degree D are given. We found (as an experimental result) that N, the order Q and the degree D of the operators in this family are related by a *linear relation*, which we called an 'ODE formula' (see section 3.1 of [1]).

Assume that we have a series $S(x)$, known modulo a prime, for which we have produced an ODE whose 'ODE formula' [1] reads

$$N = 14\,Q \;+5\,D \;-14 = (Q+1)\cdot\,(D+1) \;\;-f, \tag{E.1}$$

where Q is the order and D is the degree of $P_j(x)$, $j = 0, \ldots, Q$, of the ODE written in the form $P_Q(x)\cdot x^Q\cdot D_x^Q + \cdots + P_0(x)$, and f is the number of the independent non-minimal order ODEs with Q and D such that $(Q+1)(D+1)$ is less than the available number of the series terms of $S(x)$ (see [1] and Appendix B of [2] for the details). From the 'ODE formula' [1], we see that the minimal order ODE annihilating $S(x)$ is of order-5, and we call it $\mathcal{F}_5$. Among the many non-minimal order ODEs there is one which needs the lesser terms in $S(x)$ to be produced. This particular *non-minimal order* linear ODE is the 'optimum ODE', and has, for this example, the order-8, i.e. $(Q_0 = 8,\ D_0 = 23,\ f_0 = 3)$. In the calculations we may use this order-8 ODE and continue to call it $\mathcal{F}_5$, but in this section we call it $\mathcal{F}_5^{nm}$. Obviously, $\mathcal{F}_5$ is a right factor of $\mathcal{F}_5^{nm}$.

The local exponents at $x = 0$ of this linear ODE are $0,\ 0,\ 1,\ 2,\ 3$. These exponents are obtained whatever the order of the ODE is, in minimal order (i.e. $\mathcal{F}_5$), or in non-minimal order (i.e. $\mathcal{F}_5^{nm}$). The three exponents corresponding to the three extra solutions of the order-8 linear ODE $\mathcal{F}_5^{nm}$ appear as *non-rational numbers*. Recall that we are dealing with globally nilpotent differential equations [5]. They are therefore necessarily Fuchsian and have *rational local exponents at all of the singular points*.

The general (analytic at $x = 0$) solution $\tilde{S}(x)$ of $\mathcal{F}_5$ (or $\mathcal{F}_5^{nm}$) depends on two free coefficients, say α and β. The series $S(x)$ is a particular combination of $\tilde{S}(x)$. With $Q = Q_0 = 8$, $D = D_0 = 23$, if there are some values of $(\alpha = \alpha_0,\ \beta = \beta_0)$ in the range $[1,\ p_r]$ for which f is greater than $f_0 = 3$, then the linear differential operator $\mathcal{F}_5$ has a right factor. The solution $\tilde{S}(x)$ with $(\alpha = \alpha_0,\ \beta = \beta_0)$ gives an ODE whose 'ODE formula' reads

$$7\,Q\ \ +2\,D\ \ +2 = (Q+1)\cdot\ (D+1)\ \ -f, \tag{E.2}$$

telling us that there is indeed an order-2 right factor occurring in the linear differential operator $\mathcal{F}_5$. We call this factor F_2 when it is obtained as a minimal order ODE, and F_2^{nm} if it is obtained in non-minimal order.

Here begin the details of our remark 3. To obtain the factor (call it F_3) at the left of F_2 in $\mathcal{F}_5$, we may just use the 'rightdivision' command of DEtools in Maple, or act with F_2 on the series $S(x)$ to obtain a series and look for the linear ODE annihilating it. But assume that this is onerous, or not achievable. Either using the 'rightdivision' command is not feasible, or the series $F_2(S(x))$ no longer has enough coefficient terms to encode the remaining left factor. This what happens in the case of $L_{21}\cdot\tilde{L}_2$ when $\tilde{L}_2$ is of minimal order.

Let us give some details for the factorization $\mathcal{F}_5 = F_3\cdot F_2$. Assume that we have obtained F_3 from the series $F_2^{nm}(S(x))$. The rational solution of the symmetric square of F_3 (in minimal order or in non-minimal order) will appear as

$$\frac{P_{32}(x)}{(1-4x)^7\,(1+4x)^7\,(1-2x)\,(1+2x)}. \tag{E.3}$$

Suppose that we use the minimal order F_2 to obtain F_3. The rational solution of the symmetric square of F_3 (in minimal order or in non-minimal order) will appear as

$$\frac{P_{34}(x)}{(1-4\,x)^5\ (1+4\,x)^5}. \tag{E.4}$$

To obtain the rational solution given in (11), one has to divide by the coefficient of the higher derivative of F_2. This is because we have used F_2, in non-monic form, to mimic the situation of the large order linear differential operators for which the non-monic form is more tractable in the computations.

As far as the occurrence of a rational solution to the symmetric square of a left factor is concerned, it is irrelevant whether the right factor is of minimal order or in non-minimal order.

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