The saga of the Ising susceptibility

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Abstract

We review the many developments made since 1959 in the study of the susceptibility of the Ising model. The expressions for the form factors in terms of the nome q and the modulus k are compared and contrasted. The λ generalized correlations $C(M,N;\lambda)$ are defined and explicitly computed in terms of theta functions for $M=N=0,1$

1 Introduction

There are three important thermodynamic properties of any magnetic system in zero magnetic field: the partition function from which free energy and the specific heat are obtained; the magnetization; and the magnetic susceptibility. For the two dimensional Ising model in zero field defined by

$$\mathcal{E}_0 = -\sum_{j,k}\{E^v\sigma_{j,k}\sigma_{j+1,k} + E^h\sigma_{j,k}\sigma_{j,k+1}\} \tag{1}$$

with $\sigma_{j,k}=\pm 1$ the free energy was first computed by Onsager [1] in 1944 and the spontaneous magnetization was announced by Onsager in 1948 [2] and proven by Yang [3] in 1952. To this day a closed form for the magnetic susceptibility has never been found. We will here trace the saga of the quest for this susceptibility.

If we could solve the Ising model in the presence of a magnetic field H which interacts with the total spin of the system as

$$\mathcal{E} = \mathcal{E}_0 - H\sum_{j,k}\sigma_{j,k} \tag{2}$$

then the magnetic susceptibility would be computed as

$$\chi(H) = \frac{\partial M(H)}{\partial H} \tag{3}$$

where the magnetization is

$$M(H) = \frac{1}{Z(H)} \sum_{\sigma_{j,k}=\pm 1} \sigma_{0,0}\, e^{-\mathcal{E}/k_B T} \tag{4}$$

with the partition function defined by

$$Z(H) = \sum_{\sigma_{j,k}=\pm 1} e^{-\mathcal{E}/k_B T} \tag{5}$$

However, because the Ising model has only been solved for $H = 0$ we are forced to restrict our attention to $\chi(0)$ which from (2)-(5) is given in terms of the two point correlation functions as

$$k_B T \chi(0) = \sum_{M,N} \{\langle \sigma_{0,0}\, \sigma_{M,N} \rangle - M(0)^2\} \tag{6}$$

where $M(0)$ is the spontaneous magnetization of the system which is zero for $T > T_c$ and for $T < T_c$

$$M(0) = (1 - k^2)^{1/8} \tag{7}$$

where

$$k = \sinh 2K^v \sinh 2K^h \tag{8}$$

with $K^{v,h} = E^{v,h}/k_B T$ and T_c is defined by

$$k = 1 \tag{9}$$

The first exact result for the susceptibility was given in 1959 by Fisher [4] who used results of Kaufmann and Onsager [5] to argue that as $T \to T_c$ the susceptibility diverges as $|T - T_c|^{7/4}$. The saga may be said to begin with the concluding remark of this paper:

In conclusion we note that the relatively simple results (1) and (3) suggest strongly that there is a closed expression for the susceptibility in terms of elliptic integrals. It is to be hoped that such a formula will be discovered,$\cdots$.

To this day such a formula has not been found.

2 Form factor expansion and the λ extension

To proceed further a systematic understanding of the two point correlation function is required. For short distances the correlations are well represented by determinants [5, 6] whose size grows with the separation of the spins. However, in order to execute the sum over all separations required by (6) an alternative form of the correlations which is efficient for large distances is needed. The study of this alternative form was initiated in 1966 by Wu [7] who discovered that for the row correlation $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ that when $N|T-T_c|\gg 1$ for $T<T_c$

$$\langle\sigma_{0,0}\sigma_{0,N}\rangle \;=\; (1-t)^{1/4}\cdot\{1+f_{0,N}^{(2)}+\cdots\} \tag{10}$$

with $t=k^2$ and for $T>T_c$ we define $k=\;(\sinh 2K^v\sinh 2K^h)^{-1}$

$$\langle\sigma_{0,0}\sigma_{0,N}\rangle \;=\; (1-t)^{1/4}\cdot\{f_{0,N}^{(1)}\;+\;\cdots\} \tag{11}$$

with $t=k^2$ and $f_{0,N}^{(n)}$ is an n fold integral which exponentially decays for large N. These are the leading terms in what has become known as the form factor representation of the correlations which in general for $T<T_c$ is

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle \;=\; (1-t)^{1/4}\cdot\{1+\sum_{n=1}^{\infty}f_{M,N}^{(2n)}\} \tag{12}$$

and for $T>T_c$

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle \;=\; (1-t)^{1/4}\cdot\sum_{n=0}^{\infty}f_{M,N}^{(2n+1)} \tag{13}$$

where $f_{M,N}^{(n)}$ is an n dimensional integral. For general $M,\ N$ these $f_{M,N}^{(n)}$ were computed in 1976 by Wu, McCoy, Tracy and Barouch [8]. However, for the diagonal correlations an alternative and simpler form is available which was announced in [9] and proven in [10]. For the diagonal form factor for $T<T_c$

$$f_{N,N}^{(2n)}(t)=\frac{t^{n(N+n)}}{(n!)^2\pi^{2n}}\int_0^1\prod_{k=1}^{2n}dx_kx_k^N\prod_{j=1}^{n}\left(\frac{(1-tx_{2j})(x_{2j}^{-1}-1)}{(1-tx_{2j-1})(x_{2j-1}^{-1}-1)}\right)^{1/2}$$
$$\prod_{1\le j\le n}\prod_{1\le k\le n}\left(\frac{1}{1-tx_{2k-1}x_{2j}}\right)^2\prod_{1\le j<k\le n}(x_{2j-1}-x_{2k-1})^2(x_{2j}-x_{2k})^2 \tag{14}$$

for $T>T_c$

$$f_{N,N}^{(2n+1)}(t)\;=$$
$$\frac{t^{(n+1/2)N+n(n+1)}}{n!(n+1)!\pi^{2n+1}}\int_0^1\prod_{k=1}^{2n+1}dx_kx_k^N\prod_{j=1}^{n+1}x_{2j-1}^{-1}[(1-tx_{2j-1})(x_{2j-1}^{-1}-1)]^{-1/2}$$
$$\prod_{j=1}^{n}x_{2j}[(1-tx_{2j})(x_{2j}^{-1}-1)]^{1/2}\prod_{1\le j\le n+1}\prod_{1\le k\le n}\left(\frac{1}{1-tx_{2j-1}x_{2k}}\right)^2$$
$$\prod_{1\le j<k\le n+1}(x_{2j-1}-x_{2k-1})^2\prod_{1\le j<k\le n}(x_{2j}-x_{2k})^2 \tag{15}$$

In particular

$$f^{(1)}_{N,N}(t) \ = \ t^{N/2} \cdot \frac{\Gamma(N+1/2)}{\pi^{1/2} N!} \cdot F(\frac{1}{2}, N+\frac{1}{2}; N+1; t) \qquad (16)$$

where $F(a,b;c;t)$ is the hypergeometric function.

It is often useful and instructive to extend the form factor expansions (12) and (13) by weighting $f^{(n)}_{M,N}$ by λ^n and thus we define "λ generalized correlations"

$$C_-(M,N;\lambda) \ = \ (1-t)^{1/4} \cdot \{1 + \sum_{n=1}^{\infty} \lambda^{2n} f^{(2n)}_{M,N}\} \qquad (17)$$

and for $T > T_c$

$$C_+(M,N;\lambda) \ = \ (1-t)^{1/4} \cdot \sum_{n=0}^{\infty} \lambda^{2n+1} f^{(2n+1)}_{M,N} \qquad (18)$$

This λ extension was first introduced in 1977 by McCoy, Tracy and Wu [11] in the context of the scaling limit.

3 Leading divergence as $T \to T_c$

These form factor expansions may now be used in (6) where the sums over M, N are easily executed under the integral signs. to produce a corresponding expansion of the susceptibility [8] which we write for $T < T_c$ as

$$k_B T \chi(0) \ = \ (1-t)^{1/4} \cdot \sum_{n=1}^{\infty} \hat{\chi}^{(2n)} \qquad (19)$$

and for $T > T_c$

$$k_B T \chi(0) \ = \ (1-t)^{1/4} \cdot \sum_{n=0}^{\infty} \hat{\chi}^{(2n+1)} \qquad (20)$$

where

$$\tilde{\chi}^{(n)} = \sum_{M=-\infty}^{\infty} \sum_{N=-\infty}^{\infty} f^{(n)}_{M,N} \qquad (21)$$

In this form it is quite clear that unlike the 1959 argument of [4] that the behavior of the susceptibility as $T \to T_c$ will be different depending on whether T approaches T_c from above or below. This dramatic difference was first seen in 1973 in [12] where it is shown that for $T \to T_c \pm$

$$k_B \chi(0) \ \sim \ C_{0\pm} \cdot |T - T_c|^{-7/4} \qquad (22)$$

where

$$C_{0\pm} \ = \ 2^{-1/2} \cdot \coth 2K^v_c \coth 2K^h_c \cdot [K^v_c \coth 2K^v_c + K^h_c \coth 2K^h_c]^{-7/4} \cdot I_{\pm} \quad (23)$$

and $I_\pm$ have been numerically evaluated to 52 digits in [15]:

$$I_+ = 1.000815260440212647119476363047210236937534925597789\cdots$$
$$I_- = \frac{1}{12\pi}\cdot 1.000960328725262189480934955172097320572505951770117\cdots$$

4 The singularities of Nickel

After the computations of [8] no further work was done on the susceptibility for almost a quarter century until in 1999 Nickel [13] and [14] analyzed the singularities of the n fold integrals $\tilde{\chi}^{(n)}$. These integrals, of course, have singularities at $T = T_c$ where the individual correlation functions $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ have singularities. However, Nickel made the remarkable discovery that the integrals, for $\hat{\chi}^{(n)}$, contain many more singularities. In particular he found that, for the isotropic lattice, $\hat{\chi}^{(n)}$ has singularities in the complex temperature variable $s = \sinh 2E/k_BT$ at

$$s = s_{j,k} = e^{i\theta_{j,k}} \tag{24}$$

where

$$2\cos(\theta_{j,k}) = \cos(2\pi k/n) + \cos(2\pi j/n) \tag{25}$$

For n odd $(T > T_c)$ the behavior of $\hat{\chi}^{(n)}$ near the singularity is

$$\hat{\chi}^{(2n+1)} \sim \epsilon^{2n(n+1)-1}\cdot\ln\epsilon \tag{26}$$

with

$$\epsilon = 1 - s/s_{j,k} \tag{27}$$

and for even n $(T < T_c)$

$$\hat{\chi}^{(2n)} \sim \epsilon^{2n^2-3/2} \tag{28}$$

The discovery of Nickel singularities demonstrates that the magnetic susceptibility is a far more complicated object than either the free energy or the spontaneous magnetization and that the hope expressed in [4] of a closed form in terms of a few elliptic integrals is far too simple.

5 The theta function expressions of Orrick, Nickel, Guttmann and Perk

The next advance in the subject was made the following year by Orrick, Nickel, Guttmann and Perk [15] who studied both the form factors and the susceptibility by means of generating on the computer series of over 300 terms. From these series they then made several remarkable conjectures for the form factors.

To present these conjectures we define theta functions as

$$\theta_1(u,q) = 2\sum_{n=0}^{\infty}(-1)^n\, q^{(n+1/2)^2}\,\sin[(2n+1)u] \tag{29}$$

$$\theta_2(u,q) \;=\; 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\, \cos[(2n+1)u] \tag{30}$$

$$\theta_3(u,q) \;=\; 1+2\sum_{n=1}^{\infty} q^{n^2}\, \cos 2nu \tag{31}$$

$$\theta_4(u,q) \;=\; 1+2\sum_{n=1}^{\infty} (-1)^n\, q^{n^2}\, \cos 2nu \tag{32}$$

and for $u=0$ we use the short hand

$$\theta_2 = \theta_2(0,q), \qquad \theta_3 = \theta_3(0,q), \qquad \theta_4 = \theta_4(0,q) \tag{33}$$

The quantity q is the nome of the elliptic functions and is related to the modulus k by the relation

$$k \;=\; 4\,q^{1/2}\cdot\prod_{n=1}^{\infty}\left[\frac{1+q^{2n}}{1+q^{2n-1}}\right]^4 \tag{34}$$

In terms of these theta functions, conjectures for form factors are given in sec. 5.2 of [15] by defining an operator Φ_0 which converts a power series in z to a power series in q as

$$\Phi_0\Big(\sum_{n=0}^{\infty} c_n z^n\Big) \;=\; \sum_{n=0}^{\infty} c_n q^{n^2/4} \tag{35}$$

Conjectures are then given for $f^{(n)}_{0,0}$, $f^{(n)}_{1,1}, f^{(n)}_{1,0}$, $f^{(n)}_{2,0}$ and $f^{(n)}_{2,1}$. In particular we note

$$2^{-n}\cdot(1-k^2)^{1/4}\cdot f^{(n)}_{0,0} \;=\; \frac{(1,k^{-1/2})}{\theta_3}\cdot\Phi_0\Big(\frac{z^n(1-z^2)}{(1+z^2)^{n+1}}\Big) \tag{36}$$

and

$$2^{-n}\cdot(1-k^2)^{1/4} f^{(n)}_{1,1} = \frac{2\,(n+1)(1,k^{-1/2})}{\theta_2\theta_3^2}\cdot\Phi_0\Big(\frac{z^{n+1}(1-z^2)}{(1+z^2)^{n+2}}\Big) \tag{37}$$

where

$$(1,k^{-1/2}) = \begin{array}{lll} 1 & \text{for} & T<T_c \;\; (n \text{ even}) \\ k^{-1/2} & \text{for} & T>T_c \;\; (n \text{ odd}) \end{array} \tag{38}$$

6 Linear differential equations

A second approach to the form factors and susceptibility was initiated in 2004 in [16] and subsequently greatly developed in [17, 18, 19, 20, 21]. These studies are similar to [15] in that they expand the form factors and susceptibility in long series. However, instead of the nome q the expansion is in the (modular) variable t. The goal of these studies is to characterize the n particle contributions $\hat{\chi}^{(n)}(t)$ to the susceptibility in terms of finding a Fuchsian linear ordinary differential equation satisfied by $\hat{\chi}^{(n)}(t)$. Such a linear differential equation always exists for an n-fold integral with an algebraic integrand that depends on the parameter t. However the order and the degree of the equation rapidly become large for increasing n and it may take series of many thousands of terms to find the differential equation. Such a study can only be done by computer and the relevant software has only become available in the last 10 years.

There are several features of these differential equations to be noted. In particular the operator which annihilates $\hat{\chi}^{(n)}$ factorizes and furthermore the operator has a direct sum decomposition such that $\hat{\chi}^{(n-2j)}$ for $j = 1, \cdots [n/2]$ are "contained" in $\hat{\chi}^{(n)}$.

7 Diagonal form factors

With the observation of factorization, direct sum decomposition and Nickel singularities of the n particle contributions to the bulk susceptibility $\hat{\chi}^{(n)}$, it has become clear that the susceptibility is far more complicated that what was envisaged by Fisher [4] in 1959. Because of this complexity the question was asked if there could be a simpler object to study which would yet be able to give insight into the structures which had been observed. Several such "simplified" objects have been studied [23, 24] which consist of more or less forcibly modifying parts of the integrals for the $\hat{\chi}^{(n)}$. However, there is one "simplified" model which commands interest in its own right. In statistical language this is the "diagonal susceptibility" which is defined as restricting the sum in (6) to the correlation of spins on the diagonal

$$k_BT \cdot \chi_d \; = \sum_{N=-\infty}^{\infty} \{\langle \sigma_{0,0}\sigma_{N,N}\rangle - M^2(0)\} \tag{39}$$

In statistical language this diagonal susceptibility is the susceptibility for a magnetic field interacting only with the spins on one diagonal. However, in magnetic language this is also the susceptibility χ^x of the one dimensional quantum spin chain of the transverse Ising model

$$H_{TI} \; = \sum_{j=-\infty}^{\infty} \{\sigma_j^x \sigma_{j+1}^x + H^z \sigma_j^z\} \tag{40}$$

with

$$k_BT \cdot \chi^x \; = \sum_{j=-\infty}^{\infty} \{\langle \sigma_0^x \sigma_j^x \rangle - M_x^2\} \tag{41}$$

which is the response to a magnetic field

$$-\, H^x \sum_{j=-\infty}^{\infty} \sigma_j^x \tag{42}$$

and σ_j^i are the three Pauli spin 1/2 matrices at site j.

This interpretation gives the diagonal susceptibility a physical interpretation which the other "simplified" models do not have [23, 24]. Furthermore much more analytic information is available for the diagonal Ising correlations than for correlations off the diagonal. Firstly it is known from the work of Jimbo and Miwa [25] that the diagonal correlations are characterized by the solutions of a particular sigma form of Painlevé VI equation and secondly the integral representation of the diagonal form factors (14) and (15) is more tractable than the representation of the general off diagonal correlations.

The diagonal form factors have been been extensively studied in [9] by means of processing the differential equations obtained from long series expansion by use of Maple. Diagonal form factors $f_{N,N}^{(n)}$ for n as large as 9 and N as large as 4 have been studied and many examples are given in [9] where they have all been reduced to expressions in the elliptic integrals E and K. A few such examples are as follows:

For $n = 1$ when the hypergeometric function of (16) is reduced to the basis of E and K by use of the contiguous relations

$$f_{0,0}^{(1)} \; = \; (2/\pi) \cdot K \tag{43}$$

$$t^{1/2} \, f_{1,1}^{(1)} \; = \; (2/\pi) \cdot \{K - E\} \tag{44}$$

$$3\,t\, f_{2,2}^{(1)} \; = \; (2/\pi) \cdot \{(t+2\,)K \; - 2(t+1)\,E\} \tag{45}$$

$$15\,t^{3/2}\, f_{3,3}^{(1)} \; = \; (2/\pi) \cdot \{(4t^2 + 3t + 8)\,K \, - (8t^2 + 7t + 8)\,E\} \tag{46}$$

$$\begin{aligned} 105\,t^2\, f_{4,4}^{(1)} \; = \; & (2/\pi) \cdot \{(24t^3 + 17t^2 + 16t + 48)K \\ & - (48t^3 + 40t^2 + 40t + 48)\,E\} \end{aligned} \tag{47}$$

For $n = 2$

$$2\, f_{0,0}^{(2)} \; = \; (2/\pi)^2 \cdot K\,(K - E) \tag{48}$$

$$2\, f_{1,1}^{(2)} \; = \; 1 \; - (2/\pi)^2 \cdot K \cdot \{(t-2)\,K \; + 3E\} \tag{49}$$

$$\begin{aligned} & 6\,t\, f_{2,2}^{(2)} \; = \; 6t \\ & -(2/\pi)^2 \cdot \{6t^2 - 11t + 2)\,K^2 \; + (15t - 4)\,KE \, + 2(t+1)\,E^2\} \end{aligned} \tag{50}$$

$$90\,t^2\, f_{3,3}^{(2)} \; = \; 135t^2 - (2/\pi)^2 \cdot \{(137t^3 - 242t^2 + 52t + 8)\,K^2 \tag{51}$$

$$-(8t^3 - 319t^2 + 122t + 16)\,KE \, + 4(t+1)(2t^2 + 13t + 2)E^2\} \tag{52}$$

$$\begin{aligned} & 3150\,t^3\, f_{4,4}^{(2)} \; = \; 6300\,t^2 \\ & -(2/\pi)^2 \cdot \{(32t^5 + 6440t^4 - 1119t^3 + 2552t^2 + 464t + 128)\,K^2 \\ & -(128t^5 + 576t^4 - 14519t^3 + 548t^2 + 1056t + 256)\,KE \\ & +(1+t)(16t^4 + 58t^3 + 333t^2 + 58t + 16)\,E^2\} \end{aligned} \tag{53}$$

For $n = 3$

$$6\, f_{0,0}^{(3)} \; = \; (2/\pi) \cdot \; K \; - (2/\pi)^3 \cdot \; K^2\{(t-2)\,K \; + 3E\} \tag{54}$$

$$6\,t^{1/2}\,f^{(3)}_{1,1} \;=\; 4\,(2/\pi)\cdot(K-E) - (2/\pi)^3\cdot K\{(2t-3)\,K^2\,+6KE\,-3E^2\} \tag{55}$$

$$18\,t\,f^{(3)}_{2,2} \;=\; 7\,(2/\pi)\cdot\{(t+2)\,K\,-2\,(t+1)\,E\} \tag{56}$$

$$-(2/\pi)^3\cdot\{3(t^2-2)\,K^3\,-3\,(2t^2-11t+2)\,K^2E$$

$$-36\,(t-1)\,KE^2\,-24E^3\} \tag{57}$$

$$270\,t^{5/2}\,f^{(3)}_{3,3} \;=\; 30(2/\pi)\{(4t^2+3t+8)\,K\,-(8t^2+7t+8)\,t\,E\}$$

$$-(2/\pi)^3\cdot\{(72t^4-158t^3+189t^2-156t+8)\,K^3$$

$$-6(24t^4-108t^3+29t^2-6t+4)\,K^2E$$

$$-3\,(232t^3\,-111t^2-180\,t\,-8)\,KE^2$$

$$-4\,(t+1)(2t^2+103t+2)\,t\,E^3\} \tag{58}$$

For $n=4$

$$24\,f^{(4)}_{0,0} \;=\; 4\,(2/\pi)^2\cdot K\,(K-E)$$

$$-(2/\pi)^4\cdot K^2\{(2t-3)\,K^2\,+6KE\,-3E^2\} \tag{59}$$

$$24\,f^{(4)}_{1,1} \;=\; 9\;-(2/\pi)^2\cdot 10\,K\{(t-2)\,K\,+3E\}$$

$$+(2/\pi)^4\cdot K^2\{(t^2-6t+6)\,K^2\,+10\,(t-2)\,KE\,+15E^2\} \tag{60}$$

$$72\,t\,f^{(4)}_{2,2} \;=\; 72t$$

$$-(2/\pi)^2\cdot 16\cdot\{(6t^2-11t+2)\,K^2\,+(15t-4)\,KE\,+2(t+1)\,E^2\}$$

$$+(2/\pi)^4\cdot\{24t^3-98t^2+113t-36)\,K^4\,+2\,(74t^2-157t+66)\,K^3E$$

$$+3\,(71t-60)\,K^2E^2\,+12\,(t+9)\,KE^3\,-24E^4\} \tag{61}$$

These examples are sufficient to illustrate the following phenomena which hold for all examples considered in [9] and which are certainly true in general:

$$f^{(2n)}_{N,N} \;=\; \sum_{j=0}^{n} c^-_{j;n}\, g^{(2j)}_{N,N}(t) \tag{62}$$

$$f^{(2n+1)}_{N,N} \;=\; \sum_{j=0}^{n} c^+_{j;n}\, g^{(2j+1)}_{N,N}(t) \tag{63}$$

where $c^{\pm}_{j;n}$ are constants independent of t and $g^{(j)}_{N,N}(t)$ for even j are of the form

$$g^{(2n)}_{0,0}(t) \;=\; \sum_{j=0}^{n} P^-_{j,n;0}(t)\,K^{2n-j}E^j \tag{64}$$

$$g^{(2n)}_{1,1}(t) \;=\; \sum_{j=0}^{n} P^-_{j,n;1}(t)\,K^{2n-j}E^j \tag{65}$$

$$g^{(2n)}_{N.N}(t) \;=\; t^{-N+1}\sum_{j=0}^{2n} P^-_{j,n;N}(t)\,K^{2n-j}E^j \quad \text{for} \quad N\geq 2 \tag{66}$$

and for even j

$$g_{0,0}^{(2n+1)}(t) \;=\; \sum_{j=0}^{n} P_{j,n;0}^{+}(t)\, K^{2n+1-j} E^{j} \qquad (67)$$

$$g_{1,1}^{(2n+1)}(t) \;=\; t^{-1/2} \sum_{j=0}^{n+1} P_{j.n;1}^{+}(t)\, K^{2n+1-j} E^{j} \qquad (68)$$

$$g_{N,N}^{(2n+1)}(t) = t^{-N/2} \sum_{j=0}^{2n+1} P_{j,n;N}^{+}(t)\, K^{2n+1-j} E^{j} \quad \text{for} \quad N \geq 2 \qquad (69)$$

where $P_{j,n;m}^{\pm}(t)$ are polynomials.

The decompositions (62) and (63) represent a direct sum decomposition of the form factors [22]. The functions $g_{N,N}^{(j)}$ individually satisfy Fuchsian equations and are homomorphic to the $j+1$ symmetric power of the complete elliptic integral E (or equivalently K)

We also observe the relation between $f_{1,1}^{(n)}(t)$ and $f_{0,0}^{(n+1)}(t)$

$$(2/\pi) \cdot K \cdot f_{1,1}^{(2n)}(t) \;=\; (2n+1) \cdot f_{0,0}^{(2n+1)}(t) \qquad (70)$$

$$(2/\pi) \cdot t^{1/2} \cdot K \cdot f_{1,1}^{(2n+1)}(t) \;=\; 2\,(n+1) \cdot f_{0,0}^{(2n+2)}(t) \qquad (71)$$

8 Nome q-representation versus modulus k-representation

We will need the following identities which relate functions of the nome $q = e^{i\pi\tau}$ where $\tau = iK(k')/K(k)$ with functions of the modulus k

$$k \;=\; \frac{\theta_2^2}{\theta_3^2}, \qquad k' \;=\; (1-k^2)^{1/2} = \frac{\theta_4^2}{\theta_3^2} \qquad (72)$$

$$\frac{2}{\pi} K \;=\; \theta_3^2, \qquad \text{and:} \qquad \frac{dq}{dk} \;=\; \frac{\pi^2}{2} \frac{q}{kk'^2 K^2} \qquad (73)$$

which we will use as

$$q \frac{d}{dq} \;=\; \frac{2}{\pi^2}\, k\, k'^2 \cdot K^2 \cdot \frac{d}{dk} \qquad (74)$$

We will also use

$$\frac{dK}{dk} \;=\; \frac{E - k'^2 K}{kk'^2}, \qquad \frac{dE}{dk} \;=\; \frac{E-K}{k} \qquad (75)$$

8.1 $f_{0,0}^{(2n)}$

We first write (36) for $j = 2n$ using (72) as

$$f_{0,0}^{2n} \;=\; \frac{1}{\theta_4} \cdot \Phi_0\Big(\frac{2^{2n} z^{2n}(1-z^2)}{(1+z^2)^{2n+1}}\Big) \qquad (76)$$

Thus, by use of the elementary expansion

$$\frac{2^{2n}\cdot z^{2n}\cdot(1-z^2)}{(1+z^2)^{2n+1}} \;=\; 2\;\frac{(-1)^n}{(2n)!}\sum_{j=0}^{\infty}(-1)^j\, z^{2j}\prod_{m=0}^{n-1}4\,[j^2-m^2] \tag{77}$$

and the definition (35) of operator Φ_0 we find that in terms of the nome q

$$\begin{aligned} f_{0,0}^{(2n)} \;&=\; 2\;\frac{(-1)^n\,4^n}{\theta_4(2n)!}\cdot\sum_{j=0}^{\infty}(-1)^j\,q^{j^2}\prod_{m=0}^{n-1}[j^2-m^2] \\ &=\; 2\frac{(-1)^n\,4^n}{\theta_4(2n)!}\cdot\sum_{j=0}^{\infty}(-1)^j\prod_{m=0}^{n-1}[q\,\frac{d}{dq}-m^2]q^{j^2} \\ &=\; \frac{(-1)^n\,4^n}{\theta_4(2n)!}\cdot\prod_{m=1}^{n-1}[q\,\frac{d}{dq}\;-m^2]\cdot q\frac{d}{dq}\theta_4 \end{aligned} \tag{78}$$

To convert this to an expression in terms of the modulus k we first use (72) to write

$$\theta_4^2 \;=\; \frac{2}{\pi}\cdot k'\cdot K \tag{79}$$

and thus using (73)

$$q\frac{d}{dq}\theta_4^2 \;=\; \frac{2}{\pi^2}\cdot k\,k'^2\cdot K^2\cdot\frac{d}{dk}\left(\frac{2}{\pi}k'K\right) \tag{80}$$

which using (75) reduces to

$$q\,\frac{d}{dq}\theta_4^2 \;=\; \frac{2}{\pi^2}\cdot k'\cdot K^2\cdot\frac{2}{\pi}\{E-K\} \tag{81}$$

Using (79) on the right hand side we find

$$2\,\theta_4\cdot q\frac{d}{dq}\theta_4 \;=\; \frac{2}{\pi^2}\cdot\theta_4^2\cdot K\cdot\{E-K\} \tag{82}$$

and thus

$$\frac{1}{\theta_4}\cdot q\,\frac{d}{dq}\theta_4 \;=\; \frac{1}{\pi^2}\cdot K\cdot\{E-K\} \tag{83}$$

To evaluate $f_{0,0}^{(2)}$ we use (83) in (78) with $n=1$ to obtain

$$f_{0,0}^{(2)} \;=\; \frac{2}{\pi^2}\cdot K\cdot\{K-E\} \tag{84}$$

which is in agreement with (48). For arbitrary n the form factor $f_{0,0}^{(2n)}$ is obtained from (78) by repeated use of (83) and (75).

8.2 $f_{0,0}^{(2n+1)}$

To study $f_{0,0}^{(2n+1)}$ we first use (72) to write (36) as

$$f_{0,0}^{(2n+1)} = \frac{\theta_3}{\theta_2\theta_4} \cdot \Phi_0\Big(\frac{2^{2n+1} z^{2n+1}(1-z^2)}{(1+z^2)^{2n+2}}\Big) \tag{85}$$

and then, using the elementary expansion

$$\frac{2^{2n+1} \cdot z^{2n+1} \cdot (1-z^2)}{(1+z^2)^{2n+2}} \tag{86}$$
$$= 2\,\frac{(-1)^n}{(2n+1)!} \cdot \sum_{j=0}^{\infty} (2j+1) \cdot (-1)^j \cdot z^{2j+1} \cdot \prod_{m=0}^{n-1} [(2j+1)^2 - (2m+1)^2]$$

and the definition (35) of the operator Φ_0 we find

$$f_{0,0}^{(2n+1)} \tag{87}$$
$$= \frac{\theta_3}{\theta_2\theta_4} \frac{2(-1)^n}{(2n+1)!} \cdot \sum_{j=0}^{\infty} (2j+1) \cdot (-1)^j \cdot q^{(2j+1)^2/4} \cdot \prod_{m=0}^{n-1} [(2j+1)^2 - (2m+1)^2]$$
$$= \frac{\theta_3}{\theta_2\theta_4} \frac{2\,(-1)^n}{(2n+1)!} \cdot \prod_{m=0}^{n-1} [4\,q\frac{d}{dq} - (2m+1)^2] \sum_{j=0}^{\infty} (2j+1)(-1)^j q^{(2j+1)^2/4}$$

Thus, if we write

$$2\sum_{j=0}^{\infty} (2j+1) \cdot (-1)^j \cdot q^{(2j+1)^2/4} = \frac{\partial}{\partial u}\theta_1(u,q)|_{u=0} = \theta_2\theta_3\theta_4 \tag{88}$$

where in the last line we have used a well known identity, we find the result

$$f_{0,0}^{(2n+1)} = \frac{\theta_3}{\theta_2\theta_4} \frac{(-1)^n}{(2n+1)!} \cdot \prod_{m=0}^{n-1} [4\,q\frac{d}{dq} - (2m+1)^2]\ \theta_2\theta_3\theta_4 \tag{89}$$

We may now use (72)-(75) to reduce (89) from a function of q to a function of k.

For $n = 0$ we use (73) to find

$$f_{0,0}^{(1)} = \theta_3^2 = \frac{2}{\pi} \cdot K \tag{90}$$

which agrees with (43).

For $n \geq 3$ we need a expression analagous to (83) for the product $\theta_2\theta_3\theta_4$. From (72), (73)

$$\theta_2^2\theta_3^2\theta_4^2 = k\,k' \cdot (2\,K/\pi)^3 \tag{91}$$

and thus

$$
\begin{aligned}
q\frac{d}{dq}\theta_2^2\theta_3^3\theta_4^2 \ &= \ 2\,\theta_2\theta_3\theta_4\cdot q\frac{d}{dq}\theta_2\theta_3\theta_4 \ = \ \frac{2}{\pi^2}\cdot k\,k'^2\,K^2\cdot\frac{d}{dk}\{k\,k'(2\,K/\pi)^3\} \\
&= \ \frac{2}{\pi^2}\cdot k\,k'\cdot(2K/\pi)^3\cdot K\cdot\{(k^2-2)\,K\ +3E\} \\
&= \ \frac{2}{\pi^2}\cdot\theta_2^2\,\theta_3^2\,\theta_4^2\cdot K\cdot\{(k^2-2)\,K\ +3E\} \qquad (92)
\end{aligned}
$$

Therefore we obtain

$$
q\frac{d}{dq}\theta_2\theta_3\theta_4 \ = \ \frac{1}{\pi^2}\theta_2\theta_3\theta_4\cdot K\,\{(k^2-2)\,K\ +3E\} \qquad (93)
$$

which when used in (89) with $n=1$ gives

$$
f_{0,0}^{(3)} \ = \ \frac{1}{3!}\cdot\{(2/\pi)\cdot K\ \ -(2/\pi)^3\,K\cdot[(k^2-2)\,K\ +3E\} \qquad (94)
$$

which is in agreement with (54).

8.3 $f_{1,1}^{(n)}$

The equalities (70) and (71) which express $f_{1,1}^{(n)}$ in terms of $f_{0,0}^{(m)}$ follow immediately from (36) and (37) by use of (72) and (73).

9 The λ generalized correlations

For $N=0,1$ the diagonal λ generalized correlations defined by (17) and (18) may be obtained by using the expressions for $(1-t)^{1/4}f_{N,N}^{(n)}$ in terms of the operator Φ_0 as given by (36) and (37). In this form the sums over n are easily done as geometric series and the operator Φ_0 is then used to convert the series in z to series in the nome q which can then be expressed in terms of θ functions as was done in the previous section. Then, setting

$$
\lambda = \cos u \qquad (95)
$$

we obtain the following results

$$
\begin{aligned}
C_-(0,0;\lambda) \ &= \ \frac{\theta_3(u;q)}{\theta_3(0;q)} \qquad (96) \\
C_+(0,0;\lambda) \ &= \ \frac{\theta_2(u;q)}{\theta_2(0;q)} \qquad (97) \\
C_-(1,1;\lambda) \ &= \ \frac{-\theta_2'(u;q)}{\sin(u)\theta_2(0;q)\theta_3(0;q)^2} \qquad (98) \\
C_+(1,1;\lambda) \ &= \ \frac{-\theta_3'(u;q)}{\sin(u)\theta_2^2(0;q)\theta_3(0;q)} \qquad (99)
\end{aligned}
$$

where prime indicates the derivative with respect to u. The result (96) was first reported in [9]. The results (97)-(99) have recently been given in [26]. For u a rational multiple of π these diagonal generalized correlations reduce to algebraic functions of the modulus k. Several examples for $C_-(0,0;\lambda)$, $C_-(1,1;\lambda)$ and $C_-(2,2;\lambda)$ are given in [9].

10 Diagonal susceptibility

We may now explicitly obtain [22] the diagonal susceptibility by using the form factor expansion (12)-(15) in the definition (39) and evaluate the sum on N as a geometric series. We obtain for $T < T_c$

$$kT \cdot \chi_{d-}(t) \;=\; (1-t)^{1/4} \cdot \sum_{n=1}^{\infty} \tilde{\chi}_d^{(2n)}(t) \qquad (100)$$

with

$$\begin{aligned}
\tilde{\chi}_d^{(2n)}(t) \;=\; & \frac{t^{n^2}}{(n!)^2} \frac{1}{\pi^{2n}} \cdot \int_0^1 \cdots \int_0^1 \prod_{k=1}^{2n} dx_k \cdot \frac{1 + t^n\, x_1 \cdots x_{2n}}{1 - t^n\, x_1 \cdots x_{2n}} \\
& \times \prod_{j=1}^{n} \left(\frac{x_{2j-1}(1-x_{2j})(1-tx_{2j})}{x_{2j}(1-x_{2j-1})(1-t\,x_{2j-1})} \right)^{1/2} \\
& \times \prod_{1 \le j \le n} \prod_{1 \le k \le n} (1 - t\, x_{2j-1}\, x_{2k})^{-2} \\
& \times \prod_{1 \le j < k \le n} (x_{2j-1} - x_{2k-1})^2\, (x_{2j} - x_{2k})^2 \qquad (101)
\end{aligned}$$

and for $T > T_c$

$$kT \cdot \chi_{d+}(t) \;=\; (1-t)^{1/4} \cdot \sum_{n=0}^{\infty} \tilde{\chi}_d^{(2n+1)}(t) \qquad (102)$$

with

$$\begin{aligned}
\tilde{\chi}_d^{(2n+1)}(t) \;=\; & \frac{t^{n(n+1))}}{\pi^{2n+1} n!\,(n+1)!} \cdot \int_0^1 \cdots \int_0^1 \prod_{k=1}^{2n+1} dx_k \\
& \times \frac{1 + t^{n+1/2}\, x_1 \cdots x_{2n+1}}{1 - t^{n+1/2}\, x_1 \cdots x_{2n+1}} \cdot \prod_{j=1}^{n} \Big((1-x_{2j})(1-t\,x_{2j}) \cdot x_{2j} \Big)^{1/2} \\
& \times \prod_{j=1}^{n+1} \Big((1-x_{2j-1})(1-t\,x_{2j-1}) \cdot x_{2j-1} \Big)^{-1/2} \\
& \times \prod_{1 \le j \le n+1} \prod_{1 \le k \le n} (1 - t\, x_{2j-1}\, x_{2k})^{-2} \qquad (103) \\
& \times \prod_{1 \le j < k \le n+1} (x_{2j-1} - x_{2k-1})^2 \prod_{1 \le j < k \le n} (x_{2j} - x_{2k})^2.
\end{aligned}$$

This diagonal susceptibility has been extensively studied in [22].

The integrals for $\tilde{\chi}_d^{(1)}(t)$ and $\tilde{\chi}_d^{(2)}(t)$ are explicitly evaluated as

$$\tilde{\chi}_d^{(1)}(t) \; = \; \frac{1}{1 \; - t^{1/2}}. \qquad (104)$$

$$\tilde{\chi}_d^{(2)}(t) \; = \; \frac{1}{8\pi i} \oint dz_1 \, \frac{t}{(1 \; - t^{1/2} z_1)(z_1 \; - t^{1/2})} \; = \; \frac{t}{4\,(1-t)}. \qquad (105)$$

Fuchsian equations have been obtained for $\tilde{\chi}_d^{(3)}(t)$, $\tilde{\chi}_d^{(4)}(t)$ and $\tilde{\chi}_d^{(5)}(t)$.

From these equations we find that $\tilde{\chi}_d^{(3)}(t)$ has a direct sum decomposition into the sum of three terms. One term is just $\tilde{\chi}_d^{(1)}(t)$ a given by (104); the second is

$$\frac{1}{k-1} \cdot K \; + \frac{1}{(k-1)^2} \cdot E \qquad (106)$$

and the 3 solutions to the differential equation for the third term are two Meijer G functions and

$$\frac{(1+2k)(k+2)}{(1-k)(1+k+k^2)} \cdot \{F(1/6, 1/3; 1; Q)^2 \\ + \frac{2Q}{9} \cdot F(1/6, 1/3; 1; Q) F(7/6, 4/3; 2; Q)\} \qquad (107)$$

where

$$Q \; = \; \frac{27}{4} \frac{(1+k)^2 k^2}{(k^2+k+1)^2} \qquad (108)$$

Furthermore the $\tilde{\chi}_d^{(n)}(t)$ have singularities on $|t| = 1$ which are the analogue of the Nickel singularities for the bulk susceptibility. For $T < T_c$ the singularities in $\tilde{\chi}^{(2n)}(t)$ are at $t^n = 1$ and are of the form $\epsilon^{2n^2-1} \ln \epsilon$ and for $T > T_c$ the singularities in $\tilde{\chi}^{(2n+1)}$ are at $t^{n+1/2} = 1$ and are of the form $\epsilon^{(n+1)^2-1/2}$

11 Natural boundary

The most intriguing feature in both the bulk and the diagonal susceptibility are the singularities on the unit circle of the modular variable $k = 1$. As n increases the number of these singularities increases and becomes dense as $n \to \infty$. Therefore, unless a massive cancellation occurs the susceptibility will have a natural boundary on the circle $|k| = 1$. In terms of the nome q the circle $|k| = 1$ corresponds to the curve in Fig. 1.

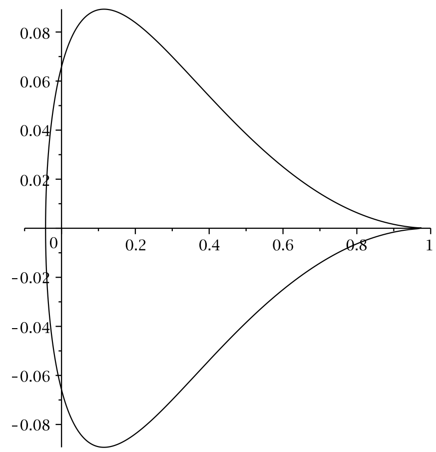

Figure 1: The curve in the plane of the nome q of the unit circle $|k| = 1$

12 Conclusion

We have seen that since 1959 a great deal of progress has been made in understanding the susceptibility of the Ising model and that the analytic structure is vastly more complicated than was envisaged 50 years ago in [4]. In particular the existence of a natural boundary is a completely new phenomena which has never before appeared in the study of critical behavior. The connections with elliptic modular functions are profound and extensive and much of the structure still remains to be discovered. It is quite remarkable that in 50 years the problem has not been solved.

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References

[1] L.Onsager, Crystal statistics,I. A two-dimensional model with an order disorder transition, Phys. Rev. 65 (1944) 117–149.

[2] L. Onsager, discussion, Nuovo Cimento 6 suppl. (1949) 261.

[3] C.N.Yang, The spontaneous magnetization of the two dimensional Ising model, Phys. Rev. 85 (1952) 808–816

[4] M.E. Fisher, The susceptibility of the plane Ising model, Physics 25 (1959) 521-524.

[5] B. Kaufmann and L. Onsager, Crystal statistics III. short range order in a binary Ising lattice, Phys. Rev. 76 (1949) 1244–1252.

[6] E.W. Montroll, R.B. Potts and J.C. Ward, Correlations and spontaneous magnetization of the two dimensional Ising model, J. Math. Phys. 4 (1963) 308-322.

[7] T.T. Wu, Theory of Toeplitz determinants and spin correlations of the two dimensional Ising model, Phys. Rev. 149 (1966) 380–401.

[8] T.T. Wu. B.M. McCoy, C.A. Tracy and E. Barouch, Spin-spin correlation functions for the two dimensional Ising model: exact theory in the scaling region, Phys. Rev. B13 (1976) 315–374.

[9] S. Boukraa, S. Hassani, J-M. Maillard,B.M. McCoy, W.P. Orrick and N. Zenine, Holonomy of the Ising model form factors, J. Phys. A40 (2007) 75-112.

[10] I. Lyberg and B.M. McCoy, Form factor expansion of the row and diagonal correlation functions of the two dimensional Ising model, J. Phys. A 40 (2007) 3329–3346.

[11] B.M. McCoy, C.A. Tracy and T.T. Wu, Painlevé equations of the third kind, J. Math. Phys. 18 (1977) 1058-1092.

[12] E. Barouch, B.M. McCoy and T.T. Wu, Zero field susceptibility of the two dimensional Ising model near T_c, Phys. Rev. Letts. 31 (1973) 1409-1411.

[13] B.G. Nickel, On the singularity structure of the 2D Ising model, J. Phys. A32 (1999) 3889–3906.

[14] B.G. Nickel, Addendum to "On the singularity structure of the 2D Ising model", J. Phys. A 33 (2000) 1693–1711.

[15] W.P. Orrick, B.G. Nickel, A. J. Guttmann and J.H.H. Perk, The susceptibility of the square lattice Ising model: new developements, J. Stat. Phys. 102 (2001) 795–841.

[16] N. Zenine, S. Boukraa, S. Hassani and J-M. Maillard, The Fuchsian differential equation of the square Ising $\chi^{(3)}$ susceptibility, J. Phys. A 37 (2004) 9651–9668.

[17] N. Zenine, S. Boukraa, S. Hassani and J-M. Maillard, Square lattice Ising model susceptibility: series expansion method and differential equation for $\chi^{(3)}$, J. Phys. A 38 (2005) 1975–1899.

[18] N. Zenine, S. Boukraa, S. Hassani and J-M. Maillard, Ising model susceptibility: The Fuchsian equation for $\chi^{(4)}$ and its factorization properties, J. Phys. A 38 (2005) 4149–4173.

[19] S. Boukraa, A.J. Guttmann, S. Hassani, I. Jensen, J-M. Maillard, B. Nickel and N. Zenine, Experimental mathematics on the magnetic susceptibility of the square lattice Ising model, J. Phys. A 41 (2008) 455202

[20] A. Bostan, S. Boukraa, A.J. Guttmann, S. Hassani, I. Jensen, J-M. Maillard and N. Zenine, High order Fuchsian equations for the square lattice Ising model: $\tilde{\chi}^{(5)}$, J. Phys. A 42 (2009) 27209

[21] S. Boukraa, S. Hassani, I. Jensen, J-M. Maillard, N. J. Zenine *High order Fuchsian equations for the square lattice Ising model:* $\chi^{(6)}$, accepted at J. Phys. **A**: Math. Theor. (2010), arXiv:0912.4968v1 [math-ph]

[22] S. Boukraa, S. Hassani, J-M. Maillard, B.M. McCoy, J-A. Weil and N. Zenine, The diagonal Ising susceptibility, J. Phys. A 40 (2007) 8219–8236.

[23] S. Boukraa, S. Hassani, J-M. Maillard, and N. Zenine, Landau singularities and singularities of holonomic integrals of the Ising class, J. Phys. A 40 (2007) 2583-2614

[24] A. Bostan, S. Boukraa, S. Hassani, J-M. Maillard, J-A. Weil and N. Zenine, Global nilpotent differential operators and the square Ising model, J. Phys. A 40 (2007) 2583–2614.

[25] M. Jimbo and T. Miwa, Studies on holonomic quantum fields XVII, Proc. Jpn. Acad. 56A (1980) 405: 7A (1981) 347.

[26] V.V. Mangazeev and A.J. Guttmann, Form factor expansions in the 2D Ising model and Painlevé VI, ArXive.1002.2490.