J. Phys. A: Math. Gen. 18 (1985) 833-846. Printed in Great Britain

$\mathscr{S}_4$-symmetry on the checkerboard Potts model

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Received 18 September 1984

Abstract. The large q expansion of the partition function Z of the checkerboard q-state Potts model with a magnetic field is given up to sixth order in $q^{-1/2}$. Up to this order, Z is shown to exhibit an unexpected $\mathscr{S}_4$-symmetry. Using various arguments (exact results, expansions, etc) we suggest the general validity of this new symmetry property of the considered model. An exact expression for the magnetisation jump is also proposed and checked on the large q expansion.

1. Introduction

Recently, a large number of new exact results relative to the Potts model has been accumulated, so that it appears, at the present time, to be one of the most tantalising problems of statistical physics, or even of the theory of critical phenomena. For instance one should mention the recent calculation of the critical exponents of the 2D Potts model from the conformal covariance property (Zamolodchikov 1984, Dotsenko and Fateev 1984). In this paper, we will concentrate on a general and fruitful model, the checkerboard Potts model, for which some progress has been made: (a) an inversion relation has been found for this model and its partition function has been obtained at criticality (Maillard and Rammal 1983), (b) an exact expression for the latent heat has been proposed and checked on the large q expansion (Rammal and Maillard 1983), and (c) more recently, an exact disorder solution has been obtained for both the partition function and the intra-row correlation functions (Baxter 1984, Jaekel and Maillard 1984). Furthermore, an intriguing property has been suggested for this model: the partition function, as well as other quantities, are symmetric functions of the four coupling constants of the model (Jaekel and Maillard 1984).

The purpose of this paper is: (i) to show this property for the magnetisation jump of the model and (ii) to present various arguments in favour of this curious symmetry. The paper is organised as follows. In § 2 we obtain the large q expansion of the checkerboard Potts model in the presence of a magnetic field. As a byproduct, the $\mathscr{S}_4$-symmetry of the partition function is checked up to sixth order in $q^{-1/2}$. An exact expression for the magnetisation jump is proposed. Making use of the $\mathscr{S}_4$-symmetry for the partition function, our suggestion is strongly supported by the q expansion (up to order six). In § 3, all available exact results on that model are collected in order to

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confirm that $\mathscr{S}_4$-symmetry. Our conclusion (§ 4) will be devoted to discussing the relation between this symmetry and more general problems such as exact integrability and universality. Technical details relative to the large q expansion are given in the appendices.

2. Large q expansion and magnetisation jump

2.1. *Large q expansion*

The partition function per site Z, of the q-state checkerboard Potts model in the presence of a magnetic field is given by

$$Z^N(a, b, c, d; h)$$
$$= \sum_{\{\sigma\}} \prod_{\langle ij\rangle} a^{\delta_{\sigma_i,\sigma_j}} \prod_{\langle jk\rangle} b^{\delta_{\sigma_j,\sigma_k}} \prod_{\langle kl\rangle} c^{\delta_{\sigma_k,\sigma_l}} \prod_{\langle li\rangle} d^{\delta_{\sigma_l,\sigma_i}} \prod_{m} h^{\delta_{\sigma_m,0}}.$$

Here a, b, c and d denote the four coupling parameters: $a = e^{K_1}$, $b = e^{K_2}$, $c = e^{K_3}$ and $d = e^{K_4}$ respectively, where K_i ($i = 1$-4) are the coupling constants of the model (see figure 1), h denotes the 'fugacity' parameter $h = e^H$, where H is the magnetic field. Each of the N spins $\{\sigma\}$ belongs to $\mathbb{Z}_q$ and the ordered sequences $\langle ij\rangle$, $\langle jk\rangle$, $\langle kl\rangle$ and $\langle li\rangle$ denote the edges of the N plaquettes.

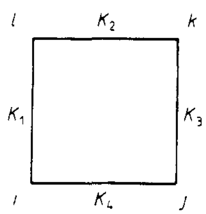


Figure 1. Elementary cell of the checkerboard Potts model.

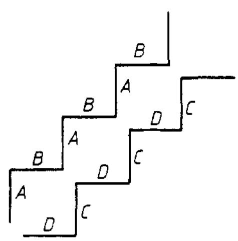


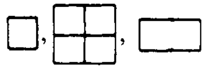
Figure 2. Diagonal transfer matrices associated with the checkerboard Potts model.

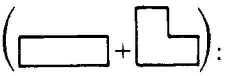
The large q expansion of the partition function Z gives the expression of the (low-temperature) normalised partition function Λ defined by $Z(a, b, c, d; h) = (abcd)^{1/2} h\Lambda(a, b, c, d; h)$.

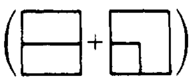
For convenience, the expansion parameters will be denoted by $A = 1/a$, $B = 1/b$, $C = 1/c$, $D = 1/d$ and $z = 1/h$. The order of each diagram appearing in the expansion will be defined by: $\theta = L - 2L'$ where L is the number of bonds and L' the number of loops. This corresponds to the order θ in $q^{-1/2}$ near criticality and to small magnetic field (A, B, C, $D \sim q^{-1/2}$, $z \sim 1$). The first terms of $\ln \Lambda$ are given by

$$\ln \Lambda\ (A, B, C, D; z)$$
$$= \square\ (q-1)zABCD \qquad (\mathrm{O}=2)$$
$$+ \square\square\ (q-1)\tfrac{1}{2}z^2(A^2B^2C^2D + A^2B^2CD^2 + A^2BC^2D^2 + AB^2C^2D^2) \qquad (\mathrm{O}=3)$$
$$+ \ldots.$$

This expansion is given up to order six ($O=6$) in appendix 1. Note that the exponent of z in this expansion is simply given by the total area of the corresponding diagram.

One remarks that this expansion is not only invariant under the action of the group of the square C_{4v} as it should be (Rammal and Maillard 1983†), but is also invariant under the action of permutation group $\mathscr{S}_4$ (symmetric group) acting on the four coupling constants A, B, C and D. This is a very curious and unexpected symmetry. In fact, such a symmetry property is obvious for many diagrams such as , ..., etc. However for the other diagrams, one has to bring them together in a specific way in order to exhibit such a symmetry. For instance

 $: (q-1)z^3(A^3B^3CD+A^3BC^3D+A^3BCD^3+AB^3C^3D+ABC^3D^3)$

 $: (q-1)(q-2)z^4(A^3B^3C^2D^2+\ldots).$

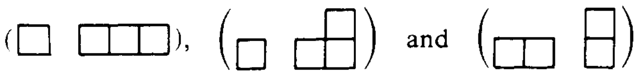
Let us mention in particular the following example of disconnected diagrams of order six. The disconnected diagrams  have a weight $x+y$ where

$$x=(-2)(q-1)^2(q-2)^2z^4(A^4B^3C^4D^3+A^3B^4C^3D^4)$$

and

$$y=(-7/4)(q-1)^2(q-2)^2z^4(A^4B^4C^4D^2+A^4B^2C^4D^4+A^4B^4C^2D^4+A^2B^4C^4D^4).$$

The y contribution is clearly $\mathscr{S}_4$-symmetric. However the x contribution has to be added to the contributions of the following disconnected diagrams



in order to get an $\mathscr{S}_4$-symmetric contribution

$$(-13)(q-1)^2(q-2)^2z^4(A^4B^4C^3D^3+\ldots).$$

One should remark that the way we are bringing the diagrams together corresponds to putting together all diagrams having the same chromatic polynomial $P(q)$ in their associated weights. This way of assembling the various diagrams can be understood as follows. If the partition function is actually $\mathscr{S}_4$-symmetric, it has to be symmetric order by order in the large q expansion, but also in the low-temperature expansion as well as in the high-field expansion. Thus, if one considers the set of diagrams having the same L (number of bonds), L' (number of loops) and S (surface) then the sum of their contributions has to be $\mathscr{S}_4$-symmetric. Note that at the lowest order, the set of triplets (L, L', S) is in one-to-one correspondence with the chromatic polynomial $P(q)$.

Due to the $\mathscr{S}_4$-symmetry, it is now convenient to introduce the canonical symmetric polynomials S_i

$$S_1=A+B+C+D \qquad S_2=AB+AC+AD+BC+BD+CD$$

$$S_3=ABC+ABD+ACD+BCD \qquad S_4=ABCD.$$

† Note a misprint in equation (2.5) of Rammal and Maillard where the partition function is written in terms of C_{4v} invariants: one should read $C_3=\frac{1}{2}S_0S_1-S_3$ instead of $\frac{1}{2}S_0-S_3$.

Using these $\mathcal{S}_4$-invariants, the expression of $\ln \Lambda$ becomes (up to the fourth order $O=4$)

$$\ln \Lambda = (q-1)zS_4 + (q-1)(q-2)\tfrac{1}{2}z^2S_3S_4 + (q-1)(q-2)(q^2-5q+7)z^4S_4^3 + (q-1)(q-2)^2z^3S_2S_4^2 + (q-1)z^2(S_3^2-2S_2S_4) - \tfrac{5}{2}(q-1)^2z^2S_4^2 + \ldots .$$

2.2. *Magnetisation discontinuity*

From the previous expression of the partition function of the checkerboard Potts model, one easily obtains the jump ΔM of the magnetisation at criticality ($T=T_c$). For $q>4$, the magnetisation M defined by

$$M = (q\langle \delta_{\sigma_0,0}\rangle - 1)/(q-1)$$

jumps discontinuously at the critical temperature T_c from zero ($M=0$ at $T>T_c$) to a positive value ($M>0$ at $T \leqslant T_c$). M is given mainly by

$$\langle \delta_{\sigma_0,0}\rangle = h\partial \ln Z/\partial h|_{h=1} = 1 - z\partial \ln \Lambda/\partial z|_{z=1}.$$

Kim (1981) has rightly conjectured that ΔM is the same for the square, triangular and honeycomb lattices. This common value of M which can be viewed as a consequence of the star-triangle relation has been calculated using corner transfer matrices (Baxter 1982). The result is

$$\Delta M = \prod_{n=1}^{\infty}\left(\frac{1-x^{2n-1}}{1-x^{2n}}\right) = \prod_{n=1}^{\infty}\left(\frac{1-x^n}{1-x^{4n}}\right) \tag{1}$$

where x is defined ($q>4$) by

$$q = x + 2 + x^{-1}, \qquad 0<x<1.$$

This exact expression (equation (1)) agrees with the large q expansion on the isotropic Potts model on the square lattice (Kim 1981)

$$\Delta M = 1 + q^{-1} - 3q^{-2} - 9q^{-3} - 27q^{-4} + \ldots .$$

It is natural to ask if this expression also holds for the checkerboard Potts model which reduces in different limits to the anisotropic (and of course isotropic) square, triangular and honeycomb lattices. Such a question is suggested also by another limit. Let us suppose that A, B, C and D are solutions of the following equations

$$(q-1)AB = 1-(A+B), \qquad (q-1)CD = 1-(C+D). \tag{2}$$

These two equations are easily seen to be consistent with the critical condition for the checkerboard Potts model

$$(q-1)(q-3)S_4 = 1-(q-2)S_3 - S_2. \tag{3}$$

A straightforward generalisation of Stephen and Mittag's argument, based on the star-triangle relation, enables us to show that the diagonal transfer matrices (see figure 2) $T(A, B)$ and $T(C, D)$ commute when equations (2) are satisfied (Stephen and

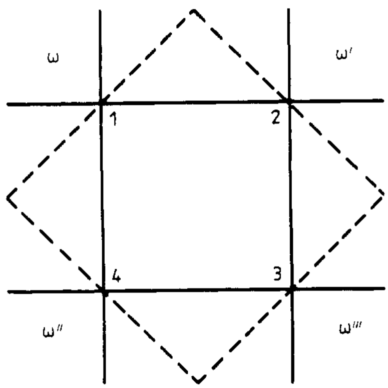


Figure 3. The staggered vertex, of respective weights ω, ω', ω'' and ω''' of the equivalent inhomogeneous six vertex model (see text).

Mittag 1972). One thus exhibits a family of commuting matrices, sharing the same eigenvector $|\psi_0\rangle$ associated with the largest eigenvalue. Obviously, this eigenvector can only be a function of q and is independent of A (or C resp. B and D). Therefore, the magnetisation jump, related to the expression $\langle\psi_0|\delta_{\sigma_0,0}|\psi_0\rangle$ is a function of q only and thus equal to ΔM given by equation (1).

From equations (2), one easily obtains the following condition

$$(q-1)^3 S_4 = (q-3) - (q-2)S_1 + (q-1)S_2 \tag{4a}$$

or when combined with the critical condition (equation (3))

$$1 + (q-3)S_1 = 2(q-1)S_2 + (q-1)^2 S_3. \tag{4b}$$

To sum up, the magnetisation jump (restricted by definition to the critical variety (equation (3))) reduces to the expression given by equation (1) on the following algebraic variety: (4*b*) and $A = 0$ or 1 (resp B, C, D). Therefore, if one believes that the partition function is $\mathcal{S}_4$-symmetric in the presence of a magnetic field, then M (viewed as a function of S_1, S_2 and S_3 after eliminating S_4) is equal to the exact expression (equation (1)), on three hyperplanes

$$1 + (q-3)S_1 = 2(q-1)S_2 + (q-1)^2 S_3 \tag{4b}$$

$$(q-3)S_3 + S_2 = 1 \tag{5}$$

and

$$(q-1)(q-3)S_1 + (q^2-3q+1)S_3 - (q^2-4q+2)S_2 = q^2 - 4q + 5. \tag{6}$$

We are now in a position to check, on the large q expansion, if ΔM is equal to the expression given by equation (1) and thus does not depend on $\{S_i\}$. The full expansion of M up to order six in $q^{-1/2}$, is given in appendix 2, in terms of $\{S_i\}$.

From (1): $\Delta M = 1 - (q-3)^{-1} + \mathrm{O}(q^{-5})$. Therefore we have to check that the expansion of $z(\partial/\partial z)\ln\Lambda(S_1, S_2, S_3, S_4)$ at $z = 1$ reduces at criticality to $(q-3)^{-1} - q^{-2} - 3q^{-3}$, up to order six in $q^{-1/2}$.

For convenience, we will denote $(q-1)^{-1}$ and $(q-3)^{-1}$ by α and β respectively. Using (3), S_4 can be replaced by $\alpha\beta[1-(q-2)S_3 - S_2]$. A somewhat tedious algebra yields, at the fourth order ($\mathrm{O} = 4$)

$$z(\partial/\partial z)\ln\Lambda|_{z=1,T=T_c}$$
$$=(q-1)\alpha\beta[1-(q-2)S_3-S_2]+(q-1)(q-2)\alpha\beta S_3[1-(q-2)S_3+\ldots]$$
$$+4(q-1)(q-2)(q^2-5q+7)\alpha^3\beta^3+3(q-1)(q-2)^2\alpha^2\beta^2S_2$$
$$+(q-1)S_3^2-2(q-1)\alpha\beta S_2-5(q-1)^2\alpha^2\beta^2$$
$$=(q-3)^{-1}-q^{-2}-14q^{-3}+\text{terms of order eight }(\mathrm{O}=8)\text{ with }S_2\text{ and }S_3^2.$$

At orders five and six ($\mathrm{O}=5,6$) the cancellation of the S_i terms is somewhat spectacular: cancellation of 58 terms! At order five, only S_1, S_2S_3 and S_3 occur but give a vanishing contribution. Furthermore there is no constant term. At order six, S_3^2, S_2^2, S_1S_3, $S_2S_3^2$ and S_2 are involved but their contributions vanish. The constant term appearing at this order is given by

$$10(q-1)(q-2)^4\alpha^4\beta^4-60(q-1)(q-2)^4\alpha^4\beta^4+65(q-1)$$
$$\times(q-2)^4\alpha^4\beta^4-28(q-1)(q-2)^2(q^2-5q+7)\alpha^5\beta^5$$
$$+70(q-1)(q-2)^2(q^2-5q+7)^2\alpha^5\beta^5+9(q-1)(q-2)^2$$
$$\times(q^2-5q+7)\alpha^5\beta^5+6(q-1)\alpha^2\beta^2+4(q-1)\alpha^2\beta^2$$
$$+16(q-1)(q-2)^2\alpha^3\beta^3+8(q-1)(q-2)^2\alpha^3\beta^3-48(q-1)$$
$$\times(q-2)^2\alpha^3\beta^3-24(q-1)(q-2)(q-3)(q^2-5q+7)\alpha^4\beta^4$$
$$+12(q-1)(q-2)^2(q-3)^2\alpha^4\beta^4-60(q-1)^2$$
$$\times(q-2)(q^2-5q+7)\alpha^4\beta^4+31(q-1)^3\alpha^3\beta^3$$
$$=11q^{-3}+\text{higher order terms.}$$

Therefore, using the $\mathcal{S}_4$-symmetry of the partition function (up to order six), we have been able to give a strong indication that equation (1) is the exact expression of the magnetisation jump on the checkerboard Potts model.

3. $\mathcal{S}_4$-symmetry

In this section we shall collect all pieces of information in favour of the $\mathcal{S}_4$-symmetry suggested in the previous sections. This information is of very different origin but is of a complementary nature.

3.1. *Exact results at $T=T_c$*

The partition function of the checkerboard Potts model has been calculated exactly at $T=T_c$ and checked on the large q expansion (Maillard and Rammal 1983). It is obviously $\mathcal{S}_4$-symmetric. Furthermore an expression for the latent heat has also been conjectured and checked on the large q expansion (Rammal and Maillard 1983). Of course, an exact $\mathcal{S}_4$-symmetric expression for the internal energy (at criticality) can also be obtained in the same spirit. These features at $T=T_c$ are to be considered on the same level as the magnetisation jump, where an exact expression has been suggested in the previous section.

3.2. *Exact results at $q=2$ and $q\to 0$*

(a) *The $q=2$ Ising case.* In the limit $q=2$ the checkerboard Potts model reduces to the so-called generalised square Ising model. The corresponding partition function in zero field has been calculated by Utiyama (see e.g. Domb and Green 1972). Furthermore, it has been shown that the partition function of this model is actually $\mathscr{S}_4$-symmetric (Jaekel and Maillard 1984). Such a symmetry is also obvious on the expression of the magnetisation (see e.g. Domb and Green 1972). Of course there is no exact expression for the magnetic susceptibility, χ, but the approximate expression for χ proposed by Syozi and Naya (1960) also exhibits $\mathscr{S}_4$-symmetry. Finally, using the star-triangle relation and considering this model as a solvable inhomogeneous eight vertex model, it can be shown that the partition function has the following form (Baxter, private communication)

$$\sum_{i=1}^{4} \phi(K_i, k(K_1, K_2, K_3, K_4))$$

where k denotes a symmetric function of the four coupling constants K_i. In this respect, the $\mathscr{S}_4$-symmetry appears in this particular case as a consequence of the star-triangle relation.

(b) *The $q\to 0$ case.* It is known that the partition function of the Potts model can be related to that of an ice-type problem on the related medial lattice (Baxter *et al* 1976). It has been noticed that the resulting ice-type model is soluble in the $q\to 0$ limit: in this limit the corresponding weights satisfy the free-fermion condition (Fan and Wu 1970). The corresponding partition function in the case of the checkerboard model has been examined in detail by Lin and Tang (1976). Therefore one can either consider Baxter's argument, based on the star-triangle relation (Z-invariance) in order to prove the $\mathscr{S}_4$-symmetry, or check this symmetry directly on the expression of the partition function as shown in appendix 3.

3.3. *Exact results on some algebraic varieties*

A disorder solution for the checkerboard Potts model has been recently obtained (Baxter 1984, Jaekel and Maillard 1984). The exact expression for the partition function as well as the intra-row correlation functions are known on the algebraic variety of codimension one

$$\frac{d-1}{(q-1)d+1}+\frac{a-1}{a+q-1}\,\frac{b-1}{b+q-1}\,\frac{c-1}{c+q-1}=0. \tag{7}$$

The corresponding partition function is clearly symmetric in the three parameters a, b and c. It is quite surprising to see that although the parameter b enters in a very different way from a and c, in Baxter's proof (1984) the final result restores that symmetry in agreement with the $\mathscr{S}_4$-symmetry of the partition function.

3.4. *Various expansions*

In addition to all these exact results, different expansions also seem to confirm the $\mathscr{S}_4$-symmetry.

(a) *Large q expansions.* As shown in § 2, it is useful to obtain the large q expansion up to order seven or eight. In this respect it is appropriate to concentrate on some families of diagrams sharing the same chromatic polynomial and check the $\mathcal{S}_4$-symmetry as outlined in § 2.

(b) *High (resp. low) temperature expansion.* A resumed high temperature expansion has been obtained recently for the checkerboard Potts model (Jaekel and Maillard 1984). Using duality it is easy to deduce the corresponding low temperature expansion. Expanding these results one gets the low T (resp. high T) expansion up to order eight in A, B, C and D (resp$(1-A)/[1+(q-1)A]$, etc.). Both expansions exhibit the $\mathcal{S}_4$-symmetry.

4. Conclusion

In this paper, we have presented different arguments in favour of the existence of an $\mathcal{S}_4$-symmetry on the checkerboard Potts model. Some of these arguments are directly related to the exact solubility of the model in certain limits ($T=T_c$, $q=2$, $q\to 0$). Therefore, the $\mathcal{S}_4$-symmetry appears first as a consequence of the star-triangle relation. However, there are also strong indications (large q expansion, disorder solution) that this symmetry also exists in general. Clearly, such a symmetry (if exact) should correspond to a very intriguing structure of the Potts model. There is no obvious proof based on some local symmetry. Furthermore, this symmetry seems to be broken on finite size (even with periodic boundary conditions) systems. It would be very interesting to understand the origin of that curious symmetry: does it come from some topological or combinatorial theorem (one can think for instance of the proof given by Sherman (1960, 1963) of Feynmann's counting rule for 2D Ising model) or is it possible to deduce it from simple and general ideas?

That symmetry, if any, could also shed some light on the analytical behaviour of the partition function and other quantities near criticality. From the universality hypothesis, the leading critical exponents should not depend on the particular lattice: square, triangular, honeycomb. Obviously, these three lattices can be viewed as appropriate limits of the checkerboard Potts model. A reasonable expression ($q<4$) for the singular part of the free energy is

$$f_{\text{sing}} \sim |z-1|^{2-\alpha}$$

where α does not depend on A, B, C, and D but only on q and z. Here z denotes a symmetric function of A, B, C and D: the fugacity variable of Hintermann *et al* (1978) (see also Rammal and Maillard 1983).

Beyond this known universality property for the partition function, we have seen that the magnetisation jump is a universal quantity ($q>4$) independent of A, B, C and D: this behaviour is to be compared with the exact result for the latent heat. In this case, by dividing this expression by some simple and $\mathcal{S}_4$-symmetric function, one gets the spontaneous staggered polarisation of the six vertex model, which depends only on q (Rammal and Maillard 1983). Because the $\mathcal{S}_4$-symmetry of the critical condition (equation (3)), these remarkable dependences of the checkerboard Potts model on the parameters A, B, C and D seem difficult to understand if the partition function (with or without field) is not $\mathcal{S}_4$-symmetric, at least near criticality.

Acknowledgments

We would like to thank Professor R Baxter and Dr M T Jaekel for helpful discussions.

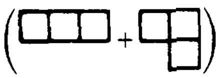
Appendix 1

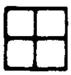
The large q expansion of the normalised partition function per site, for the checkerboard Potts model in the presence of a magnetic field up to order six in $q^{-1/2}$ (A, B, C, $D \sim q^{-1/2}$) is given by

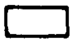
$\ln \Lambda(A, B, C, D; z)$

=  $(q-1)zABCD$ $(O=2)$

\+  $\frac{1}{2}(q-1)(q-2)z^2(AB^2C^2D^2+\dots)$ $(O=3)$

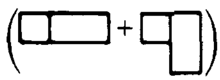
\+  $(q-1)(q-2)^2z^3z^3(A^3C^3B^2D^2+\dots)$ $(O=4)$

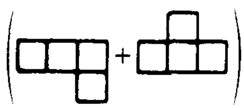
\+  $(q-1)(q-2)(q^2-5q+7)z^4A^3B^3C^3D^3$

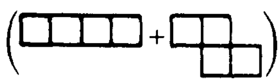
\+  $\frac{1}{2}(q-1)z^2(B^2C^2D^2+\dots)$

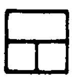
\+  $-\frac{5}{2}(q-1)^2z^2A^2B^2C^2D^2$

\+  $-4(q-1)^2(q-2)z^3(A^2B^3C^3D^3+\dots)$ $(O=5)$

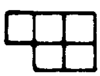
\+  $(q-1)(q-2)z^3(A^3C^3BD^2+\dots)$

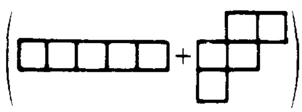
\+  $3(q-1)(q-2)^3z^4(A^3B^3C^3D^4+\dots)$

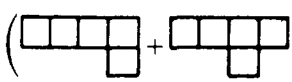
\+  $\frac{1}{2}(q-1)(q-2)^3z^4(A^4C^4B^2D^3+\dots)$

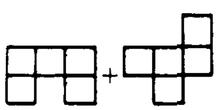
\+  $(q-1)(q-2)(q-3)z^4(A^3B^3C^3D^2+\dots)$

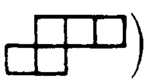
\+  $\frac{1}{2}(q-1)(q-2)(q^2-5q+7)^2z^6(A^4B^4C^4D^5+\dots)$

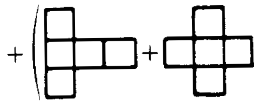
\+  $2(q-1)(q-2)^2(q^2-5q+7)z^5(A^4B^4C^4D^3+\dots)$

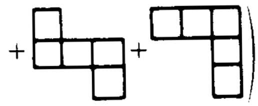
\+  $(q-1)(q-2)^4z(A^5B^5C^3D^3+\dots)$ $(O=6)$

\+ 

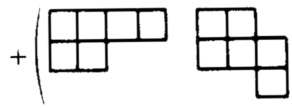
\+ 

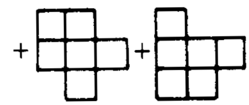
\+  $3(q-1)(q-2)^4z^5(A^4B^4C^3D^5+\dots)$

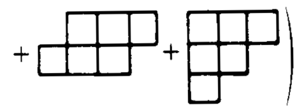




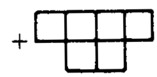
$13(q-1)(q-2)^4z^5A^4B^4C^4D^4$



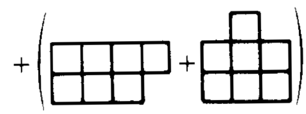




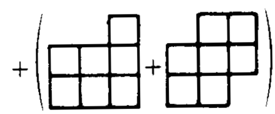
$6(q-1)(q-2)^3(q^2-5q+7)z^6(A^4B^4C^5D^5+\ldots)$



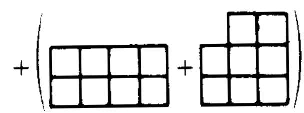
$(q-1)(q-2)^3(q^2-5q+7)z^6(A^5B^5C^5D^3+\ldots)$



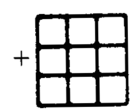
$(q-1)(q-2)^2(q^2-5q+7)^2z^7(A^5B^5C^4D^6+\ldots)$



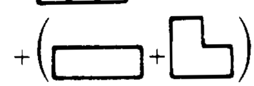
$10(q-1)(q-2)^2(q-5q+7)^2z^7A^5B^5C^5D^5$



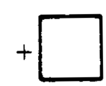
$(q-1)(q-2)(q^2-5q+7)^3z^8(A^5B^5C^6D^6+\ldots)$



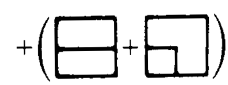
$(q-1)(q-2)(q^7+\ldots)z^9A^6B^6C^6D^6$



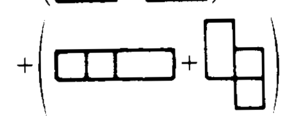
$(q-1)z^3(A^3B^3CD+\ldots)$



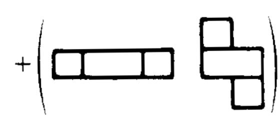
$(q-1)z^4A^2B^2C^2D^2$



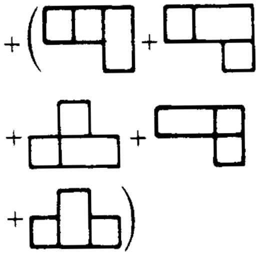
$(q-1)(q-2)z^4(A^2B^2C^3D^3+\ldots)$



$(q-1)(q-2)^2z^4(A^4B^4CD^3+\ldots)$

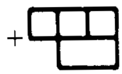


$(q-1)(q-2)^2z^4(A^4B^4C^2D^2+\ldots)$

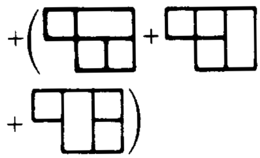


$3(q-1)(q-2)^2 z^4 (A^3B^3C^2D^4+\ldots)$

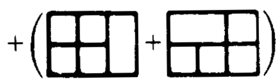
$2(q-1)(q-2)(q^2-5q+7)z^5(A^4B^4C^4D^2+\ldots)$



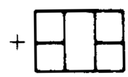
$2(q-1)(q-2)^2(q-3)z^5(A^4B^4C^4D^2+\ldots)$



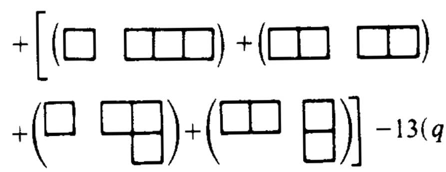
$4(q-1)(q-2)^2(q-3)z^5(A^4B^4C^3D^3+\ldots)$



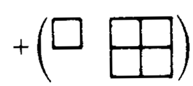
$(q-1)(q-2)(q-3)(q^2-5q+7)z^6(A^4B^4C^3D^5+\ldots)$



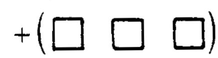
$2(q-1)(q-2)^2(q-3)^2z^6A^4B^4C^4D^4$



$-13(q-1)^2(q-2)^2z^4(A^4B^4C^3D^3+\ldots)$

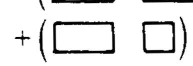


$-12(q-1)^2(q-2)(q^2-5q+7)z^5A^4B^4C^4D^4$



$\frac{31}{3}(q-1)^3z^3A^3B^3C^3D^3$

$-\frac{7}{4}(q-1)^2(q-2)^2z^4(A^4B^4C^4D^2+\ldots)$



$-4(q-1)^2z^3(A^3B^3C^3D+\ldots)$

In the above expressions the dots ... denote all the terms obtained from the first term by the action of the symmetric group $\mathcal{S}_4$.

Appendix 2

Due to the $\mathcal{S}_4$-symmetry of $\ln\Lambda$ in the variables A, B, C and D, it is possible to rewrite it in terms of the symmetric polynomials of A, B, C and D: S_1, S_2, S_3, S_4. For example

$$A^3B^3C^3D+\ldots = S_4(S_3^2-2S_2S_4)$$

$$A^5B^5C^3D^3+\ldots = S_4^3(S_2^2+2S_4-2S_1S_3)$$

$$A^4B^3C^4D+\ldots = S_4(S_2S_3^2-S_1S_3S_4-2S_2^2S_4+4S_4^2).$$

When all these calculations are performed, one gets the following expansion for $z(\partial/\partial z)\ln\Lambda|_{z=1}$

$$
\begin{aligned}
z(\partial/\partial z)\ln\Lambda|_{z=1}
&= (q-1)S_4+(q-1)(q-2)S_4+4(q-1)(q-2)(q^2-5q+7)S_4^3\\
&\quad+3(q-1)(q-2)^2S_2S_4^2+(q-1)(S_3^2-2S_2S_4)-5(q-1)^2S_4^2\\
&\quad+3(q-1)(q-2)S_4(S_2S_3-3S_1S_4)-12(q-1)^2(q-2)S_4^2S_3\\
&\quad+2(q-1)(q-2)^3S_4^2(S_2S_3-3S_1S_4)+12(q-1)(q-2)^3S_4^3S_1\\
&\quad+4(q-1)(q-2)(q-3)S_4^2S_3+3(q-1)(q-2)(q^2-5q+7)^2S_4^4S_1\\
&\quad+10(q-1)(q-2)^2(q^2-5q+7)S_4^3S_3\\
&\quad+5(q-1)(q-2)^4S_4^3(S_2^2+2S_4-2S_1S_3)\\
&\quad+15(q-1)(q-2)^4S_4^3(S_1S_3-4S_4)\\
&\quad+65(q-1)(q-2)^4S_4^4+36(q-1)(q-2)^3(q^2-5q+7)S_4^4S_2\\
&\quad+6(q-1)(q-2)^3(q^2-5q+7)S_4^3(S_3^2-2S_2S_4)\\
&\quad+7(q-1)(q-2)^2(q^2-5q+7)^2S_4^4(S_1S_3-4S_4)\\
&\quad+70(q-1)(q-2)^2(q^2-5q+7)S_4^5+8(q-1)(q-2)(q^2-5q+7)^3S_4^5S_2\\
&\quad+9(q-1)(q-2)(q^7+\ldots)S_4^6+3(q-1)S_4(S_2^2+2S_4-2S_1S_3)\\
&\quad+4(q-1)S_4^2+4(q-1)(q-2)S_4^2S_2\\
&\quad+4(q-1)(q-2)^2S_4(S_2S_3^2-S_1S_3S_4-2S_2^2S_4+4S_4^2)\\
&\quad+4(q-1)(q-2)^2S_4^2(S_2^2+2S_4-2S_1S_3)\\
&\quad+12(q-1)(q-2)^2S_4^2(S_1S_3-4S_4)\\
&\quad+20(q-1)(q-2)(q^2-5q+7)S_4^2(S_3^2-2S_2S_4)\\
&\quad+20(q-1)(q-2)^2(q-3)S_4^3S_2+6(q-1)(q-2)\\
&\quad\times(q-3)(q^2-5q+7)S_4^3(S_1S_3-4S_4)\\
&\quad+12(q-1)(q-2)^2(q-3)^2S_4^4-52(q-1)^2(q-2)^2S_4^3S_2\\
&\quad-60(q-1)^2(q-2)(q^2-5q+7)S_4^4+31(q-1)^3S_4^3\\
&\quad-12(q-1)^2S_4(S_3^2-2S_2S_4)-7(q-1)^2(q-2)^2S_4^2(S_3^2-2S_2S_4).
\end{aligned}
$$

Appendix 3

The checkerboard Potts model is equivalent to an inhomogenous ice-rule six vertex model on a square lattice. For this particular model, one has four sublattices on the corresponding vertex model, denoted by ω, ω', ω'' and ω''' (see figure 3). The associated weights are given by (Baxter *et al* 1978)

$$(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6)=(1,1,x_i,x_i,A_i,B_i)$$

for $i=1$ and 3 and respectively the vertex ω and ω'''.

$$(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6) = (x_i, x_i, 1, 1, B_i, A_i)$$

for $i=2$ and 4 and respectively the vertex ω' and ω''. Here

$$x_i = q^{-1/2}(\mathrm{e}^{K_i} - 1), \qquad A_i = t^{-1/2} + x_i t^{1/2}, \qquad B_i = t^{1/2} + x_i t^{-1/2},$$

and

$$t + t^{-1} = q^{1/2}.$$

In an appropriate limit $q \to 0$ $(x_i q^{1/2} \to 0)$ the model can be solved exactly because the free-fermion condition (Fan and Wu (1970) becomes satisfied for each vertex. The exact expression for the partitiòn function is given by (Lin and Tang 1976)

$$\frac{1}{8\pi^2} \int_{-\pi}^{\pi} \mathrm{d}\theta \int_{-\pi}^{\pi} \mathrm{d}\phi \, \ln|F(\theta, \phi)|$$

where

$$F(\theta, \phi) = a + 2b \cos\theta + 2c \cos\phi - 2f \cos(\theta + \phi) - 2g \cos(\theta - \phi)$$

and

$$a = 2(1 + x_1 x_2 x_3 x_4) + A_1 A_2 A_3 A_4 + B_1 B_2 B_3 B_4,$$

$$b = x_1 x_2 + x_3 x_4,$$

$$c = x_1 x_4 + x_2 x_3,$$

$$f = x_2 x_4, \qquad g = x_1 x_3.$$

One notes the similarity of this expression with that obtained by Utiyama (1951) for the $q=2$ limit of the checkerboard Potts model. Clearly the argument $F(\theta, \phi)$ is C_{4v} invariant (symmetry of the square). In order to prove the $\mathcal{S}_4$-symmetry of the partition function we need only the invariance under the transposition $K_1 \leftrightarrow K_4$. For this, let us rewrite $F(\theta, \phi)$ as: $\alpha + \beta \cos\theta + \gamma \sin\theta$ where

$$\alpha = a + 2c \cos\phi,$$

$$\beta = 2b - 2(f+g)\cos\phi,$$

and

$$\gamma = 2(f-g)\sin\phi.$$

The integration over θ is easily performed and gives $2\pi \ln \frac{1}{2}[\alpha + (\alpha^2 - \beta^2 - \gamma^2)^{1/2}]$. α is clearly invariant under the transposition $K_1 \leftrightarrow K_4$ and $\beta^2 + \gamma^2 = 4[(b^2 + f^2 + g^2) + 2fg \cos 2\phi - 2b(f+g)\cos\phi]$ is also invariant under that transposition.

References

Baxter R J 1982 *J. Phys. A: Math. Gen.* **15** 3329

—— 1984 *J. Phys. A: Math. Gen.* **17** L911

Baxter R J, Kelland S B and Wu F Y 1976 *J. Phys. A: Math. Gen.* **9** 397

Baxter R J, Temperley H N V and Ashley S E 1978 *Proc. R. Soc.* A **358** 535

Domb C and Green M S (eds) 1972 *Phase transitions and critical phenomena* vol 1 (London: Academic) pp 285, 323

Dotsenko V S and Fateev V A 1984 *Conformal algebra and multipoint correlation function in 2D statistical models, Nordita preprint*
Fan C and Wu F Y 1970 *Phys. Rev.* B **2** 72
Hintermann A, Kunz H and Wu F Y 1978 *J. Stat. Phys.* **19** 623
Jaekel M T and Maillard J M 1984 *J. Phys. A: Math. Gen.* **17** 2079
Kim D 1981 *Phys. Lett.* **87A** 127
Lin K Y and Tang D L 1976 *J. Phys. A: Math. Gen.* **9** 1685
Maillard J M and Rammal R 1983 *J. Phys. A: Math. Gen.* **16** 353
Rammal R and Maillard J M 1983 *J. Phys. A: Math. Gen.* **16** 1073
Sherman S 1960 *J. Math. Phys.* **1** 202
—— 1963 *J. Math. Phys.* **4** 1213
Stephen M J and Mittag L 1972 *J. Math. Phys.* **13** 1944
Syozi I and Naya S 1960 *Progr. Theor. Phys.* **24** 829
Utiyama T 1951 *Progr. Theor. Phys.* **6** 907
Zamolodchikov A B 1984 *Proceeding of Brasov Summer School (1983) (Birkhauser Editors)* to appear