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GAUSSIAN ORTHOGONAL ENSEMBLE FOR THE LEVEL SPACING STATISTICS OF THE QUANTUM FOUR-STATE CHIRAL POTTS MODEL

J.-Ch. ANGLÈS d'AURIAC and S. DOMMANGE

CRTBT, CNRS, BP 166X
25 Avenue des Martyrs, 38042 Grenoble Cedex, France

J.-M. MAILLARD and C. M. VIALLET

LPTHE, Tour 16, 1er étage, boite 126,
4 Place Jussieu, 75252 Paris Cedex 05, France

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We have performed a Random Matrix Theory (RMT) analysis of the quantum four state chiral Potts chain for different sizes of the quantum chain up to eight sites, and for different unfolding methods. Our analysis shows that one generically has a Gaussian Orthogonal Ensemble statistics for the unfolded spectrum instead of the GUE statistics one could expect. Furthermore a change from the generic GOE distribution to a Poisson distribution occurs when the hamiltonian becomes integrable. Therefore, the RMT analysis can be seen as a detector of "higher genus integrability".

Introduction : the quantum chiral Potts chain

Since the pioneering work of Wigner[1] and Dyson,[2] Random Matrix Theory (RMT) has been applied successfully in various domains of physics. One motivation is to describe, in a united universal framework, various phenomena implying chaos[3] or at least complexity. An extreme case is the emergence of integrability which manifests itself in the drastic change of the generic wignerian energy level spacing distribution into poissonian distribution. The first examples of this connection emerged when one considered simple harmonic oscillators or free fermions models. This reduction to Poisson distribution reflects nothing but the *independence* of the eigenvalues. At this point it is natural to ask whether this link between Poisson reduction and Yang–Baxter integrability still holds when the solutions of the Yang–Baxter equations are no longer parametrized in terms of abelian varieties. The perfect example to address this question is the chiral Potts model for which Au-Yang et-al have found a *higher genus* Yang–Baxter solution.[4] The Hamiltonian of the quantum chiral Potts chain first introduced by Howes, Kadanoff and den Nijs[5] and also by von Gehlen and matrices and the level spacing distribution is close to a Poissonian (exponential) distribution, $P(s) = \exp(-s)$. For non-integrable systems it can be the Gaussian Orthogonal Ensemble (GOE), the Gaussian Unitary Ensemble (GUE), or the Gaussian Symplectic Ensemble (GSE), depending on the symmetries of the model under consideration. If the hamiltonian is time reversal invariant[13] the level spacing distribution is either described by the Gaussian Orthogonal Ensemble (GOE), or by the Gaussian Symplectic Ensemble (GSE):

$$P_{\mathrm{GOE}}(s) = \frac{\pi}{2} s \exp(-\pi s^2/4) \,, \qquad P_{\mathrm{GSE}}(s) = B^3 s^4 \exp(-B s^2) \tag{4}$$

where $B = \left(\frac{8}{3}\right)^2 \frac{1}{\pi} \simeq 2.263$. Note that GOE can also occurs in a slightly more general framework ("false" time-reversal violation, A-adapted basis[12]). When one does not have any time-reversal symmetry (or "false time-reversal symmetry") the Gaussian Unitary Ensemble distribution should appear :

$$P_{\mathrm{GUE}}(s) = \frac{32}{\pi^2} s^2 \exp(-4s^2/\pi) \tag{5}$$

To quantify the "degree" of level repulsion, it may be convenient to use a parametrized distribution which interpolates between the Poisson law and the GOE Wigner law. Among the many possible distributions we have chosen the *Brody distribution*:

$$P_\beta(s) = (1+\beta)\, c_2\, s^\beta \exp(-c_2 s^{\beta+1}) \,, \quad \text{with} \quad c_2 = \left[\Gamma\left(\frac{\beta+2}{\beta+1}\right)\right]^{1+\beta} \tag{6}$$

1.1. *Representation theory*

In the presence of symmetries, one should distinguish eigenstates according to their quantum numbers. This is an *essential requirement of the method.* For instance both lattice shift and shift of colour commute with the hamiltonian H. They generate a symmetry group $S = Z_L \otimes Z_4$ which does not depend on the parameters $\alpha_i, \overline{\alpha_i}$ of the hamiltonian H. Since the group $S = Z_L \otimes Z_4$ is abelian one may diagonalize simultaneously all the elements of the group S as well as the hamiltonian H on the S-invariant spaces. This amounts to block-diagonalizing H and to split the spectrum of H into the many spectra of each block. The construction of the projectors is done with the help of the character table of irreducible representations of the symmetry group. Details can be found in Ref. 10 and Ref. 14.

In this work we concentrate on the four-state case ($N = 4$) of the quantum hamiltonian (1). For generic r and n in parametrization Eq. (3), the total symmetry group is $Z_L \otimes Z_4$. Since the characters of $Z_L \otimes Z_4$ are complex, one has to use complex numbers even though the final results are real, which increases the programming difficulties. We always restricted ourselves to hermitian hamiltonians. Consequently the blocks are also hermitian and there are only *real* eigenvalues. The diagonalization is performed using standard methods of linear algebra (contained in the LAPACK library).

various blocks (representations) is not significantly different. We also compared four different unfolding procedures, again getting similar results. We display the results on the largest size $L = 8$ for the best unfolding procedure, namely the gaussian unfolding.

Figure 1 shows the level spacing distribution $P(s)$, for the representation (0,0) and for $r = 0.5$, $n = 2.1$, and $t = 1.5$, which corresponds to $\alpha_1 = \alpha_3^\star = 1.225 + i\,0.707$, $\alpha_2 = t = 1.5$, $\overline{\alpha_1} = \overline{\alpha_3}^\star = 2.337 + i\,1.833$ and $\overline{\alpha_2} = 2.1$.

This figure shows the energy level spacing distribution and the corresponding brody fit (6) for the (least square) best value found to be $\beta_{brody} = 0.99$. On the same figure the GOE level spacing distribution is also displayed, both curves are almost indistinguishable. The GUE or GSE level spacing distribution are *clearly ruled out*, as well, of course, as the Poisson distribution. Very similar results are obtained for all the distributions corresponding to the other representations and other values away from the integrability value $\alpha_2 = t = 1$.

Let us now consider the (higher genus) integrable case which corresponds, with our parametrization, to $\alpha_2 = t = 1$.

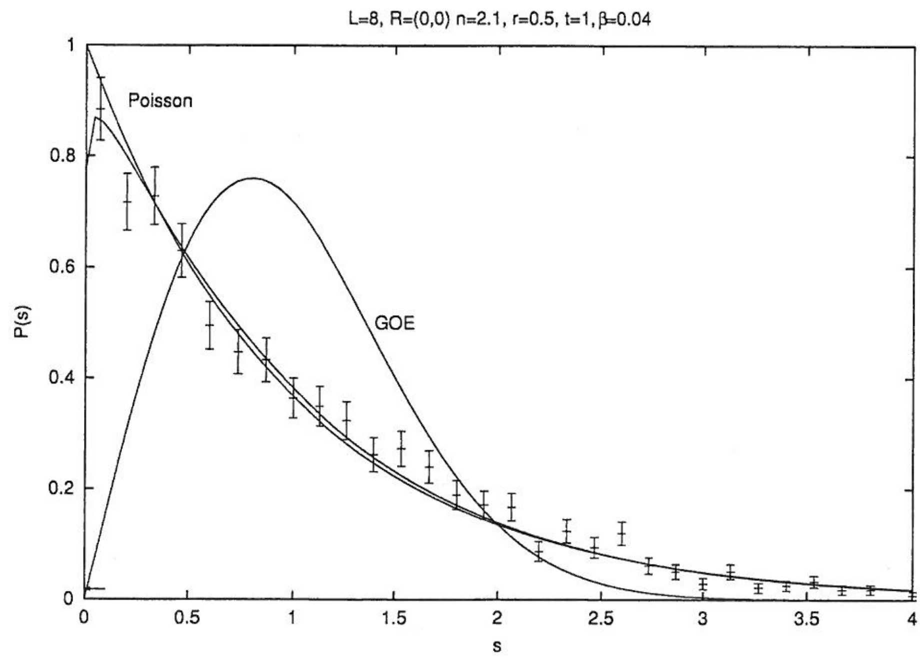

Fig. 2. Level spacing distribution on the integrability variety.

Figure 2 displays the level spacing distribution, compared to a Poisson distribution (and also to the GOE level spacing distribution), for the integrable case to the GOE value $\beta_{brody} = 1$ for every value of the parameter t, except at point $t = 1$, where the Poisson value $\beta_{brody} = 0$ should occur.

References

1. E. P. Wigner, Ann. Math. **62**, 548 (1955); **65**, 203 (1957); **67**, 325 (1958)
2. F. J. Dyson, J. Math. Phys. **3**, 140 (1962). F. J. Dyson, J. Math. Phys. **3**, 157 (1962), F. J. Dyson, J. Math. Phys. **3**, 166 (1962).
3. M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, Interdisciplinary Applied Mathematics, Springer-Verlag, New York Berlin Heidelberg, 1990.
4. R. J. Baxter, J. H. H. Perk and H. Au-Yang, Phys. Lett. **A 128**, 138 (1988)
5. S. Howes, L. P. Kadanoff and M. den Nijs, Nucl. Phys. **B 215**, 169 (1983).
6. G. von Gehlen and V. Rittenberg, Nucl. Phys. **B 257**, 351 (1985).
7. L. Dolan and M. Grady, Phys. Rev. **D 25**, 1587 (1982).
8. G. Albertini, S. Dasmahapatra and B. M. McCoy, *Exact Spectrum of the 3-state Potts Chain*, in *Yang–Baxter Equations in Paris*, edited by J-M. Maillard (World Scientific, 1992)
9. H. Au-Yang, B. M. McCoy, J. H. H. Perk, S. Tang and M-L. Yan, Phys. Lett. **A 123**, 219 (1987)
10. H. Bruus and J.-C. Anglès d'Auriac, cond-mat/9610142 (1996). H. Bruus and J.-C. Anglès d'Auriac, Phys. Rev. **B 55**, 9142 (1997).
11. D. Poilblanc, T. Ziman, J. Bellisard, F. Mila, and G. Montambaux, Europhys. Lett. **22**, 537 (1993).
12. M. Robnik and M. V. Berry, J. Phys. **A 19**, 669 (1986).
13. E. P. Wigner, *Group Theory and its application to the quantum mechanics of atomic spectra*, (Academic Press, New-York and London, 1959).
14. H. Meyer, Ph.D. thesis, Univ. J. Fourier, Grenoble, France, 1996.