R-matrices and symmetric spaces ☆

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Non-antisymmetric R-matrices which play a central role in the description of the Poisson structure of integrable models, are investigated. A wide class of such solutions is described. They are associated with the symmetric Lie algebras constructed from simple Lie algebras. This extends the classification of skew-symmetric matrices achieved by Belavin and Drinfel'd.

1. Introduction

Recent studies have enabled us to construct a number of examples of non-skew-symmetric solutions of the classical Yang–Baxter equation, describing the Poisson bracket structure of the Lax operator for an integrable classical model [1–3]. We recall that such models are described by a so-called Lax equation of motion [5]

$$\frac{\mathrm{d}L}{\mathrm{d}t}=[L,M]\,, \tag{1}$$

L, M belonging to a Lie algebra $\mathscr{G}$, for example sl(n, $\mathbb{C}$) represented by $n\times n$ matrices. The Poisson algebra of L is described by an operator R living in $\mathscr{G}\otimes\mathscr{G}$ and acting on a tensor product of vector spaces of representation $\mathscr{E}\otimes\mathscr{E}$ such that

$$\{L^{(1)}\underset{,}{\otimes}L^{(2)}\}=[R,L^{(1)}\otimes\mathbf{1}]-[R^{\Pi},\mathbf{1}\otimes L^{(2)}]\,, \tag{2}$$

where $\{L^{(1)}\underset{,}{\otimes}L^{(2)}\}^{kl}_{ij}=\{L^{(1)}{}^{k}_{i},L^{(2)}{}^{l}_{j}\}$ and Π is the permutation operator on $\mathscr{G}\otimes\mathscr{G}$: $\Pi(x\otimes y)=y\otimes x$ and $R^{\Pi}\equiv\Pi R\Pi$. As a matter of fact, eq. (2) is equivalent to the integrability of eq. (1), see ref. [6]. When L lives in a loop algebra $\mathscr{G}\otimes C[\lambda,\lambda^{-1}]$, λ being the so-called spectral parameter of the Lax pair, R depends on the two spectral parameters λ, μ of $L^{(1)}$ and $L^{(2)}$, and of course, $R^{\Pi}(\lambda,\mu)=\Pi R(\mu,\lambda)\Pi$.

The Jacobi identity on the Poisson brackets in eq. (2) induces the classical Yang–Baxter equation on R as

$$[R_{12},R_{13}]+[R_{12},R_{23}]+[R_{32},R_{13}]+(\text{terms})=0\,. \tag{3}$$

The (terms) in eq. (3) only appear when R depends on the dynamical variables of the problem, as can be seen in refs. [6,7]. We shall not consider this case here, restricting ourselves to constant R-matrices. Of course, Yang–Baxter equations also appear with permuted indexations of $\{1, 2, 3\}$, but they are equivalent to eq. (3) under conjugation by the idempotent permutation operators Π_{ij}. As an example, taking Π_{12} eq. (3) Π_{12}, noting that $(\Pi_{12})^2=\mathbf{1}$ can be inserted in the commutators, and $\Pi_{12}R_{13}\Pi_{12}=R_{23}$, etc., eq. (3) is obtained with $1\leftrightarrow 2$. When in addition R is supposed to be antisymmetric: $R^{\Pi}=-R$, the Yang–Baxter equation (3) can be rewritten as

$$[R_{12},R_{13}]+[R_{12},R_{23}]+[R_{13},R_{23}]=0\,. \tag{4}$$

It has been thoroughly studied in a series of papers by Belavin and Drinfel'd [8–10]. They showed that such an R could only depend on the difference $(\lambda-\mu)$ [8], and that the classification of rational solutions to eq. (4) was essentially based on the classification of simple Lie algebras. The trigonometric and elliptic R-matrices could be obtained by suitably defined formal series of rational solutions, given a so-called Coxeter automorphism of the associated simple Lie algebra [9,10].

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However, the antisymmetry condition is not such a natural one, as it was already emphasized in ref. [11] and more recently in refs. [7,6]. Indeed, we have already obtained a number of constant non-antisymmetric solutions associated with classically integrable models (Neumann model [12] and Adler–van Moerbeke model [13]), which we were later able to generalize [14] to a class of R-matrices of the form

$$R(\lambda,\mu)=\frac{\mathscr{C}_1 g(\mu)}{f(\lambda)-f(\mu)}+\frac{(\mathscr{C}_1-\mathscr{C}_2)g(\mu)}{f(\lambda)+f(\mu)}, \tag{5}$$

where $\mathscr{C}_1$ is the "Casimir" of the simple Lie algebra $\mathrm{sl}(n,\mathbb{C})$ constructed as $\mathscr{C}_1=\sum_{a,b}e_{ab}\otimes e_{ba}$, e_{ab} being the generators of $\mathrm{sl}(n,\mathbb{C})$ with the usual algebra $[e_{ab},e_{cd}]=\delta_{bc}e_{ad}-\delta_{ad}e_{bc}$, and $\mathscr{C}_2$ is the Casimir of the simple Lie subalgebra $\mathrm{so}(n)$ or $\mathrm{sp}(n/2)$ constructed similarly from suitably symmetrized combinations of the above generators e_{ab}. For any functions f, g and any $n\in\mathbb{N}$ the expression in (5) is a solution of eq. (3) in a purely algebraic sense, hence any representation of $\mathrm{sl}(n,\mathbb{C})$ leads to a R-matrix.

The form of eq. (5) now leads us to consider as a general ansatz for an R-matrix the following form:

$$R=\frac{A}{\lambda-\mu}+\frac{B}{\lambda+\mu}, \tag{6}$$

since in fact the multiplicative factor $g(\mu)$ describes an obvious functional dependence of R, associated with the possibility of renormalizing $L(\mu)$ in eq. (1) by an arbitrary function $1/g(\mu)$. Beware that this functional dependence appears not so trivial in the set-up of non-ultralocal integrable field theories, where g appears in the non-ultralocal supplementary term in eq. (2) $\sim g\delta'(x-y)$. Similarly $f(\lambda)$ corresponds to a redefinition of the spectral parameter and may therefore be forgotten. We shall now show that, given some assumptions on the structure of A and B, one can obtain a very interesting generalization of the Belavin–Drinfel'd results in the non-antisymmetric case.

2. Yang–Baxter equation for two-pole R-matrices

We may consider that A and B are developed on a basis $e_{a,b}\otimes e_{c,d}$ of tensor products of $\mathrm{sl}(n,\mathbb{C})$ generators; in fact this is not a restrictive assumption since one can always admit that L is represented as an $n\times n$ matrix, and then A and B are $(n\times n)\otimes(n\times n)$ matrices. In fact we can take $\mathscr{G}$ to be any Lie algebra having a *non-degenerate invariant* scalar product; in particular any simple or semi-simple Lie algebra. This freedom of choice will in fact enlarge the class of solutions which we shall describe. It is then possible to use the dualization formalism of ref. [11]. This procedure identifies elements of $\mathscr{G}\otimes\mathscr{G}$, where A and B live, with operators in $\mathscr{G}\otimes\mathscr{G}^*\simeq\mathscr{L}(\mathscr{G})$, by bijectively extending the application

$$u\otimes v\in\mathscr{G}\otimes\mathscr{G}\Rightarrow\tilde{A}:X\to(v,X)u\,, \tag{7}$$

or equivalently ${}^{\mathrm{T}}\tilde{A}:X\to(u,X)v$ such that $({}^{\mathrm{T}}\tilde{A}X,Y)=(X,\tilde{A}Y)$. Here $(\ ,\ )$ is the aforementioned non-degenerate invariant scalar product on $\mathscr{G}$, in particular the Killing form when $\mathscr{G}$ is semi-simple. The inverse application to eq. (7) is defined as

$$\tilde{A}\in\mathscr{G}\otimes\mathscr{G}^*\Rightarrow A\in\mathscr{G}\otimes\mathscr{G}\equiv\sum g^{\mu\nu}\tilde{A}(I_\mu)\otimes I_\nu \tag{8}$$

where $g^{\mu\nu}=(g_{\mu\nu})^{-1}$, $g_{\mu\nu}=(I_\mu,I_\nu)$. Now the Yang–Baxter equation for R in eq. (6) leads, looking at the poles in $\lambda_1\pm\lambda_2$, to the following equations for A and B:

$$[A_{12},A_{13}+A_{23}]=0\,,\quad [A_{12},B_{13}+B_{23}]=0\,,$$
$$[B_{12},A_{13}-B_{23}]=0\,,\quad [B_{12},B_{13}-A_{23}]=0\,. \tag{9}$$

As usual A_{ij} denotes the endomorphism in $\mathscr{G}\otimes\mathscr{G}\otimes\mathscr{G}$ acting as A on the tensor product $\mathscr{E}_i\otimes\mathscr{E}_j$ of representation spaces of $\mathscr{G}\otimes\mathscr{G}$ and $\mathbf{1}$ on $\mathscr{E}_k$, $\{i,j,k\}\equiv\{1,2,3\}$. For example, if $A=u\otimes v$, then $A_{32}=\mathbf{1}\otimes v\otimes u$. The complete Yang–Baxter equation follows from eqs. (9) together with their other forms obtained by index permutations. Such permuted forms are of course equivalent to the initial one since they follow from successive conjugations of eqs. (9) by the idempotent space-permutation operator Π_{ij} and therefore eqs. (9) are sufficient to ensure the validity of the Yang–Baxter equation.

We can now rewrite eq. (9) as a series of conditions on the dualized operators $\tilde{A}$ and $\tilde{B}$. Defining $\tilde{H}$ as $\tilde{A}-\tilde{B}$ for more convenience, we get, now skipping the tilde notation,

$\forall X,Y\in\mathscr{G}$:

$$A[X,A(Y)]=[A(X),A(Y)]\,, \tag{10}$$

$$A[X,H(Y)]=[A(X),H(Y)]\ , \tag{11}$$

$$[(A-H)(X),A(Y)]$$
$$=-(A-H)[X,(A-H)(Y)]\ , \tag{12}$$

$$[H(X),H(Y)]=H[X,H(Y)]\ . \tag{13}$$

Eqs. (10), (13) are the ordinary Yang–Baxter equations obeyed by the residue A of a single-pole rational skew-symmetric constant R-matrix as derived in ref. [9]. Eqs. (11), (12) describe the coupling between the residues at $\lambda\pm\mu$ of R.

We shall briefly recall some general properties implied by eqs. (10), (13) and then put our interest to particular subclasses of solutions for A and H. The interest of the dualized formalism here is that one considers linear operators on Lie algebras, instead of operators on tensorial products of vector spaces, thereby having at our disposal the powerful tools of Lie algebra theory. The following general properties of a solution to eq. (10) are easily demonstrated.

(i) $\mathrm{Im}\,A\subset\mathscr{G}$ is a subalgebra.

(ii) $[\mathrm{Ker}\,A,\mathrm{Im}\,A]\subset\mathrm{Ker}\,A$.

(ii) $\tilde{\mathscr{G}}^{\lambda}_{(n)}\equiv\{X\in\mathscr{G}/(A-\lambda)^nX=0\}$ is an ideal of $\mathrm{Im}\,A$ for $\lambda\neq0$, n integer.

(iv) $\mathscr{G}=\oplus_\lambda\tilde{\mathscr{G}}^\lambda$ with $\tilde{\mathscr{G}}^\lambda\equiv\bigcup_n\tilde{\mathscr{G}}^\lambda_{(n)}$.

In particular all eigenspaces $\tilde{\mathscr{G}}^\lambda_{(1)}$ are ideals of $\mathrm{Im}\,A$, hence if A is invertible and $\mathscr{G}$ is a simple Lie algebra, $A\equiv\lambda\mathbf{1}$, for some $\lambda\in\mathbb{C}$, is the only solution [9].

If A is not invertible but diagonalizable, all eigenspaces with non zero eigenvalues are mutually commuting ideals of $\mathrm{Im}\,A$, and $\mathrm{Ker}\,A$ is stabilized by $\mathrm{Im}\,A$, i.e. the space G/K, where G=exp($\mathscr{G}$), K=exp($\mathrm{Im}\,A$), is a reductive homogeneous space [15,16]. Hence to any decomposition of the considered Lie algebra $\mathscr{G}$ as a homogeneous reductive algebra $\mathscr{G}=\mathscr{K}\oplus\mathscr{M}$, followed by any decomposition of $\mathscr{K}$ into a direct sum of ideals (at least one such sum exists, i.e. $\mathscr{K}\oplus\{0\}$) is associated a set of diagonalizable A-matrices, having arbitrary eigenvalues in each of the ideals of the decomposition (see ref. [17]). This is in particular the case when $\mathscr{K}$ is semi-simple and can be decomposed as a sum of its simple ideals. If A is not diagonalizable the situation is more involved and we shall simply give here an example of such a situation.

We start from the semi-simple chiral Lie algebra $\mathscr{G}=\mathrm{su}(n)\oplus\mathrm{su}(n)$, which we decompose as a vector space into $\mathscr{K}\oplus\mathscr{D}$, $\mathscr{K}$ being one of the su(n) algebras and $\mathscr{D}$ being the diagonal subalgebra with generators $\{I_\mu+\tilde{I}_\mu\}$. One then defines the operator A which sends $\mathscr{K}$ to $\mathscr{D}$ through the canonical diagonal operation, and $\mathscr{D}$ to 0. This A is nilpotent and verifies eq. (10), hence $A/(\lambda-\mu)$ is a non-diagonalizable solution of the Yang–Baxter equation. Any chiral Lie algebra $\mathscr{S}\oplus\mathscr{S}$ allows the same construction, but this is certainly not the sole class of such matrices.

3. Analysis of the coupled equations

Let us now study the relations between A and H induced by eqs. (11), (12). We shall restrict ourselves from now on to the case when A and H are *simultaneously* diagonalizable but not necessarily invertible. This seems a rather restrictive requirement but nevertheless leads to a vast class of solutions with remarkable properties and a very simple classification which we shall now describe.

Any solution (A, H) of simultaneously diagonalizable operators corresponds to a decomposition of the algebra $\mathscr{G}$ into a direct sum (as vector space) of eigenspaces denoted as $\mathscr{G}^{\lambda_a,\lambda_h}$ where λ_a, λ_h are respectively the eigenvalues of A and H.

$$\mathscr{G}=\bigoplus_{\lambda\neq0}\{\mathscr{G}^{\lambda,0}\oplus\mathscr{G}^{\lambda,\lambda}\oplus\mathscr{G}^{\lambda,2}\}$$
$$\bigoplus_{\lambda_a,\lambda_h\neq0}\mathscr{G}^{\lambda_a,\lambda_h}\bigoplus_{\lambda_h\neq0}\mathscr{G}^{0,\lambda_h}\oplus\mathscr{G}^{0,0}\ , \tag{14}$$

where the following conditions are fulfilled:

(i) $\mathscr{G}^{0,0}$ is stabilized by adjoint action of all the other subspaces.

(ii) $\mathscr{G}^{0,\lambda_h}$ is an abelian subalgebra commuting with all other $\mathscr{G}^{\lambda_a,\lambda_h}$ whenever $\lambda_a\neq0$ or $\lambda_h\neq0$.

(iii) $\mathscr{G}^{\lambda_a,\lambda_h}$, with $\lambda_h\neq\{0,\lambda_a,2\lambda_a\}$, are abelian subalgebras commuting with all other eigenspaces except $\mathscr{G}^{0,0}$.

(iv) The particular eigenspaces $\mathscr{G}^{\lambda_a,\{0,\lambda_a,2\lambda_a\}}$ verify the following relations:

(1) $[\mathscr{G}^{\lambda_a,0},\mathscr{G}^{\lambda_a,0}]\subset\mathscr{G}^{\lambda_a,2\lambda_a}$,

(2) $[\mathscr{G}^{\lambda_a,0},\mathscr{G}^{\lambda_a,2\lambda_a}]\subset\mathscr{G}^{\lambda_a,0}$,

(3) $[\mathscr{G}^{\lambda_a,2\lambda_a},\mathscr{G}^{\lambda_a,2\lambda_a}]\subset\mathscr{G}^{\lambda_a,2\lambda_a}$.

(v) $\mathscr{G}^{\lambda_a,\lambda_a}$ is an ideal of Im A and Im H, commuting with $\mathscr{G}^{\lambda_a,0}$ and $\mathscr{G}^{\lambda_a,2\lambda_a}$.

(vi) $[\mathscr{G}^{\lambda_1,\mu_1}, \mathscr{G}^{\lambda_2,\mu_2}]=0$ whenever λ_1, $\lambda_2\neq 0$ are different, or $\mu_1, \mu_2\neq 0$ are different.

These conditions follow from the general properties which were emphasized in the previous section, concerning the commutation of two eigenvectors having *non-zero distinct eigenvalues* together with the following applications of the Yang–Baxter equations to conveniently chosen eigenvectors of A and H:

(a_1) Applying eq. (11) to $X\in\mathscr{G}^{0,0}$, $Y\in\mathscr{G}\rightarrow [X, \mathrm{Im}\, H]\subset \mathrm{Ker}\, A$.

(a_2) Applying eq. (12) to $X\in\mathscr{G}$, $Y\in\mathscr{G}\rightarrow [X, \mathrm{Im}\, A]\subset \mathrm{Ker}\, H$. This completes the proof of (i).

(b_1) Applying eq. (12) to $X\in\mathscr{G}^{0,\lambda_h}$, $Y\in\mathscr{G}^{\lambda_a,0}\rightarrow [X, Y]=0$.

(b_2) Applying eq. (11) to $X\in\mathscr{G}^{\lambda_a,\lambda_h}$, $Y\in\mathscr{G}^{0,\lambda_h}\rightarrow [X, Y]=0$.

(b_3) Applying eqs. (11), (12) to X, $Y\in\mathscr{G}^{0,\lambda_h}\rightarrow [X, Y]=0$. This completes the proof of (ii).

(c) For $\lambda_h\neq 0$ and $\lambda_a\neq\{0, \lambda_h, \lambda_h/2\}$, eq. (12) implies (iii).

(d_1) Application of eq. (12) to X and $Y\in\mathscr{G}^{\lambda_a,0}$ implies (iv) (1).

(d_2) Application of eqs. (10), (13) to X and $Y\in\mathscr{G}^{\lambda_a,2\lambda_a}$ obviously implies (iv) (2).

(d_3) Application of eq. (12) to $X\in\mathscr{G}^{\lambda_a,0}$ and $Y\in\mathscr{G}^{\lambda_a,2\lambda_a}$ finally implies (iv) (3).

(e) Application of eq. (12) to $X\in\mathscr{G}^{\lambda_a,\lambda_a}$ and $Y\in\mathscr{G}^{\lambda_a,0}$ together with the general properties of eigenvectors implies (v).

(f) The same general properties immediately imply (vi).

Let us now comment on these results. First of all the common kernel $\mathscr{G}^{0,0}$ is the tangent space to a reductive homogeneous space obtained by taking the quotient of the original group G by a subgroup K, the Lie algebra of K being the sum of all other eigenspaces which we shall denote from now on Im $\mathscr{A}\mathscr{H}$. The decomposition $\mathscr{G}=\mathscr{G}^{0,0}\oplus\mathrm{Im}\,\mathscr{A}\mathscr{H}$ represents the first step in the construction of a general R-matrix with A and B diagonalizable. Remember now that the R-matrix is obtained from the operators A and H by inversion of the dualization procedure described in (8). It follows that each eigenspace in Im $\mathscr{A}\mathscr{H}$ gives rise to a term in the R-matrix, of the form

$$\mathscr{G}^{\lambda_a,\lambda_h}\Rightarrow \sum_{I_\rho\in\mathscr{G}^{\lambda_a,\lambda_h}, I_\nu\in\mathscr{G}} g^{\rho\nu}\left\{\frac{\lambda_a}{\lambda-\mu}+\frac{\lambda_a-\lambda_h}{\lambda+\mu}\right\} I_\rho\otimes I_\nu\,. \tag{15}$$

It follows that the decomposition (14) gives rise to three types of terms.

(1) The spaces $\mathscr{G}^{0,\lambda_h}$ and $\mathscr{G}^{\lambda_a,\lambda_h}$ (without relations between λ_a and λ_h) generate almost trivial contributions to the R-matrix.

Indeed they correspond to the possibility of adding an arbitrary term containing only generators in the center of the algebra Im $\mathscr{A}\mathscr{H}$, and do not exist at all if this algebra is semi-simple, as it is the case for instance when the complete algebra $\mathscr{G}$ is semi-simple and $\mathscr{G}^{0,0}$ vanishes.

(2) The eigenspaces of the form $\mathscr{G}^{\lambda_a,\lambda_a}$ give rise to more interesting contributions: they generate a term in the R-matrix of the form $\Pi/(\lambda-\mu)$ where Π is the Casimir operator of a subalgebra of Im $\mathscr{A}\mathscr{H}$, the generators of which commute with the remaining eigenspaces in Im $\mathscr{A}\mathscr{H}$.

Obvious examples of such an occurrence are the skew-symmetric rational constant R-matrices in ref. [9]; more generally if Im $\mathscr{A}\mathscr{H}$ is semi-simple, each simple factor in it may give rise to such a term with an arbitrary weight as in the previous section.

(3) The last terms in the decomposition (14) are the subalgebras defined as

$$\mathscr{G}^{\lambda}\equiv\mathscr{G}^{\lambda,0}\oplus\mathscr{G}^{\lambda,2\lambda}\,, \tag{16}$$

with the commutation relations described by the property (iv) beforehand.

This is the new contribution to the general R-matrices and we shall comment on it in some detail.

One identifies (16) with the canonical decomposition of a symmetric Lie algebra into respectively a Lie subalgebra $\mathscr{G}^{\lambda,2\lambda}$ and the tangent space to the symmetric space $\mathscr{G}^{\lambda,0}$ (see refs. [15,16]). In particular, given a simple Lie algebra $\mathscr{G}$, any symmetric Lie algebra constructed from $\mathscr{G}$ will give rise to a double-pole R-matrix. These Lie algebras are classified in refs. [15,16]. The previously constructed examples [14] corresponded to the Cartan series A1 and A2, respectively SU(N)/SO(N) and SU($2N$)/Sp(N).

4. Conclusion

To sum up our conclusions, it follows that all double-pole ($\lambda = \pm\mu$) constant skew-symmetric diagonalizable R-matrices are combinations of (abelian contributions) plus (simple-pole contributions of simple Lie algebras à la Belavin and Drinfel'd) plus (double-pole contributions of symmetric Lie algebras). It then follows that the symmetric Lie algebras play the same role of "fundamental elements" of this class of double-pole R-matrices, as the simple Lie algebras in the case of skew-symmetric constant Lie algebras.

Open questions now are:

– To identify the integrable models corresponding to this vast new set of R-matrices and the link between the structure of these models and the symmetric Lie algebras associated with the R-matrices.

– To obtain trigonometric and elliptic R-matrices by a procedure à la Faddeev and Reshetikhin [18], an example of which is given in ref. [14]. In particular one should determine which generalization of the Coxeter automorphisms (used in the standard resummation construction [18]) is to be taken here.

– To investigate the more complicated cases of non-diagonalizable or non-simultaneously diagonalizable operators A and H. Even in the single-pole case this is a difficult question to which we have alluded in the beginning.

– To understand better the possible pole structures for non-skew-symmetric R-matrices, generalizing the theorem of Belavin and Drinfel'd [8].

We hope to address these questions in further studies.

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