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The replica momenta of a spin-glass and the phase diagram of *n*-colour Ashkin-Teller models

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Résumé. — Nous étudions les propriétés du diagramme de phase correspondant au moment de réplique d'ordre fini du verre de spin de type Ising. Nous utilisons pour cela des résultats concernant le modèle d'Ashkin Teller à n couleurs. Nous remarquons que le moment de réplique d'ordre deux en dimension deux possède des exposants critiques variant continuement et que le point tricritique de ce modèle se trouve sur la ligne de Nishimori. Nous calculons la largeur de la distribution de probabilité de la fonction de partition sur des lignes particulières du diagramme de phase. Nous exhibons en dimension quelconque une relation exacte reliant les moments de réplique d'ordre n et $n + 1$. Les propriétés de symétrie du diagramme des phases qui en résultent sont analysées. Ceci permet de vérifier l'excellent accord de certaines prédictions du « modèle des énergies aléatoires généralisé » avec ces résultats exacts.

Abstract. — Some properties of the phase diagrams of the finite replica momenta for an Ising spin-glass are studied, using exact results on n-colour Ashkin-Teller models. In the two-dimensional case the second replica momentum exhibits continuously varying critical exponents and the tricritical point is found to lie on the Nishimori's line. We obtain the width of the probability distribution of the partition function on particular lines of the phase diagram. We point out an exact relation between the n and $(n + 1)$ th replica momenta, valid for arbitrary dimensions, which implies a symmetry property of the phase diagram. This allows to verify the nice agreement of some predictions of the « generalized random energy model » with these exact results.

1. Introduction.

The finite replica problem has been considered by several authors in the development of spin-glass theory, with different motivations. A first one was to initiate a phenomenological theory of the spin-glass transition [1], based on a Landau-type study of the effective n-replica Hamiltonian for finite n, in order to shed light on phase diagrams and singularities in the $n \to 0$ limit. In the infinite dimensional case — corresponding to the mean field theory — a detailed study of the finite n replica Hamiltonian was performed by Sherrington [2].

Let us note that, apart from their interest in the spin-glass context, the models corresponding to the effective n-replica Hamiltonians, sometimes called « n-colour Ashkin-Teller models », are interesting on their own, due to the richness of their phase diagrams [3]. Indeed, they possess more than one order parameter and exhibit a multicritical point.

More recently, it has been realized that finite n studies are crucial in the understanding of the whole probability distributions $\mathcal{P}(\ln Z)$ of a spin glass, whose generating function is given by the replica momenta. Consequences of this remark in the infinite dimensional case were investigated by Derrida and Toulouse [4].

The aim of this paper is to address the finite dimensional case. Of course, one cannot expect a detailed quantitative analysis to be possible. However, one can take advantage of some available exact results on Ashkin-Teller (AT) models, and of some general symmetry properties, in order to gain some physical insights, based on analytical calculations. A special emphasis will be made on the $n = 2$ case in two dimensions. The physical outcome of this study is a quantitative evaluation of the region of the phase diagram where the annealed approximation is no longer valid due to the broadening of the distribution of the partition function. This is based in particular on an exact expression for the variance of this distribution along different lines of the phase diagram. These exact results are compared with the case of infinite dimensional models. We also point out a symmetry property of the replica momenta of a spin-glass in arbitrary dimension. This property can be confronted with a prediction of the generalized random energy model (GREM) [5]. It also supports the idea that Nishimori's line [6] plays an important role in the phase diagram of a spin-glass, as suggested in [6, 7].

2. Two replicas in two dimensions.

2.1 PHASE DIAGRAM. — Let us consider the two-dimensional Ising spin-glass on the square lattice, of Hamiltonian :

$$H = -\sum_{\langle ij \rangle} J_{ij} S_i S_j \tag{1}$$

where the random couplings J_{ij} only couple nearest neighbour sites and are distributed according to the distribution $P\left(J_{ij}\right)$. The second replica momentum is seen to be simply related to a symmetric Ashkin-Teller model, of Hamiltonian :

$$H_{\mathrm{AT}} = K_2 \sum_{\langle ij \rangle} \left(\sigma_i \sigma_j + \tau_i \tau_j\right) + \\ + K_4 \sum_{\langle ij \rangle} \sigma_i \sigma_j \tau_i \tau_j \tag{2}$$

through the relation :

$$\left\langle Z^2 \right\rangle = \mathrm{e}^{2NK_4} Z_{\mathrm{AT}}\left(K_2, K_4\right) \tag{3}$$

where N denotes the number of sites and K_2 and K_4 are coupling constants given by :

$$\mathrm{e}^{4K_2} = \frac{\left\langle \mathrm{e}^{2\beta J_{ij}} \right\rangle}{\left\langle \mathrm{e}^{-2\beta J_{ij}} \right\rangle}, \quad \mathrm{e}^{4K_4} = \left\langle \mathrm{e}^{2\beta J_{ij}} \right\rangle \left\langle \mathrm{e}^{-2\beta J_{ij}} \right\rangle . \tag{4}$$

In all these expressions, the brackets $\langle \quad \rangle$ denote the average with respect to the distribution P.

Though the Ashkin-Teller model is not integrable in its whole parameter space, a description of its phase diagram is available through exact results valid only on some particular lines (or points). It can be exactly solved on its self-dual line (SD) [8] whose equation reads :

$$\sinh 2 K_2 = \mathrm{e}^{-2K_4} \tag{5}$$

or, equivalently :

$$\left\langle \sinh 2 \beta J_{ij} \right\rangle = \pm 1 . \tag{6}$$

This SD line delimits the boundary between the ferromagnetic $\left(\left\langle \sigma_i \right\rangle, \left\langle \tau_i \right\rangle, \left\langle \sigma_i \tau_i \right\rangle \neq 0\right)$ and paramagnetic $\left(\left\langle \sigma_i \right\rangle = \left\langle \tau_i \right\rangle = \left\langle \sigma_i \tau_i \right\rangle = 0\right)$ phases, until a tricritical point is met for $K_2 = K_4 = K_{\mathrm{c}}$ given by :

$$\sinh\left(2 K_{\mathrm{c}}\right) = \exp\left(-2 K_{\mathrm{c}}\right) \tag{7}$$

that is $K_{\mathrm{c}} = \frac{\ln 3}{4}$.

A third mixed phase $\left(\left\langle \sigma_i \right\rangle = \left\langle \tau_i \right\rangle = 0, \left\langle \sigma_i \tau_i \right\rangle \neq 0\right)$ then appears, whose boundaries are not known analytically, except for the two points $\mathrm{I}_1(K_2 = 0, \ \sinh\left(2 K_4\right) = 1)$, and $\mathrm{I}_2(K_4 = \infty, K_2 = K_{\mathrm{c}}/2)$, corresponding to Ising limits of the model. It is also known [9] that the separation between these boundaries and the (SD) line is very slow (with an essential singularity), in the vicinity of the tricritical point.

These pieces of information have been used in figure 1 to obtain the phase diagram of the second

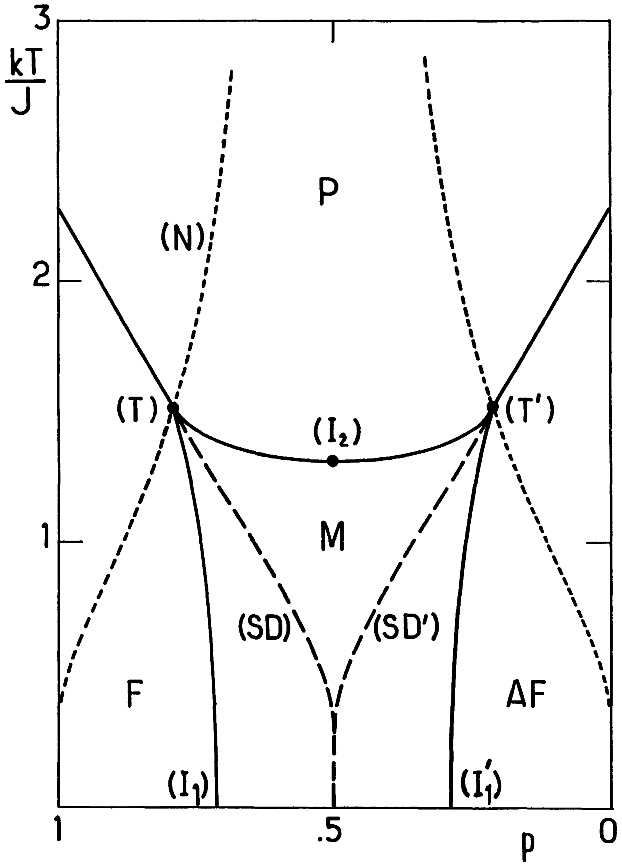

Fig. 1. — Phase diagram of the second replica momentum $\left\langle Z^2 \right\rangle$ for a two-dimensional Ising spin-glass with a binary distribution (8) in the (p, T) plane. (T) and (T') are tricritical points. (SD), (SD') are the self dual lines. (N) is the Nishimori's line. M denotes the mixed phase (whose boundaries are not known analytically, except for the points I_1, I_1', I_2).

replica momentum $\langle Z^2 \rangle$ of a 2D spin-glass, with a binary distribution for the couplings:

$$P\left(J_{ij}\right) = p\delta\left(J_{ij} - J\right) + (1-p)\,\delta\left(J_{ij} + J\right). \quad (8)$$

This diagram is given as an example, but let us emphasize that our discussion is valid for any distribution $P\left(J_{ij}\right)$.

2.2 NISHIMORI'S LINE. — The possible values of the order parameters $\langle \sigma_i \rangle$, $\langle \tau_i \rangle$ and $\langle \sigma_i \tau_i \rangle$ denoted generally by m_i^{α} and $Q_i^{\alpha\beta}$ ($\alpha, \beta = 1, 2$) in the spin-glass context, delimit three phases. It is remarkable that the corresponding tricritical point (T) lies precisely at the intersection of the SD line with the Nishimori's line of the spin-glass. Indeed, the equation of this line [6] in parameter space, reads:

$$\frac{P\left(-J_{ij}\right)}{P\left(J_{ij}\right)} = \mathrm{e}^{-2\beta J_{ij}} \Rightarrow \left\langle \mathrm{e}^{-2\beta J_{ij}} \right\rangle = 1 \quad (9)$$

which, in turn, is equivalent to $K_2 = K_4$. Thus, on Nishimori's line, the AT model (2) describing $\langle Z^2 \rangle$ reduces to the four-state scalar Potts model, and the symmetry is enhanced from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $\mathbb{Z}_4$. This is not without reminding the idea of Nishimori's line as a line of enhanced (super)-symmetry developed by the authors in reference [10].

This line plays a particular part in the critical behaviour of the spin-glass, since it can be shown [6] that the specific heat remains finite and the internal energy has no singularity on the line, despite its intersection with the ferromagnetic/paramagnetic boundary of the spin-glass, where the singularity $\ln\left(\ln\left(T - T_c\right)\right)$ is believed to hold [11]. It has been suggested [6, 7] that this apparent paradox could be solved by an intersection at a point of coexistence of three phases (this is indeed what happens in the SK model [12]; it is also strongly suggested by numerical simulations of the 3D case). Whether the mixed phase of $\langle Z^2 \rangle$ survives in 2D when $n \to 0$ to give rise to a spin-glass phase at $T \neq 0$ is an open question, but generally answered negatively [13]. However, the location of the tricritical point of $\langle Z^2 \rangle$ on Nishimori's line could provide a clue in favour of the suggestion of reference [7] in which Maynard and Rammal's proposal of a « Random Antiphase » [14] was extended in a very narrow (T, p) domain to reach Nishimori's line at a tricritical point. The fact that, in the study of $\langle Z^2 \rangle$, the critical lines split apart with an essential singularity could be in relation to a very narrow phase (and thus difficult to observe) close to the tricritical point.

2.3 THE WIDTH OF THE PROBABILITY DISTRIBUTION OF Z. — Let us now turn to more quantitative considerations. One can convert the exact expression of the free energy of the AT model [8] on its self-dual line, into an expression of $\langle Z^2 \rangle$ on the line (6). One gets two different expressions, corresponding to the two regimes above and below the tricritical point (T) at $T = T_c$ defined by (7):

$$\begin{aligned}
T > T_c \qquad & \frac{1}{N}\ln\langle Z^2 \rangle = \ln 2 + \int_{-\infty}^{+\infty} \frac{\tanh(\mu x)\,\sinh(\pi - \mu)\,x}{x \sinh \pi x}\,\mathrm{d}x \\
T = T_c \qquad & \frac{1}{N}\ln\langle Z^2 \rangle = -3\ln 2 + 4\ln\frac{\Gamma(1/4)}{\Gamma(3/4)} \\
T < T_c \qquad & \frac{1}{N}\ln\langle Z^2 \rangle = \ln 2 + \mu + 2\sum_{m=1}^{\infty}\frac{1}{m}\mathrm{e}^{-m\mu}\tanh(m\mu)
\end{aligned} \quad (10)$$

where N is the number of sites, and μ is given by $2\cos(\mu/2) = 1 + \left\langle \exp\left(-2\beta J_{ij}\right)\right\rangle$ for $T > T_c$ (cos is to be replaced by cosh for $T < T_c$). Apart from the self-dual line, it is trivial to calculate $\langle Z^2 \rangle$ when the distribution $P\left(J_{ij}\right)$ becomes invariant under $J_{ij} \to -J_{ij}$ (this corresponds to the line $p = 1/2$ for the binary distribution (8) used in Fig. 1). This is an Ising-like limit $\left(K_2 = 0\right)$ of the AT model, and we get, in this case:

$$\frac{1}{N}\ln\left\langle Z^2(\beta)\right\rangle = \ln 2\left\langle \mathrm{e}^{2\beta J_{ij}}\right\rangle + \frac{1}{N}\ln Z_{\mathrm{Ising}} \quad (11)$$

where Z_{Ising} denotes the Onsager partition function taken for the coupling $K_{\mathrm{Ising}} = \ln\left\langle \exp\left(2\beta J_{ij}\right)\right\rangle/2$. Here again, we get a transition at β_{I_2} such that $\left\langle \exp\left(2\beta_{I_2} J_{ij}\right)\right\rangle = 1 + \sqrt{2}$, corresponding to the point I_2 of figure 1. The exact expressions (9) and (10) are used in figure 2a, b to plot, as a function of temperature, the « width per site » of the distribution of the partition function:

$$\sigma = \lim_{N\to\infty}\left[\frac{\langle Z^2 \rangle}{\langle Z \rangle^2}\right]^{1/N} - 1 \quad (12)$$

along the (SD) line and the line $p = 1/2$, for the binary distribution (8). In both cases, two different regimes occur: σ stays very small in all the high temperature

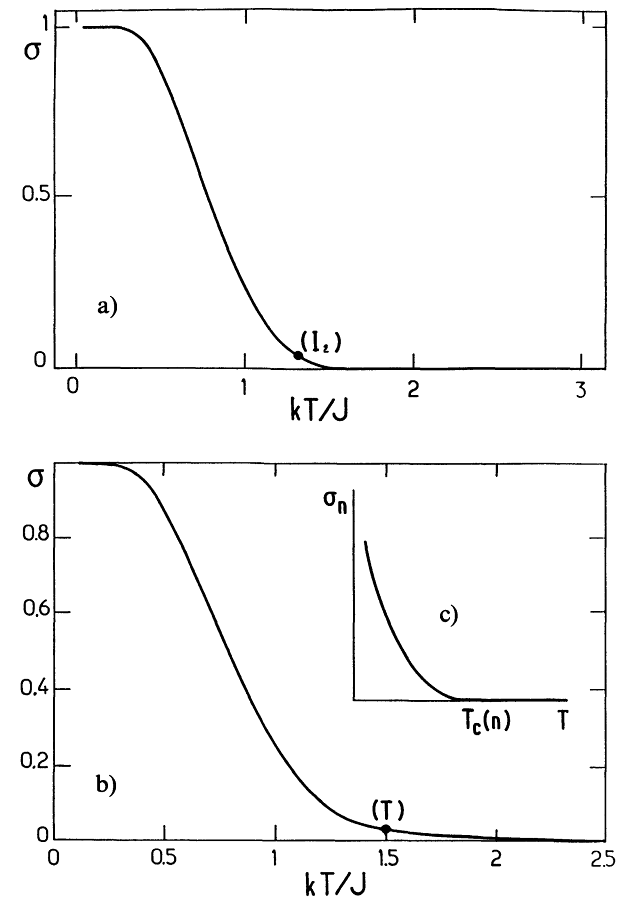

Fig. 2. — The « width per site » (Eq. (12)) of the distribution of the partition function as a function of the temperature a) along the self dual line. b) Along the $p = 1/2$ line. c) Sketch of the behaviour of as a function of the temperature in the infinite dimensional case.

phase, and increases exponentially fast when entering the mixed phase of $\langle Z^2 \rangle$.

This is very similar to what happens in infinite dimensional models, such as SK [4, 12] and random energy model (REM) [15]. In these models, the phase diagram of a given replica momentum has a similar structure to that of the diagram of figure 1. For symmetric couplings ($J_0 = 0$ in the SK model), the paramagnetic/mixed phase transition for $\langle Z^n \rangle$ takes place at a critical temperature $T_c(n)$, above which $\langle Z^n \rangle = \langle Z \rangle^n$ in the thermodynamic limit. Below $T_c(n)$, $\sigma_n \equiv \left(\langle Z^n \rangle / \langle Z \rangle^n \right)^{1/N} - 1$ starts increasing as $\exp\left(a/T^2\right)$. This behaviour is sketched in figure 2c. As expected, for the finite dimensional case, σ does not strictly vanish in the high temperature phase. However, its smallness indicates slight deviations from the annealed behaviour in all the paramagnetic phase of $\langle Z^2 \rangle$. In this phase, the spin-glass free energy is certainly well approximated by: $2 \ln \langle Z \rangle - 1/2 \ln \langle Z^2 \rangle$. This expression, which can be shown to be the first correction to the annealed approximation at high temperature, is also the expected one for a strictly log-normal distribution of the partition function.

The mechanism responsible for the exponential growth of the width of the distribution function of Z can be easily clarified, making reference to the simplified model of random independent energies (REM) [15]. There, the correlations between energies being neglected, this growth can be entirely related to the non-zero probability for two energy levels to coincide. Indeed, one has, in the thermodynamic limit:

$$\langle Z^2(T) \rangle_{\text{REM}} = \langle Z(T) \rangle^2 + \langle Z(T/2) \rangle \quad (13)$$

where the last term comes for overlapping energies. This reflects the highly degenerate character of the landscape of energy valleys at low temperature. This picture, while modified by correlations, remains qualitatively valid in the finite-dimensional case. A non-zero value for the order parameter $Q^{\alpha\beta}\,(1 \leqslant \alpha, \beta \leqslant n)$ associated with $\langle Z^n \rangle$ reveals the weight of pairs of overlapping levels among n energy levels of the spin-glass. (Indeed, $Q^{\alpha\beta}$ appears with a factor $n(n-1)/2$ in a symmetric mean field approach). The phase diagram of figure 1 and the plot of σ in figure 2 thus give information on the energy landscape of the 2D spin-glass as a function of T and p.

It is interesting to notice that the width σ exhibits continuously varying critical exponents as a function of the concentration p of impurities. Indeed [8], the exponent α of the associated AT model is given by $\alpha = (2 - 4\,\mu/\pi)/(3 - 4\,\mu/\pi)$ and thus changes from $\alpha = 0$ at $p = 0$ to $\alpha = 2/3$ at $p = p_{\mathrm{T}}$ where p_{T} corresponds to the tricritical point.

2.4 Generalized binary distributions: low temperature behaviour of the replica momenta. — We conclude this section by making some remarks concerning the quantity best-suited for a replica method treatment. This remark can be conveniently illustrated on the example of a distribution over random couplings of the form:

$$P\left(J_{ij}\right) = p\delta\left(J_{ij} - J\right) + (1-p)\ \delta\left(J_{ij} - aJ\right) \quad (14)$$

where a is an arbitrary parameter. The limits $a = -1$ and $a = 0$ correspond to the spin-glass problem (8) and to the bond-diluted problem, respectively.

It is easy to see that the phase diagram associated with a given finite replica momentum $\langle Z^n \rangle$ has rather pathological properties at low temperature, except for the spin-glass case $a = -1$. Indeed, a ferromagnetic (resp. antiferromagnetic) phase exists at $T = 0$, independently of the value of the concentration p, for $a > -1$ (resp. $a < -1$). This behaviour has certainly no relevance for the quenched problem. The shape of such a pathological phase diagram is illustrated in the case of $\langle Z^2 \rangle$ in figure 3a for $a = -1/2$.

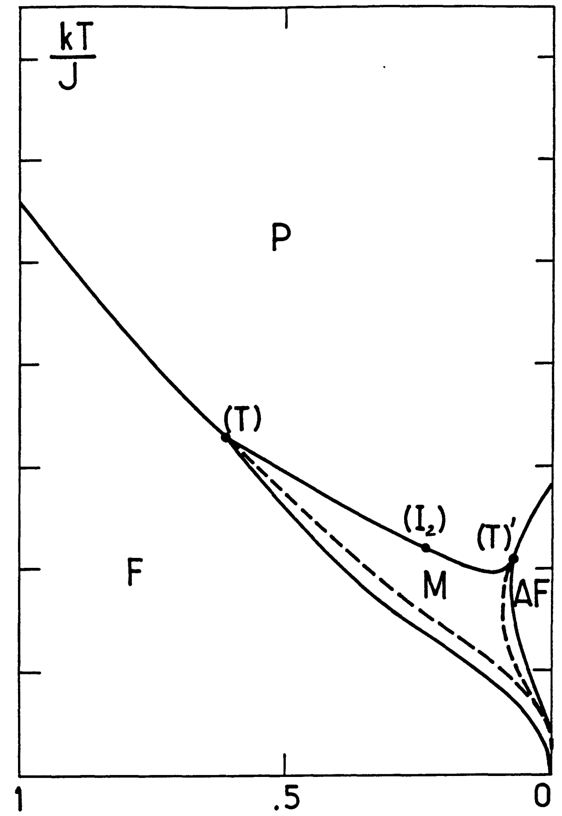

Fig. 3a. — Phase diagram of the second replica momentum $\langle Z^2 \rangle$ for a two-dimensional Ising spin-glass with a binary distribution (14) in the (p, T) plane for $a = -1/2$.

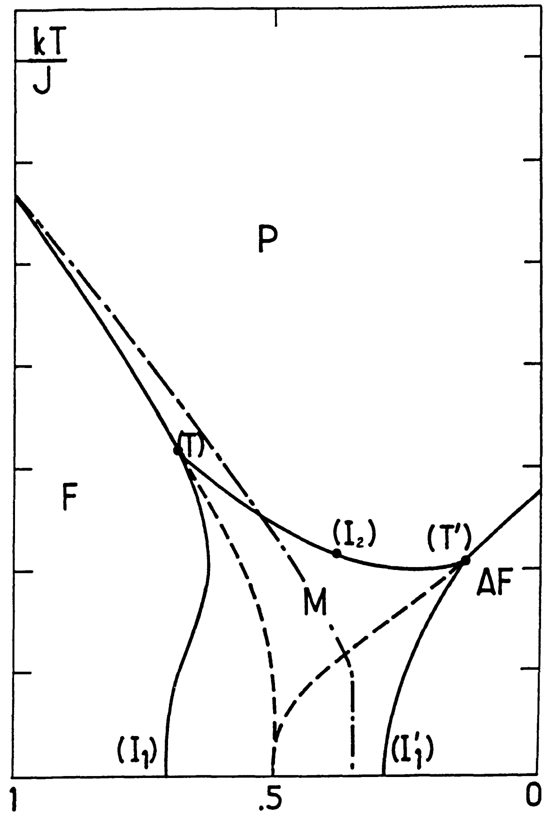

Fig. 3b. — Phase diagram of $\langle \tilde{Z}^2 \rangle$ (Eq. (15)) for a two-dimensional Ising spin-glass with a binary distribution (Eq. (14)) for $a = -1/2$ together with the dilute bond limit $a = 0$.

This trivialization of the phase diagram has a simple explanation : at low temperature, the correspondence between $\langle Z^2 \rangle$ and the n-colour AT model is dominated by contributions of terms like $\langle \cosh^n (\beta J_{ij}) \rangle^{2N}$. This is apparent for $n = 2$ in equation (3). This problem can be easily cured by extracting these large contributions from the partition function before averaging : for a given configuration of couplings, one writes :

$$Z[\{J_{ij}\}] = \left(\prod_{\langle ij \rangle} \cosh \beta J_{ij} \right) \tilde{Z}[\{J_{ij}\}] . \quad (15)$$

The replica momenta of $\tilde{Z}$ do not suffer from the previous pathologies. In the case of $\langle Z^2 \rangle$ one has again a mapping on a symmetric AT model, which is detailed in the Appendix. Note that when $a = -1$, the approach of this section and of the previous ones coincide. The resulting phase diagram in the case of distribution (14) with $a = -1/2$ is depicted in figure 3b, together with the bond diluted limit $a = 0$ where the mixed phase disappears. Note the existence of reentrant phases in the first case. It is important to remark that the Ising-like critical point I_1 $(T = 0)$ corresponds to $p = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$ for all values of $a < 0$. For $a = 0$, I_1 jumps to $p = 1/\sqrt{2}$ (a similar behaviour exists for I_1'). This remark holds for all finite replica momenta $\langle Z^n \rangle$.

These remarks explain in particular why the annealed lower-bounds on the free energy obtained from $\tilde{Z}$ are far more relevant than those obtained from Z. Some developments along this line have been achieved in reference [16].

3. A symmetry property of $\langle Z^n \rangle$ and of Ashkin-Teller models in arbitrary dimension.

3.1 A RELATION BETWEEN n AND $n + 1$ REPLICAS. — In this section, we point out a symmetry property of the replica momenta for an arbitrary spin-glass (arbitrary dimension, arbitrary lattice...), and investigate its consequences on the phase diagram. Let us consider an Ising spin-glass, with the Hamiltonian (1) (but on an arbitrary lattice this time) ; the random couplings are distributed according to a given probability distribution $P(J_{ij})$. It is well known that H is invariant under a local spin reversal $S_i \to S_i \tau_i$, provided the couplings J_{ij} are changed accordingly : $J_{ij} \to J_{ij} \tau_i \tau_j$. This remark naturally leads to introduce the concept of a frustrated configuration of couplings J_{ij} [17], as a configuration which cannot be reduced to a purely ferromagnetic one (all $J_{ij} > 0$) by a « gauge » transformation ; the configuration J_{ij} then admits a non trivial ground-state. It is thus important to distinguish between configurations which can be « gauged away », and truly disordered configurations, giving to the spin-glass problem all its complexity. In this respect, it is natural to extract in the distribution $P\left(J_{ij}\right)$ a part which preserves the gauge symmetry, i.e. invariant under $J_{ij} \to -J_{ij}$. We assume that $P\left(J_{ij}\right)$ can be written as :

$$P\left(J_{ij}\right) = C\, \mathrm{e}^{\beta_f J_{ij}}\, \phi\left(\left|J_{ij}\right|\right) \tag{16}$$

where β_f is a parameter depending on the moments of P, and C is a constant such that ϕ is normalized to one.

Using the gauge symmetry of H, one can write the n-th replica momentum with respect to the distribution P under the following form :

$$\left\langle Z^n(\beta)\right\rangle_P = C^{N_B} \int \prod_{\langle ij\rangle} \mathrm{d}J_{ij}\, \phi\left(\left|J_{ij}\right|\right) \sum_{\{S_i^\alpha\}} \mathrm{e}^{\beta_f \sum_{\langle ij\rangle} J_{ij}\tau_i\tau_j + \beta \sum_{\alpha=1}^{n} \sum_{\langle ij\rangle} J_{ij} S_i^\alpha S_j^\alpha} \tag{17}$$

where N_B is the number of bonds of the lattice. Summing over all the possible choices for the $\left\{\tau_i\right\}$, which play the part of an additional replica, one gets :

$$\left\langle Z^n\left(\beta = \beta_f\right)\right\rangle_p = 2^{-N}\, C^{N_B} \left\langle Z^{n+1}\left(\beta_f\right)\right\rangle_\phi . \tag{18}$$

In the particular case of the binary distribution (8), this reads (with obvious notations) :

$$\left\langle Z^n\left(\beta_f, p\right)\right\rangle = \frac{\left\langle Z^{n+1}\left(\beta_f, p = 1/2\right)\right\rangle}{2^N\left(\cosh \beta_f J\right)^{N_B}} \tag{19}$$

where β_f is given by $\tanh \beta_f J = 2p - 1$.

Let us note that $\beta = \beta_f$ is precisely Nishimori's condition. Equation (18) thus relates the n-th replica momentum on Nishimori's subspace to the $(n+1)$-th one for a symmetric distribution of the couplings. In particular, this relates the critical point of $\left\langle Z^n\right\rangle$ on the subspace $\beta = \beta_f$ to the one of $\left\langle Z^{n+1}\right\rangle$ for a symmetric distribution.

As an illustration, let us note that, in general, Nishimori's subspace intersects for $\beta = \beta_f$ the ferromagnetic/paramagnetic critical line of the quenched system, i.e. of the derivative of $\left\langle Z^n\right\rangle$ with respect to n at $n = 0$. Relation (18) thus implies that the *derivative* of $\left\langle Z^{n+1}\right\rangle$ with respect to n, at $n = 1$, for the symmetric distribution ϕ has a transition at $\beta = \beta_f$ between a paramagnetic and a mixed phase. This remark is perhaps not academic, since it has been proposed [18] that the mixed phase of $\left.\dfrac{\partial\left\langle Z^n\right\rangle}{\partial n}\right|_{n=1}$ (sometimes called « hidden Mattis phase ») could be associated with dynamical properties (slow relaxation) of the true $(n \to 0)$ quenched problem. The existence of such a transition at $J_0 = 0$, $\beta J = 1$ in the case of the SK model, demonstrated in [19], follows directly from the symmetry relation (18) and the location of the tricritical point of the quenched system at $J_0/J = \beta J = 1$.

Let us note finally that a more precise meaning can be given to the parameter β_f defined by (16). Actually, it has been shown in reference [21] that the average over the configurations of the couplings in the spin-glass problem can be replaced by an average over the frustrations themselves, interacting through an effective Hamiltonian. The thermodynamical state of these frustration variables is characterized by an inverse temperature which is precisely β_f. When $\beta \ll \beta_f$, the spin degrees of freedom dominate over the degrees of freedom associated to frustrations, while when $\beta \gg \beta_f$, the frustration disorder becomes relevant. Nishimori's line, on which $\beta = \beta_f$, thus appears as a natural separation between these two regimes for a spin-glass.

3.2 A COMPARISON WITH THE GREM. — An interesting application of the exact result (18) is that it provides a test of one of the consequences of the GREM approach to the finite dimensional spin-glass problem [5]. In this approach, one includes correlations between energies by using as an input the first two replica momenta of the true spin-glass. The other replica momenta are then completely specified by the model. In particular, this gives an estimation of $\left\langle Z^3\right\rangle$. For a Gaussian symmetric distribution of the couplings :

$$P\left(J_{ij}\right) = \frac{1}{J}\sqrt{\frac{2}{\pi}}\, \mathrm{e}^{-2J_{ij}^2/J^2} \tag{20}$$

one gets [21] :

$$\left\langle Z^3(\beta J)\right\rangle^{\mathrm{GREM}} = \left\langle Z(\beta J)\right\rangle^3 \frac{\left\langle Z^2\left(\sqrt{\frac{3}{2}}\beta J\right)\right\rangle^2}{\left\langle Z\left(\sqrt{\frac{3}{2}}\beta J\right)\right\rangle^4} \tag{21}$$

which exhibits a critical point separating a mixed and paramagnetic phase at

$$\beta J_c^{\mathrm{GREM}} = \sqrt{4/3 \ln\left(1 + \sqrt{2}\right)} = 1.084\ldots$$

The value of $\langle Z^3 \rangle^{\mathrm{GREM}}$ at this critical point is thus :

$$\frac{1}{N} \ln \langle Z^3 \rangle_{\mathrm{crit}}^{\mathrm{GREM}} = 2 \ln 2 + \ln \left(1 + \sqrt{2}\right) + + \frac{4}{\pi} G \simeq 3.434 \quad (22)$$

where $G = 0.915965...$ is Catalan's constant.

On the other hand, the symmetry property (19) allows to obtain the exact expression of $\langle Z^3 \rangle$ at criticality, together with the corresponding critical temperature :

$$\beta J_{\mathrm{c}}^{\mathrm{exact}} = \sqrt{\ln 3} \simeq 1.048$$

$$\frac{1}{N} \ln \langle Z^3 \rangle_{\mathrm{c}}^{\mathrm{exact}} = \frac{\ln 3}{4} - 2 \ln 2 + 4 \ln \frac{\Gamma(1/4)}{\Gamma(3/4)} \simeq 3.227 \, . \quad (23)$$

One can thus see that the GREM leads to a remarkably good approximation both for the location of the critical point of $\langle Z^3 \rangle$ and for its value at criticality in two dimensions. Including the pair correlations between energies has lowered this critical point from the REM result [15] $\beta J_{\mathrm{c}}^{\mathrm{REM}} = 4 \sqrt{\ln 2/3} = 1.93...$ to less than 4 % of the exact value.

3.3 A SYMMETRY PROPERTY OF n-COLOUR ASHKIN-TELLER MODELS. — The symmetry property (19) can be turned into a similar relation for n-colour AT models ; this corresponds to the case of a Gaussian distribution :

$$p\left(J_{ij}\right) = \frac{1}{\sqrt{2\,\pi}\,J} \mathrm{e}^{-\frac{\left(J_{ij} - J_0\right)^2}{2 J^2}} \quad (24)$$

for which $\beta_{\mathrm{f}} = J_0/J^2$. After performing the Gaussian integration, the n-th replica momentum corresponds to the effective Hamiltonian of an n-colour AT model :

$$H_{\mathrm{AT}}^{(n)} = \sum_{\langle ij \rangle} \left[K_2 \sum_{\alpha} S_i^{\alpha} S_j^{\alpha} + \frac{K_4}{2} \sum_{\alpha\beta} S_i^{\alpha} S_j^{\alpha} S_i^{\beta} S_j^{\beta} \right] \quad (25)$$

with $K_2 = \beta J_0$ and $K_4 = \beta^2 J^2$. The previous relation between the n-th and $(n+1)$-th moment can be recovered directly if one applies a gauge transformation $S_i^{\alpha} \to \tau_i S_i^{\alpha}$ to the Hamiltonian H. This leads to the following relation between partition functions :

$$Z_{\mathrm{AT}}^{(n)} \left(K_2 = K_4 = K\right) = 2^{-N} Z_{\mathrm{AT}}^{(n+1)} \times \times \left(K_2 = 0, K_4 = K\right) . \quad (26)$$

This shows in particular that the critical point of the Hamiltonian $H_{\mathrm{AT}}^{(n)}$ for $K_2 = K_4$ coincides with the critical point of the Hamiltonian $H_{\mathrm{AT}}^{(n+1)}$ for $K_2 = 0$. This can be sketched on the phase diagrams in the $\left(1/K_4, K_2/K_4\right)$ plane, for different values of n (see Fig. 4) : the intersection of the axis $K_2 = K_4$ with the paramagnetic/ferromagnetic critical line for the n-colour AT model arises at the same value of $1/K_4$ as the intersection of the paramagnetic/mixed critical line of the $(n+1)$ colour model, with the axis $K_2 = K_4$. This symmetry property of the phase diagrams of n-colour AT models is illustrated, in the infinite dimensional case, by the mean-field results of Sherrington : see the figure 2 of reference [2] on which it clearly appears. It can also be checked for a finite dimension on the results of reference [22] where a Monte-Carlo simulation of the three-colour problem for $D = 2$ and 3 has been performed. Using the known results on the critical point for $n = 2$, $K_2 = K_4$ (four-state Potts model), one gets critical values for K_4 (when $K_2 = 0$) in the $n = 3$ case which are in good agreement with the values asymptotically deduced from the figures 2 and 3 of reference [22].

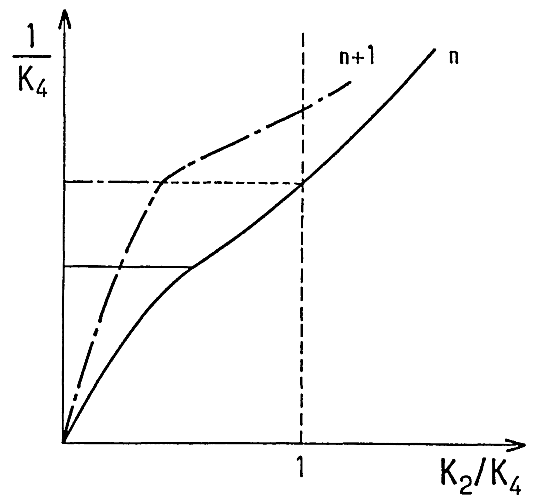

Fig. 4. — Correspondance between the phase diagram of the Ashkin-Teller model for n and $(n+1)$ colours. The tricritical point of the $(n+1)$ Ashkin-Teller model occurs for the same $1/K_4$ value as the intersection of the $K_2 = K_4$ line with the paramagnetic-ferromagnetic critical line for the n colour Ashkin-Teller model.

It is clear on equation (26) that, on the line $K_2 = K_4$, the « gauge symmetry » of the n-colour AT model is restored. Indeed, this symmetry, which amounts to flip the replicas in a *site-dependent way* : $S_i^{\alpha} \to \tau_i S_i^{\alpha}$, is broken in general by the K_2-term of the Hamiltonian H_{AT} (this term originates, in the spin-glass case, from the non-invariance of $P\left(J_{ij}\right)$ under $J_{ij} \to -J_{ij}$). Relation (26) expresses precisely that, on the line $K_2 = K_4$, $H_{\mathrm{AT}}^{(n)}$ is equivalent to $H_{\mathrm{AT}}^{(n+1)}$ with $K_2 = 0$, an Hamiltonian which is gauge-invariant. For the spin-glass also, the gauge symmetry is restored on Nishimori's line $\beta = \beta_{\mathrm{f}}$, since $\langle Z^n \rangle_p$ becomes equivalent to $\langle Z^{n+1} \rangle_{\phi}$, for a symmetric ϕ. Moreover, one can show [6] that the quenched mean-value of any gauge-invariant operator is equal on Nishimori's line to its *annealed* mean value with respect to the distribution ϕ.

These symmetry considerations could give some arguments in favour of the location of the tricritical point of the $\langle Z^n \rangle$'s (or alternatively of the $H_{AT}^{(n)}$'s) on this particular line. Indeed, the occurrence of a tricritical point on a line of enhanced symmetry has already been encountered in reference [23] for some generalizations of spin models. It is in fact what happens for $n = 2$, $D = 2$ and for the quenched limit $n \to 0$ in $D = \infty$ (and presumably also in $D = 3$). Furthermore, it is interesting to note that, while it is true that the mean-field analysis of reference [2] shows that the tricritical point for $D = \infty$ is *not* on the line $K_2 = K_4$ (except in the spin-glass limit $n \to 0$), this line still plays some part in the phase diagram of the model. Indeed, it has been shown in reference [24] that an analysis of the minima of the free energy in the infinite dimensional $n = 2$ case leads to a separation between three regions whose boundaries do intersect on the line $K_2 = K_4$. It is only the stability analysis of these minima which leads to modify this picture, and to separate the true tricritical point from the line $K_2 = K_4$. As a consequence, three exists a region of the phase diagram where a relative and an absolute minimum coexist. The fact that the tricritical point moves again to the line $K_2 = K_4$ in the limit $n \to 0$ could be in relation with the fact that the system chooses the relative minimum of the free-energy, a well-known feature of spin-glasses.

4. Conclusion.

In this paper, we have used some exact results for the two-dimensional Ashkin-Teller model in order to study the second replica momentum of a 2D spin-glass. The calculation of $\langle Z^2 \rangle$ along two directions of its phase-diagram strongly suggests that the behaviour of the width of the probability distribution of the partition function (and thus the nature of the correlations between energy levels) is close to the infinite dimensional case. This delimits the domain of the phase diagram of the spin-glass where the annealed approximation is no longer relevant. The different phases of $\langle Z^2 \rangle$ meet at a tricritical point which lies precisely on Nishimori's line. Indeed, this line is a line of enhanced symmetry which appears to play a special part in the understanding of the phase diagrams both for the finite replica momenta and for the quenched system. Moreover, some comparisons between these exact and the predictions of the « generalized random energy model » indicate that taking only the pair correlations between energies into account is a very good approximation, at leat for certain thermodynamical quantities.

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Appendix

Having defined a modified partition function $\tilde{Z}$ through equation (15), the quenched free energy is given by :

$$\langle \ln Z \rangle = N_{\text{Bonds}} \langle \ln \cosh \beta J_{ij} \rangle + \langle \ln \tilde{Z} \rangle \qquad (27)$$

$\langle \tilde{Z}^2 \rangle$ is again related to a symmetric AT model :

$$\langle \tilde{Z}^2 \rangle = c\left(\tilde{K}_2, \tilde{K}_4\right) Z_{\text{AT}}\left(\tilde{K}_2, \tilde{K}_4\right) \qquad (28)$$

where $\tilde{K}_2$ and $\tilde{K}_4$ are given by (denoting $t_{2,4} = \tanh\left(\tilde{K}_{2,4}\right)$, $t = \tanh\left(\beta J_{ij}\right)$) :

$$\frac{t_2\left(1 + t_4\right)}{1 + t_4 t_2^2} = \langle t \rangle, \quad \frac{t_2^2 + t_4}{1 + t_4 t_2^2} = \langle t^2 \rangle \qquad (29)$$

and with :

$$c\left(\tilde{K}_2, \tilde{K}_4\right) = \left[\cosh \tilde{K}_4 \cosh^2 \tilde{K}_2 \left(1 + t_4 t_2^2\right)\right]^{-2N} . \qquad (30)$$

The remarkable lines and points of the phase diagram of $\langle \tilde{Z}^2 \rangle$ are given by :

(SD) line :

$$2\langle t \rangle + \langle t^2 \rangle = 1 \qquad (31)$$

4-state Potts model line :

$$\langle t \rangle = \langle t^2 \rangle$$

tricritical point :

$$(\text{T}) \quad \langle t \rangle = \langle t^2 \rangle = 1/3$$

Ising-like points :

$$\text{I}_2 \quad \langle t^2 \rangle = \sqrt{2} - 1 \quad \langle t \rangle = 0$$

$$\text{I}_1, \text{I}_1 \quad \langle t^2 \rangle = 1 \quad \langle t \rangle = 1/\sqrt{2}$$

to be compared with (6), (7). These formulae are valid for an arbitrary distribution $P\left(J_{ij}\right)$. When this distribution allows for the existence of Nishimori's line, its location in the phase diagram coincides with the 4-state Potts model line and has the same properties as in section 2.2 (in particular its intersection with the (SD) line occurs at the tricritical point (T)).

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