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Rational dynamical zeta functions for birational transformations

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Abstract

We propose a conjecture for the exact expression of the unweighted dynamical zeta function for a family of birational transformations of two variables, depending on two parameters. This conjectured function is a simple rational expression with integer coefficients. This yields an algebraic value for the topological entropy. Furthermore, the generating function for the Arnold complexity is also conjectured to be a rational expression with integer coefficients with the same singularities as for the dynamical zeta function. This leads, at least in this example, to an equality between the asymptotic of the Arnold complexity and the exponential of the topological entropy. We also give a semi-numerical method to effectively compute the Arnold complexity. © 1999 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

To study the complexity of continuous, or discrete, dynamical systems, a large number of concepts have been introduced [1,2]. A non-exhaustive list includes the Kolmogorov–Sinai metric entropy [3,4], the Adler–Konheim–McAndrew topological entropy [5], the Arnold complexity [6], the Lyapounov characteristic exponents, the various fractal dimensions, [7,8] the Feigenbaum's numbers of period-doubling cascades [9,10], etc. Many authors have tried to study and discuss the relations between these various notions in an abstract framework [11,12]. Inequalities have been shown, for instance the metric entropy is bounded by the topological entropy, let us also mention the Kaplan–Yorke relation [13,14]. Furthermore, many specific dynamical systems have been introduced enabling to see these notions at work. Some of the most popular are the Lorentz system [15], the baker map [16], the logistic map [17], the Henon map [18]. Each of these systems has been useful to understand and exemplify the previous complexity measures.

Here, we introduce another two-parameter family of mapping of two variables, originating from lattice statistical mechanics, for which much can be said. In particular, we will conjecture an exact algebraic value for the exponential of the topological entropy and for the asymptotic of the Arnold complexity. Furthermore, these two measures of complexity will be found to be equal for all the values of the two parameters, generic or not (the notion of genericity is explained below). A fundamental distinction must be made between the previously mentioned complexity measures according to their invariance under certain classes of transformations. One should distinguish, at least, two different sets of complexity measures, the ones which are invariant under the larger classes of variables transformations, like the *topological entropy* or the *Arnold complexity* [6], and the other measures of complexity which also have invariance properties, but under a " less large" set of transformations, and are therefore more sensitive to the details of the mapping (for instance they will depend on the metric).

We now introduce the following two parameters family of birational transformations $k_{\alpha,\varepsilon}$:

$$\begin{aligned} u_{n+1} &= 1 - u_n + u_n/v_n \,, \\ v_{n+1} &= \varepsilon + v_n - v_n/u_n + \alpha \cdot (1 - u_n + u_n/v_n) \end{aligned} \tag{1}$$

which can also be written projectively:

$$\begin{aligned} u_{n+1} &= (v_n t_n - u_n v_n + u_n t_n) \cdot u_n \,, \\ v_{n+1} &= \varepsilon \cdot u_n \cdot v_n \cdot t_n + (u_n - t_n) \cdot v_n^2 + \alpha \cdot (v_n t_n - u_n v_n + u_n t_n) \cdot u_n \,, \\ t_{n+1} &= u_n \cdot v_n \cdot t_n \,. \end{aligned} \tag{2}$$

As far as complexity calculations are concerned, the $\alpha = 0$ case is singled out. In that case, it is convenient to use a change of variables (see Appendix A) to get the very simple form k_ε:

$$\begin{aligned} y_{n+1} &= z_n + 1 - \varepsilon \,, \\ z_{n+1} &= y_n \cdot \frac{z_n - \varepsilon}{z_n + 1} \end{aligned} \tag{3}$$

or on its homogeneous counterpart:

$$y_{n+1} = (z_n + t_n - \varepsilon \cdot t_n) \cdot (z_n + t_n) ,$$
$$z_{n+1} = y_n \cdot (z_n - \varepsilon \cdot t_n) , \qquad (4)$$
$$t_{n+1} = t_n \cdot (z_n + t_n) .$$

These transformations derive from a transformation acting on a $q \times q$ matrices M [20]:

$$K_q = t \circ I, \qquad (5)$$

where t permutes the entries $M_{1,2}$ with $M_{3,2}$ and I is the homogeneous inverse: $I(M) = det(M) \cdot M^{-1}$. Transformations of this type, generated by the composition of permutations of the entries and matrix inverse, naturally emerge in the analysis of lattice statistical mechanics symmetries [19].

These transformations turn out to provide a set of examples for which various conjectures can be made. This is the aim of this paper which is organized as follows: in the first part of the paper we exactly compute the growth of the complexity of the first successive iterations (degree of the successive expressions). From these integers, we conjecture various algebraic values for the complexity. Different cases, corresponding respectively to $\alpha = 0$ and $\alpha \neq 0$, are distinguished in two subsections. The results of these sections are confirmed by a semi-numerical method we introduce. In the second part of the paper we address the problem of evaluating another measure of the complexity, namely the *topological entropy*. This is done computing formally the first terms of the expansion of the generating function of the number of fixed points. This leads us to conjecture rational expressions for these generating functions. The *same* singled out (α, ε)-values as for the complexity growth appear, and are separately analyzed also in two subsections. The last section is devoted to a discussion about a possible " diffeomorphism of the torus" interpretation for the rationality of the generating functions we conjecture.

2. The complexity growth

The correspondence [20] between transformations K_q and $k_{\alpha,\varepsilon}$, more specifically between K_q^2 and $k_{\alpha,\varepsilon}$, is given in Appendix A. It will be shown below that, beyond this correspondence, K_q^2 and $k_{\alpha,\varepsilon}$ share properties concerning the complexity. Transformation K_q is homogeneous and of degree $(q-1)$ in the q^2 homogeneous entries. When performing the nth iterate one expects a growth of the degree of each entry as $(q-1)^n$. It turns out that, at each step of the iteration, some factorization of all the entries occurs. The common factor can be factorized out in each entry leading to a reduced matrix M_n, which is taken as the representent of the nth iterate in the projective space. Due to these factorizations, the growth of the calculation is not $(q-1)^n$ but rather λ^n where generically λ is the largest root of $1 + \lambda^2 - \lambda^3 = 0$ (i.e. $1.46557123 < q - 1$) [20,21] as detailed in Appendix B. We call λ the complexity growth or simply the complexity. This result is a consequence of a *stable factorization scheme* [3] given in Appendix B, from which two generating functions $\alpha(x)$ and $\beta(x)$ can be constructed. The stability of the factorization scheme yields rational expressions for these generating functions. All the results of this section are based on the assumption that this stability actually holds. This assumption is verified, up to high orders, for many cases (see also [44]), but a completely rigorous proof is lacking. Generating function $\alpha(x)$ keeps track respectively of the degrees of the determinants of the successive reduced matrices and $\beta(x)$ of the degrees of the successive common factors. [4] The actual value of λ is the inverse of the pole of $\beta(x)$ (or $\alpha(x)$) of smallest modulus. The algebraicity of the complexity is, in fact, a straight consequence of the rationality of functions $\alpha(x)$ and $\beta(x)$ with integer coefficients [20]. The same calculations have also been performed on transformations (1) and (2). In that case, factorizations also occur at each step, and generating functions can be calculated. These generating functions are, of course, different from the generating functions for K_q^2 (see [20]) but they have the *same* poles, and consequently the same complexity growth. One sees that, remarkably, the complexity λ does not depend on the birational representation considered: K_q^2 for any value of q, $k_{\alpha,\varepsilon}$ or the homogeneous transformation Eq. (2). It will be useful to define various degree generating functions $G(x)$:

$$G(x) = \sum_n d_n \cdot x^n \,, \tag{6}$$

where d_n is the degree of some quantities we look at, at each iteration step (numerators or denominators of the two components of k^n, degree of the entries of the " reduced" matrices M_n's, degree of the extracted polynomials f_n's in Appendix B ...). The complexity growth λ is the inverse of the pole of smallest modulus of any of these degree generating functions $G(x)$:

$$\log \lambda = \lim_{m \to \infty} \frac{\log d_m}{m} \,. \tag{7}$$

2.1. Complexity growth for $\alpha = 0$

In the $\alpha = 0$ case, which corresponds to a codimension one variety of the parameter space (see Appendices A and C), additional factorizations, compared to the $\alpha \neq 0$ factorization scheme (B.2) and (B.3) of Appendix B, occur reducing further the growth of the complexity. The new complexity is given, for K_q, by the equation $1 - \lambda^2 - \lambda^4 = 0$ i.e. $\lambda \simeq 1.27202\ldots$. For k_ε, which corresponds to K_q^2, the equation reads

$$1 - \lambda - \lambda^2 = 0 \tag{8}$$

leading to the complexity $\lambda \simeq 1.61803 \cdots \simeq (1.27202 \cdots)^2$. Not surprisingly, the complexity of the mappings $k_{\alpha,\varepsilon}$ for $\alpha = 0$ (see (1)) and mapping k_ε (see (3)), are the

[3] Complexity growth can also be understood from a singularity point of view [47], or through recurrence relations associated with the geometry of the singularities of the evolution [48]. This is not the approach developed here.

[4] The function $\alpha(x)$ should not be confused with the parameter α.

same: complexity λ corresponds to the asymptotic behavior of the degree of the successive quantities encountered in the iteration (see (7)). Clearly, this behavior remains unchanged under simple changes of variables. Note that this complexity growth [5] analysis can be performed directly on transformation k_ε, or its homogeneous counterpart Eq. (4). The number of generating functions in the two cases is not the same, but all these functions lead to the same complexity. In fact, complexity λ is linked to the *Arnold complexity* [6], known to be invariant under transformations corresponding to a change of variables (like the change of variables from Eq. (1) (for $\alpha = 0$) to Eq. (3) or to Eq. (4)). Let us also recall that the Arnold complexity counts the number of intersections between a fixed line [6] and its nth iterate, which clearly goes as λ^n. Conversely, all these calculations can be seen as a handy way of calculating the Arnold complexity.

All these considerations allow us to design a semi-numerical method to get the value of the complexity growth λ for any value of the parameter ε. The idea is to iterate, with (3) (or (1)), a generic *rational* initial point (y_0, z_0) and to follow the magnitude of the successive numerators and denominators. During the first few steps, some accidental simplifications may occur, but, after this transient regime, the integer denominators (for instance) grow like λ^n where n is the number of iterations. Typically, a best fit of the logarithm of the numerator as a linear function of n, between $n = 10$ and 20, gives the value of λ within an accuracy of 0.1%. An integrable mapping corresponds to a polynomial growth of the calculations: the value of the complexity λ has, will be numerically very close to 1. Fig. 1 shows the values of the complexity as a function of the parameter ε. The calculations have been performed using an infinite-precision C-library. [7]

For most of the values of ε we have found $\lambda \simeq 1.618$, in excellent agreement with the value predicted in Eq. (8). In [26], it has been shown that the simple rational values $\varepsilon = -1, 0, 1/3, 1/2, 1$ yield integrable mappings. For these special values one gets $\lambda \approx 1$ corresponding to a *polynomial* growth [26]. In addition, Fig. 1 singles out two sets of values $\{1/4, 1/5, 1/6, \ldots, 1/13\}$ and $\{3/5, 2/3, 3/7\}$, suggesting two infinite sequences $\varepsilon = 1/n$ and $\varepsilon = (m-1)/(m+3)$ [8] for n and m integers such that $n \geqslant 4$ and $m \geqslant 7$ and m odd. We call " non-generic" the values of ε of one of the two forms above (together with the integrable values), and " generic" the others. To confirm this set of values, we go back to (the matrix) transformation K_q, for $q = 3$, to get a generating function of the degrees of some factors (the f_n's in Appendix B) extracted at each step of iteration, namely, with the notations of [19,21,44] and of Appendix B, function $\beta(x)$. From now on we will give, instead of $\beta(x)$, the expression of the following complexity

[5] Growth of the calculations related with factorizations were also introduced by Veselov for some particular Cremona transformations [22–24].

[6] Or the intersection of the nth iterate of any fixed algebraic curve together with any other possibly different but fixed algebraic curve.

[7] The multi-precision library gmp is part of the GNU project.

[8] Note that $m \longrightarrow (m+3)/(m-1)$ is an involution.

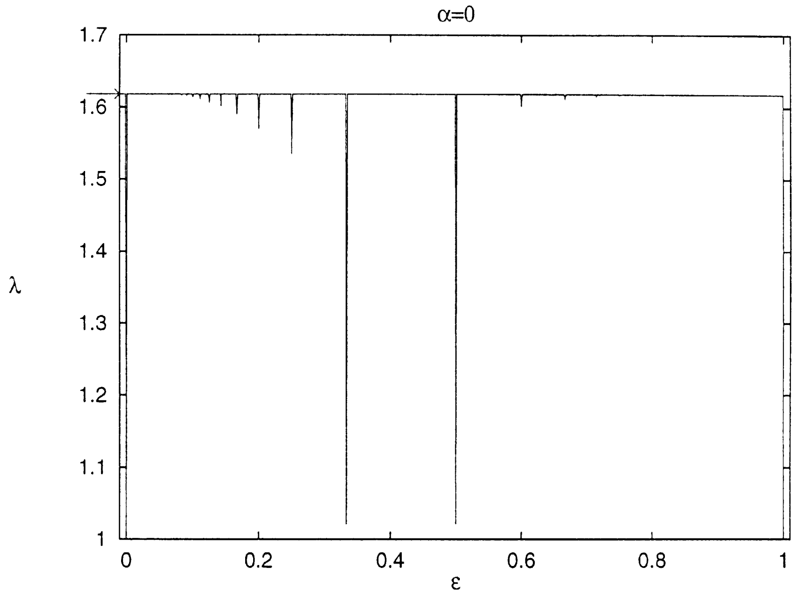

Fig. 1. Complexity λ as a function of ε taken of the form $M/720$ plus the special values $1/7, 1/11, 1/13$ and $5/7$ for $\alpha = 0$. The arrow indicates the expected value.

generating function defined, for $q \times q$ matrices, as

$$G_\varepsilon^\alpha(q,x) = \frac{\beta(x)}{q \cdot x} . \qquad (9)$$

In the following the calculations are displayed for 3×3 matrices and $G_\varepsilon^\alpha(q,x)$ will simply be denoted $G_\varepsilon^\alpha(x)$. Let us recall that the value of the complexity λ is the inverse of the root of smallest modulus of the denominator of this rational function. Examples of these calculations in order to get the corresponding factorization scheme and deduce the generating function $\beta(x)$ or $G_\varepsilon^\alpha(x)$, are given in Appendix B. In Appendix C, we show how to choose an initial matrix to iterate which satisfies $\alpha = 0$ and $\varepsilon = p/q$ for any integers p and q. First, we have obtained (see Appendix B) the generating function $G_\varepsilon(x)$ in the generic case for $\alpha = 0$:

$$G_\varepsilon(x) = \frac{1 + x + x^3}{1 - x^2 - x^4} . \qquad (10)$$

We also got the generating function $G_\varepsilon(x)$ for the different non-generic cases:

$$G_{1/m}(x) = \frac{1 + x + x^3 - x^{2m+1} - x^{2m+3}}{1 - x^2 - x^4 + x^{2m+4}} \quad \text{with } m \geqslant 4 , \qquad (11)$$

$$G_{(m-1)/(m+3)}(x) = \frac{1 + x + x^3 - x^{2m+6}}{1 - x^2 - x^4 + x^{2m+4}} \quad \text{with } m \geqslant 7 \ m \text{ odd} \qquad (12)$$

and

$$G_{\mathrm{int}}(x) = \frac{1 + x + x^3 + x^4 + x^8 + x^{12}}{1 - x^2 - x^6 + x^8 - x^{10} + x^{12} + x^{16} - x^{18}}$$
$$= \frac{1 + x \cdot (1 + x^2) + x^4 \cdot (1 + x^4 + x^8)}{1 - x^2 \cdot (1 - x^{12}) - x^6 \cdot (1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12})} \qquad (13)$$

for the two integrable values $\varepsilon = 1/2$ and $\varepsilon = 1/3$. For $\varepsilon = 1/m$ ($m \geqslant 4$) and $\varepsilon = (m-1)/(m+3)$ ($m \geqslant 7$ and m odd), the corresponding complexities are the inverse of the roots of smallest modulus of polynomial

$$1 - x^2 - x^4 - x^{2m+4} = 0 \qquad (14)$$

in agreement with the values of Fig. 1. In this figure the ε-axis has been discretized as $M/720$ (M integer) and the extra values $1/7, 1/11, 1/13$ and $5/7$ have been added. This semi-numerical method acts as an " integrability detector" and, further, provides a simple and efficient way to determine the complexity of an algebraic mapping. Applied to mappings (1), (5), or (3), it shows that the complexity is, generically, *independent* of the value of the parameter ε, except for the four integrable points, and for two denombrable sets of points. [9]

It is worth noticing that these results are not specific to 3×3 matrices, for example relation Eq. (10) is actually valid simply replacing $G_\varepsilon^\alpha(x)$ by $G_\varepsilon^\alpha(q,x)$.

2.2. Complexity growth for $\alpha \neq 0$

These complexity growth calculations can straightforwardly be generalized to $\alpha \neq 0$. As explained in Appendix B, the generic generating function associated to the factorization schemes B.2 and B.3 is

$$G_\varepsilon^\alpha(x) = \frac{1 + x^2}{1 - x - x^3} \; . \qquad (15)$$

The pole of smallest modulus of Eq. (15) gives 1.46557... for the value of the complexity for the matrix transformation K. The complexity for the transformation $k_{\alpha,\varepsilon}$ is the square of this value: $\lambda = 2.14790\ldots$. Fig. 2 shows, for $\alpha = 1/100$, complexity λ as a function of the parameter ε, obtained with the semi-numerical method previously explained. Even with such a " small value" of α the expected drastic change of value of the complexity (namely $1.61803 \rightarrow 2.14790$) is non-ambiguously seen. Moreover, Fig. 2 clearly shows that, besides the value $\varepsilon = 0$ known to be integrable whatever α [26], at least the following values $\varepsilon = 1/2$, $\varepsilon = 1/3$ and $\varepsilon = 3/5$ are associated with a significantly smaller complexity, at least for the discretization in ε we have investigated. From these numerical results and by analogy with $\alpha = 0$, one could figure out that all the $\varepsilon = 1/m$ are also non-generic values of ε. In fact a factorization scheme analysis (like the one depicted in Appendix B) shows that $\varepsilon = 1/4$ or $\varepsilon = 1/7$ actually correspond to the generic generating function (15). We got similar results for other

[9] These two sets of points also appear naturally in the framework of a " singularity confinement analysis" [25].

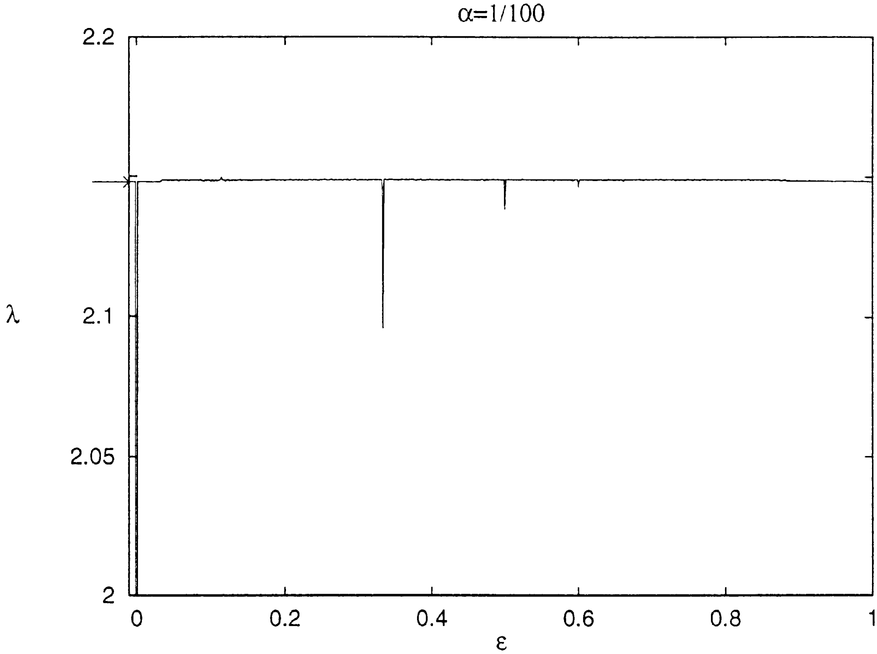

Fig. 2. Complexity λ as a function of ε taken of the form $M/720$ for $\alpha = 1/100$. The arrow indicates the expected value.

values of $\alpha \neq 0$. However, when varying α and keeping ε fixed, new values of the complexity λ occur, λ being some " stair-case" function of α. We will not exhaustively describe the rather involved " stratified" space in the (α, ε) plane, corresponding to the various non-generic complexities. Let us just keep in mind that, besides $\varepsilon = 0$ and $\varepsilon = -1$, at least $\varepsilon = 1/2, 1/3$ and $3/5$ are singled out for $\alpha \neq 0$ in our semi-numerical analysis. The generic expression (for 3×3 matrices) for the generating function $G(x)$, (15), is replaced, for the non-generic $\varepsilon = 1/2$ (with $\alpha \neq 0$), by

$$G^{\alpha}_{1/2}(x) = \frac{1 + x + x^3 - x^{16}}{(1 - x^2) \cdot (1 - x^2 - x^4 - 2x^6 - x^8 - 2x^{10} - x^{12} - x^{14})} \,. \tag{16}$$

For the other non-generic value of ε, $\varepsilon = 1/3$, the complexity generating function reads

$$G^{\alpha}_{1/3}(x) = \frac{1 + x + x^3 - x^{12}}{(1 - x^2) \cdot (1 - x^2 - x^4 - 2x^6 - x^8 - x^{10})} \,. \tag{17}$$

For the non-generic value $\varepsilon = 3//5$, the complexity generating function reads

$$G^{\alpha}_{3/5}(x) = \frac{1 + x + x^3 - x^{20}}{(1 - x^2) \cdot (1 - x^2 - x^4 - 2x^6 - x^8 - 2x^{10} - x^{12} - 2x^{14} - x^{16} - x^{18})} \,. \tag{18}$$

3. Dynamical zeta function and topological entropy

It is well known that the fixed points of the successive powers of a mapping are extremely important in order to understand the complexity of the phase space. A lot of work has been devoted to study these fixed points (elliptic or saddle fixed points, attractors, basin of attraction, etc.), and to analyze related concepts (stable and unstable manifolds, homoclinic points, etc.). We will here follow another point of view and study the generating function of the *number* of fixed points. By analogy with the Riemann ζ function, Artin and Mazur [27] introduced a powerful object – the so-called *dynamical zeta function*: [10]

$$\zeta(t) = \exp\left(\sum_{m=1}^{\infty} \#\text{fix}(k^m) \cdot \frac{t^m}{m} \right) , \tag{19}$$

where $\#\text{fix}(k^m)$ denotes the number of fixed points of k^m. The generating functions

$$H(t) = \sum \#\text{fix}(k^m) \cdot t^m \tag{20}$$

can be deduced from the ζ function

$$H(t) = t \frac{\mathrm{d}}{\mathrm{d}t} (\log \zeta(t)) . \tag{21}$$

The topological entropy h is related to the singularity of the dynamical ζ function

$$\log h = \lim_{m \to \infty} \frac{\log(\#\text{fix}(k^m))}{m} . \tag{22}$$

If the dynamical zeta function can be interpreted as the ratio of two characteristic polynomials of two linear operators [11] A and B, namely $\zeta(t) = det(1 - t \cdot B)/det(1 - t \cdot A)$, then the number of fixed points $\#\text{fix}(k^m)$ can be expressed from $Tr(A^n) - Tr(B^n)$. In this linear operators framework, the *rationality* of the ζ function, and therefore the algebraicity of the exponential of the topological entropy, amounts to having a *finite dimensional representation* of the linear operators A and B. In the case of a rational ζ function, the exponential of the topological entropy is the inverse of the pole of smallest modulus. Since the number of invariant points remains unchanged under topological conjugacy (see Smale [35] for this notion), the ζ function is also a topologically invariant function, invariant under a large set of transformations, and does not depend on a specific choice of variables. Such invariances were also noticed for the complexity growth λ. It is then tempting to make a connection between the *rationality* of the complexity generating function previously given, and a possible *rationality* of the dynamical ζ function. We will also compare the growth complexity λ and the exponential of the topological entropy h.

[10] Other dynamical zeta functions, taking a weighted counting into account, have also been introduced see for instance [28,30]. For the scope of our paper, we only need to use the unweighed Artin and Mazur dynamical zeta function.

[11] For more details on these Perron–Frobenius, or Ruelle–Araki transfer operators, and other shifts on Markov's partition in a symbolic dynamics framework, see for instance [31–34].

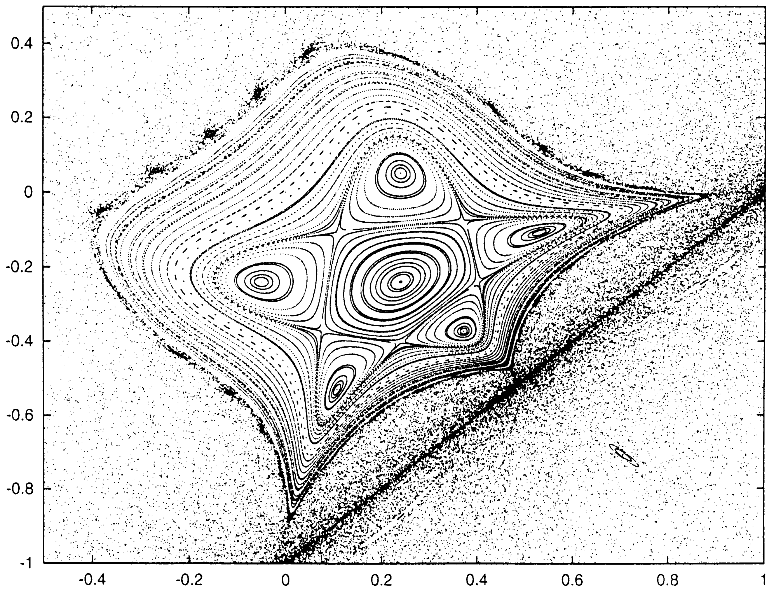

Fig. 3. Phase portrait of k_ε for $\alpha = 0$ and $\varepsilon = 13/25$: 550 orbits of length 1000 have been generated. Fifty orbits start from points randomly chosen near a fixed point of order 5 of $k_\varepsilon = k_{13/25}$, and 500 others orbits start from randomly chosen points outside the elliptic region. Only the points inside the frame are shown.

3.1. Dynamical zeta function for $\alpha = 0, \varepsilon$ generic

We try here to get the expansion of the dynamical zeta function of the mapping k_ε (see Eq. (3)), for generic values of ε which are neither of the form $1/m$, nor of the form $(m-1)/(m+3)$. We concentrate on the value $\varepsilon = 13/25 = 0.52$. This values is close to the value 1/2 where the mapping is integrable [26]. One can gain an idea of the number, and localization, of the (real) fixed points looking at the phase portrait of Fig. 3. The elliptic fixed points $(y_0, z_0) = (0.24, -0.24)$ is well seen, as well as the five elliptic points and the five saddle points of k_ε^5. Many points of higher degree are also seen. Transformation k_ε has a single fixed point for any ε. This fixed point is elliptic for $\varepsilon \geqslant 0$ and localized at $(y_0, z_0) = ((1-\varepsilon)/2, (\varepsilon - 1)/2)$. Transformation k_ε^2 has only the fixed point inherited from k_ε. The new fixed points of k_ε^3 are $(2-\varepsilon, (\varepsilon-1)/2), (-1, 1)$ and $((1-\varepsilon)/2, \varepsilon - 2)$. Transformation k_ε^4 has four new fixed points. At this point the calculations are a bit too large to be carried out with a literal ε, and we particularize $\varepsilon = 13/25$. For k_ε^5 we have five new elliptic points and five new saddles points. The coordinates z and y of these points are roots of the two polynomials

$$P(z) = (4375z^2 + 1550z - 89)(175z^2 + 106z + 7)(25z^2 + 12z + 1)^2(25z + 6)^3 , \tag{23}$$

$$Q(y) = P(-y) . \tag{24}$$

The five pairings of the seven roots of Eqs. (23) and (24), giving the five elliptic points, are $(0.530283, -0.107335)$, $(-0.050283, -0.24)$, $(0.372665, -0.372665)$, $(0.107335, -0.530283)$, $(0.24, 0.050283)$ and the five pairings giving the five hyperbolic-saddle points are $(0.372665, -0.075431)$, $(0.107335, -0.107335)$, $(0.404568, -0.24)$, $(0.075431, -0.372665)$, $(0.24, -0.404568)$. This is clearly seen on Fig. 3 where the occurrence of five " petals" corresponding to five elliptic points are obvious, the five hyperbolic points being located between the petals.

For transformation k_ε^6, beyond the fixed points of k and k^3, one gets two complex saddle fixed points, i.e. transformation k has two 6-cycles. For transformation k_ε^7, one obtains one elliptic real fixed point, one saddle real fixed point and two complex saddle fixed points. For transformation k_ε^8, one obtains one saddle real fixed point and four complex saddle fixed points. For transformation k_ε^9, one obtains one elliptic real fixed point, three saddle real fixed points and four complex saddle fixed points. For transformation k_ε^{10}, one obtains one elliptic real fixed point, one saddle real fixed point and three complex elliptic fixed points and six saddle complex fixed points. The two elliptic fixed points of k_ε^{10} $(0.24, -0.874)$ and $(0.874, -0.24)$ are seen as " ellipse" in Fig. 3. For transformation k_ε^{11}, one obtains one elliptic real fixed point, five saddle real fixed points and 12 complex saddle fixed points. In Fig. 3 a fixed point of k_ε^{12} lying on $y+z=0$ is seen near $y=-13/25$. The polynomials, similar to Eqs. (23) and (24) (or to Eq. (D.1) given in Appendix D), as well as the specific pairing of roots, for the successive iterates k^n, are available through e-mail. [12]

It is worth noticing, that among the 53 cycles of k_ε of length smaller, or equal, to 11, as much as 44 have a representent on the line $y+z=0$, six have one on the line $y+\bar{z}=0$. Two of the three remaining cycles are of length 11, while the last is of length eight. The particular role played by the $y+z=0$ line can be simply understood. Let us calculate the inverse of the birational transformation (3). It has a very simple form

$$z_{n+1} = y_n - (1-\varepsilon), \qquad y_{n+1} = z_n \cdot \frac{y_n+\varepsilon}{y_n - 1} \tag{25}$$

which is nothing but transformation (3) where y_n and $-z_n$ have been permuted. The $y_n \leftrightarrow -z_n$ symmetry just corresponds to the *time-reversal symmetry* $k_\varepsilon \leftrightarrow k_\varepsilon^{-1}$ transformation. The $y+z=0$ line is the *time-reversal invariant line.* Also note that only one of the 31 complex cycles is of the form $Z_0, Z_1, \ldots, Z_p, \bar{Z}_0, \bar{Z}_1, \ldots, \bar{Z}_p$ where $Z_i=(y_i, z_i)$ and $\bar{Z}_i$ is the complex conjugate. The 30 remaining complex cycles are actually 15 cycles and their complex conjugates.

Eventually, we observe an *area preserving* [36] property *in the neighborhood of all the fixed points of* k_ε^n: the product of the modulus of the two eigenvalues of the Jacobian (i.e. the determinant) of k_ε^n, at all fixed points for $n \leqslant 11$, is equal to 1. This *local* property is rather non-trivial: the determinant of the product of the Jacobian over an *incomplete* cycle is very complicated and only when one multiplies by the last Jacobian does the product of the determinants shrink to 1.

[12] anonymous@crtbt.polycnrs-gre.fr.

The total number of fixed points of k_ε^N for N running from 1 to 11, yields the following expansion, up to order eleven, for the generating function $H(t)$ of the number of fixed points:

$$H_\varepsilon(t) = t + t^2 + 4t^3 + 5t^4 + 11t^5 + 16t^6 + 29t^7 \\ +45t^8 + 76t^9 + 121t^{10} + 199t^{11} + \cdots \, . \tag{26}$$

This expansion coincides with one of the *rational* function

$$H_\varepsilon(t) = \frac{t \cdot (1 + t^2)}{(1 - t^2)(1 - t - t^2)} \tag{27}$$

which corresponds to a very *simple rational* expression for the dynamical zeta function

$$\zeta_\varepsilon(t) = \frac{1 - t^2}{1 - t - t^2} \, . \tag{28}$$

Expansion (26) remains unchanged for all the other generic values of ε we have also studied.

We conjecture that: *The simple rational expression* (28) *is the actual expression of the dynamical zeta function for any generic value of* ε.

The simplicity of the rational expression (28) suggests " a diffeomorphism of the torus" interpretation which seems to indicate that there should exist a topological conjugacy enabling to establish this conjecture. This will be discussed in Section 4.

Comparing the expression (8) with Eq. (28), one sees that the singularities of the dynamical zeta function happen to coincide with the singularities of the generating functions of the Arnold complexity. In particular, the complexity growth λ and the exponential of the topological entropy h are *equal.*

When mentioning zeta functions, it is tempting to seek for simple *functional relations* relating $\zeta(t)$ and $\zeta(1/t)$. Let us introduce the following " avatar" of the dynamical zeta function:

$$\hat{\zeta}(t) = \frac{\zeta(t)}{\zeta(t) - 1} \, . \tag{29}$$

The transformation $z \to z/(z-1)$ is an involution. One immediately verifies that $\hat{\zeta}_\varepsilon(t)$ corresponding to (28) verifies two extremely simple and remarkable functional relations

$$\hat{\zeta}_\varepsilon(t) = -\hat{\zeta}_\varepsilon(1/t) \quad \text{and} \quad \hat{\zeta}_\varepsilon(t) = \hat{\zeta}_\varepsilon(-1/t) \, , \tag{30}$$

or on the zeta function $\zeta(t)$

$$\zeta_\varepsilon(1/t) = \frac{\zeta_\varepsilon(t)}{2 \cdot \zeta_\varepsilon(t) - 1} \quad \text{and} \quad \zeta_\varepsilon(-1/t) = \zeta_\varepsilon(t) \, . \tag{31}$$

The generating function (27) verifies

$$H_\varepsilon(-1/t) = -H_\varepsilon(t) \, . \tag{32}$$

An alternative way of writing the dynamical zeta functions relies on the decomposition of the fixed points into *cycles* which corresponds to the Weil conjectures [37]. Let us introduce N_r the number of irreducible cycles of k_ε^r: for instance for N_{12} we count the number of fixed points of k_ε^{12}, that are not fixed points of $k_\varepsilon, k_\varepsilon^3, k_\varepsilon^4$ or k_ε^6, and divide by 12. One can write the dynamical zeta function as

$$\zeta_\varepsilon(t) = \frac{1}{(1-t)^{N_1}} \cdot \frac{1}{(1-t^2)^{N_2}} \cdot \frac{1}{(1-t^3)^{N_3}} \cdots \frac{1}{(1-t^r)^{N_r}} \cdots . \tag{33}$$

The combination of the N_r's, inherited from the product (33), automatically takes into account the fact that the total number of fixed points of k_ε^r can be obtained from fixed points of k_ε^p, where p divides r, and from irreducible fixed points of k_ε^r itself (see [37] for more details). A detailed analysis of this cycle decomposition (33) for generic values of ε will be detailed elsewhere [38]. The previous exhaustive list of fixed points (up to order 12) can be revisited in this irreducible cycle decomposition point of view. The results of footnote 12 yield: $N_1=1$, $N_2=0$, $N_3=1$, $N_4=1$, $N_5=2$, $N_6=2$, $N_7=4$, $N_8=5$, $N_9=8$, $N_{10}=11$, $N_{11}=18$. One actually verifies easily that (28) and (33) have the same expansion up to order 12 with these values of the N_r's. The next N_r's should be $N_{12}=25$, $N_{13}=40$, $N_{14}=58$, $N_{15}=90, \ldots$.

It should be noticed that if one introduces some generating function of the *real* fixed points of k^N, this generating function has the following expansion, up to order 11, for $\varepsilon = 0.52$:

$$H_\varepsilon^{\mathrm{real}} = t + t^2 + 4t^3 + 5t^4 + 11t^5 + 4t^6 + 15t^7 + 13t^8 + 40t^9 + 31t^{10} + 67t^{11} + \cdots . \tag{34}$$

This series is irregular, furthermore its coefficients depend very much on parameter ε. In contrast to generating function (20), the generating function $H_\varepsilon^{\mathrm{real}}$ has *no unverisality property* in ε. This series does not take into account the topological invariance in complex projective space: it just tries to describe the dynamical system in the real space. This series $H_\varepsilon^{\mathrm{real}}$ corresponds to the " complexity" as seen on the phase portrait of Fig. 3. One sees here the quite drastic opposition between the notions well suited to describe transformations in complex projective spaces and the ones aiming at describing transformations in real variables.

3.2. Dynamical zeta functions for $\alpha = 0, \varepsilon$ non-generic

To further investigate the identification of these two notions (Arnold complexity-topological entropy), we now perform similar calculations (of fixed points and associated zeta dynamical functions) for $\varepsilon = 1/m$ with $m \geqslant 4$ and $\varepsilon = (m-1)/(m+3)$ with $m \geqslant 7$ odd.

The calculations have been performed for $\varepsilon = 1/m$ for $m = 4, 5, 7$ and 9, giving the expansion of $H_\varepsilon(t)$ up to order 11. For $m = 4$ this gives

$$H_{1/4}(t) = t + t^2 + 4t^3 + 5t^4 + 11t^5 + 10t^6 + 22t^7 + 29t^8 + 49t^9 + 71t^{10} + 111t^{11} + \cdots , \tag{35}$$

for $m=5$:

$$H_{1/5}(t)=t+t^2+4t^3+5t^4+11t^5+16t^6+22t^7 +37t^8+58t^9+91t^{10}+144t^{11}+\cdots\,, \tag{36}$$

for $m=7$:

$$H_{1/7}(t)=t+t^2+4t^3+5t^4+11t^5+16t^6+29t^7 +45t^8+67t^9+111t^{10}+177t^{11}+\cdots\,, \tag{37}$$

and for $m=9$:

$$H_{1/9}(t)=t+t^2+4t^3+5t^4+11t^5+16t^6+29t^7 +45t^8+76t^9+121t^{10}+188t^{11}+\cdots\,, \tag{38}$$

All these expressions are compatible with this single expression of the ζ function

$$\zeta_{1/m}(t)=\frac{1-t^2}{1-t-t^2+t^{m+2}}\,. \tag{39}$$

We conjecture that this expression is exact at every order and for every value of $m\geqslant 4$. Again all the singularities of this expression coincide with those of generating function corresponding to the Arnold complexity (see Eq. (14)).

As far as functional relations relating $\zeta(t)$ and $\zeta(\pm 1/t)$ are concerned, recalling (29), one immediately verifies that $\hat{\zeta}(t)$ corresponding to (39) verifies the simple functional relation

$$t^{m+1}\cdot\hat{\zeta}_{1/m}(t)=\hat{\zeta}_{1/m}(1/t)\quad\text{or}\quad \zeta_{1/m}(1/t)=\frac{t^{m+1}\cdot\zeta_{1/m}(t)}{t^{m+1}\cdot\zeta_{1/m}(t)-\zeta_{1/m}(t)+1}\,. \tag{40}$$

Actually $\hat{\zeta}_{1/m}(t)$ has a very simple nth root of unity form:

$$\hat{\zeta}_{1/m}(t)=\frac{1-t^2}{t\cdot(1-t^{m+1})}\,. \tag{41}$$

Also note that when m is odd, and only in that case, $\hat{\zeta}_{1/m}(t)$ *also satisfies* the functional relation

$$t^{m+1}\cdot\hat{\zeta}_{1/m}(t)=-\hat{\zeta}_{1/m}(-1/t)\,. \tag{42}$$

No simple functional relation, similar to (32), can be deduced on $H(t)$.

Similar calculations can also be performed for the second set of non-generic values of ε, namely $\varepsilon=(m-1)/(m+3)$ with $m\geqslant 7$, m odd. For $m=7$, that is $\varepsilon=3/5$, one gets, up to order 11, the *same expansion* as Eq. (37):

$$H_{3/5}(t)=t+t^2+4t^3+5t^4+11t^5+16t^6+29t^7 +45t^8+67t^9+111t^{10}+177t^{11}+\cdots \tag{43}$$

suggesting, again, the dynamical zeta function

$$\zeta_{3/5}(t)=\frac{1-t^2}{1-t-t^2+t^9}\,. \tag{44}$$

For $m=9$, that is $\varepsilon=2/3$, one gets

$$H_{2/3}(t)=t+t^2+4t^3+5t^4+11t^5+16t^6+29t^7+45t^8 +76t^9+121t^{10}+177t^{11}+\cdots$$

A compatible zeta function could be [13]

$$\zeta_{2/3}(t)=\frac{1-t^2-t^{11}-t^{12}-t^{13}}{1-t-t^2+t^{11}}\ . \qquad (45)$$

This form is not the same as Eq. (39), however it has the same poles.

Comparing these rational expressions for the dynamical zeta function ((28), (39),...), and the rational expressions for the generating functions of the Arnold complexity ((11), (12), (13),...) for the generic, and non-generic, values of ε, one sees that one actually has the same singularities in these two sets of generating functions. [14] The identification between the growth complexity λ and the (exponential of the) topological entropy is thus valid *for generic values of* ε, and *even for non-generic ones.*

It is worth noticing that, due to the topological character of the dynamical zeta function, these results are of course not specific of the y and z representation of the mapping (3) but are also valid for the (u,v) representation (1): in particular the exact expressions of the dynamical zeta functions (namely (28), (39)), remain unchanged and, of course, the denominators of the complexity generating functions are also the same for generic, or non-generic, values of ε.

The *local area preserving* property *in the neighborhood of all the fixed point of* k_ε^n previously noticed for $\alpha=0$, ε generic, is also verified for these non-generic values of ε.

3.3. Dynamical zeta functions for $\alpha \neq 0$

This (generic) identification is not restricted to $\alpha=0$. One can also consider mapping (1) for arbitrary values of α and ε and calculate the successive fixed points. Of course, as a consequence of the higher complexity of the $\alpha \neq 0$ situation (as shown in section IIB, the complexity immediately jumps from $1.61803\cdots$ to $2.14789\cdots$) the number of successive fixed points is drastically increased and the calculations cannot be performed up to order 11 anymore. In the generic case, the expansion of the generating function $H(t)$ of the number of fixed points can be obtained up to order 7:

$$H_\varepsilon^\alpha=2t+2t^2+11t^3+18t^4+47t^5+95t^6+212t^7+\cdots\ . \qquad (46)$$

One has two fixed points for k, no new fixed points for k^2, three sets of three new fixed points for k^3 (giving $3\times 3+2=11$ fixed points), four sets of four new fixed points for k^4 (giving $4\times 4+2=18$ fixed points), nine sets of five new fixed points for k^5 (giving $9\times 5+2=47$ fixed points), 14 sets of six new fixed points for k^6 (giving

[13] The series is not large enough to confirm this form. A first simple analysis seems to show that the next terms are $\cdots+296t^{12}+469t^{13}+785t^{14}+\cdots$.

[14] Note that t has to be replaced by x^2 since k_ε is associated to transformation K^2 and not to K.

$14 \times 5 + 3 \times 3 + 2 = 95$ fixed points). This expansion corresponds to the following order 7 expansion for the dynamical function:

$$\zeta_\varepsilon^\alpha(t) = 1 + 2t + 3t^2 + 7t^3 + 15t^4 + 32t^5 + 69t^6 + 148t^7 + \cdots \tag{47}$$

thus yielding to the following rational expression for the dynamical zeta function:

$$\zeta_\varepsilon^\alpha(t) = \frac{(1-t^2)\cdot(1+t)}{1 - t\cdot(1+t)^2} = \frac{(1-x^2)\cdot(1+x^2)^2}{(1-x-x^3)\cdot(1+x+x^3)} \quad \text{with } t = x^2 \ . \tag{48}$$

Let us recall the " alternative" zeta function (29), but for $\zeta = \zeta_\varepsilon^\alpha(t)/(1+t)$. It verifies the simple functional relation

$$t^2 \cdot \hat{\zeta}_\varepsilon^\alpha(t) = -\hat{\zeta}_\varepsilon^\alpha(1/t) \ . \tag{49}$$

This new rational conjecture (48) corresponds to the following expression for $H(t)$:

$$H_\varepsilon^\alpha(t) = \frac{t\cdot(2+3t^2+t^3)}{(1-t^2)\cdot(1-t-2t^2-t^3)} \ . \tag{50}$$

Comparing the denominators of Eqs. (48) and (15), one sees that, like for the case $\alpha = 0$, there is an identification between the growth complexity and the (exponential of the) topological entropy

$$\lambda = h \ . \tag{51}$$

Heuristically, this identification can be understood as follows. The components of k^N, namely y_N and z_N, are of the form $P_N(y,z)/Q_N(y,z)$ and $R_N(y,z)/S_N(y,z)$, where $P_N(y,z)$, $Q_N(y,z)$, $R_N(y,z)$ and $S_N(y,z)$ are polynomials of degree asymptotically growing like λ^N. The Arnold complexity amounts to taking the intersection of the Nth iterate of a line (for instance a simple line like $y = y_0$ where y_0 is a constant) with another simple (fixed) line (for instance $y = y_0$ itself or any other simple line or any *fixed* algebraic curve). For instance, let us consider the Nth iterate of the $y = y_0$ line, which can be parameterized as

$$y_N = \frac{P_N(y_0,z)}{Q_N(y_0,z)}, \qquad z_N = \frac{S_N(y_0,z)}{T_N(y_0,z)}, \tag{52}$$

with line $y = y_0$ itself. The number of intersections, which are the solutions of $P_N(y_0,z)/Q_N(y_0,z) = y_0$, grows like the degree of $P_N(y_0,z) - Q_N(y_0,z)\cdot y_0$: asymptotically it grows like $\simeq \lambda^N$. On the other hand, the calculation of the topological entropy corresponds to the number of fixed points of k^N, that is to the number of intersection of the two curves

$$P_N(y,z) - Q_N(y,z)\cdot y = 0, \qquad R_N(y,z) - S_N(y,z)\cdot z = 0 \tag{53}$$

which are two curves of degree growing asymptotically like $\simeq \lambda^N$. The number of fixed points is obviously bounded by $\simeq \lambda^{2N}$ but one can figure out that it should (generically) grow like $\simeq \lambda^N$. This is fully confirmed by our exact calculations.

The Eulerian product Weil-decomposition (33) of the dynamical zeta function (48) corresponds to the following numbers of r-cycles: $N_1 = 2$, $N_2 = 0$, $N_3 = 3$, $N_4 = 4$, $N_5 = 9$, $N_6 = 14$, $N_7 = 30$, $N_8 = 54$, $N_9 = 107$, $N_{10} = 204$, $N_{11} = 408$, $N_{12} = 25$, $N_{13} = 1593$, $N_{14} = 3162$.

3.4. Dynamical zeta functions for $\alpha \neq 0$ with ε non-generic

For a non-generic value of ε when $\alpha \neq 0$, namely $\varepsilon = 1/2$, the expansion of the generating function $H(t)$ and of the dynamical zeta function read, respectively,

$$H^{\alpha}_{1/2}(t) = 2t + 2t^2 + 11t^3 + 18t^4 + 47t^5 + 95t^6 + 198t^7 + \cdots ,$$

$$\zeta^{\alpha}_{1/2}(t) = 1 + 2t + 3t^2 + 7t^3 + 15t^4 + 32t^5 + 69t^6 + 146t^7 + \cdots .$$

A possible rational expression for the dynamical zeta function is for instance

$$\zeta^{\alpha}_{1/2}(t) = \frac{1 + t - t^7}{1 - t - t^2 - 2t^3 - t^4 - 2t^5 - t^6 - t^7} = \frac{1 + t \cdot (1 - t^6)}{1 - t \cdot (1 - t + t^2) \cdot (1 + t + t^2)^2} . \quad (54)$$

This last result has to be compared with (16).

For another non-generic value of ε when $\alpha \neq 0$, namely $\varepsilon = 1/3$, the expansion of the generating function $H(t)$ and of the dynamical zeta function read, respectively,

$$H^{\alpha}_{1/3}(t) = 2t + 2t^2 + 11t^3 + 18t^4 + 42t^5 + 83t^6 + 177t^7 + \cdots ,$$

$$\zeta^{\alpha}_{1/3}(t) = 1 + 2t + 3t^2 + 7t^3 + 15t^4 + 31t^5 + 65t^6 + 136t^7 + \cdots .$$

A possible rational expression for the dynamical zeta function is, for instance,

$$\zeta^{\alpha}_{1/3}(t) = \frac{1 + t}{1 - t - t^2 - 2t^3 - t^4 - t^5} = \frac{1 + t}{1 - t \cdot (1 + t^2) \cdot (1 + t + t^2)} . \quad (55)$$

This last result has to be compared with (17). These results [15] are again in agreement with an Arnold-complexity-topological-entropy identification.

The *local area preserving* property in the neighborhood of all the fixed points of $k^n_{\alpha,\varepsilon}$ previously noticed for $\alpha = 0$, is also verified for $\alpha \neq 0$ for (1) *for generic values of* ε *generic as well as these non-generic values of* ε.

4. Comments and speculations

Based on analytical and semi-numerical calculations we have conjectured rational expressions with integer coefficients for the generating functions of the complexity and for the dynamical zeta functions for various values of the parameters of a family of birational transformations. According to these conjectures, the growth complexity and the exponential of the topological entropy are algebraic numbers. Moreover, these two numbers are *equal* for all the values of the parameters.

From a general point of view, rational dynamical zeta functions (see for instance [34,39,40]) are known in the literature through theorems where the dynamical systems are asked to be *hyperbolic*, or through combinatorial proofs using symbolic dynamics

[15] However for the non-generic value of ε, $\varepsilon = 3/5$, we do not have enough coefficients in the expansion of the dynamical zeta function to actually compare it with (18).

arising from Markov partition [41] and even, far beyond these frameworks [42], for the so-called " isolated expansive sets" (see [42,43] for a definition of the isolated expansive sets). There also exists an explicit example of a rational zeta dynamical function but only in the case of an *explicit linear* dynamics on the torus R^2/Z^2, deduced from an $SL(2,Z)$ matrix, namely the cat map [2,45] (diffeomorphisms of the torus see for instance [29]):

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \zeta(z) = \frac{det(1 - z \cdot B)}{det(1 - z \cdot A)} = \frac{(1-z)^2}{1 - 3 \cdot z + z^2} \ . \tag{56}$$

Note that golden number singularities for complexity growth generating functions have already been encountered (see Eq. (7.28) in [44] or Eq. (5) in [46]). In our examples, we are not in the context where the known general theorems can apply straightforwardly. The question of the demonstration of the rationality of zeta functions we have conjectured remains open.

In the framework of a " diffeomorphisms of the torus" interpretation, the degree of the denominator of a rational dynamical zeta function gives a lower bound of the dimension g of this " hidden" torus C^g/Z^g. On expression (39) valid for $\alpha = 0$ and $\varepsilon = 1/m$, one notes that dimension g grows linearly with m. For these values of ε one would like to barter the two natural variables y and z for g variables. Such a uniformization is however known to be extremely difficult, even in integrable cases. The iteration of some birational transformations which densify Abelian surfaces (resp. varieties) has been seen to correspond to polynomial growth of the calculations [19]. Introducing well-suited variables θ_i $(i=1,\ldots,g)$ to uniformize the Abelian varieties the iteration of these birational transformations just corresponds to a shift [16] $\theta_i \to \theta_i + n \cdot \eta_i$. For such polynomial growth situations, matrix A can be thought as the Jordan matrix associated with this translation, its characteristic polynomial yielding eigenvalues equal to 1.

To sum up, may only seek for (a certainly involved) topological conjugacy between a two-dimensional torus and the (y,z) plane only for the *generic* values of ε, since they do not exclude, at first sight, two-dimensional torus.

Many denominators of rational zeta functions encountered here are of the form $1 - t \cdot Y(t)$ where $Y(t)$ is product of cyclotomic polynomials [50,51]. We have encountered

$$Y(t) \; = \; (1+t) \text{ (resp. } (1+t)^2) \quad \text{for } \alpha = 0 \text{ (resp. } \alpha \neq 0) \text{ and } \varepsilon \text{ generic },$$

$$Y(t) \; = \; \frac{(1+t^3)}{1+t} \cdot \frac{(1-t^3)^2}{(1-t)^2} \text{ (resp. } (1+t^2) \cdot \frac{(1-t^3)}{1-t}) \quad \text{for } \alpha \neq 0 \text{ and } \varepsilon = \tfrac{1}{2} \text{ (resp. } \varepsilon = \tfrac{1}{3})$$

More generally, the rational dynamical zeta functions, or the rational functions $G(q,x)$ encountered here, are of the form: $(1 + X(z))/(1 - Y(z))$ (for G) or $(1 - X(z))/(1 - Y(z))$ (for ζ) where $X(z)$ and $Y(z)$ have some kind of decomposition on *cyclotomic*

[16] This " diffeomorphisms of the torus" interpretation is quite obvious in Fig. 3 of [44].

polynomials

$$X(z) = \sum_r z^r \cdot \Pi_m P_m^{(r)}(z) \quad \text{with } P_m^{(r)}(z) \text{ cyclotomic polynomials} . \qquad (57)$$

This is particularly obvious on expressions (13) but also on expressions (12), or (54), or even (39). We do not have yet any l-adic cohomology interpretation (see for instance p. 453 of [37]) of this cyclotomic polynomials " encoding" of the zeta functions or the complexity functions $G(q,x)$. Most of these rational expressions for zeta functions satisfy very simple functional relations but one also expects, for (54) or (55) for instance, more involved but, still simple, functional relations similar to the ones obtained by Voros in [52]. Many of the generating functions $G(q,x)$ can also be seen to satisfy simple functional relations relating $G(q,x)$ and $G(q,1/x)$. This will be detailed elsewhere. [17]

In practice, it is numerically easier to get the generating functions of Arnold complexity than getting the dynamical zeta functions. If one assumes the rationality of the dynamical zeta function and the identification between growth complexity and (exponential of the) topological entropy, getting the generating functions of Arnold complexity is a simpler way to " guess" the denominator of the dynamical zeta functions.

The analysis developed here can be applied to a very large set of birational transformations of an *arbitrary* number of variables, always leading *rational* generating functions [44,49]. Moreover, these generating functions are always simple rational expressions with integer coefficients (thus yielding *algebraic numbers* for the growth complexity λ). They even have the previously mentioned " cyclotomic encoding". At this point, the question can be raised [18] to see if the iteration of *any* birational transformation of an arbitrary number of variables always yields rational generating functions for the growth complexity. We have even found rational generating functions of Arnold complexity for rational transformations *which are not birational* (see (7.7) and (7.28) in [44]): any proof of these rationalities should not depend too heavily on a naive *reversibility* of the mapping [53].

We have also calculated Lyapunov exponents [54] in order to study the *metric entropy*. These numerical calculations will be detailed elsewhere [54] for transformation (3) for $\varepsilon = 0.52$. These results give quite small values of the Lyapunov exponents, the largest of which being much smaller than the topological entropy. We thus infer that, in this very example, the metric entropy is much smaller than the topological entropy. We have here an opposition between topological concepts originating from complex protective spaces and the metric concepts of real analysis. The " non-topological" complexity measures do not seem to be able to identify with the previous topological and algebraic quantities. On the birational examples studied here, the metric entropy does not seem to share the same algebraic values as the topological complexity measures. One could also calculate (the expansion of) various weighted dynamical zeta functions to see if these expansions are again compatible with rational expressions. If so, one

[17] For instance the generating function of the degrees $g(x)$ given by Eq. (5) in [46] verifies $g(x)+g(1/x)=1$.
[18] After [44].

could see if their poles can be linked with various entropy concepts (order q entropies [55], ...) and in particular the metric entropy.

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Appendix A. Correspondence between transformation K acting on $Q \times Q$ matrices and the (U, V) transformation $K_{\alpha,\varepsilon}$

In previous papers [20,21], it has been noticed that the successive *even* iterates of K, acting on an initial matrix M, actually belong to a plane which contains matrices M, $K^2(M)$ and $K^4(M)$, or equivalently a fixed matrix P (see [20]):

$$K^{2n}(M) = c_0 \cdot M + c_1 \cdot K^2(M) + c_2 \cdot P$$

for any integer n (even relative integer). In fact one even has the following property: any point of the plane containing M, $K^2(M)$ and P is transformed, by the even iterates K^{2n}, into another point of this plane. This can be used to define the two-dimensional mapping $k_{\varepsilon,\alpha}$ compatible with mapping K. Let M be an arbitrary $q \times q$ matrix, and let us define [19]

$$u_n(M) = x_2 \cdot x_4 \cdot x_6 \cdots x_{2n-2} \cdot u_1(M),$$
$$v_n(M) = x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6 \cdots x_{2n-1} \cdot v_1(M)\,, \qquad \text{(A.1)}$$

where

$$u_1(M) = \frac{x_0}{\rho_2}, \qquad v_1(M) = -\frac{x_0 x_1}{\lambda_2} \quad \text{and} \quad \alpha(M) = -\frac{\rho_1 \rho_2}{\lambda_2}, \qquad \varepsilon(M) = \frac{\lambda_1 - \lambda_2}{\lambda_2}\,, \qquad \text{(A.2)}$$

where $x_n = \det(\hat{K}^m(M)) \cdot \det(\hat{K}^{n+1}(M))$ with $\hat{K} = t \cdot \hat{I}$, and $\hat{I}(M) = M^{-1}$ and: [20]

$$\lambda_1 = \frac{x_0 x_2 \cdot (x_1 x_3 - 1)}{x_2 - 1}, \qquad \lambda_2 = \frac{x_1 \cdot (x_0 x_2 - 1)}{x_1 - 1}\,,$$

$$\rho_1 = \frac{x_1 x_2 x_3 + x_1 x_2 - x_1 - 1}{x_2 - 1}, \qquad \rho_2 = \frac{x_0 x_1 x_2 + x_0 x_1 - x_0 - 1}{x_1 - 1}\,.$$

[19] Note a miss-print in [20]: in Eq. (6.35) u_n/v_n should be replaced by v_n/u_n yielding (see (A.1) and (A.2)): $v_n/u_n = x_3 \cdots x_5 \cdot x_7 \cdots x_{2n-1} \cdot v_1/u_1$.

[20] Note that the λ_i's and ρ_i's are not exactly the same as the ones given in [20], in Eqs. (6.13) and (6.14): the λ_i's and ρ_i's in [20] are homogeneous expressions and the λ_i's and ρ_i's we introduce here are inhomogeneous true invariants which can be deduced from the ones in [20] dividing them be q_0 or q_1.

Then one has the K^2 invariance of α and ε:

$$\alpha(M) = \alpha(K^2(M)), \qquad \varepsilon(M) = \varepsilon(K^2(M)) \tag{A.3}$$

and $k_{\alpha,\varepsilon}$ can be seen as a representation of K^2:

$$(u(M), v(M)) = k_{\alpha,\varepsilon}(u(K^2(M)), v(K^2(M))\ , \tag{A.4}$$

where α and ε are precisely the values given by Eq. (A.2). Transformation $k_{\alpha,\varepsilon}$ reads

$$k_{\alpha,\varepsilon} : (u, v) \to (U, V) = \left(\frac{v + u - u \cdot v}{v}, U \cdot \alpha + 1 + \varepsilon - \frac{u + v - u \cdot v}{u} \right)\ . \tag{A.5}$$

In the $\alpha = 0$ case, this transformation simplifies and one can introduce new variables y and z given by

$$y = v - 1, \qquad z = \frac{(1-u)\cdot(1-v)\cdot v}{u\cdot(v-1)}\ . \tag{A.6}$$

With these new variables, $k_{\alpha,\varepsilon}$ reads

$$k_\varepsilon : (y, z) \to \left(z + 1 - \varepsilon, y \cdot \frac{z - \varepsilon}{z + 1} \right)\ . \tag{A.7}$$

For $\alpha = \varepsilon = 0$, transformation $k_{\alpha,\varepsilon}$ is integrable [20] the invariant being (see (6.38) in [20])

$$\mathcal{I} = \frac{(1-u)\cdot(1-v)\cdot v}{u}\ . \tag{A.8}$$

This algebraic expression is of course only well-suited for ε=0. The variable z amounts to considering $\mathcal{I}/(v-1)$ for arbitrary ε's.

Appendix B. Factorization scheme for (5)

For $q \times q$ matrices ($q \geqslant 3$) the factorizations corresponding to the iterations of K read

$$f_1 = det(M_0)\,, \qquad M_1 = K(M_0)\,, \qquad f_2 = \frac{det(M_1)}{f_1^{q-2}}, \qquad M_2 = \frac{K(M_1)}{f_1^{q-3}}$$

$$f_3 = \frac{det(M_2)}{f_1 \cdot f_2^{q-3}}, \qquad M_3 = \frac{K(M_2)}{f_2^{q-3}}\ ,$$

$$f_4 = \frac{det(M_3)}{f_1^{q-1} \cdot f_2 \cdot f_3^{q-2}}\ , \qquad M_4 = \frac{K(M_3)}{f_1^{q-2} \cdot f_3^{q-3}}\ , \qquad f_5 = \frac{det(M_4)}{f_1^2 \cdot f_2^{q-1} \cdot f_3 \cdot f_4^{q-2}}\ ,$$

$$M_5 = \frac{K(M_4)}{f_1 \cdot f_2^{q-2} \cdot f_4^{q-3}}\ ,$$

$$f_6 = \frac{det(M_5)}{f_1^{q-2} \cdot f_2^2 \cdot f_3^{q-1} \cdot f_4 \cdot f_5^{q-2}}\ , \qquad M_6 = \frac{K(M_5)}{f_1^{q-3} \cdot f_2 \cdot f_3^{q-2} \cdot f_5^{q-3}}\ ,$$

$$f_7 = \frac{det(M_6)}{f_1 \cdot f_2^{q-2} \cdot f_3^2 \cdot f_4^{q-1} \cdot f_5 \cdot f_6^{q-2}}\ ,$$

$$M_7 = \frac{K(M_6)}{f_2^{q-3} \cdot f_3 \cdot f_4^{q-2} \cdot f_6^{q-3}} \cdots \tag{B.1}$$

and for arbitrary n

$$det(M_n) = f_{n+1} \cdot (f_n^{q-2} \cdot f_{n-1} \cdot f_{n-2}^{q-1} \cdot f_{n-3}^2) \times (f_{n-4}^{q-2} \cdot f_{n-5} \cdot f_{n-6}^{q-1} \cdot f_{n-7}^2) \cdots f_1^{\delta_n} \,, \tag{B.2}$$

$$K(M_n) = M_{n+1} \cdot (f_n^{q-3} \cdot f_{n-2}^{q-2} \cdot f_{n-3}) \times (f_{n-4}^{q-3} \cdot f_{n-6}^{q-2} \cdot f_{n-7}) \cdots f_1^{\mu_n} \,, \tag{B.3}$$

where $\mu_n = q-3$ for $n = 1 \pmod 4$, $\mu_n = 0$ for $n = 2 \pmod 4$, $\mu_n = q-2$ for $n = 3 \pmod 4$ and $\mu_n = 1$ for $n = 0 \pmod 4$ and δ_n also depends on the truncation. Factorization relations *independent of* q, *occur*:

$$\frac{K(M_n)}{det(M_n)} = \frac{M_{n+1}}{f_{n+1} \cdot f_n \cdot f_{n-1} \cdot f_{n-2} \cdot f_{n-3} \cdot f_{n-4} \cdots} \,. \tag{B.4}$$

Let us introduce [20,21] the generating functions $\alpha(x)$ and $\beta(x)$ of the degree of the $det(M_n)$'s and f_n's. Their exact expressions read

$$\alpha(x) = \frac{q}{1+x} + \frac{q^2 \cdot x \cdot (1+x^2)}{(1-x)(1+x)(1-x-x^3)}, \qquad \beta(x) = \frac{q \cdot x \cdot (1+x^2)}{1-x-x^3} \,. \tag{B.5}$$

It is clear that one has an exponential growth of exponents α_n's, β_n's, μ_n's and v_n's: these coefficients grow like λ^n where $\lambda \sim 1.465\cdots$.

This displays the generic factorization scheme. However, on various subvarieties like codimension one subvariety $\alpha = 0$, the factorization scheme can be modified as a consequence of additional factorizations occurring at each iteration step, thus yielding a smaller value for the complexity λ.

B.1. Factorization scheme for $\alpha = 0$, ε generic

For $\alpha = 0$ the previous factorization scheme becomes for 3×3 matrices[21]

$$f_1 = det(M_0) \,, \qquad M_1 = K(M_0) \,, \qquad f_2 = \frac{det(M_1)}{f_1} \,, \qquad M_2 = K(M_1) \,,$$

$$f_3 = \frac{det(M_2)}{f_1^2 \cdot f_2} \,, \qquad M_3 = \frac{K(M_2)}{f_1} \,,$$

$$f_4 = \frac{det(M_3)}{f_1 \cdot f_2 \cdot f_3}, \qquad M_4 = K(M_3), \qquad f_5 = \frac{det(M_4)}{f_1^2 \cdot f_2^2 \cdot f_3^2 \cdot f_4},$$

$$M_5 = \frac{K(M_4)}{f_1 \cdot f_2 \cdot f_3} \,,$$

[21] These results can straightforwardly be generalized to $q \times q$ matrices, they are just a bit more involved.

$$f_6 = \frac{det(M_5)}{f_1 \cdot f_2^2 \cdot f_3 \cdot f_4 \cdot f_5}, \qquad M_6 = \frac{K(M_5)}{f_2}, \qquad f_7 = \frac{det(M_6)}{f_1^2 \cdot f_2 \cdot f_3^2 \cdot f_4^2 \cdot f_5^2 \cdot f_6}$$

$$M_7 = \frac{K(M_6)}{f_1 \cdot f_3 \cdot f_4 \cdot f_5} ,$$

$$f_8 = \frac{det(M_7)}{f_1 \cdot f_2 \cdot f_3 \cdot f_4^2 \cdot f_5 \cdot f_6 \cdot f_7}, \qquad M_8 = \frac{K(M_7)}{f_4},$$

$$f_9 = \frac{det(M_8)}{f_1^2 \cdot f_2^2 \cdot f_3^2 \cdot f_4 \cdot f_5^2 \cdot f_6^2 \cdot f_7^2 \cdot f_8},$$

$$M_9 = \frac{K(M_8)}{f_1 \cdot f_2 \cdot f_3 \cdot f_5 \cdot f_6 \cdot f_7}, \qquad f_{10} = \frac{det(M_9)}{f_1 \cdot f_2^2 \cdot f_3 \cdot f_4 \cdot f_5 \cdot f_6^2 \cdot f_7 \cdot f_8 \cdot f_9} , \cdots \tag{B.6}$$

and for arbitrary n

$$det(M_n) = f_{n+1} \cdot (f_n \cdot f_{n-1}^2 \cdot f_{n-2}^2 \cdot f_{n-3}^2) \cdot (f_{n-4} \cdot f_{n-5}^2 \cdot f_{n-6}^2 \cdot f_{n-7}^2) \cdots , \tag{B.7}$$

$$K(M_n) = M_{n+1} \cdot (f_{n-1} \cdot f_{n-2} \cdot f_{n-3}) \cdot (f_{n-5} \cdot f_{n-6} \cdot f_{n-7}) \cdots \tag{B.8}$$

for n even and

$$det(M_n) = f_{n+1} \cdot (f_n \cdot f_{n-1} \cdot f_{n-2} \cdot f_{n-3}^2) \cdot (f_{n-4} \cdot f_{n-5} \cdot f_{n-6} \cdot f_{n-7}^2) \cdots , \tag{B.9}$$

$$K(M_n) = M_{n+1} \cdot f_{n-3} \cdot f_{n-7} \cdot f_{n-11} \cdot f_{n-15} \cdot f_{n-19} \cdots \tag{B.10}$$

for n odd.

The exact expressions of the generating functions $\alpha(x)$ and $\beta(x)$ read,[22]

$$\alpha(x) = \frac{3}{1+x} + \frac{3 \cdot \beta(x)}{1-x^2} \quad \text{where}$$

$$\beta(x) = 3 \cdot \frac{x \cdot (1+x+x^3)}{1-x^2-x^4} = -3 + 3 \cdot (1+x)/(1-x^2-x^4) . \tag{B.11}$$

It is important to note that factorization scheme (B.6) is *actually stable*, *but of a slightly more general form*, as compared to (B.1), or the ones described in [44]: recalling the generating functions $\eta(x)$ and $\phi(x)$ of the exponents that occur in the factorization scheme (see Eqs. (8.6) and (8.10) in [44]), one must now introduce *two sets* of such exponents generating functions, $\eta_1, \phi_1, \eta_2, \phi_2$, in order to keep track of the parity of n, and even split these four functions into their odd and even parts:

$$\eta_{12} = (\eta_1(x) + \eta_1(-x))/2, \qquad \eta_{11} = (\eta_1(x) - \eta_1(-x))/2 ,$$
$$\eta_{22} = (\eta_2(x) + \eta_2(-x))/2, \qquad \eta_{21} = (\eta_2(x) - \eta_2(-x))/2, \qquad \phi_{12} = \cdots . \tag{B.12}$$

[22] Result (B.11) corresponds to a very simple expression for another generating function introduced in [44], namely the function $\rho(x)$ (see for instance Eq. (8.12) in [44]).

We must also decompose $\alpha(x)$ and $\beta(x)$ in odd and even parts:

$$\alpha_1(x)=\frac{\alpha(x)-\alpha(-x)}{2}, \qquad \alpha_2(x)=\frac{\alpha(x)+\alpha(-x)}{2},$$
$$\beta_1(x)=\frac{\beta(x)-\beta(-x)}{2}, \qquad \beta_2(x)=\frac{\beta(x)+\beta(-x)}{2}$$

namely

$$\beta_2(x)=\frac{3\cdot x^2\cdot(x^2+1)}{1-x^2-x^4}, \qquad \beta_1(x)=\frac{3\cdot x}{1-x^2-x^4},$$

$$\alpha_2(x)=\frac{3\cdot(1+2x^2+2x^4)}{(1-x^2)(1-x^2-x^4)}, \qquad \alpha_1(x)=\frac{3\cdot x\cdot(2+x^2+x^4)}{(1-x^2)(1-x^2-x^4)}.$$

Instead of functional relations (8.6) and (8.10) in [44], one now has the following relations:

$$\alpha_1(x)-2\cdot x\cdot\alpha_2(x)+3\cdot x\cdot(\eta_{12}(x)\cdot\beta_2(x)+\eta_{11}(x)\cdot\beta_1(x))=0,$$

$$\alpha_2(x)-2\cdot x\cdot\alpha_1(x)-3+3\cdot x\cdot(\eta_{22}(x)\cdot\beta_1(x)+\eta_{21}(x)\cdot\beta_2(x))=0,$$

$$x\cdot\alpha_1(x)-\beta_2(x)-(\phi_{21}(x)\cdot\beta_1(x)+\phi_{22}(x)\cdot\beta_2(x))=0,$$

$$x\cdot\alpha_2(x)-\beta_1(x)-(\phi_{11}(x)\cdot\beta_2(x)+\phi_{12}(x)\cdot\beta_1(x))=0, \tag{B.13}$$

where the odd and even parts of the exponents generating functions $\eta_1(x), \phi_1(x), \eta_2(x), \phi_2(x)$, read

$$\eta_{12}(x)=\frac{x^2}{1-x^4}, \qquad \eta_{11}(x)=\frac{x}{1-x^2}, \qquad \eta_{22}(x)=0, \qquad \eta_{21}(x)=\frac{x^3}{1-x^4},$$

$$\phi_{11}(x)=\frac{x\cdot(2x^2+1)}{1-x^4}, \qquad \phi_{12}(x)=2\frac{x^2}{1-x^2},$$

$$\phi_{21}(x)=\frac{x}{1-x^2}, \qquad \phi_{22}(x)=\frac{x^2\cdot(2x^2+1)}{1-x^4}.$$

Period four in the factorization scheme (B.6) corresponds to the occurrence of a $1-x^4=0$ singularity for these exponents generating functions.

The " stability" of factorization scheme (B.1) corresponds to the following ($n\to n+1$)-property: the exponents of the f_n's occurring at the mth step of iteration are also the one's at $(m+1)$th step of iteration, the f_n's being changed into f_{n+1}: at each new iteration step one only needs to find the exponent of f_1 (if any). The " stability" of factorization scheme (B.6) is a straight generalization mod. 2. of the previous property: the exponents of the f_n's occurring at the mth step of iteration are also the one's at $(m+2)$th step of iteration the f_n's being changed into f_{n+2}.

B.2. Factorization scheme for $\alpha \neq 0, \varepsilon$ non-generic

Let us come back to $\alpha \neq 0$ with the non-generic value $\varepsilon = 1.2$. We consider here $\alpha = 396/6095 \simeq 0.06497128$. The factorization scheme reads

$$f_1 = det(M_0), \qquad M_1 = K(M_0), \qquad f_2 = \frac{det(M_1)}{f_1}, \qquad M_2 = K(M_1)\,,$$

$$f_3 = \frac{det(M_2)}{f_1 \cdot f_2}, \qquad M_3 = K(M_2)\,,$$

$$f_4 = \frac{det(M_3)}{f_1^2 \cdot f_2 \cdot f_3}, \qquad M_4 = \frac{K(M_3)}{f_1}, \qquad f_5 = \frac{det(M_4)}{f_1^2 \cdot f_2^2 \cdot f_3 \cdot f_4}, \qquad M_5 = \frac{K(M_4)}{f_1 \cdot f_2}\,,$$

$$f_6 = \frac{det(M_5)}{f_1 \cdot f_2^2 \cdot f_3^2 \cdot f_4 \cdot f_5}, \qquad M_6 = \frac{K(M_5)}{f_2 \cdot f_3}\,,$$

$$f_7 = \frac{det(M_6)}{f_1 \cdot f_2 \cdot f_3^2 \cdot f_4^2 \cdot f_5 \cdot f_6}, \qquad M_7 = \frac{K(M_6)}{f_3 \cdot f_4}\,,$$

$$f_8 = \frac{det(M_7)}{f_1^2 \cdot f_2 \cdot f_3 \cdot f_4^2 \cdot f_5^2 \cdot f_6 \cdot f_7}, \qquad M_8 = \frac{K(M_7)}{f_1 \cdot f_4 \cdot f_5}\,,$$

$$f_9 = \frac{det(M_8)}{f_1^2 \cdot f_2^2 \cdot f_3 \cdot f_4 \cdot f_5^2 \cdot f_6^2 \cdot f_7 \cdot f_8}\,,$$

$$M_9 = \frac{K(M_8)}{f_1 \cdot f_2 \cdot f_5 \cdot f_6}, \qquad f_{10} = \frac{det(M_9)}{f_1 \cdot f_2^2 \cdot f_3^2 \cdot f_4 \cdot f_5 \cdot f_6^2 \cdot f_7^2 \cdot f_8 \cdot f_9}\,,$$

$$M_{10} = \frac{K(M_9)}{f_2 \cdot f_3 \cdot f_6 \cdot f_7}\,,$$

$$f_{11} = \frac{det(M_{10})}{f_1 \cdot f_2 \cdot f_3^2 \cdot f_4^2 \cdot f_5 \cdot f_6 \cdot f_7^2 \cdot f_8^2 \cdot f_9 \cdot f_{10}}, \ldots$$

$$M_{19} = \frac{K(M_{18})}{f_3 \cdot f_6 \cdot f_7 \cdot f_8 \cdot f_{11} \cdot f_{12} \cdot f_{15} \cdot f_{16}}\,,$$

$$f_{20} = \frac{det(M_{19})}{f_1^2 \cdot f_3 \cdot f_5^2 \cdot f_7 \cdot f_8^2 \cdot f_9^2 \cdot f_{10} \cdot f_{11} \cdot f_{12}^2 \cdot f_{13}^2 \cdot f_{14} \cdot f_{15} \cdot f_{16}^2 \cdot f_{17}^2 \cdot f_{18} \cdot f_{19}}\,,$$

$$M_{20} = \frac{K(M_{19})}{f_1 \cdot f_5 \cdot f_8 \cdot f_9 \cdot f_{12} \cdot f_{13} \cdot f_{16} \cdot f_{17}}\,,$$

$$f_{21} = \frac{det(M_{20})}{f_1^2 \cdot f_3 \cdot f_5^2 \cdot f_7 \cdot f_8^2 \cdot f_9^2 \cdot f_{10}^2 \cdot f_{11} \cdot f_{12} \cdot f_{13}^2 \cdot f_{14}^2 \cdot f_{15} \cdot f_{16} \cdot f_{17}^2 \cdot f_{18}^2 \cdot f_{19} \cdot f_{20}}, \ldots \qquad \text{(B.14)}$$

Up to the 13th iteration one has the previously described $(n \to n+1)$-property, but this property is broken with f_{15} in favour of the $(n \to n+2)$-property encountered with (B.6). The previously introduced odd–even-parity dependent exponents generating functions $\eta_{ij}(x)$ and $\phi_{ij}(x)$ now read

$$\eta_{12}(x) = x^2 + x^6 + x^{10} + x^{12}, \qquad \eta_{11}(x) = x^3 + x^7 + x^{11} + \frac{x^{15}}{1-x^4},$$

$$\eta_{22}(x) = x^2 + x^6 + x^{10} + \frac{x^{14}}{1-x^4}, \qquad \eta_{21}(x) = x^3 + x^7 + x^{11},$$

$$\phi_{11}(x) = x + 2x^3 + 2x^7 + x^9 + 2x^{11} + x^5 + 2x^{13},$$

$$\phi_{12}(x) = \frac{(1+2x^2)\cdot x^{14}}{1-x^4} + x^2 + 2x^4 + x^6 + 2x^8 + x^{10} + 2x^{12},$$

$$\phi_{21}(x) = x + 2x^3 + 2x^7 + x^9 + 2x^{11} + x^5 + \frac{(1+2x^2)\cdot x^{13}}{1-x^4},$$

$$\phi_{22}(x) = x^2 + 2x^4 + x^6 + 2x^8 + x^{10} + 2x^{12},$$

from which one deduces, from relations (B.13), the rational expressions of the α_i's and β_i's:

$$\beta_2(x) = \frac{3\cdot x^2\cdot(1+x^2)}{(1-x^2)\cdot(1-x^2-x^4-2x^6-x^8-2x^{10}-x^{12}-x^{14})},$$

$$\beta_1(x) = \frac{3\cdot(1+x^2)\cdot(1+x^4)\cdot(1+x^8)\cdot x}{1-x^2-x^4-2x^6-x^8-2x^{10}-x^{12}-x^{14}},$$

$$\alpha_2(x) = 3\cdot\frac{1+2x^2+5x^4+4x^6+5x^8+4x^{10}+5x^{12}+5x^{14}+3x^{16}}{(1-x^2)\cdot(1-x^2-x^4-2x^6-x^8-2x^{10}-x^{12}-x^{14})},$$

$$\alpha_1(x) = 3\cdot x\cdot\frac{(2+4x^2+4x^4+5x^6+4x^8+5x^{10}+4x^{12}+4x^{14})}{(1-x^2)\cdot(1-x^2-x^4-2x^6-x^8-2x^{10}-x^{12}-x^{14})}$$

yielding the rational expression for $\beta(x)$:

$$\beta(x) = \frac{3\cdot x\cdot(1+x+x^3-x^{16})}{1-2x^2-x^6+x^8-x^{10}+x^{12}+x^{16}}$$
$$= 3\cdot\frac{x\cdot(1+x^2)\cdot(1+x-x^2+x^4-x^6+x^8-x^{10}+x^{12}-x^{14})}{(1-x^2)\cdot(1-x^2-x^4-2x^6-x^8-2x^{10}-x^{12}-x^{14})}. \qquad \text{(B.15)}$$

The complexity growth corresponds to the (smallest) root of

$$1-x^2-x^4-2x^6-x^8-2x^{10}-x^{12}-x^{14} = 0. \qquad \text{(B.16)}$$

These results have also been checked using the previously depicted semi-numerical complexity growth evaluation method for $\varepsilon = 1/3$ and $\alpha = 396/6095 \simeq 0.06497\cdots$. The following value for the complexity has been obtained: $\lambda \simeq 1.46199$ in good agreement with the exact algebraic value deduced from (B.16), namely: $\lambda \simeq 1.46188\cdots$ (to be compared with the generic algebraic value of $\lambda, \lambda \simeq 1.4655\cdots$ associated with $1-x-x^3=0$).

The singularities of (B.15) are in agreement with the dynamical zeta function calculated for these values of α and ε:

$$\zeta(t) = \frac{1+t-t^7}{1-t-t^2-2t^3-t^4-2t^5-t^6-t^7} = \frac{1+t\cdot(1-t^6)}{1-t\cdot(1-t+t^2)\cdot(1+t+t^2)^2} \,. \tag{B.17}$$

These calculations can also be performed, for $\alpha \neq 0$, for the other non-generic value of $\varepsilon : \varepsilon = 1/3$. As far as the factorization scheme is concerned one gets exactly the same scenario as the one described in (B.14), the breaking of the $(n \to n+1)$-property and the occurrence of a $(n \to n+2)$-property taking place with f_{11} instead of f_{15} previously. For $\varepsilon = 1/3$ and, for instance, for $\alpha = 237/6095 \simeq 0.038884\cdots$, one gets the following expression for $\beta(x)$:

$$\beta(x) = \frac{3\cdot x\cdot(1+x+x^3-x^{12})}{1-2x^2-x^6+x^8+x^{12}} = \frac{3\cdot x\cdot(1+x^2)\cdot(1+x-x^2+x^4-x^6+x^8-x^{10})}{(1-x^2)\cdot(1-x^2-x^4-2x^6-x^8-x^{10})} \,. \tag{B.18}$$

Again these results have been compared with the complexity growth deduced from the semi-numerical method, for $\varepsilon = 1/3$ and $\alpha = 237/6095 \simeq 0.038884\cdots$. We have obtained the following value for the complexity: $\lambda \simeq 1.44865$ in good agreement with the exact algebraic value deduced from (B.18), namely: $\lambda \simeq 1.44717\cdots$.

The singularities of (B.18) are in agreement with the dynamical zeta function calculated for these values of α and ε:

$$\zeta(t) = \frac{1+t}{1-t-t^2-2t^3-t^4-t^5} = \frac{1+t}{1-t\cdot(1+t^2)\cdot(1+t+t^2)} \,. \tag{B.19}$$

Appendix C. Choice of a initial matrix corresponding to given values of ε and α

We present, in this section, a possible choice of an initial 3×3 matrix corresponding to a prescripted value of α and ε. From the results of Appendix A, one has

$$\alpha = \frac{(x_3x_1x_2+x_2x_1-x_1-1)\cdot(x_2x_0x_1+x_1x_0-x_0-1)}{(x_2-1)\cdot(x_0x_2-1)\cdot x_1} \tag{C.1}$$

and

$$\varepsilon = \frac{(x_1x_3-1)\cdot(1-x_1)\cdot x_0\cdot x_2}{(1-x_2)\cdot(x_0x_2-1)\cdot x_1} - 1 \,. \tag{C.2}$$

In order to perform our complexity growth calculations to get the factorization scheme of the transformation, one needs to iterate a non-trivial, initial matrix as simple as possible, in the $\alpha = 0$ case and for non-generic values of ε ($\varepsilon = 0.52$, $\varepsilon = 1/m, \ldots$).

Actually, let us consider a matrix of the form

$$M_0 = \begin{bmatrix} 1 & 3 & x \\ 5 & 2 & y \\ -4 & 8 & z \end{bmatrix} . \tag{C.3}$$

The $\alpha = 0$ condition factorizes as follows:

$$\alpha = -5\frac{(y+5)(x+3+z)(2x+61-11y+2z)(x-y-z)}{(y-5)^2(z+4x)(2x-11y-61+2z)} . \tag{C.4}$$

On the other hand, expression of $1+\varepsilon$ is also very simple since it also factorizes:

$$1+\varepsilon = -2\frac{(x-z-5)(5x+5z+3y)^2}{(y-5)^2(z+4x)(2x-11y-61+2z)} . \tag{C.5}$$

Appendix D. The polynomial to find the fixed points of K_ε^9

The fixed points of k_ε^N can be found as a suitable pair of roots of two polynomials $P(z)$ and $Q(y)$. The number of pairs of roots being relatively small (degree(P) $\times$ degree(Q)), it is straightforward to check which are the admissible pairs. For $\varepsilon = 13/25$ and $N = 9$, the two polynomials happen to verify $P(x) = Q(-x)$. We give below the expression of $P(z)$:

$$\begin{aligned} P(z) = {} & 31441432237625122070 3125z^{18} + 13582698726654052734375 00z^{17} \\ & + 7526890525245666503906 2500z^{16} + 28193916758642578125000000z^{15} \\ & + 47123542720804879760742187 50z^{14} + 1470245177129130834960937 5000z^{13} \\ & + 11545929550378045706713867 1875z^{12} \\ & + 2891620682990942742246093 75000z^{11} \\ & + 2350390744951453728523113 28125z^{10} \\ & - 284231908640546035318198 12500z^{9} \\ & - 1293918964637044949045506 98750z^{8} - 474688418556648700047025 80000z^{7} \\ & + 97685207019298617577561447 00z^{6} + 8841684508557014424153308400z^{5} \\ & + 1497468490621088327339020023z^{4} + 774177918347949394432093 20x^{3} \\ & + 14196266775922682562956676z^{2} - 525991376147246600507280z \\ & + 46021743292264609877 28. \end{aligned} \tag{D.1}$$

The actual value of z for the fixed point on $y+\bar{z}=0$ is: $z \simeq -0.4956845+0.003449852 \cdot I$. Polynomials $P(z)$ and their partners $Q(y)$ corresponding to the fixed points of $k_\varepsilon^{10}, k_\varepsilon^{11}$, and k_ε^{12}, are available in footnote 12 as well as their respective pairings of roots.

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