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Three-dimensional integrable models based on modified tetrahedron equations and quantum dilogarithm

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Abstract

Simple solutions of modified tetrahedron equation are given. These solutions are related to the known solutions of the tetrahedron equation. Each R-matrix contains one (or two) "spectral" parameter(s). Using these simple solutions one gets two (resp. four) parameters commuting sets of two-layer transfer matrices, the whole number of parameters in each two-layer transfer matrix being five (resp. ten). © 1997 Elsevier Science B.V.

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1. Introduction

The subject of this paper is the modified tetrahedron equation in the vertex form [1,2]:

$$R_{123} \cdot R'_{145} \cdot R''_{246} \cdot R'''_{356} = \widetilde{R}'''_{356} \cdot \widetilde{R}''_{246} \cdot \widetilde{R}'_{145} \cdot \widetilde{R}_{123}, \tag{1}$$

where the tildes mean that the parameters in the R-matrices in the left and right hand sides differ.

As it is known, with the help of the modified tetrahedron equation, one can actually construct a family of commuting transfer matrices, which Boltzmann weights are combinations of at least eight "primary" weights R and/or $\widetilde{R}$. This is explained in detail for the W-form of weights in Refs. [1–3], where solutions of the modified tetrahedron equation of the W-type were given. Those solutions are related to the Zamolodchikov-Bazhanov-Baxter model [4,5].

Recently a vertex formulation of the Zamolodchikov-Bazhanov-Baxter model has been obtained [6], and the duality between W and vertex weights has been established with the help of three-dimensional analogue

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of the Baxter's Ψ-vectors (Baxter's vertex-IRF correspondence) [7]. An advantage of the vertex approach is that the R-matrix, being an operator, can be written in a basis-independent form in terms of so-called *quantum dilogarithms* [8–10], so that the proof of the tetrahedron equation reduces to applying several times the *pentagon relation* for the quantum dilogarithms [11,12]. Moreover, in the operator approach, the modified tetrahedron equation appears naturally. This observation will be used in the following.

In the operator approach it also appeared that some hierarchy of solutions of tetrahedron equations naturally emerges. This hierarchy consists of the complete R-matrix as the "parent" and several of its limits as the "descendants". These descendants correspond to the Hietarinta-type R-matrix [13,15], and to the L-operator of Ref. [7]. The investigations of the descendants are much more simple than the investigation of the parent. In this paper we will only deal with the descendants, leaving the complete case for a separate paper.

These limits of the complete R-matrix (or W-weight) have not been investigated in Refs. [2,3], so the results of this very paper are new.

In this paper we use the quantum dilogarithms and operator solutions of the tetrahedron equations rather as a useful trick to simplify some tedious calculations that would appear if we take N finite.

The paper is organized as follows. In Section 2 the hierarchy of the operator solutions is recalled and a method of derivation of *finite-dimensional* modified tetrahedron equations is described. In Section 3 the explicit forms of the functional transformations responsible for the parameters of finite-dimensional R-matrices are given and useful parameterizations of simple R-matrices for the *modified tetrahedron equation* are written. Section 3 contains the final results: "transmutation-permutation" relations for transfer matrices and construction of a commuting set of *two-layers* transfer matrices.

2. Operator solutions

Let $\mathcal{W}$ be the Weyl algebra over C generated by invertible elements $\hat{u}_i, \hat{v}_i$ such that

$$\hat{u}_i \hat{v}_i = q \cdot \hat{v}_i \hat{u}_i, \tag{2}$$

$\hat{u}_i, \hat{v}_i$ and $\hat{u}_j, v_j$ commute when $i \neq j$. Introduce also the notation:

$$\hat{w}_i \stackrel{\text{def}}{=} -q^{-1/2} \hat{u}_i \hat{v}_i. \tag{3}$$

Denote the set of monomials of $\mathcal{W}$ as $\mathcal{W}_0$:

$$\mathcal{W}_0 = \{\ldots \cdot \hat{v}_i^{n_i} \cdot \hat{u}_i^{m_i} \cdot \ldots\}, \quad n_i, m_i \in Z\,. \tag{4}$$

Note that the map $\hat{u}_i \rightarrow \hat{u}_i^{-1}$, $\hat{v}_i \rightarrow \hat{v}_i^{-1}$, for all i, simultaneously, is an obvious isomorphism of $\mathcal{W}$.

Further denote $\widehat{\mathcal{W}}$ a completion of this Weyl algebra (see [10–12] for details[3], in particular Eqs. (1.15), (1.16), (1.17) in [12]). In general, functions in $\widehat{\mathcal{W}}$ are defined by their permutation relations with the elements of $\mathcal{W}_0$. Note that for any $\hat{a}, \hat{x} \in \mathcal{W}_0$, $\hat{a}\,\hat{x} = q^f \cdot \hat{x}\,\hat{a}$. Let us introduce two important functions in $\widehat{\mathcal{W}}$: At first, let a function $\psi(\hat{a})$ be defined as

$$\psi(\hat{a})\,\hat{x} = \hat{x}\,(q^{1/2}\hat{a}; q)_f^{-1}\,\psi(\hat{a}), \tag{5}$$

where

$$(a;q)_n = \prod_{\sigma=0}^{n-1}(1 - q^{\sigma} a) = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{6}$$

[3] The reader interested in the precise formulation of operator product in a purely algebraic setting can refer to the theory of vertex operator algebra [14].

The function $\psi(\hat{a})$ has a universal realization as q-exponent:

$$\psi(\hat{a}) = (q^{1/2}\hat{a}; q)_\infty. \tag{7}$$

$\psi(\hat{a})$ is often called "quantum dilogarithm function"[4], and its main property is the *pentagon relation* [11]:

$$\psi(\hat{v})\,\psi(\hat{u}) = \psi(\hat{u})\,\psi(\hat{w})\,\psi(\hat{v}). \tag{8}$$

The other element is the function $P(\hat{a},\hat{b})$,

$$P(\hat{a},\hat{b}) = P(\hat{a}^{-1},\hat{b}^{-1}) = P(\hat{b},\hat{a}^{-1}) = P(q\hat{a}\hat{b}^{-1},\hat{b}) = \ldots, \quad P(\hat{a},\hat{b})^2 = 1, \tag{9}$$

where $\hat{a},\hat{b} \in \mathcal{W}_0$, $\hat{a}\,\hat{b} = q^2 \cdot \hat{b}\,\hat{a}$. Let $\hat{x} \in \mathcal{W}_0$ be such that $\hat{a}\,\hat{x} = q^{f_a} \cdot \hat{x}\,\hat{a}$ and $\hat{b}\,\hat{x} = q^{f_b} \cdot \hat{x}\,\hat{b}$. Then $P(\hat{a},\hat{b})$ is defined via its permutation relation:

$$P(\hat{a},\hat{b})\;\hat{x} = q^{f_a f_b}\,\hat{x}\,\hat{a}^{f_b}\,\hat{b}^{-f_a}\,P(\hat{a},\hat{b}). \tag{10}$$

Note that the permutation operator of spaces i,j is a particular case of the P function:

$$P(\hat{v}_i^{-1}\hat{v}_j, \hat{u}_i\hat{u}_j^{-1}) = P_{i,j}. \tag{11}$$

Realization of P, necessary for our purposes, will be given below.

Define now

$$\widehat{P}_{123} = P(\hat{v}_1\hat{v}_2^{-1}\hat{v}_3, \hat{v}_1\hat{u}_2\hat{u}_3^{-1}), \tag{12}$$

and similarly for any $\widehat{P}_{ijk}$[5].

Then one has the following list of solutions of the tetrahedron equation (up to the above-mentioned isomorphism and some other isomorphisms)

Table 1 of R-matrices:

(o) $\widehat{P}_{123}$,

(i) $\widehat{r}^{+}_{123} = \psi(\hat{w}_1^{-1}\hat{u}_3)\;\widehat{P}_{123}$,

(ii) $\widehat{r}^{-}_{123} = \widehat{P}_{123}\;\psi(\hat{w}_1^{-1}\hat{u}_3)^{-1}$,

(iii) $\widehat{R}^{\epsilon}_{123} = \psi(\hat{w}_1^{-1}\hat{u}_3)\;\widehat{P}_{123}\,\psi((\hat{w}_1^{-1}\hat{u}_3)^{\epsilon})^{-1}$,

(iv) $\widehat{R}^{c}_{123} = \psi(\hat{w}_2^{-1}\hat{w}_3)\;\psi(\hat{w}_1^{-1}\hat{u}_3)\;\widehat{P}_{1,23}\,\psi(\hat{w}_1\hat{u}_3^{-1})^{-1}\;\psi(\hat{w}_2\hat{w}_3^{-1})^{-1}$.

In this table $\epsilon = \pm 1$ and the superscript c in $\widehat{R}^c$ means "complete".

In the $q^N \to 1$ limit,

$$q = q_0 \cdot \omega\,, \quad q_0 = \exp\{-\tau/N^2\}\,, \quad \omega^N = 1, \tag{13}$$

it is useful to "split" each operator in $\mathcal{W}$ into "finite – and infinite – dimensional parts". For each $\hat{a} \in \mathcal{W}$, one can write

$$\hat{a} = a\,A, \quad \text{where} \quad A^N = 1, \tag{14}$$

[4] The reader interested in the precise definitions of the quantum dilogarithm as a pure mathematical object can refer to [12] and also [10].

[5] The notations for the indices of $\hat{u}, \hat{v}, \hat{w}$, as well as for the indices of R-matrices, are not crucial. In this section the indices will take the values $1,\ldots,6$, which is conventional for the tetrahedron equation. Later, constructing transfer matrices, one deals with large number of spaces, so it will be necessary to change the numerical notations for the indices to more "symbolic" notations.

where A and a can be seen as commuting elements, so that, if $\hat{a}\,\hat{b} = q^f\,\hat{b}\,\hat{a}$, then $a\,b = q_0^f\,b\,a$ and $A\,B = \omega^f\,B\,A$. Note that $\hat{a}^N = a^N$, etc., are the key relations for the extracting a from $\hat{a}$, etc. Obviously, a and b become numbers when $q^N \longrightarrow 1$.

The functions ψ and $\widehat{P}_{123}$ also split, for the leading $1/\tau$-term, into *product of finite and infinite* dimensional parts (see [12] for more details), namely,

$$\psi(\hat{a}) = \exp\{-\frac{\mathrm{Li}_2(-a^N)}{\tau}\} \cdot d(aA) \cdot (1 + O(\tau)), \tag{15}$$

where

$$d(x) = (1 + x^N)^{(N-1)/2N} \cdot \prod_{k=1}^{N-1} (1 - \omega^{k+1/2}x)^{-k/N}. \tag{16}$$

The matrix elements of the function d, in any finite[6] dimensional basis, can be easily calculated via

$$d(\omega x) = \frac{\Delta(x)}{1 - \omega^{1/2}x} \cdot d(x), \tag{17}$$

where

$$\Delta(x)^N = 1 + x^N, \tag{18}$$

Any formal expansion like $f(X)$ means

$$f(X) = \sum_{n,m\in Z_N} f(\omega^n) \cdot \omega^{-nm} \cdot X^m. \tag{19}$$

The pre-exponent[7] in (15) yields functional transformations (Poisson action see Eqs. (2.6), (2.7), (2.8) in [12] for details). Actually, for any $\hat{a}, \hat{b} \in \mathcal{W}$, such that $\hat{a}\hat{b} = q^f\,\hat{b}\,\hat{a}$, one gets, for the $1/\tau$-leading term[8]

$$\exp\{-\frac{\mathrm{Li}_2(-a^N)}{\tau}\} \cdot f(b) \cdot \exp\{\frac{\mathrm{Li}_2(-a^N)}{\tau}\} = f(b(1 + a^N)^{-f/N}) \stackrel{\mathrm{def}}{=} \exp\{-\mathrm{Li}_2(-a^N)\} \circ f(b). \tag{20}$$

One also has the decomposition

$$\widehat{P}_{123} = p_{123}\ \pi_{123}, \tag{21}$$

where

$$p_{123} = \sum_{n,m\in Z_N} \omega^{-nm} \cdot (V_1 V_2^{-1} V_3)^n \cdot (V_1 U_2 U_3^{-1})^m, \tag{22}$$

and where, fixing notations for the decomposition of $\hat{u}$, $\hat{v}$, $\hat{w}$,

$$\hat{u}_i = u_i\,U_i\,, \quad \hat{v}_i = v_i\,V_i\,, \quad \hat{w}_i = w_i\,W_i\,, \quad v_i \longrightarrow w_i/u_i. \tag{23}$$

[6] A finite-dimensional basis for $U\,V = \omega\,V\,U$ is, for instance, $V\,|j\rangle = \omega^j\,|j\rangle$ and $U\,|j\rangle = |j-1\rangle$. Formula (18) is just a way of calculating $d(\hat{x}) = d(x\,X)$ without roots (and up to some multipliers), in a finite-dimensional basis.: $\langle i|\,d(x\,X)\,|j\rangle = \sum d(x\omega^n)\ \omega^{-nm}\,\langle i|X^m|j\rangle$.

[7] For $\hat{x} = x\,X$ one has (see [12] for such a leading-term-splitting relation for functions like the quantum dilogarithm) $f(\hat{x}) = \exp(\phi(x)/\tau) \cdot F(x\,X) \cdot (1 + O(\tau))$. We call the "pre-exponent" such a $\exp(\phi(x))$ term.

[8] The adjoint action is denoted by a circle.

$\pi_{1,23}$ reads

$$\pi_{123} \circ f(u_1,u_2,u_3,w_1,w_2,w_3) = f(\frac{u_1w_2}{w_3}, \frac{u_1u_3}{w_1}, \frac{w_1u_2}{u_1}, \frac{w_1w_2}{w_3}, w_3, w_2). \tag{24}$$

Let us now consider the tetrahedron equation for any R-operator deduced from Table 1 above

$$\widehat{R}_{123} \cdot \widehat{R}_{145} \cdot \widehat{R}_{246} \cdot \widehat{R}_{356} = \widehat{R}_{356} \cdot \widehat{R}_{246} \cdot \widehat{R}_{145} \cdot \widehat{R}_{123}. \tag{25}$$

Each $\widehat{R}$ splits into product of two factors, the finite-dimensional R, depending on some parameters, and the "functional-map part" $\mathcal{F}$:

$$\widehat{R} = R \cdot \mathcal{F}. \tag{26}$$

The $\mathcal{F}$'s obey the "functional tetrahedron equation"[9], so they can be quicked out from (25). The finite-dimensional tetrahedron equation[10] remains[11]

$$\begin{aligned} &R_{123}(\mathcal{F}_{123} \circ R_{145})(\mathcal{F}_{123} \circ \mathcal{F}_{145} \circ R_{246})(\mathcal{F}_{123} \circ \mathcal{F}_{145} \circ \mathcal{F}_{246} \circ R_{356}) \\ &\quad = R_{356}(\mathcal{F}_{356} \circ R_{246})(\mathcal{F}_{356} \circ \mathcal{F}_{246} \circ R_{145})(\mathcal{F}_{356} \circ \mathcal{F}_{246} \circ \mathcal{F}_{145} \circ R_{123}). \end{aligned} \tag{27}$$

Obviously, this equation is *not* the tetrahedron equation, but the *modified tetrahedron equation* [1,2]. Moreover, one obtains, at once, a *rational parameterization* of the modified tetrahedron equation (because $\exp\{-\mathrm{Li}_2\}$ and π act rationally on the N-th powers of the parameters).

3. Functional maps and finite-dimensional equations

Now let us see how $\mathcal{F}$ and R look like. For this purpose let us introduce the independent operators on which $\hat{R}_{123}$ generally depends. They read

$$\hat{a} = \hat{w}_2^{-1}\hat{w}_3, \quad \hat{b} = \hat{u}_1^{-1}\hat{u}_2, \quad \hat{c} = \hat{w}_1^{-1}\hat{u}_3, \tag{28}$$

and similarly $\hat{a}', \hat{b}', \hat{c}'$ for $\widehat{R}_{145}$, $\hat{a}'', \hat{b}'', \hat{c}''$ for $\widehat{R}_{246}$, and $\hat{a}''', \hat{b}''', \hat{c}'''$ for $\widehat{R}_{356}$. Generically there are *eight* independent operators (and hence eight independent variables for the finite-dimensional tetrahedron equation, at least for the case (iv)), because there are four angle-like relations for these twelve $a^\#$, $b^\#$, $c^\#$:

$$a'' = a'\, a''', \quad b' = b\, b'', \quad c'' = a\, c''', \quad c' = c\, b'''. \tag{29}$$

Let

$$\mathcal{F}_{123} \circ f(u_1,u_2,u_3,w_1,w_2,w_3) = f(u_1',u_2',u_3',w_1',w_2',w_3'). \tag{30}$$

Then one obtains for (i)

$$\frac{u_1'}{u_1} = \frac{1}{a}, \quad \frac{w_1'}{w_1} = \frac{\Delta(c)}{a}, \quad \frac{u_2'}{u_2} = \frac{c}{b\,\Delta(c)}, \quad \frac{w_2'}{w_2} = \frac{a}{\Delta(c)}, \quad \frac{u_3'}{u_3} = \frac{b\,\Delta(c)}{c}, \quad \frac{w_3'}{w_3} = \frac{1}{a}, \tag{31}$$

[9] Note, that such solutions for the functional tetrahedron equation have already been described in Ref. [16] for the cases (i), (ii) and (iii), with $\epsilon = 1$.

[10] More details can be found in [8]. Heuristically the functional tetrahedron equation for the $\mathcal{F}$'s can be seen (at the leading $1/\tau$ order) as the tetrahedron equation for $\hat{R}_{ijk}$ for $N = 1$.

[11] The adjoint action is denoted by a circle: $FRF^{-1} = (F \circ R)$. The finite-dimensional tetrahedron Eq. (28) corresponds to a rewriting of the LHS and RHS of the tetrahedron equation (namely for the LHS: $R_{123}\,F_{123}\,R_{145}\,F_{145}\;R_{246}\,F_{246}\,R_{356}\,F_{356}$ becomes $R_{123}\,F_{123}\,R_{145}F_{123}^{-1}\,F_{123}\,F_{145}\cdots$).

and for (ii)

$$\frac{u_1'}{u_1}=\frac{\Delta(ab)}{a},\quad \frac{w_1'}{w_1}=\frac{1}{a},\quad \frac{u_2'}{u_2}=\frac{c}{b},\quad \frac{w_2'}{w_2}=a,\quad \frac{u_3'}{u_3}=\frac{b}{c},\quad \frac{w_3'}{w_3}=\frac{\Delta(ab)}{a},\tag{32}$$

and for (iii)

$$\frac{u_1'}{u_1}=\frac{\Delta((ab)^\epsilon)^\epsilon}{a},\quad \frac{w_1'}{w_1}=\frac{\Delta(c)}{a},\quad \frac{u_2'}{u_2}=\frac{c}{\Delta(c)\,b},\quad \frac{w_2'}{w_2}=\frac{a}{\Delta(c)},\quad \frac{u_3'}{u_3}=\frac{\Delta(c)\,b}{c},\quad \frac{w_3'}{w_3}=\frac{\Delta((ab)^\epsilon)^\epsilon}{a},\tag{33}$$

for (iv)

$$\begin{aligned}&\frac{u_1'}{u_1}=\frac{b}{\Delta(a\Delta(b))},\quad \frac{w_1'}{w_1}=\frac{\Delta(c\Delta(a))}{a},\quad \frac{u_2'}{u_2}=\frac{c}{\Delta(b\Delta(c))},\\ &\frac{w_2'}{w_2}=\frac{a}{\Delta(c\Delta(a))},\quad \frac{u_3'}{u_3}=\frac{\Delta(b\Delta(c))}{c},\quad \frac{w_3'}{w_3}=\frac{b}{\Delta(a\Delta(b))}.\end{aligned}\tag{34}$$

The R's read (Table 2)

(o) $p_{1,23}=\sum_{n,m\in Z_N}\omega^{-nm}\cdot A^n\cdot(BC^{-1})^m$,

(i) $r^+_{123}=d(cC)\ p_{1,23}$,

(ii) $r^-_{123}=p_{1,23}\ d(abC)^{-1}$,

(iii) $R^\epsilon_{123}=d(cC)\ p_{1,23}\ d((abC)^\epsilon)^{-1}$,

(iv) $R^c_{123}=d(aA)\ d(c\Delta(a)\,C)\ p_{1,23}\ d(\frac{\Delta(a)}{ab}\ C^{-1})^{-1}\ d(\frac{\Delta(a\Delta(b))}{b\Delta(c\Delta(a))}\ A^{-1})^{-1}$.

In the finite-dimensional-R-matrices case simpler expressions can be obtained from the more complicated. Namely, taking the limit when $a\to 0$, ab and c being fixed, one obtains, from the "parent" R^c-matrix (iv), the R^--matrix (iii). In the limit $1/ab\to 0$, c being fixed, R^- becomes r^+, and in the limit $c\to 0$, $1/ab$ being fixed, R^ϵ becomes r^- (up to isomorphism $\mathcal{W}$ above-mentioned).

Let us now consider case (i) in details. One sees, from Table 2 above, that the only parameter in r^+ is c. It is thus convenient to express this dependence as

$$r^+=r^+(\Delta(c)).\tag{35}$$

Let us see what happens for the four $c^\#$ when one cancels functional parts. Then the (modified) tetrahedron equation one obtains (the calculations are simple but cumbersome) is

$$\begin{aligned}&r^+_{123}(\Delta(c))\ r^+_{145}(\frac{\Delta(c\Delta(ab'''))}{\Delta(c)})\ r^+_{246}(\Delta(c'''\Delta(c)))\ r^+_{356}(\frac{\Delta(c)\Delta(ab'''\Delta(c'''))}{\Delta(c\Delta(ab'''))})\\ &=r^+_{356}(\Delta(c'''))\ r^+_{246}(\Delta(ab'''\Delta(c''')))\ r^+_{145}(\frac{\Delta(c'''\Delta(c))}{\Delta(c''')})\ r^+_{123}(\frac{\Delta(c''')\Delta(c\Delta(ab'''))}{\Delta(c'''\Delta(c))})\end{aligned}\tag{36}$$

This is the first, and the last time, parameterization of the modified tetrahedron equation in terms of $a^\#, b^\#, c^\#$ is written out.

Eq. (36) can more nicely be parameterized by

$$\Delta(c)^N=t\alpha,\quad \Delta(c''')^N=\frac{\widetilde{\alpha}(t\alpha-1)}{t(\alpha\widetilde{\alpha}-1)},\quad \Delta(ab''')^N=\frac{t^2-1}{t\alpha-1}.\tag{37}$$

It is convenient to use N-th powers of $\Delta(c)$ as the parameter of r^+:

$$r^+(\Delta(c)) \longrightarrow r^+(\Delta(c)^N)\,. \tag{38}$$

Such parameterization will be used everywhere below. To avoid all the problems concerning the branches of N-th power roots when one restores c and $\Delta(c)$ for given $r^+(x)$, $x = \Delta(c)^N = 1 + c^N$, one will consider c and $\Delta(c)$ as positive real numbers.

Thus Eq. (36) can be written as

$$r^+_{123}(t\alpha)\, r^+_{145}(\frac{t}{\alpha})\, r^+_{246}(\frac{t\alpha-1}{\alpha\widetilde{\alpha}-1})\, r^+_{356}(\frac{\alpha(t\widetilde{\alpha}-1)}{t(\alpha\widetilde{\alpha}-1)}) = r^+_{356}(\frac{\widetilde{\alpha}(t\alpha-1)}{t(\alpha\widetilde{\alpha}-1)})\, r^+_{246}(\frac{t\widetilde{\alpha}-1}{\alpha\widetilde{\alpha}-1})\, r^+_{145}(\frac{t}{\widetilde{\alpha}})\, r^+_{123}(t\widetilde{\alpha})\,. \tag{39}$$

The parameterization of modified tetrahedron equation for r^- looks similar, as well as that for R^ϵ. Let us parameterize R^ϵ as follows:

$$R^\epsilon_{123}(\Delta(c)^N, \Delta((ab)^\epsilon)^N) = d(cC)\cdot p_{1,23}\cdot d((abC)^\epsilon)^{-1}\,. \tag{40}$$

Then the parameterization of the modified tetrahedron equation reads

$$\begin{aligned} &R^\epsilon_{123}(t\alpha, t'\alpha')\cdot R^\epsilon_{145}(\frac{t}{\alpha},\frac{t'}{\alpha'}) R^\epsilon_{246}(\frac{t\alpha-1}{\alpha\widetilde{\alpha}-1},\frac{t'\alpha'-1}{\alpha'\widetilde{\alpha}'-1})\cdot R^\epsilon_{356}(\frac{\alpha(t\widetilde{\alpha}-1)}{t(\alpha\widetilde{\alpha}-1)},\frac{\alpha'(t'\widetilde{\alpha}'-1)}{t'(\alpha'\widetilde{\alpha}'-1)}) \\ &= R^\epsilon_{356}(\frac{\widetilde{\alpha}(t\alpha-1)}{t(\alpha\widetilde{\alpha}-1)},\frac{\widetilde{\alpha}'(t'\alpha'-1)}{t'(\alpha'\widetilde{\alpha}'-1)})\cdot R^\epsilon_{246}(\frac{t\widetilde{\alpha}-1}{\alpha\widetilde{\alpha}-1},\frac{t'\widetilde{\alpha}'-1}{\alpha'\widetilde{\alpha}'-1}) R^\epsilon_{145}(\frac{t}{\widetilde{\alpha}},\frac{t'}{\widetilde{\alpha}'})\cdot R^\epsilon_{123}(t\widetilde{\alpha}, t'\widetilde{\alpha}')\,. \end{aligned} \tag{41}$$

For R^ϵ one can easily see that the parameterization is just *two copies* of the parameterization of r^+. Switching off one of the two copies, one obtains r^+ or r^-. So only the case r^+ will be considered for simplicity, but all formulae will be valid for r^- and R^ϵ as well.

4. Transmutation relation for transfer matrices

In the modified tetrahedron equation we shall interpret R_{246} and R_{356} as the elements of transfer matrices, and R_{123} and R_{145} as the intertwiners. $K\times L$ transfer matrix is thus defined as

$$T_{\{c_{i,j}\}}(\{z_{i,j}\}) = \mathrm{Trace}_{\{a_i\},\{b_j\}} \overrightarrow{\prod_j}\, \overrightarrow{\prod_i}\, R_{a_i, b_j, c_{i,j}}(z_{i,j})\,, \tag{42}$$

where $i = 1,\dots,K$, $j = 1,\dots,L$, and "Trace" implies that cyclic boundary conditions are imposed: $a_{K+1} = a_1$, $b_{L+1} = b_1$. For a graphical interpretation of the transfer matrix see Fig. 1.

Now the spaces are marked by the letters a_i, b_j, $c_{i,j}$, instead of numbers, (see the footnote in Section 2), and the "permutation-transmutation" relation for two transfer matrices, namely

$$T_{\{c_{i,j}\}}(\{z_{i,j}\})\cdot T_{\{c_{i,j}\}}(\{w_{i,j}\}) = T_{\{c_{i,j}\}}(\{\widetilde{w}_{i,j}\})\cdot T_{\{c_{i,j}\}}(\{\widetilde{z}_{i,j}\}) \tag{43}$$

is provided by the system of the modified tetrahedron relations,

$$\begin{aligned} &R_{o,a_i,a'_i}(x_{i,j})\cdot R_{o,b_j,b'_j}(y_{i,j})\cdot R_{a_i,b_j,c_{i,j}}(z_{i,j})\cdot R_{a'_i,b'_j,c_{i,j}}(w_{i,j}) \\ &= R_{a'_i,b'_j,c_{i,j}}(\widetilde{w}_{i,j})\cdot R_{a_i,b_j,c_{i,j}}(\widetilde{z}_{i,j})\cdot R_{o,b_j,b'_j}(y_{i+1,j})\cdot R_{o,a_i,a'_i}(x_{i,j+1})\,, \end{aligned} \tag{44}$$

where o is the auxiliary space, periodic boundary conditions being assumed.

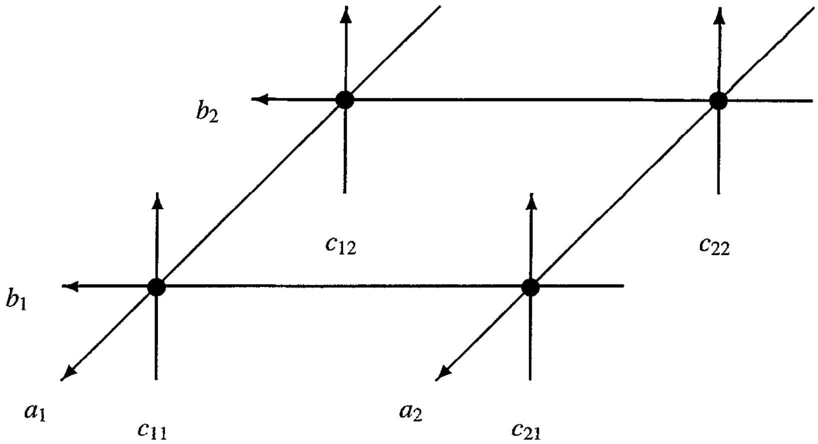

Fig. 1. Fragment of the transfer matrix.

Parameterization (39), (41) is well-suited for the solution of Eqs. (44) because the parameters of the intertwiners are the *independent variables*. However, finally, one has to exclude them, and the final system for $\{z_{i,j}, w_{i,j}, \widetilde{z}_{i,j}, \widetilde{w}_{i,j}\}$ one obtains reads

$$(1 - z_{i,j}^{-1}) \cdot (1 - w_{i,j}^{-1}) = (1 - \widetilde{z}_{i,j}^{-1}) \cdot (1 - \widetilde{w}_{i,j}^{-1}), \quad (1 - w_{i,j}) \cdot (1 - z_{i,j+1}) = (1 - \widetilde{z}_{i,j}) \cdot (1 - \widetilde{w}_{i,j+1}),$$
$$z_{i,j} \cdot w_{i+1,j} = \widetilde{w}_{i,j} \cdot \widetilde{z}_{i+1,j}. \tag{45}$$

For the construction of an integrable lattice statistical model it is convenient to consider the case when $z_{i,j} = z_{i+2,j} = z_{i,j+2}$ and similarly for the other transfer matrices. Then the number of independent variables in each transfer matrix reduces to four, namely $z_{11}, z_{12}, z_{21}, z_{22}$. Parameterizing them as follows:

$$z_{11} = \frac{(1+m_2)m_1}{m_1 - m_2} - g\,\frac{(1+m_2)m_1}{m_1 - m_2}, \quad z_{22} = \frac{1+m_1}{m_1 - m_2} - g\,\frac{(1+m_2)(k+m_2)m_1}{(m_1 - m_2)(k+m_1)m_2},$$
$$z_{12} = -\frac{k+m_2}{m_1 - m_2} + g^{-1}\,\frac{(1+m_1)(k+m_1)m_2}{(m_1 - m_2)(1+m_2)m_1}, \quad z_{21} = -\frac{(k+m_1)m_2}{(m_1 - m_2)k} + g^{-1}\,\frac{(k+m_1)m_2}{(m_1 - m_2)k}, \tag{46}$$

and simply denoting the corresponding transfer matrix by $T(k, m_1, m_2, g)$, one obtains the following form for relation (43):

$$T(k, m_1, m_2, g) \cdot T(k, m_2, m_1, f) = T(k, m_1, m_2, f^{-1}) \cdot T(k, m_2, m_1, g^{-1}). \tag{47}$$

The two-layers transfer matrices are to be defined as

$$T_2(k, m_1, m_2, a, b) = T(k, m_1, m_2, a) \cdot T(k, m_2, m_1, b), \tag{48}$$

They actually form a two-parameters commuting family:

$$[T_2(k, m_1, m_2, a, b),\, T_2(k, m_1, m_2, a', b')] = 0. \tag{49}$$

All these relations for the transfer matrices simplify in the "nice" limit $z_{11} = z_{22} = z_1$, $z_{12} = z_{21} = z_2$ (this limit corresponds to $k = m_1 m_2 = 1$) and similarly for other one-layer transfer matrices. In this case the relation

$$T(z_1, z_2) \cdot T(w_1, w_2) = T(w_2, w_1) \cdot T(z_2, z_1) \tag{50}$$

is verified provided that

$$\frac{1 - z_1^{-1}}{1 - z_2^{-1}} = \frac{1 - w_2^{-1}}{1 - w_1^{-1}} . \tag{51}$$

Note that the *two-layer* transfer matrices (48) also obey the "transmutation" relation (43). This fact is actually the very condition for integrability: this "transmutation" relation [12] can be regarded as Baxter's local condition of the integrability, known also as the "Star-Star relation" [5,17,18] and corresponding to the spatial symmetry of three-dimensional models with the body-centered-cube structure [19].

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[12] Relations (50), (51) for the three-dimensional model were written originally by R.J. Baxter even before 1984.