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Integrable mappings and polynomial growth ★

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Abstract

We describe birational representations of discrete groups generated by involutions, having their origin in the theory of exactly solvable vertex-models in lattice statistical mechanics. These involutions correspond respectively to two kinds of transformations on $q \times q$ matrices: the inversion of the $q \times q$ matrix and an (involutive) permutation of the entries of the matrix. We concentrate on the case where these permutations are elementary transpositions of two entries. In this case the birational transformations fall into six different classes. For each class we analyze the factorization properties of the iteration of these transformations. These factorization properties enable to define some canonical homogeneous polynomials associated with these factorization properties. Some mappings yield a polynomial growth of the complexity of the iterations. For three classes the successive iterates, for $q = 4$, actually lie on elliptic curves. This analysis also provides examples of integrable mappings in arbitrary dimension, even infinite. Moreover, for two classes, the homogeneous polynomials are shown to satisfy non trivial non-linear recurrences. The relations between factorizations of the iterations, the existence of recurrences on one or several variables, as well as the integrability of the mappings are analyzed.

1. Introduction

In previous papers, we have analyzed birational representations of discrete groups generated by involutions, having their origin in the theory of exactly

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solvable models in lattice statistical mechanics [1–6].

The group of birational transformations first studied in [7–10] is generated by the so-called *inversion relations* [11,12] which amount to combine two very simple algebraic transformations: the matrix inversion and permutations [1] of the entries of a matrix [1]. In such a general framework, the dimension of the lattice and the lattice itself, only occur through the number of inversion relations and the permutations of the entries introduced to generate the birational transformations [4,5,13].

This justifies considering the following problem: to generalize to $q \times q$ matrices and analyze *birational* transformations generated by the matrix inverse and a permutation of the entries of the matrix, and finally find the permutations of the matrix for which the corresponding birational transformations yield *integrable* mappings [2]. This analysis is interesting for itself for the theory of mappings of several variables and the theory of discrete dynamical systems, disregarding the relation with integrable lattice models. Such an analysis is performed in a series of parallel publications [14–16].

In [14], a particular transposition of the entries was analyzed. For this very transposition, it has been shown that the iteration of the associated birational transformations *present some remarkable factorization properties.* Actually *the entries of the successive $q \times q$ matrices corresponding to the iteration of our transformation, as well as the determinants of these matrices, do factorize into homogeneous polynomials of all the entries of the initial $q \times q$ matrix.* These factorization properties explain [17] why the "complexity" of these iterations (degree of the successive iterates), instead of having the exponential growth one expects at first sight, actually has a *polynomial growth* [18,19]. It has also been shown that the homogeneous polynomial factors occurring in these factorizations *do satisfy remarkable non-linear recurrences* and that these recurrences *were actually integrable recurrences yielding algebraic elliptic curves* [14].

We will concentrate here on simple heuristic examples of permutations: in fact all the transpositions of *two* entries of a $q \times q$ matrix. In [16] it has been shown that the analysis of the birational transformations corresponding to *all the transpositions of two entries,* actually reduces to the study of *six classes* of such mappings.

The transposition analyzed in [14] corresponds to the first class with respect to this classification. We will revisit here the analysis performed in [14] (occurrence of factorizations, recurrences, ...) for the five remaining classes. The mappings associated with three of these five classes are not integrable,

[1] In the framework of vertex lattice models. For spin models, the groups of birational transformations are also generated by similar simple involutions but slightly different [1,2,10].

[2] There are many definitions of the key word "integrability" in the literature (integrability *à la Liouville,* integrability in the sense of commuting discrete flows...). Here a mapping is called integrable if the successive iteration points belong to algebraic elliptic (or rational) curves.

even for $q = 4$ [16]. This will clarify the relations between all these structures and the integrability. In particular, it will help to understand *to what extend factorizations yield integrability.* In fact, it will be shown that the occurrence of factorizations is a *quite general phenomenon*: it does occur even outside the framework of integrability. The existence of factorization of our transformations yields a growth of the complexity of the iterations, even when exponential, smaller than the generic $(q-1)^n$ growth. On the other hand, *integrable mappings only occur with a polynomial growth* of the complexity of the iteration. The relation between integrability and polynomial growth has already been discussed by several authors [18,20,21] with some emphasis on the Cremona transformations [19]. Let us note that the framework of the analysis performed here is slightly different, in particular we deal with birational transformations acting in projective spaces of arbitrary dimension (odd or even: *no simplectic structure* is needed). As far as recurrences are concerned, it is tempting, at first sight, to see a close connection between the occurrence of *recurrences* and the integrability of the birational transformations, since this integrability yields *curves.* The detailed analysis of the five remaining classes rules out such naive connections, and will make clear the actual relations between these various structures. *As a byproduct it will provide, with the integrable subcase of one of these classes* (class IV), *an example of integrable mapping in arbitrary dimension, even infinite.*

2. Notations

Let us consider the following $q \times q$ matrix:

$$R_q = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & \cdots \\ m_{21} & m_{22} & m_{23} & m_{24} & \cdots \\ m_{31} & m_{32} & m_{33} & m_{34} & \cdots \\ m_{41} & m_{42} & m_{43} & m_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2.1}$$

Let us introduce the following transformations, the matrix inverse $\widehat{I}$, the homogeneous matrix inverse I:

$$\widehat{I} : R_q \longrightarrow R_q^{-1} \tag{2.2}$$

$$I : R_q \longrightarrow R_q^{-1} \cdot \det(R_q) \tag{2.3}$$

The homogeneous inverse I is a *polynomial* transformation on each of the entries m_{ij} which associates to each m_{ij} its corresponding cofactor.

In the following, t will denote an arbitrary transposition of two entries of the $q \times q$ matrix, and t_{ij-kl} will denote the transposition exchanging m_{ij} and m_{kl}.

The two transformations t and $\widehat{I}$ are *involutions* whereas the homogeneous inverse verifies
$I^2 = (\det(R_q))^{q-2} \cdot \mathcal{I}d$, where $\mathcal{I}d$ denotes the identity transformation.
We also introduce the (generically *infinite order*) transformations $K = t \cdot I$ and $\widehat{K} = t \cdot \widehat{I}$.
K is a (homogeneous) *polynomial* transformation on the entries m_{ij}, while transformation $\widehat{K}$ is clearly a rational transformation on the entries m_{ij}. In fact $\widehat{K}$ is a *birational* transformation since its inverse transformation is $\widehat{I} \cdot t$.

3. Recalls

3.1. Six equivalence classes

Let us first recall that, as far as the analysis of transformation K is concerned, *all the transpositions can actually be reduced to six different classes* [16] *of transpositions* [3]. One can thus study a single mapping in each class and directly deduce the results concerning all the other transformations of the same class. A first step to prove it amounts to giving an equivalence relation on these 120 transpositions, which does not modify the structure of the corresponding transformations [16]. This equivalence up to relabelling conjugations, does not modify the properties of the mappings and yields *seven* equivalence classes (with the notation $[m_{ij} - m_{kl}]$ denoting the transposition exchanging the two entries m_{ij} and m_{kl} of matrix (2.1)):

- Class C_1 corresponds to all the 6 transpositions of the form $[m_{ij} - m_{ji}]$
- Class C_2 corresponds to all the 6 transpositions of the form $[m_{ii} - m_{jj}]$
- Class C_3 corresponds to all the 12 transpositions of the form $[m_{ij} - m_{kl}]$
- Class C_4 corresponds to all the 24 transpositions of the form $[m_{ij} - m_{jk}]$ or $[m_{ji} - m_{kj}]$
- Class C_5 corresponds to all the 24 transpositions of the form $[m_{ij} - m_{ik}]$ or $[m_{ji} - m_{ki}]$
- Class C_6 corresponds to all the 24 transpositions of the form $[m_{ii} - m_{jk}]$
- Class C_7 corresponds to all the 24 transpositions of the form $[m_{ii} - m_{ij}]$ or $[m_{ii} - m_{ji}]$

where the various indices i, j, k and l are all different.

Moreover, one can actually show [16] that classes C_1 and C_2 *yield the same behavior* as far as the iterations of their associated birational mappings are concerned: the transformations K^2 respectively associated to classes C_1 and C_2 *are conjugated.* Therefore classes C_1 and C_2 *can be brought together in the same class*, we will denote class I, as far as the analysis of the birational transforma-

[3] At first sight, one has to study as many mappings as there are transpositions t, of two elements among the sixteen entries of the matrix, that is $\binom{2}{16} = 120$.

tion is concerned. The five other classes ($C_3, \ldots, C_7$) will be relabelled classes (II,..., VI) in the same order.

It is important to note that this classification in six classes *holds for $q \times q$ matrices, for any value of $q \geq 4$*. For $q = 3$ one remarks that class II no longer exists and similarly, for $q = 2$, classes III, IV, V do not exist anymore.

Let us also remark that any transposition of two entries $m_{i_1 j_1}$ and $m_{i_2 j_2}$ of a $q \times q$ matrix can be associated with a transposition exchanging $m_{\sigma(i_1)\sigma(j_1)}$ and $m_{\sigma(i_2)\sigma(j_2)}$, where $\sigma(i_1), \sigma(j_1), \sigma(i_2)$ and $\sigma(j_2)$ run into $\{1, 2, 3, 4\}$. One can thus restrict the transposition to one in the 4×4 block-matrix corresponding to the first four rows and columns.

3.2. Class I

Let us recall the factorization properties and the recurrences obtained for transposition t_{12-21} [14] which represents one transposition among a set of transpositions which has been denoted class I in the exhaustive classification given in the previous section. Let us also recall that this transformation corresponds, for $q = 4$, to *integrable mappings* and yields a foliation of CP_{15} in *algebraic elliptic curves* given as intersections of quadrics [16].

Let us first consider the successive matrices obtained by iteration of the homogeneous transformation K, associated with t_{12-21}, on generic $q \times q$ matrix R_q and their determinants:

$$M_0 = R_q, \quad M_1 = K(M_0), \quad f_1 = \det(M_0)$$

The determinant of matrix M_1 *remarkably factorizes* enabling to introduce a homogeneous polynomial f_2:

$$f_2 = \frac{\det(M_1)}{f_1^{q-3}} \tag{3.1}$$

Moreover, f_1^{q-4} also *factorizes in all the entries* of matrix $K(M_1)$, leading to introduce a "reduced" matrix M_2:

$$M_2 = \frac{K(M_1)}{f_1^{q-4}} \tag{3.2}$$

Again, $\det(M_2)$ factorizes enabling to define a new polynomial f_3:

$$f_3 = \frac{\det(M_2)}{f_1^3 \cdot f_2^{q-3}} \tag{3.3}$$

Calculating $K(M_2)$, one can see that $f_1^2 \cdot f_2^{q-4}$ factorizes in all the entries of this matrix $K(M_2)$, leading to a new matrix:

$$M_3 = \frac{K(M_2)}{f_1^2 \cdot f_2^{q-4}} \tag{3.4}$$

Again, its determinant factorizes $f_1^{q-1} \cdot f_2^3 \cdot f_3^{q-3}$, yielding the homogeneous polynomial f_4:

$$f_4 = \frac{\det(M_3)}{f_1^{q-1} \cdot f_2^3 \cdot f_3^{q-3}} \tag{3.5}$$

Calculating $K(M_3)$, one sees that $f_1^{q-2} \cdot f_2^2 \cdot f_3^{q-4}$ factorizes in all the entries of this matrix $K(M_3)$, leading to introduce a new matrix:

$$M_4 = \frac{K(M_3)}{f_1^{q-2} \cdot f_2^2 \cdot f_3^{q-4}} \tag{3.6}$$

The factorization properties are now stabilized and they reproduce similarly at any order n. Generally, for $n \geq 1$ and $q \geq 4$, one has the factorizations:

$$M_{n+3} = \frac{K(M_{n+2})}{f_n^{q-2}\, f_{n+1}^2\, f_{n+2}^{q-4}} \tag{3.7}$$

$$f_{n+3} = \frac{\det(M_{n+2})}{f_n^{q-1}\, f_{n+1}^3\, f_{n+2}^{q-3}} \tag{3.8}$$

giving the following relation *independent of* q:

$$\frac{K(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{f_n f_{n+1} f_{n+2} f_{n+3}} \tag{3.9}$$

Note that $K(M_{n+2})/\det(M_{n+2})$ is nothing but $\widehat{K}(M_{n+2})$.

This defines the (left) action of the homogeneous transformation K on matrices M_n and on the set of polynomials f_n. These polynomials are closely related to determinants of these matrices, and are actually the (generically) "optimal" factorizations corresponding to the iterations of the (left) action of K [14].

One can also introduce a right-action of K on the matrices M_n: the entries m_{ij} of M_0 are replaced by the corresponding entries of $K(M_0)$, i.e. $(K(M_0))_{ij}$ (and similarly for any algebraic expression of these entries such as the f_n's for instance). Amazingly, the *right-action* of K on the f_n's and the matrices M_n's yields a *remarkable factorization* of f_1 (and only f_1):

$$(f_n)_K = f_{n+1} \cdot f_1^{\mu_n} \tag{3.10}$$

and

$$(M_n)_K = M_{n+1} \cdot f_1^{\nu_n} \tag{3.11}$$

In order to relate the right and left action of K, one can also introduce the matrices $\widehat{M_n}$ which corresponds to n-times the left (or right) action of K on M_0:

$$\widehat{M}_n = K^n(M_0) = \Big(M_0\Big)_{K^n} \tag{3.12}$$

One has the following relations:

$$\widehat{M}_1 = K(M_0), \qquad \widehat{M}_2 = K(\widehat{M}_1) = f_1^{(q-4)} \cdot M_2, \cdots$$

$$\vdots$$

$$\begin{aligned}\widehat{M}_n &= K(\widehat{M}_{n-1}) \\ &= \left(f_1^{(q-1)^{n-4}} \cdot f_2^{(q-1)^{n-5}} \cdot f_3^{(q-1)^{n-6}} \cdots f_{n-4}^{(q-1)} \cdot f_{n-3}\right)^{(q-2)^3} \\ &\quad \cdot f_{n-2}^{(q-2)(q-3)} \cdot f_{n-1}^{q-4} \cdot M_n \end{aligned} \tag{3.13}$$

Denoting α_n the degree of the determinant of matrix M_n and β_n the degree of the polynomial f_n, one immediately gets from Eqs. (3.7), (3.8), (3.10) and (3.11) the following linear relations (with integer coefficients):

$$\begin{aligned} &\alpha_{n+2} = (q-1)\,\beta_n + 3\,\beta_{n+1} + (q-3)\,\beta_{n+2} + \beta_{n+3}, \\ &(q-1)\,\alpha_{n+2} = \alpha_{n+3} + q(q-2)\,\beta_n + 2q\,\beta_{n+1} + q(q-4)\,\beta_{n+2}, \\ &(q-1)\,\beta_n = \beta_{n+1} + q\,\mu_n, \\ &(q-1)\,\alpha_n = \alpha_{n+1} + q^2\,\nu_n \end{aligned} \tag{3.14}$$

Let us introduce $\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$ the generating functions of the α_n's, β_n's, μ_n's and ν_n's:

$$\begin{aligned} \alpha(x) &= \sum_{n=0}^{\infty} \alpha_n \cdot x^n, \qquad & \beta(x) &= \sum_{n=1}^{\infty} \beta_n \cdot x^n, \\ \mu(x) &= \sum_{n=1}^{\infty} \mu_n \cdot x^n, \qquad & \nu(x) &= \sum_{n=1}^{\infty} \nu_n \cdot x^n \end{aligned} \tag{3.15}$$

From the right-action of K (see factorizations (3.10) and (3.11)) one also gets linear relations on the α_n's, β_n's, μ_n's and ν_n's:

$$(q-1)\,\beta_n = \beta_{n+1} + q\,\mu_n, \quad (q-1)\,\alpha_n = \alpha_{n+1} + q^2\,\nu_n \tag{3.16}$$

as well as the corresponding linear relations on the generating functions:

$$\begin{aligned} ((q-1)x - 1) \cdot \beta(x) &= q\,x\,\mu(x) - q\,x, \\ ((q-1)x - 1) \cdot \alpha(x) &= q^2\,x\,\nu(x) - q \end{aligned} \tag{3.17}$$

The explicit expressions of these generating functions read respectively:

$$\alpha(x) = \frac{q\,(1 + (q-3)x + 3x^2 + (q-1)x^3)}{(1+x)(1-x)^3},$$

$$\beta(x) = \frac{q\,x}{(1+x)(1-x)^3},$$
$$\mu(x) = \frac{(q-3)+2x^2-x^3)}{(1+x)(1-x)^3},$$
$$\nu(x) = \frac{x\,((q-4)+2x+(q-2)x^2)}{(1+x)(1-x)^3} \tag{3.18}$$

giving on the α_n's, β_n's, μ_n's and ν_n's:

$$\alpha_n = q\Big(\frac{q\,n^2}{2}+\frac{q}{4}-\frac{(-1)^n q}{4}+(-1)^n\Big),$$
$$\beta_n = \frac{q}{8}\Big(2\,n\,(n+2)+1-(-1)^n\Big),$$
$$\mu_n = \beta_n - \frac{(n+1)\,(n+2)}{2},$$
$$\nu_n = \frac{\alpha_n}{q} - (1+n+n^2) \tag{3.19}$$

On the explicit expressions (3.19) of these degrees and exponents, one sees that the iteration of the homogeneous transformation K yields, as a consequence of factorizations (3.7), (3.11), ... a *polynomial growth* of the complexity of the calculations: the degree of all the homogeneous expressions appearing in the iterations (the entries of successive matrices M_n, their determinants, ...) grows like n^2. On the generating functions $\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$ (relations (3.18)) this corresponds to the fact that one *only has* $x = \pm 1$ *singularities.*

Another important consequence of these factorizations is to introduce the (optimal) homogeneous polynomials f_n. Remarkably, these polynomials *do verify, independently of q, a whole hierarchy of non-linear recurrences* [14] such as:

$$\frac{f_n f_{n+3}^2 - f_{n+4} f_{n+1}^2}{f_{n-1} f_{n+3} f_{n+4} - f_n f_{n+1} f_{n+5}} = \frac{f_{n-1} f_{n+2}^2 - f_{n+3} f_n^2}{f_{n-2} f_{n+2} f_{n+3} - f_{n-1} f_n f_{n+4}} \tag{3.20}$$

or for instance, among many other recurrences:

$$\frac{f_{n+1} f_{n+4}^2 f_{n+5} - f_{n+2} f_{n+3}^2 f_{n+6}}{f_{n+2}^2 f_{n+3} f_{n+7} - f_n f_{n+4} f_{n+5}^2} = \frac{f_{n+2} f_{n+5}^2 f_{n+6} - f_{n+3} f_{n+4}^2 f_{n+7}}{f_{n+3}^2 f_{n+4} f_{n+8} - f_{n+1} f_{n+5} f_{n+6}^2} \tag{3.21}$$

These recurrences are of course compatible with the linear recurrences on the β_n's (Eqs. (3.14)) and also with the right-action of K on the f_n's (see factorization (3.10)). Moreover these recurrences do have a *three parameter symmetry group.* Introducing the variables g_n which are the product of two consecutive polynomials f_n, $g_n = f_n \cdot f_{n+1}$, one can simply verify that all these recurrences are actually invariant under the three-parameter symmetry group:

$$g_n \;\rightarrow\; a^{n^2} \cdot b^n \cdot c \cdot g_n \tag{3.22}$$

It is therefore tempting to introduce new variables taking these symmetries into account (more precisely, variables *invariant* under this three-parameter group):

$$x_n \;=\; \frac{f_{n-1}^2\, f_{n+2}}{f_{n+1}^2\, f_{n-2}} \tag{3.23}$$

In fact, these variables can directly be obtained from the inhomogeneous transformation $\widehat{K}$ (see Section 5) and read:

$$x_n \;=\; \det\Big(\widehat{K}^n(R_q) \cdot \widehat{K}^{n+1}(R_q)\Big) \tag{3.24}$$

The equivalence of these two definitions for x_n, (3.23) and (3.24), has already been explained in [14] and will not be recalled here.

With these new variables, recurrence (3.20) becomes:

$$\frac{x_{n+2} - 1}{x_{n+1}\, x_{n+2}\, x_{n+3} - 1} \;=\; \frac{x_{n+1} - 1}{x_n\, x_{n+1}\, x_{n+2} - 1} \cdot x_n\, x_{n+2} \tag{3.25}$$

Similarly to the f_n's, one has a *whole hierarchy of recurrences* on the x_n's. The analysis of this hierarchy of *compatible* non-linear recurrences has been performed in [14] and will not be detailed here.

It is important to note that these recurrences can be extended to *any relative integer n*. Even more, *these recurrences are invariant under the "time-reversal" transformation*:

$$x_n \;\longrightarrow\; \frac{1}{x_{-n}} \tag{3.26}$$

This is a consequence of the fact that the transformations one considers, are *birational (hence reversible) transformations.*

All these factorizations and recurrences have been proved in [14], *even for arbitrary q.*

Moreover, it has been shown that these recurrences yield algebraic *elliptic curves* [16]. This can be shown relating them to *biquadratic* relations, introducing the (homogeneous) variables q_n:

$$q_n \;=\; \frac{f_n\, f_{n+3}}{f_{n+1}\, f_{n+2}} \tag{3.27}$$

Eq. (3.25) becomes the *biquadratic relation*:

$$(\rho - q_n - q_{n+1})(q_n\, q_{n+1} + \lambda) \;=\; \mu \tag{3.28}$$

where λ, ρ and μ are constants of integrations [4].

[4] Many examples of integrable mappings related to biquadratic elliptic curves have recently been obtained by several authors [22–25].

Let us also recall the finite order conditions for recurrence (3.25). Recalling the biquadratic relation (3.28) and in particular the three parameters λ, μ and ρ, recurrence (3.25) can be seen to yield finite order orbits, which can be written as algebraic conditions bearing alternatively on the x_n's, or the q_n's, or even on the three parameters λ, μ and ρ. These algebraic conditions are given in [14] for order four, five, six and seven. Let us, for instance, just recall here the conditions of order six respectively in the variables q_n, x_n or λ, μ and ρ:

$$-x_n\, x_{n+1}\, x_{n+2} - x_n\, x_{n+1}\, x_{n+2}^2 + x_n^2 x_{n+2}^2 x_{n+1} - 1 + x_{n+2} + x_n\, x_{n+2} = 0 \tag{3.29}$$

or

$$\lambda^3 + \rho^2\lambda^2 - 3\,\mu\lambda\rho + 2\,\mu^2 = 0 \tag{3.30}$$

or

$$\begin{aligned} &-q_n\, q_{n+2}\, q_{n+3} - q_n\, q_{n+3}^2 + q_{n+1}\, q_{n+3}^2 - q_n^2\, q_{n+2} + q_n^2\, q_{n+3} \\ &\quad + q_n\, q_{n+1}\, q_{n+3} = 0 \end{aligned} \tag{3.31}$$

The relations between these various properties and structures (factorization properties, existence of recurrences, integrability, ...) have been sketched in [14]. The fact that products of a *fixed number* of f_n's occur in relation (3.20) is related to the fact that products of a fixed number of f_n's also occur in the factorizations (3.7), (3.8). The polynomial growth of these iterations is, at first sight, in good agreement with a framework of products of fixed number of polynomials [5]. To some extend the integrability of the mappings, or more precisely the occurrence of (algebraic) elliptic curves, for arbitrary q, yield such *polynomial growth* of the iterations (see [17,26]).

Transformation K can be seen as a homogeneous transformation bearing on q^2 entries of the $q \times q$ matrix. For small values of q ($q = 3$, $q = 4$, $q = 5$, ...), one can actually look at the images of the iteration of K and see that these orbits yield *curves* [16]. For $q = 4$ it is possible to show that these curves are elliptic curves given as the *intersections of fourteen quadrics* in $\mathbb{C}P_{15}$ [16]. These quadrics can be obtained as "Plücker-like" well-suited sums and differences of 2×2 minors of the 4×4 matrix [16], in a very similar way as it occurs in the sixteen vertex model [6]. These elliptic curves have been seen to be closely related to biquadratic relations [16] which is not surprising recalling [22–25]. This situation *can probably be generalized to* $q \times q$ matrices, the elliptic curve in $\mathbb{C}P_{q^2-1}$ being now the intersection of $q^2 - 2$ algebraic expressions of higher degree [16] [6]. The relation between these elliptic curves and the elliptic curves

[5] However it will be shown in forthcoming publications that polynomial growth may occur even with more involved factorizations [15].

[6] Our proof of this statement for arbitrary q is not complete at the present moment.

associated with the recurrence on the f_n's or x_n's (see (3.20) and (3.25)) has been analyzed in detail in [14].

Let us finally mention that, for a given initial matrix M_0, the successive iterates of M_0 under transformation K^2 move in a *five-dimensional affine projective space*:

$$K^{2n}(M_0) = a_0^n \cdot M_0 + a_1^n \cdot P + a_2^n \cdot M_2 + a_3^n \cdot M_4 + a_4^n \cdot M_6 + a_5^n \cdot M_8 \tag{3.32}$$

$$K^{2n+1}(M_0) = b_0^n \cdot M_1 + b_1^n \cdot P + b_2^n \cdot M_3 + b_3^n \cdot M_5 + b_4^n \cdot M_7 + b_5^n \cdot M_9 \tag{3.33}$$

where matrix P is a fixed matrix, *independent of the initial matrix* M_0, of entries $P_{i,j} = \delta_{i,1} \cdot \delta_{j,2} - \delta_{i,2} \cdot \delta_{j,1}$. Considering the points in $\mathbb{C}P_{q^2-1}$ associated to the successive $q \times q$ matrices corresponding to the iteration of M_0 under transformation K (instead of K^2), one thus gets sets of points which belong to *two* five dimensional affine subspace of $\mathbb{C}P_{q^2-1}$, *which depend on the initial matrix* M_0. Fig. 1 gives one orbit corresponding to the iteration of transformation $\widehat{K}$ for a 5×5 matrices, that is in a 24-dimensional space. One has apparently a foliation of this 24-dimensional space in terms of elliptic curves.

4. The results for the five other classes

The analysis performed in Section 3.2 of the iteration of transformation K for the transposition t_{12-21} representing class I can similarly be performed for the five other classes.

4.1. Classes II

For $q = 4$, the analysis of the iteration of the homogeneous transformation K for the transpositions of class II *yield exactly the same factorizations* (and therefore the same generating functions $\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$) *as for class* I. However, *the homogeneous polynomials* f_n (see Eqs. (3.7) and (3.8)) *do not satisfy any simple recurrence* like (3.20). In fact, it will be shown in Section 5.2 that there actually exist recurrences on a finite set of variables which enable, after elimination, to get algebraic relations between two variables (namely x_n and another one). One does not have simple recurrences like (3.25) but still a quite (involved) algebraic relation on these variables. The orbits of K yield, for $q = 4$, *algebraic elliptic curves* which can be seen as intersections of fourteen "Plücker-like quadrics" in $\mathbb{C}P_{15}$ [16].

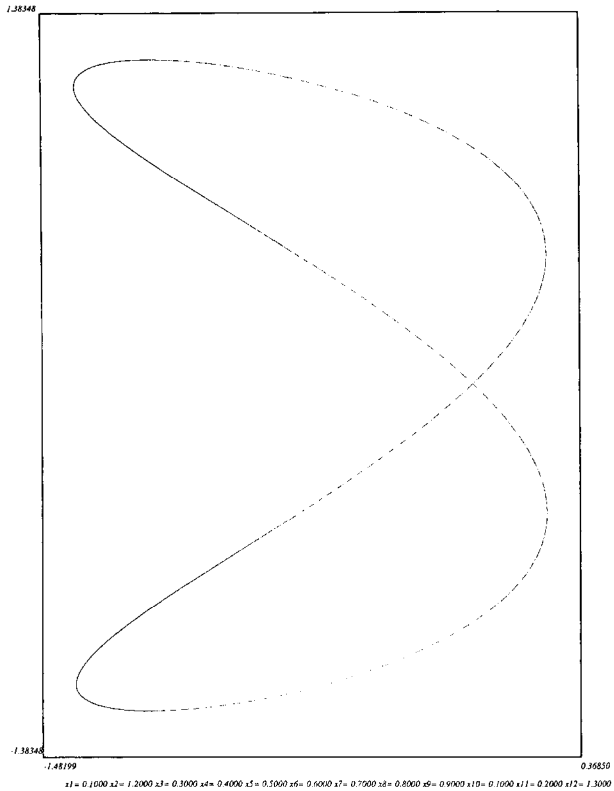

Fig. 1. Projection of the iteration of $\widehat{K}$, for class I, in a 24-dimensional space associated to 5×5 matrices.

For class II, the factorizations corresponding to the iterations of transformation K detailed in Section 3.2 (see Eqs. (3.1), (3.2), (3.3), (3.4), (3.5), ...) for class I , are drastically different, when $q \geq 5$, already after two iterations:

$$\begin{cases} f_1 = \det(M_0) \\ M_1 = K(M_0) \end{cases} \qquad \begin{cases} f_2 = \det(M_1)/f_1^{q-3} \\ M_2 = K(M_1)/f_1^{q-4} \end{cases}$$

$$\begin{cases} f_3 = \det(M_2)/(f_1^2 \cdot f_2^{q-3}) \\ M_3 = K(M_2)/(f_1 \cdot f_2^{q-4}) \end{cases} \qquad \begin{cases} f_4 = \det(M_3)/(f_1^{q-2} \cdot f_2^2 \cdot f_3^{q-3}) \\ M_4 = K(M_3)/(f_1^{q-3} \cdot f_2 \cdot f_3^{q-4}) \end{cases}$$

$$f_5 = \frac{\det(M_4)}{f_1^3 \cdot f_2^{q-2} \cdot f_3^2 \cdot f_4^{q-3}}, \qquad M_5 = \frac{K(M_4)}{f_1^2 \cdot f_2^{q-3} \cdot f_3 \cdot f_4^{q-4}},$$

$$f_6 = \frac{\det(M_5)}{f_1^{q-3} \cdot f_2^3 \cdot f_3^{q-2} \cdot f_4^2 \cdot f_5^{q-3}}, \qquad M_6 = \frac{K(M_5)}{f_1^{q-4} \cdot f_2^2 \cdot f_3^{q-3} \cdot f_4 \cdot f_5^{q-4}},$$

$$f_7 = \frac{\det(M_6)}{f_1^2 \cdot f_2^{q-3} \cdot f_3^3 \cdot f_4^{q-2} \cdot f_5^2 \cdot f_6^{q-3}},$$
$$M_7 = \frac{K(M_6)}{f_1 \cdot f_2^{q-4} \cdot f_3^2 \cdot f_4^{q-3} \cdot f_5 \cdot f_6^{q-4}} \tag{4.1}$$

yielding the following factorizations for arbitrary n:

$$\det(M_n) = f_{n+1} \cdot (f_n^{q-3} \cdot f_{n-1}^2 \cdot f_{n-2}^{q-2} \cdot f_{n-3}^3) \cdot (f_{n-4}^{q-3} \cdot f_{n-5}^2 \cdot f_{n-6}^{q-2} \cdot f_{n-7}^3) \cdots f_1^{\delta_n} \tag{4.2}$$

where δ_n depends on the truncation, and

$$K(M_n) = M_{n+1} \cdot (f_n^{q-4} \cdot f_{n-1} \cdot f_{n-2}^{q-3} \cdot f_{n-3}^2) \cdot (f_{n-4}^{q-4} \cdot f_{n-5} \cdot f_{n-6}^{q-3} \cdot f_{n-7}^2) \cdots f_1^{\zeta_n} \tag{4.3}$$

where $\zeta_n = q-4$ for $n = 1 \pmod 4$, $\zeta_n = 1$ for $n = 2 \pmod 4$, $\zeta_n = q-3$ for $n = 3 \pmod 4$ and $\zeta_n = 2$ for $n = 0 \pmod 4$.

For factorization (4.3), one has periodically (with period four) the sequence [(q-4)(1)(q-3)(2)] for the exponents of the f_n's of the "string-like" factor in the right-hand side of (4.3), while for factorization (4.2), one has periodically (again with period four) the sequence [(q-3)(2)(q-2)(3)] for the exponents of the f_n's of the "string-like" factor in the right-hand side of relation (4.2).

One notes that the following factorization *independent of* q occurs, which is actually different from relation (3.9):

$$\frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{f_1 f_2 \cdots f_n f_{n+1}} \tag{4.4}$$

These factorizations (4.2) and (4.4) yield linear recurrences on the α_n's and β_n's:

$$\alpha_n = \beta_{n+1} + (q-3)\beta_n + 2\beta_{n-1} + (q-2)\beta_{n-2} + 3\beta_{n-3} + (q-3)\beta_{n-4} + 2\beta_{n-5} + (q-2)\beta_{n-6} + 3\beta_{n-7} + \cdots + \delta_n \beta_1 \tag{4.5}$$

and

$$q\,(\beta_1 + \beta_2 + \cdots + \beta_{n+1}) = \alpha_n + \alpha_{n+1} \tag{4.6}$$

From relation (4.5), one gets on the generating functions $\alpha(x)$ and $\beta(x)$:

$$x\,\alpha(x) = \beta(x) \cdot \left(1 + \frac{(q-3)\,x + 2\,x^2 + (q-2)\,x^3 + 3\,x^4}{1 - x^4}\right) \tag{4.7}$$

and from relation (4.6):

$$\frac{q\,\beta(x)}{(1-x)} = (1+x) \cdot \alpha(x) - q \tag{4.8}$$

The generating functions $\alpha(x)$ and $\beta(x)$ read:

$$\alpha(x) = \frac{q\left(1 + 2x^4 + xq - 3x + 2x^2 + x^3 q - 2x^3\right)}{(1-x)(1+x)(1-2x-2x^3)},$$

$$\beta(x) = \frac{qx\,(1+x^2)}{1-2x-2x^3} \tag{4.9}$$

Again (see Eqs. (3.10) ...) the *right* action of K on the f_n's and on matrices M_n's *factorizes* f_1 *and only* f_1:

$$(f_n)_K = f_{n+1} \cdot f_1^{\mu_n} \quad \text{and} \quad (M_n)_K = M_{n+1} \cdot f_1^{\nu_n} \tag{4.10}$$

One deduces again, from factorizations (4.10), the linear recurrences (3.16) and relations (3.17) on the generating functions. The generating functions $\mu(x)$ and $\nu(x)$ read respectively:

$$\mu(x) = \frac{x\,\left((q-3) - x + (q-3)\,x^2\right)}{1-2x-2x^3},$$

$$\nu(x) = \frac{x\,\left(q - 4 + x + (q-3)\,x^2 + 2x^3\right)}{(1-x)\,(1+x)\left(1-2x-2x^3\right)} \tag{4.11}$$

From Eqs. (4.9) and (4.11), it is clear that one has an *exponential growth* of exponents α_n, β_n, μ_n and ν_n. They grow like λ^n where $\lambda = 2.359304086$ ... is the largest root of $2 + 2z^2 - z^3$.

Let us underline that, for a given initial matrix M_0, the successive iterates of M_0 under transformation K^2 move, in a *three-dimensional affine matrix space*:

$$K^{2n}(M_0) = a_0^n \cdot M_0 + a_1^n \cdot P + a_2^n \cdot M_2 + a_3^n \cdot M_4 \tag{4.12}$$

$$K^{2n+1}(M_0) = b_0^n \cdot M_1 + b_1^n \cdot P + b_2^n \cdot M_3 + b_3^n \cdot M_5 \tag{4.13}$$

where matrix P is a fixed matrix representing the transposition of class II one considers (here $m_{1,2} - m_{3,4}$), *independent of the initial matrix* M_0, of entries $P_{i,j} = \delta_{i,1} \cdot \delta_{j,2} - \delta_{i,3} \cdot \delta_{j,4}$.

Figs. 2a and 2b show (the projection of) two orbits corresponding to the iteration of a transformation of class II for 5×5 matrices. For class II, though one often gets curves (similar to Fig. 1), one sees, with Fig. 2b for instance, that some orbits may lie on higher dimensional varieties [7].

Moreover, a careful look at Fig. 2b, shows the occurrence of a "small island" in a quite uniform density of points. This situation is reminiscent of the one encountered with the Henon-Heiles mappings or the "almost" integrable mappings [16,27].

The drastically different behavior, one encounters, for $q = 4$ and for $q > 4$, shows that the q-generalization of transformation K (we introduce in Section 3.2) is certainly non-trivial.

[7] This gives a strong indication that, when one gets curves, these curves are not algebraic.

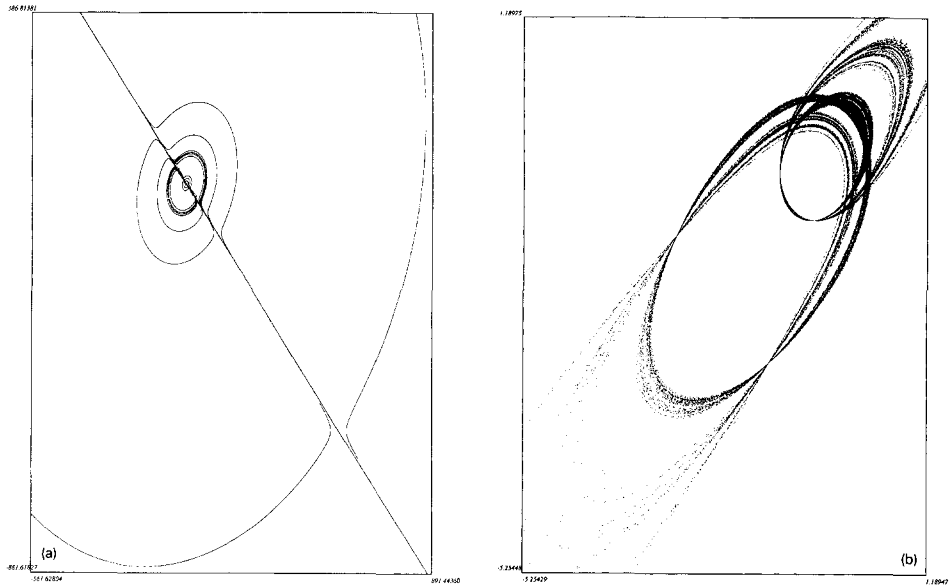

Fig. 2. (a) Projection of the iteration of $\widehat{K}$, for class II, in a 24-dimensional space associated to 5×5 matrices. (b) Another orbit corresponding to the iteration of $\widehat{K}$, for class II, in a 24-dimensional space.

4.2. Class III

Remarkably, the analysis of the iteration of the homogeneous transformation K for the transpositions of class III *yield exactly the same factorizations for arbitrary* q (and therefore the same generating functions $\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$) *as for class* I (see Section 3.2). However, *the homogeneous polynomials* f_n (see Eqs. (3.7) and (3.8)) *do not satisfy any simple recurrence* like (3.20). In fact, it will be shown in Section 5.2 that there actually exist recurrences on a finite set of variables which enable, after elimination, to get algebraic relations between two variables (namely x_n and another one), and finally on a *single variable.* The elimination happens to be less involved than for class II. Nevertheless one does not have simple recurrences like (3.25) but still a quite (involved) algebraic relation on these variables.

Again, the orbits of K yield, for $q = 4$, *algebraic elliptic curves* which can be seen as intersections of fourteen "Plücker-like quadrics" in $\mathbb{C}P_{15}$ [16].

As for class I, the successive iterates of a given initial matrix M_0, under transformation K^2, move in a *five-dimensional affine projective space*:

$$K^{2n}(M_0) = a_0^n \cdot M_0 + a_1^n \cdot P + a_2^n \cdot M_2 + a_3^n \cdot M_4 + a_4^n \cdot M_6 + a_5^n \cdot M_8 \tag{4.14}$$

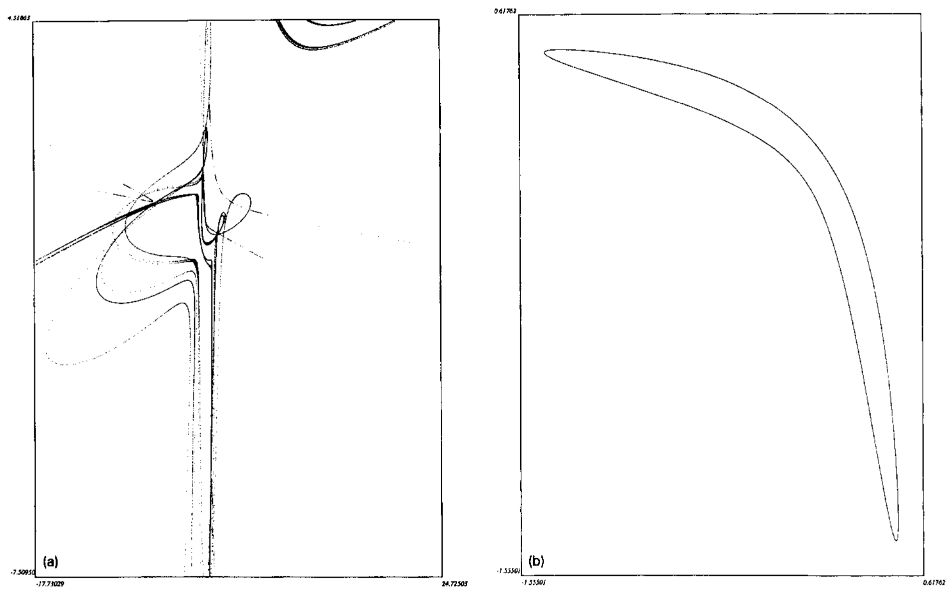

Fig. 3. (a) Iteration of $\widehat{K}$, for class III, for 5 × 5 matrices. (b) Another orbit corresponding to the iteration of $\widehat{K}$, for class III, for 5 × 5 matrices.

$$K^{2n+1}(M_0) = b_0^n \cdot M_1 + b_1^n \cdot P + b_2^n \cdot M_3 + b_3^n \cdot M_5 + b_4^n \cdot M_7 + b_5^n \cdot M_9 \tag{4.15}$$

where matrix P is a fixed matrix representing the transposition of class III one considers.

Figs. 3a and 3b show (the projection of) two orbits corresponding to the iteration of a transformation of class III for 5 × 5 matrices.

Very often the iteration of a transformation of class III for 5 × 5 matrices yields curves similar to Fig. 1. Fig. 3b *however looks like a set of curves lying on a surface.* This seems to rule out, for class III, a foliation of the 24-dimensional parameter space in curves, but it does not rule out the fact that these orbits could be algebraic surfaces or "nice" higher dimensional algebraic varieties, like abelian varieties (which is suggested by the polynomial growth).

The occurrence of integrable recurrences, independent of q, on the f_n's associated with algebraic elliptic curves probably explains the better regularity of mappings of class I, compared to mappings of class III.

4.3. Class IV

The factorizations corresponding to the iterations of transformation K detailed in Section 3.2 (see Eqs. (3.1), (3.2), (3.3), (3.4), (3.5), ...) for class I (and also, for classes II and III), now read for class IV:

$$\begin{cases} f_1 = \det(M_0) \\ M_1 = K(M_0) \end{cases} \qquad \begin{cases} f_2 = \det(M_1)/f_1^{q-2} \\ M_2 = K(M_1)/f_1^{q-3} \end{cases}$$

$$\begin{cases} f_3 = \det(M_2)/(f_1 \cdot f_2^{q-2}) \\ M_3 = K(M_2)/f_2^{q-3} \end{cases} \qquad \begin{cases} f_4 = \det(M_3)/(f_1^{q-1} \cdot f_2 \cdot f_3^{q-2}) \\ M_4 = K(M_3)/(f_1^{q-2} \cdot f_3^{q-3}) \end{cases}$$

$$f_5 = \frac{\det(M_4)}{f_1^2 \cdot f_2^{q-1} \cdot f_3 \cdot f_4^{q-2}}, \quad M_5 = \frac{K(M_4)}{f_1 \cdot f_2^{q-2} \cdot f_4^{q-3}},$$
$$f_6 = \frac{\det(M_5)}{f_1^{q-2} \cdot f_2^2 \cdot f_3^{q-1} \cdot f_4 \cdot f_5^{q-2}}, \quad M_6 = \frac{K(M_5)}{f_1^{q-3} \cdot f_2 \cdot f_3^{q-2} \cdot f_5^{q-3}},$$
$$f_7 = \frac{\det(M_6)}{f_1 \cdot f_2^{q-2} \cdot f_3^2 \cdot f_4^{q-1} \cdot f_5 \cdot f_6^{q-2}}, \quad M_7 = \frac{K(M_6)}{f_2^{q-3} \cdot f_3 \cdot f_4^{q-2} \cdot f_6^{q-3}} \tag{4.16}$$

yielding the following factorizations for arbitrary n:

$$\det(M_n) = f_{n+1} \cdot (f_n^{q-2} \cdot f_{n-1} \cdot f_{n-2}^{q-1} \cdot f_{n-3}^2) \cdot (f_{n-4}^{q-2} \cdot f_{n-5} \cdot f_{n-6}^{q-1} \cdot f_{n-7}^2) \cdots f_1^{\delta_n} \tag{4.17}$$

where δ_n depends on the truncation, and

$$K(M_n) = M_{n+1} \cdot (f_n^{q-3} \cdot f_{n-2}^{q-2} \cdot f_{n-3}) \cdot (f_{n-4}^{q-3} \cdot f_{n-6}^{q-2} \cdot f_{n-7}) \cdot (f_{n-8}^{q-3} \cdot f_{n-10}^{q-2} \cdot f_{n-11}) \cdots f_1^{\zeta_n} \tag{4.18}$$

where $\zeta_n = q-3$ for $n = 1 \pmod 4$, $\zeta_n = 0$ for $n = 2 \pmod 4$, $\zeta_n = q-2$ for $n = 3 \pmod 4$ and $\zeta_n = 1$ for $n = 0 \pmod 4$.

For factorization (4.18), one has periodically (with period four) the sequence $[(q-3)(0)(q-2)(1)]$ for the exponents of the f_n's of the "string-like" factor in the right-hand side of (4.18), while for factorization (4.17), one has periodically (again with period four) the sequence $[(q-2)(1)(q-1)(2)]$ for the exponents of the f_n's of the "string-like" factor in the right-hand side of (4.17).

One notes that the following factorization *independent of* q occurs, which is different from relation (3.9), but actually identifies with relation (4.4):

$$\frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{f_1 f_2 \cdots f_n f_{n+1}} \tag{4.19}$$

These factorizations (4.17) and (4.19) yield linear recurrences on the α_n's and β_n's:

$$\alpha_n = \beta_{n+1} + (q-2)\,\beta_n + \beta_{n-1} + (q-1)\,\beta_{n-2} + 2\,\beta_{n-3} + (q-2)\,\beta_{n-4}$$
$$+\beta_{n-5} + (q-1)\,\beta_{n-6} + 2\,\beta_{n-7} + \cdots + \delta_n\,\beta_1 \qquad (4.20)$$

and

$$q\,(\beta_1 + \beta_2 + \cdots + \beta_{n+1}) = \alpha_n + \alpha_{n+1} \qquad (4.21)$$

From relation (4.20), one gets on the generating functions $\alpha(x)$ and $\beta(x)$:

$$x\,\alpha(x) = \beta(x)\cdot\left(1 + \frac{(q-2)\,x + x^2 + (q-1)\,x^3 + 2\,x^4}{1-x^4}\right) \qquad (4.22)$$

and from relation (4.21) one recovers relation (4.8):

$$\frac{q\,\beta(x)}{(1-x)} = (1+x)\cdot\alpha(x) - q \qquad (4.23)$$

The generating functions $\alpha(x)$ and $\beta(x)$ read:

$$\alpha(x) = \frac{q\,(1 + x^4 + xq - 2x + x^2 + x^3q - x^3)}{(1-x)(1+x)(1-x-x^3)}, \quad \beta(x) = \frac{q\,x\,(1+x^2)}{1-x-x^3} \qquad (4.24)$$

Again (see Eqs. (3.10), (4.10) ...) the *right* action of K on the f_n's and on matrices M_n's *factorizes* f_1 *and only* f_1:

$$(f_n)_K = f_{n+1}\cdot f_1^{\mu_n} \quad \text{and} \quad (M_n)_K = M_{n+1}\cdot f_1^{\nu_n} \qquad (4.25)$$

One deduces again, from factorizations (4.25), the linear recurrences (3.16) and the relations (3.17) on the generating functions. The generating functions $\mu(x)$ and $\nu(x)$ read respectively:

$$\mu(x) = \frac{x\left((q-2)\,(1+x^2) - x\right)}{1-x-x^3},$$
$$\nu(x) = \frac{x\left(q - 3 + (q-2)\,x^2 + x^3\right)}{(1-x)\,(1+x)\,\left(1-x-x^3\right)} \qquad (4.26)$$

From Eqs. (4.24) and (4.26), it is clear that one has an *exponential growth* of exponents α_n, β_n, μ_n and ν_n. They grow like λ^n where $\lambda = 1.465571226$ ... is the largest root of $z^3 - z^2 - 1$. One remarks that some homogeneous polynomials, similar to the numerators, or denominators appearing in recurrences like (3.20), do *satisfy some additional factorization properties*:

$$(f_4 - f_2f_3),\quad (f_5 - f_2f_4),\quad (f_6 - f_3f_5),\quad (f_7 - f_4f_6),\quad (f_8 - f_5f_7)$$
$$(f_1f_5 - f_4f_3),\quad (f_2f_6 - f_5f_4),\quad (f_3f_7 - f_6f_5f_1),\quad (f_4f_8 - f_7f_6f_2),\cdots$$
$$(f_2f_6f_{10}f_{14} - f_4f_8f_{12}f_{13}),\cdots \qquad (4.27)$$

The polynomials f_n for class IV not only satisfy this additional factorization but *actually satisfy, for arbitrary q, exact relations* where the new polynomials (4.27) play a key role:

$$
\begin{aligned}
&(-f_6 + f_3\, f_5)\,(-f_3 + f_1\, f_2) + (f_1\, f_5 - f_4\, f_3)\,(-f_4 + f_1\, f_3) = 0,\\
&f_3\, f_7\,(-f_{15} + f_{12}\, f_{14})\,(-f_4\, f_8\, f_{12} + f_2\, f_6\, f_{10}\, f_{11})\\
&\quad + f_1\, f_5\, f_9\,(f_2\, f_6\, f_{10}\, f_{14} - f_4\, f_8\, f_{13}\, f_{12})\,(-f_{13} + f_{10}\, f_{12}) = 0\\
&\vdots
\end{aligned}
$$

In fact, the f_n's do not satisfy simple recurrences like (3.20), *but "pseudo-recurrences", where products from f_1 to f_n occur.* One of these "pseudo-recurrences" can be written as follows:

$$
\begin{aligned}
&\frac{(f_{n+2} - f_{n-1} f_{n+1})}{(f_n - f_{n-3} f_{n-1})} \cdot \frac{f_{n-6}\, f_{n-10}\, f_{n-14}\, \cdots}{f_{n-4}\, f_{n-8}\, f_{n-12}\, \cdots}\\
&= \frac{f_n\,(f_{n-1}\, f_{n-5}\, f_{n-9}\, \cdots) - (f_{n+1}\, f_{n-3}\, f_{n-7}\, \cdots)}{f_{n-2}\,(f_{n-3}\, f_{n-7}\, f_{n-11}\, \cdots) - (f_{n-1}\, f_{n-5}\, f_{n-9}\, \cdots)}
\end{aligned}
\tag{4.28}
$$

The polynomials occurring in the numerator and the denominator of the "pseudo-recurrence" (4.28) suggests the following recurrence on the β_n's:

$$
\beta_{n+3} - \beta_n - \beta_{n+2} = 0 \tag{4.29}
$$

This recurrence would have suggested, since the beginning, a $1 - x - x^3 = 0$ singularity (see relations (4.24) and (4.26)).

Though, one does not have recurrences on the f_n's but pseudo-recurrences such as (4.28), the previous variables x_n (see (3.24)), which can always be defined, *remarkably satisfy very simple recurrences* (see the demonstration in Section 5.2). As for class I, the recurrences on the x_n's are *independent of q*: this independence will be understood in Section 5.2. One of these recurrences reads:

$$
\frac{x_{n+3} - 1}{x_{n+2}\, x_{n+4} - 1} = \frac{x_{n+1} - 1}{x_n\, x_{n+2} - 1} \cdot x_n\, x_{n+3} \tag{4.30}
$$

Studying the iteration of $\widehat{K}$ in the $q^2 - 1$-dimensional space $\mathbb{C}P_{q^2-1}$, one can show that these orbits actually belong to remarkable two dimensional subvarieties (given by intersection of quadrics in $\mathbb{C}P_{15}$ [16]), namely *planes* (see Section 5.2.3 in the following and see also [16]). Inside these planes, *which depend on the initial point* in the $q^2 - 1$-dimensional space (that is the initial matrix), the orbits look like curves for many of the trajectories (see [16]).

4.4. Class V

The factorizations corresponding to the iterations of transformation K read for class V:

$$\begin{cases} f_1 = \det(M_0) \\ M_1 = K(M_0) \end{cases} \quad \begin{cases} f_2 = \det(M_1)/f_1^{q-3} \\ M_2 = K(M_1)/f_1^{q-4} \end{cases}$$

$$\begin{cases} f_3 = \det(M_2)/(f_1 \cdot f_2^{q-3}) \\ M_3 = K(M_2)/(f_2^{q-4}) \end{cases} \quad \begin{cases} f_4 = \det(M_3)/(f_1^{q-1} \cdot f_2 \cdot f_3^{q-3}) \\ M_4 = K(M_3)/(f_1^{q-2}\, f_3^{q-4}) \end{cases}$$

The factorizations are now stabilized, yielding for arbitrary n:

$$\det(M_{n+2}) = f_n^{q-1} \cdot f_{n+1} \cdot f_{n+2}^{q-3} \cdot f_{n+3} \tag{4.31}$$

$$K(M_{n+2}) = f_n^{q-2} \cdot f_{n+2}^{q-4} \cdot M_{n+3} \tag{4.32}$$

One notes that again, *as well as for classes* I and III (see Eq. (3.9)), the following factorizations, *independent of* q, occur:

$$\frac{K(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{f_n f_{n+1} f_{n+2} f_{n+3}} \tag{4.33}$$

Factorizations (4.31) and (4.32) yield linear recurrences on the α_n's and β_n's:

$$\alpha_{n+2} = (q-1)\,\beta_n + \beta_{n+1} + (q-3)\,\beta_{n+2} + \beta_{n+3} \tag{4.34}$$

$$(q-1)\,\alpha_{n+2} = \alpha_{n+3} + q\,(q-2)\,\beta_n + q\,(q-4)\,\beta_{n+2} \tag{4.35}$$

The two generating functions $\alpha(x)$ and $\beta(x)$ read:

$$\alpha(x) = \frac{q\,(1 + (q-3)\,x + x^2 + (q-1)\,x^3)}{(1+x)(1-3x+x^2-x^3)},$$
$$\beta(x) = \frac{q\,x}{(1+x)(1-3x+x^2-x^3)} \tag{4.36}$$

Remarkably, similarly to what happened for classes I, II, III and IV , the right-action of K on the f_n's, or the M_n's, factorizes f_1 and only f_1: the factorizations (3.10), (3.11), the linear relations (3.16) on the exponents α_n, β_n, μ_n and ν_n, as well as the linear relations (3.17) on the generating functions, *are still valid for* class V. The two generating functions $\mu(x)$ and $\nu(x)$ read:

$$\mu(x) = \frac{x\,((q-3) - 2\,x - x^3)}{(1+x)(1-3\,x+x^2-x^3)},$$

$$\nu(x) = \frac{x\,(q-4+(q-2)x^2)}{(1+x)(1-3x+x^2-x^3)} \tag{4.37}$$

One notes that the *roots of the denominator* of $\alpha(x)$, $\beta(x)$, $\mu(x)$ and $\nu(x)$ are *not on the unit circle.* Thus one has an *exponential growth of the complexity* of the calculation since the degree of all the polynomials one deals with (that is the exponents α_n, β_n, μ_n and ν_n) *grow exponentially* with n, like λ^n with $\lambda = 2.769292354\ ...$ [8]. For instance, expanding $\beta(x)$, one gets:

$$\beta(x) = qx + 2qx^2 + 6qx^3 + 16qx^4 + 45qx^5 + 124qx^6 + 344qx^7 \\ +952qx^8 + 2637qx^9 + \cdots$$

On this example, one sees that it is possible to *have factorizations involving products of a fixed number of polynomials f_n* and, in the same time, an *exponential growth of the calculations* of the iterations.

Again, one can study the iteration of $\widehat{K}$ seen as a birational transformation in $\mathbb{C}P_{q^2-1}$. These orbits look like *curves* in some domain of $\mathbb{C}P_{q^2-1}$ [16]. For $q = 4$, these orbits can be seen to lie on a subvariety which is the intersection of *twelve* Plücker-like quadrics in $\mathbb{C}P_{15}$ and, more generally, at most, $q^2 - 4$ algebraic expressions for $q \times q$ matrices [16].

4.5. Class VI

For class VI, the factorizations corresponding to the iterations of the homogeneous transformation K read as follows:

$$\begin{cases} f_1 = \det(M_0) \\ M_1 = K(M_0) \end{cases} \quad \begin{cases} f_2 = \det(M_1)/f_1^{q-2} \\ M_2 = K(M_1)/f_1^{q-3} \end{cases}$$

$$\begin{cases} f_3 = \det(M_2)/(f_1 \cdot f_2^{q-2}) \\ M_3 = K(M_2)/(f_2^{q-3}) \end{cases} \quad \begin{cases} f_4 = \det(M_3)/(f_1^{q-2} \cdot f_2 \cdot f_3^{q-2}) \\ M_4 = K(M_3)/(f_1\, f_3)^{q-3} \end{cases}$$

$$f_5 = \frac{\det(M_4)}{f_1 \cdot f_2^{q-2} \cdot f_3 \cdot f_4^{q-2}}, \quad M_5 = \frac{K(M_4)}{(f_2\, f_4)^{q-3}},$$

$$f_6 = \frac{\det(M_5)}{f_1^{q-2} \cdot f_2 \cdot f_3^{q-2} \cdot f_4 \cdot f_5^{q-2}}, \quad M_6 = \frac{K(M_5)}{(f_1\, f_3\, f_5)^{q-3}}, \ \cdots$$

yielding the following *"string-like" factorizations for arbitrary n*:

[8] This value of λ is the largest root of $P(z) = -1 + z - 3z^2 + z^3$ (let us note the change $x \to 1/z$).

$$K(M_n) = M_{n+1} \cdot (f_n \cdot f_{n-2} \cdot f_{n-4} \cdot f_{n-6} \cdots f_{\xi_n})^{q-3} \tag{4.38}$$

where $\xi_n = 1$ for n odd and $\xi_n = 2$ for n even.

$$\det(M_n) = f_{n+1} \cdot f_n^{q-2} \cdot f_{n-1} \cdot f_{n-2}^{q-2} \cdot f_{n-3} \cdot f_{n-4}^{q-2} \cdots f_1^{\zeta_n} \tag{4.39}$$

where $\zeta_n = 1$ for n even and $\zeta_n = q-2$ for n odd [9].

Eqs. (4.38) and (4.39) yield the following simple "string-like" relation *independent of* q, which amazingly happens to be *the same relation as for class* IV (see Eq. (4.19)) and the same relation as for class II for $q \geq 5$ (see factorization (4.4)):

$$\frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{f_1 \cdot f_2 \cdots f_n \cdot f_{n+1}} \tag{4.40}$$

In fact, one notices the occurrence of "string-like" factorization relations (like (4.39) or (4.40)), instead of factorizations *with a fixed number of polynomials* (see relations (3.10), (3.11), or (3.8)), for the two classes IV and VI, for which the transposition *permutes entries belonging to the same column or to the same row*, but also for class II (for $q \geq 5$), the transposition of which involving two rows and two columns.

Eqs. (4.39) and (4.40) also yield the following linear recurrences on the α_n's and β_n's:

$$\alpha_n = \beta_{n+1} + (q-2)\,\beta_n + \beta_{n-1} \\ + (q-2)\,\beta_{n-2} + \beta_{n-3} + (q-2)\,\beta_{n-4} + \cdots + \zeta_n\,\beta_1 \tag{4.41}$$

and

$$q\,(\beta_1 + \beta_2 + \cdots + \beta_{n+1}) = \alpha_n + \alpha_{n+1} \tag{4.42}$$

Introducing the "odd" and "even" generating functions $\alpha_{odd}(x)$, $\alpha_{even}(x)$ and $\beta_{odd}(x)$, $\beta_{even}(x)$:

$$\alpha_{odd}(x) = \alpha_1\,x + \alpha_3\,x^3 + \alpha_5\,x^5 + \cdots,$$
$$\alpha_{even}(x) = \alpha_0 + \alpha_2\,x^2 + \alpha_4\,x^4 + \cdots$$

and similarly:

$$\beta_{odd}(x) = \beta_1\,x + \beta_3 x^3 + \beta_5 x^5 + \cdots,$$
$$\beta_{even}(x) = \beta_0 + \beta_2 x^2 + \beta_4 x^4 + \cdots$$

One deduces from (4.41) the following relations on the partial generating functions $\alpha_{even}(x)$ and $\alpha_{odd}(x)$, $\beta_{even}(x)$ and $\beta_{odd}(x)$:

[9] Note that one has no factorization for $q = 3$ for $K(M_n)$.

$$\alpha_{even}(x) = \frac{1}{1-x^2} \cdot \left((q-2) \quad \beta_{even}(x) + \frac{\beta_{odd}(x)}{x}) \right) \tag{4.43}$$

$$\alpha_{odd}(x) = \frac{1}{1-x^2} \cdot \left((q-2) \quad \beta_{odd}(x) + \frac{\beta_{even}(x)}{x}) \right) \tag{4.44}$$

yielding on the generating functions $\alpha(x) = \alpha_{even}(x) + \alpha_{odd}(x)$ and $\beta(x) = \beta_{even}(x) + \beta_{odd}(x)$:

$$\alpha(x) = \frac{1}{1-x^2} \cdot \left((q-2) \quad \beta(x) + \frac{\beta(x)}{x}) \right) \tag{4.45}$$

One also recovers (4.8) or (4.23) from relation (4.42):

$$\frac{q\,\beta(x)}{(1-x)} = (1+x) \cdot \alpha(x) - q \tag{4.46}$$

Then, the generating functions $\alpha(x)$ and $\beta(x)$ read:

$$\alpha(x) = \frac{q\,(1+(q-2)\,x)}{(1+x)\,(1-2x)}, \quad \beta(x) = \frac{q\,x\,(1-x)}{1-2x} \tag{4.47}$$

Similarly to what happened for all the other classes, the *right-action of K* on the f_n's, or the M_n's, factorizes f_1 *and only* f_1: the factorizations (3.10), (3.11), the linear relations on the exponents α_n, β_n, μ_n and ν_n (3.16) as well as the linear relations on the generating functions (3.17) *are still valid for* class VI. In particular one still has the two functional relations:

$$\begin{aligned} ((q-1)\,x-1) \cdot \beta(x) &= q\,x\,\mu(x) - q\,x, \\ ((q-1)\,x-1) \cdot \alpha(x) &= q^2\,x\,\nu(x) - q \end{aligned} \tag{4.48}$$

yielding the following expressions for $\mu(x)$ and $\nu(x)$:

$$\mu(x) = \frac{(q-2-(q-1)\,x)\,x}{1-2x}, \quad \nu(x) = \frac{(q-3)\,x}{(1+x)\,(1-2x)} \tag{4.49}$$

Since $z = 1/x = 2$ is the only root of all these generating functions which is not on the unit circle α_n, β_n, μ_n and ν_n clearly *grow exponentially* like 2^n. For instance $\beta(x)$ reads:

$$\beta(x) = q\,x \left(1 + \sum_{n=0}^{\infty} 2^n x^{n+1} \right) \tag{4.50}$$

Let us also note, for example, that $\mu_{n+1} = 2\,\mu_n$ (for $n \geq 2$). The fact that "string-like" factorizations occur is, at first sight, not compatible with the existence of simple recurrences on the f_n's like (3.20) where products of a fixed number of f_n's occur. Actually, we have not been able to find any simple recurrences on the f_n's [10]. One should however note the following point: expressions, similar to the numerators or denominators appearing in recurrences like (3.20), do *satisfy some nice additional factorization properties*, which enable to introduce new polynomials $f_n^{(1)}$, $f_n^{(2)}$, $f_n^{(3)}$, $f_n^{(4)}$:

$$
\begin{aligned}
f_4^{(1)} &= (f_4 - f_3^2), \quad f_5^{(1)} = (f_5 - f_4^2)/f_1, \quad f_6^{(1)} = (f_6 - f_5^2)/(f_1 f_2), \\
f_7^{(1)} &= (f_7 - f_6^2)/(f_1^2 f_2 f_3), \quad f_8^{(1)} = (f_8 - f_7^2)/(f_1^2 f_2^2 f_3 f_4), \\
f_5^{(2)} &= (f_5 - f_4 f_3^2), \quad f_6^{(2)} = (f_6 - f_5 f_4^2)/f_1, \\
f_7^{(2)} &= (f_7 - f_6 f_5^2)/(f_1 f_2), \quad f_8^{(2)} = (f_8 - f_7 f_6^2)/(f_1^2 f_2 f_3), \\
f_6^{(3)} &= (f_6 - f_4^4)/f_1, \quad f_7^{(3)} = (f_7 - f_5^4)/(f_1 f_2), \\
f_8^{(3)} &= (f_8 - f_6^4)/(f_1^2 f_2 f_3), \\
f_7^{(4)} &= (f_7 - f_4^8)/f_1, \ \cdots
\end{aligned}
\tag{4.51}
$$

Moreover, there does exist other additional factorizations. For example the following polynomials *do factorize* but their factors are not the polynomials f_n, and not even the new polynomials (4.51):

$$
\begin{aligned}
f_4^{(5)} &= (f_2 f_4 - f_3^2 f_1), \qquad f_5^{(5)} = (f_3 f_5 - f_4^2 f_1 f_2), \\
f_6^{(5)} &= (f_4 f_6 - f_5^2 f_1 f_2 f_3), \\
f_4^{(6)} &= (f_4 - f_1 f_2 f_3), \qquad f_5^{(6)} = (f_5 - f_1 f_2 f_3 f_4), \\
f_6^{(6)} &= (f_6 - f_1 f_2 f_3 f_4 f_5), \ \cdots
\end{aligned}
\tag{4.52}
$$

However, though the situation seems very similar to the one encountered for class IV (see Eqs. (4.27)), we have not been able to find pseudo recurrences like (4.28), neither recurrences on the x_n's. These factorizations (4.51) and (4.52) suggest the following linear recurrences on the β_n's *only valid for* $n \geq 2$:

$$\beta_{n+1} - 2\,\beta_n = 0, \qquad \beta_{n+2} - \beta_{n+1} - 2\,\beta_n = 0 \tag{4.53}$$

$$\beta_{n+1} = \beta_1 + \beta_2 + \cdots + \beta_n \tag{4.54}$$

For $n = 1$ one has $\beta_1 = \beta_2 = q$ which is not in agreement with (4.53). From relation (4.54), one gets:

$$\beta(x) = \frac{x\,\beta(x)}{1-x} + q\,x \tag{4.55}$$

which is satisfied by the exact expression of $\beta(x)$, namely Eq. (4.47).

---

[10] One may also think, at first sight, that such unpleasant "string-like" factorizations rule out any possible polynomial growth and automatically yield exponential growth: in fact *this is not true* [15].

Many more compatibilities between linear recurrences on the exponents and factorizations (4.38), (4.39), (4.40) or "additional" factorizations (4.51) and (4.52) can be verified. In particular, despite the fact that the iteration corresponding to class VI seems to be involved, it is nevertheless possible to associate to these iterations recurrences bearing on a *fixed number* of variables including the variable x_n (see Section 5.2 in the following).

Again, one can study the iteration of $\widehat{K}$ seen as a birational transformation in $\mathbb{C}P_{q^2-1}$. For $q = 4$ these orbits look like *curves* in some domain of $\mathbb{C}P_{15}$ [16]. In fact, these orbits can be seen to belong to a three-dimensional subvariety which is the *intersection of only twelve Plücker-like quadrics* in $\mathbb{C}P_{15}$ and more generally the intersection of, at most, $q^2 - 4$ algebraic expressions in $\mathbb{C}P_{q^2-1}$ for $q \times q$ matrices [16].

5. Demonstration

Let us prove here all the results given previously, in particular the *factorization results* and the existence of *recurrences on a fixed number of variables* and sometimes, on a single variable.

Let t denote the transposition exchanging $m_{i_2j_2}$ and $m_{i_1j_1}$. Let P be a fixed matrix associated to t, for which all entries are equal to zero, except the two entries which are permuted by t:

$$P_{i_1j_1} = 1, \quad P_{i_2j_2} = -1 \tag{5.1}$$

Δ_0 will denote the difference between the two entries $m_{i_2j_2}$ and $m_{i_1j_1}$:

$$\Delta_0 = m_{i_2j_2} - m_{i_1j_1} \tag{5.2}$$

Δ_1 denotes the difference between the two entries $\widehat{K}(R_q)_{i_2j_2}$ and $\widehat{K}(R_q)_{i_1j_1}$, and generally Δ_n denotes the difference between the two entries $\widehat{K}^n(R_q)_{i_2j_2}$ and $\widehat{K}^n(R_q)_{i_1j_1}$. With these notations transposition t reads on a generic matrix R_q:

$$t(R_q) = R_q + \Delta_0 \cdot P \tag{5.3}$$

Replacing in (5.3) matrix R_q by matrix $\widehat{I}(R_q)$, the inhomogeneous transformation $\widehat{K}$, can also be seen as a "deformation" of the matricial inverse $\widehat{I}$:

$$\widehat{K}(R_q) = \widehat{I}(R_q) - \Delta_1 \cdot P \tag{5.4}$$

Noticing that:

$$\Delta_0(\widehat{I}(R_q)) = -\Delta_0(\widehat{K}(R_q)) = -\Delta_1 \tag{5.5}$$

Let us introduce matrix $U = R_q \cdot \widehat{K}(R_q)$, which is, by construction, close from the identity matrix. We will first assume that $j_1 \neq j_2$ (and of course $i_1 \neq i_2$ [11]):

$$U = \mathcal{I}d_q - \Delta_1 R_q \cdot P =$$

$$\begin{bmatrix}
1 & 0 & \cdots 0 & -\Delta_1 m_{1i_1} & 0 \cdots\cdots 0 & \Delta_1 m_{1i_2} & 0 \cdots\cdots 0 \\
0 & 1 & \ddots 0 & -\Delta_1 m_{2i_1} & 0 \cdots\cdots 0 & \Delta_1 m_{2i_2} & 0 \cdots\cdots 0 \\
\vdots & \vdots & \ddots \vdots & \vdots & \vdots \cdots\cdots \vdots & \vdots & \vdots \cdots\cdots \vdots \\
0 \cdots & 0 & 1 & -\Delta_1 m_{(j_1-1)i_1} & 0 \cdots\cdots 0 & \Delta_1 m_{(j_1-1)i_2} & 0 \cdots\cdots 0 \\
0 \cdots & \cdots & 0 & 1-\Delta_1 m_{j_1i_1} & 0 \cdots\cdots 0 & \Delta_1 m_{j_1i_2} & 0 \cdots\cdots 0 \\
0 \cdots & \cdots & 0 & -\Delta_1 m_{(j_1+1)i_1} & 1 \quad 0 \quad \cdots 0 & \Delta_1 m_{(j_1+1)i_2} & 0 \cdots\cdots 0 \\
0 \cdots & \cdots & 0 & -\Delta_1 m_{(j_1+2)i_1} & 0 \quad 1 \quad \ddots 0 & \Delta_1 m_{(j_1+2)i_2} & 0 \cdots\cdots 0 \\
\vdots \cdots & \cdots & \vdots & \vdots & \vdots \ddots \ddots \vdots & \vdots & \vdots \cdots\cdots \vdots \\
0 \cdots & \cdots & 0 & -\Delta_1 m_{(j_2-1)i_1} & 0 \quad 0 \quad \cdots 1 & \Delta_1 m_{(j_2-1)i_2} & 0 \cdots\cdots 0 \\
0 \cdots & \cdots & 0 & -\Delta_1 m_{j_2i_1} & 0 \cdots\cdots 0 & 1+\Delta_1 m_{j_2i_2} & 0 \cdots\cdots 0 \\
0 \cdots & \cdots & 0 & -\Delta_1 m_{(j_2+1)i_1} & 0 \cdots\cdots 0 & \Delta_1 m_{(j_2+1)i_2} & 1 \quad 0 \quad \cdots 0 \\
0 \cdots & \cdots & 0 & -\Delta_1 m_{(j_2+2)i_1} & 0 \cdots\cdots 0 & \Delta_1 m_{(j_2+2)i_2} & 0 \quad 1 \quad \ddots 0 \\
\vdots \cdots & \cdots & \vdots & \vdots & \vdots \cdots\cdots \vdots & \vdots & \vdots \ddots \ddots \vdots \\
0 \cdots & \cdots & 0 & -\Delta_1 m_{qi_1} & 0 \cdots\cdots 0 & \Delta_1 m_{qi_2} & 0 \quad 0 \quad \ddots 1
\end{bmatrix} \tag{5.6}$$

This expression of U gives, at once, the determinant:

$$\begin{aligned}
\det(U) = x_0 &= (1 - \Delta_1 m_{j_1i_1})(1 + \Delta_1 m_{j_2i_2}) + \Delta_1^2 m_{j_1i_2} m_{j_2i_1} \\
&= 1 + (m_{j_2i_2} - m_{j_1i_1})\Delta_1 + (m_{j_1i_2} m_{j_2i_1} - m_{j_1i_1} m_{j_2i_2})\Delta_1^2 \\
&= 1 + T_0 \Delta_1 + N_0 \Delta_1^2 \qquad (5.7)
\end{aligned}$$

where $N_0 = (m_{j_1i_2} m_{j_2i_1} - m_{j_1i_1} m_{j_2i_2})$ (that is the 2×2 minor corresponding to rows j_1 and j_2 and columns i_1 and i_2 of matrix R_q) and $T_0 = m_{j_2i_2} - m_{j_1i_1}$ (T_0 corresponds to the difference of the two entries exchanged by t for the transposed matrix). One is now able to easily calculate the second step of the iteration:

$$\widehat{K}^2(R_q) = t(\widehat{I}(\widehat{K}(R_q))) = t(\widehat{I}(U).R_q) \tag{5.8}$$

where $\widehat{I}(U)$ also differs from the identity matrix by the two columns j_1 and j_2. Each entry of these columns is the ratio by x_0 of a polynomial quadratic in Δ_1. Let us calculate explicitly $\widehat{I}(U)$ as a polynomial in Δ_1. From relation (5.6) one directly gets:

[11] If $i_1 = i_2$ one can choose another element of the same class, which satisfies $j_1 = j_2$.

$$\widehat{I}(U) = \sum_{n=0}^{\infty} \Delta_1^n (R_q \cdot P)^n \tag{5.9}$$

Matrix $(R_q \cdot P)$ being of a quite simple form, it is easy to calculate its minimal polynomial which reads:

$$x \cdot (x^2 + T_0 x + N_0) \tag{5.10}$$

One can thus obtain the expression of the matrices $(R_q \cdot P)^n$ in terms of $(R_q \cdot P)^2$, of $(R_q \cdot P)$ and of the identity matrix. After straightforward calculations one gets:

$$\widehat{I}(U) = \mathcal{I}d + \frac{\Delta_1 (1 + T_0 \Delta_1)}{x_0} \cdot (R_q \cdot P) + \frac{\Delta_1^2}{x_0} (R_q \cdot P)^2 \tag{5.11}$$

Let us now revisit these equations when $j_1 = j_2 = j$ (or equivalently $i_1 = i_2$), that is for classes IV and VI. In this $j_1 = j_2 = j$ case, U reads:

$$U = \begin{bmatrix}
1 & 0 & \cdots & 0 & \Delta_1 (m_{1i_2} - m_{1i_1}) & 0 & \cdots & \cdots & 0 \\
0 & 1 & \ddots & 0 & \Delta_1 (m_{2i_2} - m_{2i_1}) & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\
0 & \cdots & 0 & 1 & \Delta_1 (m_{(j-1)i_2} - m_{(j-1)i_1}) & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 1 + \Delta_1 (m_{ji_2} - m_{ji_1}) & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & \Delta_1 (m_{(j+1)i_2} - m_{(j+1)i_1}) & 1 & 0 & \cdots & 0 \\
0 & \cdots & \cdots & 0 & \Delta_1 (m_{(j+2)i_2} - m_{(j+2)i_1}) & 0 & 1 & \ddots & 0 \\
\vdots & \cdots & \cdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & \Delta_1 (m_{qi_2} - m_{qi_1}) & 0 & 0 & \cdots & 1
\end{bmatrix}$$

This expression of U gives at once the determinant:

$$\begin{aligned} \det(U) = x_0 &= 1 + \Delta_1 (m_{ji_2} - m_{ji_1}) \\ &= 1 + T_0 \Delta_1 \end{aligned} \tag{5.12}$$

where $T_0 = m_{ji_2} - m_{ji_1}$ (T_0 still correspond to the difference of the two entries exchanged by t for the transposed matrix). Eq. (5.11) becomes:

$$\widehat{I}(U) = \mathcal{I}d + \frac{\Delta_1}{x_0} \cdot (R_q \cdot P) \tag{5.13}$$

Relation (5.8) is still valid and enables to calculate the second step of the iteration.

Let us now give a proof of the factorization properties for the various classes defined in Sections 3.2 and 4.

5.1. Demonstration of the factorizations

The demonstration of the factorizations has been done for class I in [14]: we will here just recall (and generalize to all the other classes) the main steps.

Factorization properties are obviously associated with the homogeneous matrices $K^n(R_q)$ (instead of matrices $\widehat{K}^n(R_q)$, which do not have polynomial entries):

$$K(R_q) = \det(R_q) \cdot \widehat{K}(R_q)$$

K being a homogeneous transformation of degree $q-1$, one obtains:

$$K^2(R_q) = x_0 \det(R_q)^{q-2} \cdot \widehat{K}^2(R_q) \qquad (5.14)$$

Let us first study classes I, II, III and V for which $j_1 \neq j_2$ and $i_1 \neq i_2$. Eqs. (5.8) and (5.11) give the form for $x_0 \widehat{K}^2(R_q)$. One remarks that its entries are polynomials in the entries of the matrix R_q and *quadratic* in Δ_1. The definition of Δ_1 straightforwardly shows that its denominator is $\det(R_q)$. Thus matrix $\widehat{K}^2(R_q)$ reads:

$$\widehat{K}^2(R_q) = \frac{M_2}{x_0 \cdot \det(R_q)^2} \qquad (5.15)$$

where M_2 is a matrix with polynomial entries. Eq. (5.14) thus proves the first step of the factorization:

$$K^2(R_q) = \det(R_q)^{q-4} \cdot M_2 \qquad (5.16)$$

A similar demonstration can be performed on $\det(K(R_q))$ and yields:

$$\det(K(R_q)) = x_0 \cdot \det(R_q)^{q-1} \qquad (5.17)$$

The expression of x_0, namely (5.7), is also quadratic in Δ_1. One thus has the following factorization:

$$\det(K(R_q)) = \det(R_q)^{q-3} \cdot f_2 \qquad (5.18)$$

As far as classes IV and VI are concerned, *for which* $j_1 = j_2$ *or* $i_1 = i_2$, x_0, as well as the entries of matrix $x_0 \widehat{K}^2(R_q)$, given by Eqs. (5.8) and (5.13), are polynomials in the entries of the matrix R_q and *linear* in Δ_1. Matrix $\widehat{K}^2(R_q)$ reads:

$$\widehat{K}^2(R_q) = \frac{M_2}{x_0 \cdot \det(R_q)} \qquad (5.19)$$

where M_2 has polynomial entries. Eq. (5.14) thus proves the first step of the factorization:

$$K^2(R_q) = \det(R_q)^{q-3} \cdot M_2 \tag{5.20}$$

From relation (5.17) one also gets:

$$\det(K(R_q)) = \det(R_q)^{q-2} \cdot f_2 \tag{5.21}$$

Notice that factorization (5.16) is only valid for $q > 3$, and (5.18) for $q > 2$, while (5.20) is valid for $q > 2$, and (5.21) for $q > 1$.

Considering successively the explicit expressions of $K^n(R_q)$ and of their determinants, one notices that there are further factorizations (see for instance Eq. (3.8)), that could be obtained the same way. However these further factorizations depend on the class one considers. We will thus just assume these factorizations (however the first steps of the factorizations have been strictly obtained by formal computer calculations and their general form has been proved recursively in [14]). Their generic form reads:

$$f_n(K) = f_1^{\mu_n} \cdot f_{n+1} \tag{5.22}$$

$$\det(M_n) = f_1^{v_n} \cdot f_2^{v_{n-1}} \cdot f_3^{v_{n-2}} \cdots f_{n-1}^{v_2} \cdot f_n^{v_1} \cdot f_{n+1} \tag{5.23}$$

$$\Big(M_n\Big)_K = f_1^{\nu_n} \cdot M_{n+1} \tag{5.24}$$

$$K(M_n) = M_{n+1} \cdot f_1^{u_n} \cdot f_2^{u_{n-1}} \cdot f_3^{u_{n-2}} \cdots f_{n-1}^{u_2} \cdot f_n^{u_1} \tag{5.25}$$

with the following relations between the different exponents:

$$\nu_{n+1} = (q-1)\,\nu_n + u_{n+1} - (u_1\,\mu_n + u_2\,\mu_{n-1} + \cdots + u_n\,\mu_1) \tag{5.26}$$

$$\mu_{n+1} = v_{n+1} + q\,\nu_n - (v_n\,\mu_1 + v_{n-1}\,\mu_2 + \cdots + \mu_n\,v_1) \tag{5.27}$$

Moreover, it can be shown that (5.25), the factorization relation on $K(M_n)$, necessarily yields relation (5.23), the factorization of the determinant (and also the inequalities $v_n \geq 1 + u_n$ when $u_n \neq 0$). The left factorizations, (5.23) and (5.25), and the right factorizations, (5.22) and (5.24), are equivalent when assuming (5.26) and (5.27). The proof is given in [14].

5.2. *Demonstration of the recurrences*

Let us briefly sketch the demonstration of the existence of recurrences *independent of* q (like recurrences (3.25) or (4.30)), on a *finite* set of variables including variable x_n (Eq. (3.25)).

Such a demonstration has already been performed for class I in [14]. Therefore one will not recall this demonstration but one will only sketch the demonstration for the other classes.

5.2.1. *Demonstration of the recurrences for class II*

In order to represent class II, let us take the transposition t exchanging m_{12} and m_{34}. Definition (5.2) now reads:

$\Delta_0 = [R_q]_{34} - [R_q]_{12} = m_{34} - m_{12}$, relation (5.1) becomes:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5.28}$$

and Δ_1 reads:

$$\Delta_1 = [\widehat{K}(R_q)]_{34} - [\widehat{K}(R_q)]_{12} = [\widehat{I}(R_q)]_{12} - [\widehat{I}(R_q)]_{34} \tag{5.29}$$

Here matrix $U = R_q \cdot \widehat{K}(R_q)$ reads:

$$U = \begin{bmatrix} 1 & -\Delta_1 m_{11} & 0 & \Delta_1 m_{13} & 0 & 0 & \dots \\ 0 & 1-\Delta_1 m_{21} & 0 & \Delta_1 m_{23} & 0 & 0 & \dots \\ 0 & -\Delta_1 m_{31} & 1 & \Delta_1 m_{33} & 0 & 0 & \dots \\ 0 & -\Delta_1 m_{41} & 0 & 1+\Delta_1 m_{43} & 0 & 0 & \dots \\ \vdots & -\Delta_1 m_{51} & \vdots & \Delta_1 m_{53} & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & 0 & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

This transposition involving two columns (and two rows), the determinant x_0 of matrix U is given by relation (5.7), as a *quadratic* expression of Δ_1:

$$\det(U) = x_0 = 1 + T_0 \Delta_1 + N_0 \Delta_1^2 \tag{5.30}$$

where $N_0 = (m_{41} m_{23} - m_{43} m_{21})$ and $T_0 = m_{43} - m_{21}$.

Relation (5.11) yields:

$$x_0 \cdot \widehat{I}(U) =$$

$$\begin{bmatrix} x_0 & \Delta_1 (\Delta_1 m_{11} m_{43} + m_{11} - \Delta_1 m_{41} m_{13}) & 0 & -\Delta_1 (\Delta_1 m_{11} m_{23} - \Delta_1 m_{13} m_{21} + m_{13}) & 0 & 0 & \dots \\ 0 & 1 + \Delta_1 m_{43} & 0 & -\Delta_1 m_{23} & 0 & 0 & \dots \\ 0 & \Delta_1 (\Delta_1 m_{31} m_{43} + m_{31} - \Delta_1 m_{33} m_{41}) & x_0 & \Delta_1 (\Delta_1 m_{33} m_{21} - m_{33} - \Delta_1 m_{23} m_{31}) & 0 & 0 & \dots \\ 0 & \Delta_1 m_{41} & 0 & 1 - \Delta_1 m_{21} & 0 & 0 & \dots \\ \vdots & \Delta_1 m_{51} & \vdots & -\Delta_1 m_{53} & x_0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & 0 & x_0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \tag{5.31}$$

Matrix $\widehat{K}^2(R_q)$ is obtained from relation (5.8). Its explicit form is quite involved, and will not be given here.

We will just concentrate on a *fixed finite* number of variables, enabling to understand the evolution of T_0, N_0 and Δ_0, the action of $\widehat{K}^2$ preserving this set of variables. Namely $\widehat{K}^2$ transforms the following variables as follows:

$$
\begin{aligned}
m_{21} &\to \frac{m_{21} - \Delta_1 N_0}{x_0}, \\
m_{23} &\to \frac{m_{23}}{x_0}, \\
m_{41} &\to \frac{m_{41}}{x_0}, \\
m_{43} &\to \frac{m_{43} + \Delta_1 N_0}{x_0}
\end{aligned}
\tag{5.32}
$$

From (5.32) one gets the equations:

$$
\begin{aligned}
T_2 &= \frac{T_0 + 2\Delta_1 N_0}{x_0}, \\
N_2 &= \frac{N_0 - \Delta_1 N_0 (m_{21} - m_{43}) + \Delta_1^2 N_0^2}{x_0^2} = \frac{N_0 (1 + \Delta_1 T_0 + \Delta_1^2 N_0)}{x_0^2} = \frac{N_0}{x_0}
\end{aligned}
\tag{5.33}
$$

Let us introduce the two variables:

$$F_0 = m_{11}\, m_{22}^{<2>} + m_{33}\, m_{44}^{<2>} - m_{13}\, m_{42}^{<2>} - m_{31}\, m_{24}^{<2>} \tag{5.34}$$

$$G_0 = m_{22}\, m_{11}^{<2>} + m_{44}\, m_{33}^{<2>} - m_{42}\, m_{13}^{<2>} - m_{24}\, m_{31}^{<2>} \tag{5.35}$$

where $m_{ij}^{<2>}$ denotes the entries of matrix $\widehat{K}^2(R_q)$. These two variables happen to be equal. As a consequence of this remarkable equality, Δ_2 satisfies the following relation:

$$\frac{\Delta_2 + \Delta_0}{\Delta_1} = F_0 = G_0 = \frac{F_0 + G_0}{2} \tag{5.36}$$

One thus has to calculate the action of $\widehat{K}^2$ on many new entries $m_{\alpha,\beta}$ occurring in the right-hand side of (5.34) and (5.35). Remarkably *this action is the same on all these $m_{\alpha,\beta}$'s*, namely:

$$m_{\alpha,\beta} \to m_{\alpha,\beta}^{<2>} = \frac{m_{\alpha,\beta} + \Delta_1 P_0^{(\alpha,\beta)}}{x_0} \tag{5.37}$$

where the $P_0^{(\alpha,\beta)}$'s are 2×2 minors, which, remarkably, also transform similarly under $\widehat{K}^2$:

$$P_0^{(\alpha,\beta)} \quad \rightarrow \quad P_2^{(\alpha,\beta)} \;=\; \frac{P_0^{(\alpha,\beta)}}{x_0} \tag{5.38}$$

yielding:

$$x_2 \cdot m_{\alpha,\beta}^{<4>} \;=\; m_{\alpha,\beta}^{<2>}\, \frac{\Delta_1 + \Delta_3}{\Delta_1} \;-\; m_{\alpha,\beta}\, \frac{\Delta_3}{x_0\, \Delta_1} \tag{5.39}$$

From relation (5.39) one can get the action of $\widehat{K}^2$ on $(F_0 + G_0)/2$:

$$\frac{F_0 + G_0}{2} \;\rightarrow\; \frac{F_2 + G_2}{2} \;=\; \frac{\Delta_1 + \Delta_3}{x_2\, \Delta_1}\, E_2 \;-\; \frac{\Delta_3}{x_0\, x_2\, \Delta_1}\, \frac{F_0 + G_0}{2} \tag{5.40}$$

where $E_0 \;=\; m_{11}\, m_{22} + m_{33}\, m_{44} - m_{13}\, m_{42} - m_{31}\, m_{24}$, the other E_n's being deduced from E_0 by the successive right-action of $\widehat{K}^2$.

Recalling Eqs. (5.36) one gets:

$$\frac{(\Delta_{n+2} + \Delta_{n+4})}{\Delta_{n+3}} \;+\; \frac{\Delta_{n+3}\, (\Delta_n + \Delta_{n+2})}{\Delta_{n+1}^2\; x_n\; x_{n+2}} \;-\; \frac{(E_{n+2}\, (\Delta_{n+1} + \Delta_{n+3}))}{x_{n+2}\, \Delta_{n+1}} \;=\; 0 \tag{5.41}$$

One also needs the right-action of $\widehat{K}^2$ on E_0. It can be deduced from relation (5.39):

$$x_{n+2}\, E_{n+4} - \frac{(\Delta_{n+2} + \Delta_{n+4})\, (\Delta_{n+1} + \Delta_{n+3})}{\Delta_{n+1}\, \Delta_{n+3}}$$
$$-\frac{\Delta_{n+3}}{\Delta_{n+1}\; x_n\; x_{n+2}} \left(\frac{\Delta_{n+3}\, E_n}{\Delta_{n+1}\, x_n} - \frac{(\Delta_{n+1} + \Delta_{n+3})\, (\Delta_n + \Delta_{n+2})}{\Delta_{n+1}^2} \right) \;=\; 0 \tag{5.42}$$

Eqs. (5.41) and (5.42) enable to eliminate the E_n's. Introducing the well-suited variables $\delta_n \;=\; \Delta_{n+2}/\Delta_n$ one gets:

$$\frac{x_{n+4}\, x_{n+6}\; (1 + \delta_{n+6})}{1 + \delta_{n+5}} + \frac{\delta_{n+5}^2\; (\delta_{n+4} + 1)}{\delta_{n+4}\; (1 + \delta_{n+5})}$$
$$-\frac{\delta_{n+5}\; (1 + \delta_{n+3})\; (\delta_{n+4} + 1)}{\delta_{n+4}}$$
$$-\frac{\delta_{n+3}^3 \delta_{n+5}\; (1 + \delta_{n+2})}{\delta_{n+2}\, \delta_{n+4}\, x_{n+2}\, x_{n+4}\; (1 + \delta_{n+1})}$$
$$-\frac{\delta_{n+3}^3 \delta_{n+5}\, \delta_{n+1}^2\; (\delta_n + 1)}{\delta_n\, \delta_{n+2}\, \delta_{n+4}\, x_n\, x_{n+2}^2 x_{n+4}\; (1 + \delta_{n+1})}$$
$$+\frac{\delta_{n+3}^2 \delta_{n+5}\; (1 + \delta_{n+3})\, (1 + \delta_{n+2})}{\delta_{n+2}\, \delta_{n+4}\, x_{n+2}\, x_{n+4}} \;=\; 0 \tag{5.43}$$

Finally, coming back to Eqs. (5.30) and (5.33), one can eliminate the T_n's and N_n's, and get, with the same variables δ_n's, another relation between the x_n's and the δ_n's:

$$\left(\frac{(x_{n+4}-1)}{\delta_{n+3}} - \frac{(x_{n+2}-1)}{x_{n+2}}\right) = \frac{(1+\delta_{n+1})(1+\delta_{n+3})}{x_{n+2}} \times \left(\frac{(x_{n+2}-1)}{\delta_{n+1}} - \frac{(x_n-1)}{x_n}\right) \tag{5.44}$$

One can in principle eliminate x_n between (5.43) and (5.44): it yields "huge" calculations. In contrast the elimination of δ_n seems out of range.

Let us just note that, though such a system of recurrences is quite involved, one can however get some *finite order conditions* for these recurrences of class II, namely the orbits of order three and four:

- order three:

$$x_n\, x_{n+1}\, x_{n+2} - 1 = 0\,, \quad \text{or} \quad 1 + \delta_{n+1} + \delta_n\, \delta_{n+1} = 0 \tag{5.45}$$

- order four:

$$x_n\, x_{n+2} - 1 = 0\,, \quad \text{or} \quad \delta_n\, \delta_{n+2} + 1 = 0 \tag{5.46}$$

5.2.2. Demonstration of the recurrences for class III

One will just sketch here briefly the demonstration of the recurrences for class III. Let us represent class III, with transposition t exchanging m_{12} and m_{31}. Then (5.2) reads: $\Delta_0 = [R_q]_{31} - [R_q]_{12} = m_{31} - m_{12}$, and matrix P defined by (5.1) becomes:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ -1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5.47}$$

Δ_1 reads:

$$\Delta_1 = [\widehat{K}(R_q)]_{31} - [\widehat{K}(R_q)]_{12} = [\widehat{I}(R_q)]_{12} - [\widehat{I}(R_q)]_{31} \tag{5.48}$$

Here matrix $U = R_q \cdot \widehat{K}(R_q)$ has a simple form similar to (5.30). The determinant x_0 of matrix U is again given by Eq. (5.7), namely:

$$\det(U) = x_0 = 1 + T_0\Delta_1 + N_0\Delta_1^2 \tag{5.49}$$

with $N_0 = (m_{11}\, m_{23} - m_{13}\, m_{21})$ and $T_0 = m_{13} - m_{21}$.

Thus one can calculate the explicit form of $\widehat{I}(\widehat{K}(R_q))$:

$$x_0 \cdot \widehat{I}(\widehat{K}(R_q)) = \begin{bmatrix} m_{11} & m_{21} + \Delta_1\,(m_{11}\,m_{22} - m_{12}\,m_{21}) & m_{13} + \Delta_1\,N_0 & \cdots \\ m_{21} - \Delta_1\,N_0 & m_{22} + \Delta_1\,(m_{13}\,m_{22} - m_{12}\,m_{23}) & m_{23} & \cdots \\ m_{31} + \Delta_1\,(m_{31}\,m_{13} - m_{33}\,m_{11}) & w & m_{33} + \Delta_1\,(m_{31}\,m_{23} - m_{33}\,m_{21}) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (5.50)$$

where

$$w = m_{32}\,x_0 + \Delta_1\,[m_{22}\,(m_{31} + \Delta_1\,(m_{31}\,m_{13} - m_{33}\,m_{11})) \\ -m_{12}\,(m_{33} + \Delta_1\,(m_{31}\,m_{23} - m_{33}\,m_{21}))]$$

Matrix $\widehat{K}^2(R_q)$ is obtained permuting entries m_{12} and m_{31} in relation (5.50). This yields the following expression for Δ_2:

$$\Delta_2 = \frac{-\Delta_0 + \Delta_1\,(m_{11}\,m_{22} - m_{12}\,m_{21} + m_{33}\,m_{11} - m_{31}\ ,m_{13})}{x_0} \quad (5.51)$$

Let us denote P_0 the expression appearing in the right-hand side of Δ_2:

$$P_0 = m_{11}\,m_{22} - m_{12}\,m_{21} + m_{33}\,m_{11} - m_{31}\,m_{13} \quad (5.52)$$

Similarly N_0, T_0 and P_0 transform, under K^2, as follows:

$$T_0 \rightarrow T_2 = \frac{T_0 + 2\,\Delta_1\,N_0}{x_0},$$
$$N_0 \rightarrow N_2 = \frac{N_0}{x_0},$$
$$P_0 \rightarrow P_2 = \frac{P_0}{x_0} - \Delta_2\,T_2 \quad (5.53)$$

Since these results have been calculated on a generic matrix R_q, they can be applied successively on each matrix $\widehat{K}^n(R_q)$ and thus all the equations given above are *actually recurrence relations*. The expressions of $\widehat{K}(R_q)$ and $\widehat{K}^2(R_q)$ again allow us to prove the recurrence on x_n's and Δ_n's.

Introducing the well-suited variable δ_n:

$$\delta_n = \frac{\Delta_{n+2}}{\Delta_n} \quad (5.54)$$

and eliminating N_n's, T_n's and P_n's, one gets the following relations on the x_n's and δ_n's:

$$x_n\,x_{n+2}\,x_{n+4} + x_n\,x_{n+2}\,x_{n+4}\,\delta_{n+1} - x_{n+2}\,x_n - x_n\,x_{n+2}\,\delta_{n+1}$$

$$
\begin{aligned}
&-x_n\,\delta_{n+3}\,x_{n+2} - 2\,x_n\,\delta_{n+3}\,x_{n+2}\,\delta_{n+1} + x_n\,\delta_{n+3} + 2\,x_n\,\delta_{n+3}\,\delta_{n+1} \\
&-\delta_{n+3}^2\delta_{n+1}\,x_n\,x_{n+2} + \delta_{n+3}^2\delta_{n+1}\,x_n + \delta_{n+3}\,\delta_{n+1}^2\,x_n \\
&+\delta_{n+3}^2\delta_{n+1}^2 x_n - \delta_{n+3}\,\delta_{n+1}^2 - \delta_{n+3}^2\delta_{n+1}^2 = 0
\end{aligned}
\tag{5.55}
$$

and

$$
x_n\,x_{n+2} = \frac{\delta_{n+1}\,(\delta_{n+1}\,\delta_n + \delta_{n+1} + 1)}{\delta_n\,(\delta_{n+2}\,\delta_{n+1} + \delta_{n+2} + 1)}
\tag{5.56}
$$

In order to write down this last equation in a more handable way, let us introduce a new variable R_n:

$$
R_n = \frac{1 + \delta_{n+1} + \delta_n\,\delta_{n+1}}{\delta_n}
\tag{5.57}
$$

With these new variables R_n, Eq. (5.56) read:

$$
x_n\,x_{n+2} = \frac{R_n}{R_{n+1}}
\tag{5.58}
$$

From this last equation one can get x_{n+2} in terms of x_n, as well as x_{n+4} in terms of x_{n+2} (and therefore of x_n). Let us use, in Eq. (5.55), these expressions of x_{n+2} and x_{n+4}. Remarkably one obtains x_n as a function of the δ_n's. This expression of x_n takes a very simple form when introducing some new variables Q_n:

$$
Q_n = \frac{1 + \delta_{n+1} + \delta_{n+1}\delta_{n+3} + \delta_n\delta_{n+1} + \delta_n\delta_{n+1}\delta_{n+3} + \delta_n\,\delta_{n+1}\,\delta_{n+2}\delta_{n+3}}{\delta_n\delta_{n+1}}
\tag{5.59}
$$

x_n reads:

$$
x_n = \frac{R_{n+3}\,Q_n}{R_{n+1}\,Q_{n+1}}
\tag{5.60}
$$

One straightforwardly obtains x_{n+2} shifting n by two in Eq. (5.60):

$$
x_{n+2} = \frac{R_{n+5}\,Q_{n+2}}{R_{n+3}\,Q_{n+3}}
\tag{5.61}
$$

One can now get the product $x_n\,x_{n+2}$ from (5.60) and (5.61):

$$
x_n\,x_{n+2} = \frac{R_{n+5}\,Q_n\,Q_{n+2}}{R_{n+1}\,Q_{n+1}\,Q_{n+3}}
\tag{5.62}
$$

The two equations (5.58) and (5.62) yield an *algebraic relation* between the δ_n's, namely between $\delta_n, \delta_{n+1}, \ldots, \delta_{n+6}$:

$$
Q_{n+1}\,Q_{n+3}\,R_n - Q_n\,Q_{n+2}\,R_{n+5} = 0
\tag{5.63}
$$

This relation can be written in terms of an invariant expression, yielding a first integration:

$$\frac{Q_{n+1}\,Q_{n+3}}{R_{n+1}\,R_{n+2}\,R_{n+3}\,R_{n+4}\,R_{n+5}} = \frac{Q_n\,Q_{n+2}}{R_n\,R_{n+1}\,R_{n+2}\,R_{n+3}\,R_{n+4}} = \lambda \tag{5.64}$$

Similarly to what has been recalled in Section 3.2, the first *finite order conditions* for the recurrences of class III happen to be simple relations when written in terms of these new variables R_n and Q_n. These finite order conditions read respectively for order three, four, five, six and eight:

$$R_n = 0, \quad R_n = 1, \quad Q_n = 1, \quad Q_n = R_{n+1} \quad \text{and} \quad Q_n = 0 \tag{5.65}$$

5.2.3. Demonstration of the recurrences for class IV

Transposition t for class IV permutes two entries of the same column (or of the same row). Let us, for instance, represent class IV by transposition t which exchanges m_{12} and m_{32}.

One has $\Delta_0 = [R_q]_{32} - [R_q]_{12} = m_{32} - m_{12}$, and matrix P reads:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5.66}$$

$\Delta_1(R_q)$ still denotes $\Delta_0(\widehat{K}(R_q))$:

$$\Delta_1 = [\widehat{K}(R_q)]_{32} - [\widehat{K}(R_q)]_{12} = [\widehat{I}(R_q)]_{12} - [\widehat{I}(R_q)]_{32} \tag{5.67}$$

Recalling relation (5.12), the determinant of matrix U reads:

$$\det(U) = x_0 = 1 + \Delta_1\, T_0 \tag{5.68}$$

where $T_0 = m_{23} - m_{21}$. One has the following explicit form for $\widehat{K}^2(R_q)$ yielding Δ_2 and T_2:

$$\widehat{K}^2(R_q) = \frac{1}{x_0} \begin{bmatrix} m_{11} - \Delta_1\,(m_{11}\,m_{23} - m_{21}\,m_{13}) & x_0\,m_{32} - \Delta_1\,m_{22}\,(m_{31} - m_{33}) & m_{13} - \Delta_1\,(m_{11}\,m_{23} - m_{21}\,m_{13}) & x_0\,m_{14} - \Delta_1\,m_{24}\,(m_{11} - m_{13}) & \dots \\ m_{21} & m_{22} & m_{23} & m_{24} & \dots \\ m_{31} - \Delta_1\,(m_{31}\,m_{23} - m_{21}\,m_{33}) & x_0\,m_{12} - \Delta_1\,m_{22}\,(m_{11} - m_{13}) & m_{33} - \Delta_1\,(m_{31}\,m_{23} - m_{21}\,m_{33}) & x_0\,m_{34} - \Delta_1\,m_{24}\,(m_{31} - m_{33}) & \dots \\ m_{41} - \Delta_1\,(m_{41}\,m_{23} - m_{21}\,m_{43}) & x_0\,m_{42} - \Delta_1\,m_{22}\,(m_{41} - m_{43}) & m_{43} - \Delta_1\,(m_{41}\,m_{23} - m_{21}\,m_{43}) & x_0\,m_{44} - \Delta_1\,m_{24}\,(m_{41} - m_{43}) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5.69}$$

$$
\begin{aligned}
\Delta_2 &= \frac{\Delta_1\, m_{22}\, (m_{33} - m_{31} + m_{11} - m_{13})}{x_0} + m_{32} - m_{12} \\
&= \frac{\Delta_1\, m_{22}\, (m_{33} - m_{31} + m_{11} - m_{13})}{x_0} - \Delta_0 , \\
T_2 &= \frac{T_0}{x_0}
\end{aligned} \tag{5.70}
$$

Similarly, one can write how various quantities such as $(m_{21} - m_{23})$, m_{22}, $(m_{11} - m_{13})$ and $(m_{33} - m_{31})$ transform under $\widehat{K}^2$:

$$
\begin{aligned}
m_{22} \qquad &\rightarrow \frac{m_{22}}{x_0} , \\
m_{33} - m_{31} &\rightarrow \frac{m_{33} - m_{31}}{x_0} , \\
m_{13} - m_{11} &\rightarrow \frac{m_{13} - m_{11}}{x_0}
\end{aligned} \tag{5.71}
$$

These various quantities can easily be eliminated, yielding:

$$
\frac{(\Delta_4 + \Delta_2)\, x_2}{\Delta_3} = \frac{(\Delta_2 + \Delta_0)}{x_0\, \Delta_1} \tag{5.72}
$$

and

$$
x_2 = 1 + \frac{\Delta_3\, (x_0 - 1)}{\Delta_1\, x_0} \tag{5.73}
$$

Let us now introduce a new variable δ_n:

$$
\delta_n = \frac{\Delta_{n+2}}{\Delta_n} \tag{5.74}
$$

Eqs. (5.72) and (5.73) respectively read:

$$
\delta_{n+1} = \frac{x_n\, (x_{n+2} - 1)}{x_n - 1} \tag{5.75}
$$

$$
\frac{\delta_{n+2} + 1}{\delta_{n+1}} = \frac{1 + \delta_n}{x_n\, x_{n+2}\, \delta_n} \tag{5.76}
$$

Eliminating the variable δ_n in relations (5.75) and (5.76), *one recovers recurrence* (4.30) bearing on a single variable x_n:

$$
\frac{x_{n+3} - 1}{x_{n+2}\, x_{n+4} - 1} = \frac{x_{n+1} - 1}{x_n\, x_{n+2} - 1} \cdot x_n\, x_{n+3} \tag{5.77}
$$

Eliminating the variable x_n in Eqs. (5.75) and (5.76), *one also gets a recurrence bearing on another single variable*:

$$
\begin{aligned}
&(\delta_n + 1)\, (\delta_{n+1} + 1)\, (\delta_{n+3} + 1)\, (\delta_{n+4} + 1)\, \delta_{n+2} \\
&\quad = \delta_{n+3}\, (\delta_n\, \delta_{n+2} + 2\, \delta_n + 1)\, (\delta_{n+4}\, \delta_{n+2} + 2\, \delta_{n+2} + 1)
\end{aligned} \tag{5.78}
$$

One can now study finite order orbits. They read respectively for order four and order six:

- orbit of order four:

$$x_n \, x_{n+2} - 1 \; = \; 0 \tag{5.79}$$

and

$$1 + x_n + x_{n+1} - x_n \, x_{n+1} \; = \; 0 \tag{5.80}$$

that is:

$$\delta_n \, \delta_{n+1} + \delta_n - \delta_{n+1} + 1 \; = \; 0 \tag{5.81}$$

- orbit of order six:

$$\delta_{n+1} \, \delta_{n+4} - \delta_n \, \delta_{n+3} \; = \; 0 \, , \quad \text{or} \quad x_n \, x_{n+2} - x_{n+1} \; = 0 \tag{5.82}$$

5.2.4. Demonstration of the recurrences for class V

Let us represent class V, by transposition t exchanging m_{11} and m_{23}. Then (5.2) reads

$\Delta_0 = [R_q]_{23} - [R_q]_{11} = m_{23} - m_{11}$, and P given by (5.1) becomes:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & -1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5.83}$$

Moreover Δ_1 reads:

$$\Delta_1 \; = \; [\widehat{K}(R_q)]_{23} - [\widehat{K}(R_q)]_{11} \; = \; [\widehat{I}(R_q)]_{11} - [\widehat{I}(R_q)]_{23} \tag{5.84}$$

This transposition perturbing two columns (and two rows), x_0 is a quadratic expression in terms of Δ_1 and reads:

$$\det(U) \; = \; x_0 \; = \; 1 + T_0 \, \Delta_1 + N_0 \, \Delta_1^2 \tag{5.85}$$

with $N_0 = (m_{31} \, m_{12} - m_{11} \, m_{32})$ and $T_0 = m_{32} - m_{11}$.

Relation (5.11) yields:

$$x_0 \cdot \widehat{I}(U) =$$

$$\begin{bmatrix} \Delta_1 m_{32} + 1 & 0 & -\Delta_1 m_{12} & 0 & \dots \\ \Delta_1\,(\Delta_1 m_{21} m_{32} + m_{21} - \Delta_1 m_{31} m_{22}) & x_0 & \Delta_1\,(\Delta_1 m_{22} m_{11} - m_{22} - \Delta_1 m_{21} m_{12}) & 0 & \dots \\ \Delta_1 m_{31} & 0 & 1 - \Delta_1 m_{11} & 0 & \dots \\ -\Delta_1\,(\Delta_1 m_{31} m_{42} - \Delta_1 m_{41} m_{32} - m_{41}) & 0 & \Delta_1\,(\Delta_1 m_{42} m_{11} - m_{42} - \Delta_1 m_{41} m_{12}) & x_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (5.86)$$

Matrix $\widehat{K}^2(R_q)$ is obtained from relation (5.8). Its explicit form is quite involved, and will not be given here.

We will just concentrate on a *fixed* number of variables, enabling to understand the evolution of T_0, N_0 and Δ_0, the action of $\widehat{K}^2$ preserving this set of variables:

$$\begin{aligned} m_{11} &\to \frac{m_{11} - \Delta_1 N_0}{x_0} - \Delta_2\,, \\ m_{12} &\to \frac{m_{12}}{x_0}\,, \\ m_{31} &\to \frac{m_{31}}{x_0}\,, \\ m_{32} &\to m_{32}^{<2>} = \frac{m_{32} + \Delta_1 N_0}{x_0} \end{aligned} \quad (5.87)$$

From (5.87) one gets the following equations:

$$T_2 = \frac{T_0 + 2\Delta_1 N_0}{x_0} + \Delta_2 \quad (5.88)$$

Finally, coming back to Eqs. (5.85) and (5.88), one can eliminate the T_n's and get the following equation:

$$\frac{1 - x_{n+2}}{\Delta_{n+3}} - \frac{1 - x_n}{x_n \Delta_{n+1}} + \Delta_{n+2} - \Delta_{n+3} N_{n+2} - \frac{\Delta_{n+1} N_n}{x_n} = 0 \quad (5.89)$$

On the other hand N_0 transforms as follows:

$$N_2 = \frac{N_0\,(1 + \Delta_1 T_0 + \Delta_1^2 N_0)}{x_0^2} + \Delta_2 m_{32}^{<2>} = \frac{N_0}{x_0} + \Delta_2 m_{32}^{<2>} \quad (5.90)$$

From (5.87) and relation (5.90) one can eliminate the $m_{32}^{<n>}$'s and get:

$$\frac{N_{n+4}}{\Delta_{n+3}\Delta_{n+4}} - N_{n+2}\left(1 + \frac{1}{\Delta_{n+3}\Delta_{n+4}} + \frac{1}{\Delta_{n+2}\Delta_{n+3}}\right) x_{n+2}^{-1}$$

$$+\frac{N_n}{x_n\, x_{n+2}\, \Delta_{n+3}\, \Delta_{n+2}} = 0 \tag{5.91}$$

Moreover Δ_{n+2} satisfies the following relation:

$$\frac{x_n\, \Delta_{n+2} + \Delta_n}{\Delta_{n+1}} - N_n - P_n - Q_n + \Delta_{n+1}\, R_n = 0 \tag{5.92}$$

where:

$$P_0 = \det(M_1), \quad Q_0 = \det(M_2), \quad R_0 = \det(M_3) \tag{5.93}$$

with:

$$M_1 = \begin{bmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{bmatrix}, \quad M_2 = \begin{bmatrix} m_{11} & m_{13} \\ m_{21} & m_{23} \end{bmatrix}, \quad M_3 = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \tag{5.94}$$

$$Q_{n+2} - \frac{Q_n}{x_n} + \frac{\Delta_{n+1}\, R_n}{x_n} = 0 \tag{5.95}$$

$$R_{n+2} - \frac{R_n}{x_n} + \frac{\Delta_{n+2}\, N_n}{x_n} + \Delta_{n+2}\, P_{n+2} = 0 \tag{5.96}$$

$$P_{n+2} - \frac{P_n}{x_n} + \frac{\Delta_{n+1}\, R_n}{x_n} + \frac{N_n}{x_n} - N_{n+2} = 0 \tag{5.97}$$

One directly gets from relations (5.95) and (5.97) a covariant expression, enabling to easily eliminate one variable:

$$Q_{n+2} + N_{n+2} - P_{n+2} = \frac{Q_n + N_n - P_n}{x_n} \tag{5.98}$$

The elimination of the variables P_n, Q_n and R_n in this set of recurrences ((5.92), (5.95) and (5.96), (5.97)) can be performed and yield a third equation between the x_n's, Δ_n's and N_n's (the two other equations being (5.89) and (5.91)). The elimination of the N_n's yield two equations, relating the x_n's and the Δ_n's, of the form:

$$A_n + B_n \cdot x_n + C_n \cdot x_n\, x_{n+2} + D_n \cdot x_n\, x_{n+2}\, x_{n+4} \\ + E_n \cdot x_n\, x_{n+2}\, x_{n+4}\, x_{n+6} = 0 \tag{5.99}$$

and

$$F_n + G_n \cdot x_n + H_n \cdot x_n\, x_{n+2} + I_n \cdot x_n\, x_{n+2}\, x_{n+4} + J_n \cdot x_n\, x_{n+2}\, x_{n+4}\, x_{n+6} \\ + K_n \cdot x_n\, x_{n+2}\, x_{n+4}\, x_{n+6}\, x_{n+8} = 0 \tag{5.100}$$

where the $A_n, B_n, \cdots, K_n$ are polynomials in terms of the Δ_n's. One can, in principle, eliminate the x_n's to get an algebraic relation between the Δ_n's only.

The coefficients of these last two equations are too involved expressions to enable to perform such an elimination.

Let us just note that, though such a system of recurrences (5.99) and (5.100) is quite involved, one can however get some finite order conditions for these recurrences of class V, namely the orbits of order four:

$$\Delta_{n+1}\Delta_{n+2} + \Delta_{n+1}\Delta_n + \Delta_{n+2}\Delta_{n+3} + \Delta_{n+3}\Delta_n + \Delta_{n+1}\Delta_{n+3}\Delta_n\Delta_{n+2} = 0, \quad \text{or} \quad x_{n+2}x_n - 1 = 0 \tag{5.101}$$

5.2.5. Demonstration of the recurrences for class VI

Let us now consider a transposition of class VI. It permutes two entries in the same column, or row. Let us take for example a transposition perturbing only one row: t denotes the transposition exchanging the entries m_{11} and m_{12} of matrix R_q. Δ_0 now reads: $\Delta_0 = [R_q]_{21} - [R_q]_{11} = m_{21} - m_{11}$, and P denotes the following matrix:

$$P = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5.102}$$

For this transposition $T_0 = m_{12} - m_{11}$, and since t perturbes a row (and not a column as in the previous demonstration) matrix U will denote: $U = \widehat{K}(R_q) \cdot R_q$ which reads (instead of $U = R_q \cdot \widehat{K}(R_q)$):

$$U = \mathcal{I}d_q - \Delta_1 \cdot \left(P \cdot R_q\right) \tag{5.103}$$

The action of $\widehat{I}$ on U reads:

$$\widehat{I}(U) = \mathcal{I}d_q + \frac{\Delta_1 \cdot \left(R_q \cdot P\right)}{x_0} \tag{5.104}$$

and Eq. (5.8) becomes:

$$\widehat{K}^2(R_q) = t(\widehat{I}(\widehat{K}(R_q))) = t(R_q.\widehat{I}(U)) \tag{5.105}$$

From relations (5.103), (5.104) and (5.105) one gets the explicit form of $\widehat{K}^2(R_q)$, namely:

$$x_0 \widehat{K}^2(R_q) = \begin{bmatrix} (m_{12} - \Delta_1 P_0) & m_{11} & (m_{13} - \Delta_1 Q_0) & \dots \\ m_{21} & (m_{22} - \Delta_1 P_0) & (m_{23} - \Delta_1 Q_0) & \dots \\ m_{31} & m_{32}x_0 + \Delta_1 m_{31}(m_{12} - m_{22}) & m_{33}x_0 + \Delta_1 m_{31}(m_{13} - m_{23}) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5.106}$$

where P_0 and Q_0 read:

$$P_0 = m_{11}\, m_{22} - m_{12}\, m_{21}, \quad Q_0 = m_{11}\, m_{23} - m_{13}\, m_{21} \tag{5.107}$$

From definition (5.12), x_0, the determinant of matrix U, reads :

$$x_0 = 1 + \Delta_1\, T_0 \tag{5.108}$$

One thus gets the following action of $\widehat{K}^2$ on a *finite set* of homogeneous variables:

$$\begin{aligned} m_{11} &\to \frac{m_{12} - \Delta_1\, P_0}{x_0}\,, \\ m_{12} &\to \frac{m_{11}}{x_0}\,, \\ m_{21} &\to \frac{m_{21}}{x_0}\,, \\ m_{22} &\to \frac{m_{22} - \Delta_1\, P_0}{x_0} \end{aligned} \tag{5.109}$$

Relation (5.109) yields:

$$\Delta_2 = \frac{-\Delta_0 + \Delta_1\, P_0}{x_0} \tag{5.110}$$

Relation (5.108) yields with a shift of two (that is replacing matrix R_q by $\widehat{K}^2(R_q)$):

$$\begin{aligned} x_2 &= 1 + \Delta_3\, \frac{m_{21} - m_{12} + \Delta_1\, P_0}{x_0} \\ &= 1 + \Delta_3\, \frac{m_{21} - m_{11} + m_{11} - m_{12} + \Delta_1\, P_0}{x_0} \\ &= 1 + \Delta_2\, \Delta_3 + \frac{\Delta_3\,(x_0 - 1)}{x_0\, \Delta_1} \end{aligned} \tag{5.111}$$

Similarly Eq. (5.107) gives after a shift of two:

$$\begin{aligned} P_2 &= \frac{1}{x_0^2} \cdot \Big((m_{11} + \Delta_0 - \Delta_1\, P_0)\, (m_{22} - \Delta_1\, P_0) - (m_{12} - \Delta_0)\, m_{21} \Big) \\ &= \frac{1}{x_0^2} \cdot \Big(P_0\, (1 - \Delta_1\, (m_{11} - m_{21})) + (m_{21} + m_{22} - \Delta_1\, P_0)\, (\Delta_0 - \Delta_1\, P_0) \Big) \\ &= \frac{1}{x_0^2} \cdot \Big(P_0\, x_0 - \Delta_2\, x_0\, (m_{21} + m_{22} - \Delta_1\, P_0) \Big) \\ &= \frac{1}{x_0} \cdot \Big(P_0 - \Delta_2\, (m_{21} + m_{22} - \Delta_1\, P_0) \Big) \end{aligned} \tag{5.112}$$

One immediately gets from Eq. (5.109):

$$m_{21} + m_{22} \to \frac{m_{21} + m_{22} - \Delta_1 P_0}{x_0} \tag{5.113}$$

and from Eq. (5.112):

$$\begin{aligned} P_4 &= \frac{1}{x_2}\left(P_2 - \Delta_4\,((\frac{m_{21} + m_{22} - \Delta_1 P_0}{x_0}) - \Delta_3 P_2)\right) \\ &= \frac{1}{x_2}\left(P_2\,(1 + \frac{\Delta_4}{\Delta_2}) + \frac{\Delta_4 P_0}{\Delta_2 x_0} - \Delta_4\,(\Delta_2 + \Delta_4 x_2)\right) \end{aligned} \tag{5.114}$$

Eq. (5.110) can be written as follows:

$$P_0 \;=\; \frac{x_0 \Delta_2 + \Delta_0}{\Delta_1} \tag{5.115}$$

Relation (5.115) shifted by two, or four, respectively gives P_2 and P_4 in terms of x_n's and Δ_n's. In fact the Δ_n's only appear through their products $\Delta_n \Delta_{n+1}$, we will denote p_n in the following. Replacing P_0, P_2 and P_4 in relation (5.114) one gets for arbitrary n:

$$\begin{aligned} &x_{n+4}\, p_{n+5} + p_{n+4} - p_{n+4}^2\left(1 + \frac{p_{n+2}}{p_{n+3}\, x_{n+2}}\right)\left(p_{n+2}^{-1} + p_{n+3}^{-1} + 1\right) \\ &+ \frac{p_{n+4}^2 p_{n+2}\,(x_n\, p_{n+1} + p_n)}{p_{n+3}\, p_{n+1}^2 x_n\, x_{n+2}} = 0 \end{aligned} \tag{5.116}$$

Besides, relation (5.111) can also be written in terms of the p_n's and reads:

$$\frac{1 - x_{n+2}}{p_{n+2}} \;=\; \frac{1 - x_n}{x_n\, p_{n+1}} - 1 \tag{5.117}$$

The elimination of the x_n's in Eqs. (5.116) and (5.117) can be performed but it yields a "huge" algebraic relation between the p_n's. In principle one should also be able to get another algebraic relation between the x_n's, but the calculations are too large.

Let us note that these algebraic relations between the p_n's, or x_n's, are not (at first sight !) recurrences, however for particular conditions on the initial p_n's or x_n's one can actually get a (quite involved) recurrence on the x_n's.

Let us now write the finite order orbits in terms of the variables p_n's or x_n's for order three and four respectively:

- order three:

$$p_{n+2}\, p_n\, p_{n+1} \,+\, p_{n+2}\, p_{n+1} \,+\, p_{n+2}\, p_n \,+\, p_n\, p_{n+1} \;=\; 0 \tag{5.118}$$

and

$$\begin{aligned} &2\, x_n\, x_{n+1}\, x_{n+2} - x_n\, x_{n+2} - x_{n+1}\, x_{n+2} + 1 - x_n\, x_{n+1} \;=\; 0 \\ &\text{or} \quad x_n\, x_{n+1}\, x_{n+2} - 1 \;=\; 0 \end{aligned} \tag{5.119}$$

- order four:

$$p_{n+3}\,p_{n+2}\,p_n\,p_{n+1} + p_{n+3}\,p_{n+2}\,p_n + p_{n+3}\,p_n\,p_{n+1}$$
$$+p_{n+3}\,p_{n+2}\,p_{n+1} + p_{n+2}\,p_n\,p_{n+1} = 0 \qquad (5.120)$$

or

$$x_n\,x_{n+1}\,x_{n+2}\,x_{n+3} - 1 = 0 \qquad (5.121)$$

We have given in this last section examples of conditions corresponding to finite orbits for the iteration of K, for various classes, even when no simple recurrence on a single variable exists. These expressions and these calculations very much depend on the specific birational transformations one considers. In fact it can be shown that finite order conditions can be obtained in a quite general framework not associated to transpositions of two entries anymore. In this respect the example of finite order conditions of order four, given in Appendix A, is quite illuminating.

6. Class IV revisited

Among the classes with exponential growth, class IV is singled out, as far as two particular properties are concerned. On one hand, the mappings of this class have $q^2 - 3$ algebraic invariants. More precisely the algebraic varieties, the equations of which correspond to these invariants, are actually *planes* *(depending of course of the initial point of* CP_{q^2-1}*)*. On the other hand, simple recurrences, bearing on the x_n's, emerge. These recurrences have been shown to have an integrable subcase [16].

We will now study in detail this class. In particular, we will show *how to associate with these mappings in* CP_{q^2-1}*, mappings in* CP_2, which are closely related to the recurrences in the x_n's. Finally, we will show how the *integrability modifies the factorization properties* and provides an example of integrable mapping for arbitrary q.

6.1. Class IV as a mapping in two variables

From Eqs. (5.77) and (5.78) or (5.82), one may have the "prejudice" that the orbits of transformation K in CP_{15} (or CP_{q^2-1}) should be curves [16]. In fact it has been shown in [16] that, in some domain of the parameter space CP_{15} (or CP_{q^2-1}), these orbits look like *curves* which may explode in some algebraic surface.

These *algebraic surfaces are actually planes* [16] (depending on the initial matrix one iterates). This can be shown as follows, coming back to the action of $\widehat{K}^2$ on a generic matrix R_q (see Eq. (5.69)), which can be written in the following way:

$$\widehat{K}^2(R_q) \; = \; \frac{1}{x_0} \cdot (R_q + \Delta_1 F + b_1 P) \tag{6.1}$$

where matrix P still denotes the constant matrix given in (5.66) and b_1 reading:

$$b_1 \; = \; \frac{x_0 \Delta_2 - \Delta_0}{2} \tag{6.2}$$

and F denotes a $q \times q$ matrix, quadratic in the entries of matrix R_q:

$$F =$$

$$\begin{pmatrix} m_{21}m_{13} - m_{11}m_{23} & ((m_{12} + m_{32})(m_{21} - m_{23}) + m_{22}(m_{13} - m_{11} + m_{33} - m_{31}))/2 & m_{21}m_{13} - m_{11}m_{23} & m_{14}(m_{21} - m_{23}) + m_{24}(m_{13} - m_{11}) & \dots \\ 0 & 0 & 0 & 0 & \dots \\ m_{21}m_{33} - m_{31}m_{23} & ((m_{12} + m_{32})(m_{21} - m_{23}) + m_{22}(m_{13} - m_{11} + m_{33} - m_{31}))/2 & m_{21}m_{33} - m_{31}m_{23} & m_{34}(m_{21} - m_{23}) + m_{24}(m_{33} - m_{31}) & \dots \\ m_{21}m_{43} - m_{41}m_{23} & m_{42}(m_{21} - m_{23}) + m_{22}(m_{43} - m_{41}) & m_{21}m_{43} - m_{41}m_{23} & m_{44}(m_{21} - m_{23}) + m_{24}(m_{43} - m_{41}) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

One will recursively show that the successive iterates of $\widehat{K}^2$ read:

$$\widehat{K}^{2n}(R_q) \; = \; \frac{1}{x_0 \, x_2 \dots x_{2n-2}} \cdot (R_q + a_n F + b_n P) \tag{6.3}$$

F denotes the same matrix as in relation (6.1): *F does depend on R_q, but not on the order n of the iteration.* In other words all the iterates of *$\widehat{K}^2$ lie in a plane which depends on the initial matrix R_q* (or equivalently, on any other "even" iterates of R_q). This plane is led by two vectors, namely a *fixed vector* P and another one F, *depending on the initial matrix.*

In order to show recursively relation (6.3), let us perform the *right-action* of $\widehat{K}^2$ on Eq. (6.3). One gets:

$$\widehat{K}^{2n+2}(R_q) \; = \; \frac{1}{x_2 \, x_4 \dots x_{2n}} \cdot \left(\widehat{K}^2(R_q) + (a_n)_{\widehat{K}^2} (F)_{\widehat{K}^2} + (b_n)_{\widehat{K}^2} P\right) \tag{6.4}$$

Matrix $(\widehat{K}^2)(R_q)$ is given by Eq. (6.1). Straight calculations show that:

$$(F)_{\widehat{K}^2} \; = \; \frac{F}{x_0} \tag{6.5}$$

The right hand side of (6.4) thus reads:

$$\widehat{K}^{2n+2}(R_q) = \frac{1}{x_0 \, x_2 \dots x_{2n}} \cdot \left(R_q + (\Delta_1 + (a_n)_{\widehat{K}^2}) F + (b_1 + x_0 \, (b_n)_{\widehat{K}^2}) P\right) \tag{6.6}$$

which is of the same form as (6.3) with the following definition of a_{n+1} and b_{n+1}:

$$a_{n+1} = \Delta_1 + (a_n)_{\widehat{K}^2}, \quad b_{n+1} = b_1 + (b_n)_{\widehat{K}^2} \cdot x_0 \tag{6.7}$$

Defining $a_1 = \Delta_1$, the successive a_n's read:

$$a_n = \Delta_1 + \Delta_3 + \ldots + \Delta_{2n-1} \tag{6.8}$$

With this expression of the a_n's, one notices that $1 + T_0 a_n$ is directly related to the x_n's:

$$1 + T_0 a_n = x_0 x_2 \ldots x_{2n-2} \tag{6.9}$$

One can also give the expression of the successive b_n's:

$$b_n = \frac{((1 + a_n T_0)\Delta_{2n} - \Delta_0)}{2} \tag{6.10}$$

In order to obtain a_{n+1} and b_{n+1} in terms of a_n and b_n, one can consider a generic matrix:

$$A = \frac{1}{(1 + a T_0)} \cdot (R_q + a F + b P)$$

and get $\widehat{K}^2(A)$. Since it is necessarily of the form:

$$\widehat{K}^2(A) = \frac{1}{(1 + a' T_0)} \cdot (R_q + a' F + b' P)$$

one obtains a' and b' in terms of a and b. In fact, these calculations are quite heavy (see for instance Appendix B) and it is simpler to use the recurrences on the x_n's or more precisely the recurrences on the homogeneous variables q_n's:

$$x_n = \frac{q_{n+2}}{q_n} \tag{6.11}$$

From recurrence (4.30) bearing on the x_n's, one recovers the "almost integrable" recurrence studied in Section 8 of [16]:

$$q_{n+3} q_{n+5} \frac{q_{n+6} - q_{n+2}}{(q_{n+3} - q_{n+5})} = q_{n+1} q_{n+3} \frac{q_{n+4} - q_n}{(q_{n+1} - q_{n+3})} \tag{6.12}$$

which can be partially integrated (see Eq. (8.18) in [16]) as follows:

$$q_{2n+2} + q_{2n} + \frac{\lambda_2}{q_{2n+1}} = \rho_2 \tag{6.13}$$

$$q_{2n+3} + q_{2n+1} + \frac{\lambda_1}{q_{2n+2}} = \rho_1 \tag{6.14}$$

Let us first give the correspondence between the variables of the mapping in CP_{q^2-1}, and the variables of the recurrences (the q_n's, λ_1 and λ_2).

From relation (6.9) one directly gets the q_n's, for n even:

$$q_{2n} = x_0 x_2 \ldots x_{2n-2} \cdot q_0 = (1 + T_0 a_n) \cdot q_0 \tag{6.15}$$

One also has $q_{2n+1} = x_1 x_3 \ldots x_{2n-1} \cdot q_1$, and recalling the definition of the x_n's:

$$x_n = \det(\widehat{K}^n(R_q)) \cdot \det(\widehat{K}^{n+1}(R_q))$$

one can write the q_n's, for n odd, as:

$$\begin{aligned} q_{2n+1} &= x_0 x_2 \ldots x_{2n-2} \cdot \frac{\det(\widehat{K}^{2n}(R_q))}{\det(R_q)} \cdot q_1 \\ &= (1 + T_0 a_n) \cdot \frac{\det(\widehat{K}^{2n}(R_q))}{\det(R_q)} \cdot q_1 \end{aligned} \tag{6.16}$$

Moreover one has the following remarkable relation:

$$\det(\widehat{K}^{2n}(R_q)) = \frac{1}{(1 + T_0 a_n)^2} \cdot \left(\det(R_q) + a_n \mathcal{P}_1 + b_n \mathcal{P}_2\right) \tag{6.17}$$

It is not surprising, since the rank of matrix P is one, that Eq. (6.17) is *linear* in term of b_n. In contrast its expression in term of a_n is anything but obvious.

The q_n's, for n odd, finally read:

$$q_{2n+1} = \frac{1}{(1 + T_0 a_n)} \cdot \left(1 + a_n \frac{\mathcal{P}_1}{\det(R_q)} + b_n \frac{\mathcal{P}_2}{\det(R_q)}\right) \cdot q_1 \tag{6.18}$$

One now has to get explicit expressions for $\mathcal{P}_1$ and $\mathcal{P}_2$.

Let us first recall $T_1 = T_0(\widehat{K}(R_q))$, one remarks that one has:

$$\frac{\mathcal{P}_2}{\det(R_q)} = -T_1 \tag{6.19}$$

On the other hand, one notices that recurrences (6.13) and (6.14) can also be written, eliminating ρ_1 and ρ_2, as:

$$\lambda_2 = q_{2n} q_{2n+1} \cdot \frac{(x_{2n} x_{2n+2} - 1)\, x_{2n+1}}{1 - x_{2n+1}} \tag{6.20}$$

$$\lambda_1 = q_{2n+1} q_{2n+2} \cdot \frac{(x_{2n+1} x_{2n+3} - 1)\, x_{2n+2}}{1 - x_{2n+2}} \tag{6.21}$$

In order to write these expressions only in terms of the entries of the matrix R_q, one has to get rid of these factors $q_n q_{n+1}$, which correspond to an artifact of the homogeneity. In this respect let us remark that $T_n q_n = T_{n+2} q_{n+2}$: $w_0 = T_n T_{n+1} q_n q_{n+1}$ *is thus a constant.* Introducing:

$$E_n = \frac{x_{n-1} x_{n+1} - 1}{(x_n - 1)\, T_{n-1} T_n} \cdot x_n \tag{6.22}$$

one obtains $\lambda_2 = w_0 E_{2n+1}$ (and similarly $\lambda_1 = w_0 E_{2n+2}$).

Relation $E_{n+2} = E_n$ is always satisfied, and recurrences (6.13) and (6.14) are finally equivalent to:

$$\lambda_1 = w_0 E_0, \quad \lambda_2 = w_0 E_1 \tag{6.23}$$

Notice that:

$$E_0 = \frac{(\Delta_0 + \Delta_2)\, x_0}{\Delta_1\, T_0^2} \tag{6.24}$$

or, else, written directly in the entries of matrix R_q:

$$E_0 = \frac{m_{22}\,(m_{33} - m_{31} + m_{11} - m_{13})}{(m_{23} - m_{21})^2} \tag{6.25}$$

With these expressions, one first notices the following relation between $\mathcal{P}_1$ and $\mathcal{P}_2$:

$$\frac{\mathcal{P}_1}{\det(R_q)} = \left(1 + \frac{\lambda_1}{2\, q_0\, q_1} + \frac{\mathcal{P}_2\, \Delta_0}{2\, \det(R_q)}\right) \cdot T_0 \tag{6.26}$$

One also gets from relations (6.23) and (6.24):

$$\frac{(\Delta_0 + \Delta_2)}{\Delta_1\, T_0} = \frac{\lambda_1}{q_1\, T_1} \tag{6.27}$$

Recalling $x_n = q_{n+2}/q_n = 1 + T_n\, \Delta_{n+1}$, one gets:

$$\Delta_1 = \frac{q_2 - q_0}{q_0\, T_0}, \quad \Delta_2 = \frac{q_3 - q_1}{q_1\, T_1} \tag{6.28}$$

yielding an explicit expression for $\mathcal{P}_2 = -\, T_1 \cdot \det(R_q)$:

$$\frac{\mathcal{P}_2}{\det(R_q)} = \frac{q_0\, q_2\, (q_3 - q_1) + \lambda_1\, (q_0 - q_2)}{q_0\, q_1\, q_2\, \Delta_0} \tag{6.29}$$

Recalling the two relations between the q_n's, namely recurrences (6.13) and (6.14), and substituting, from (6.15) and (6.18), the q_n's, in terms of the a_n's and b_n's, one gets a_{n+1} and b_{n+1} in terms of a_n and b_n (see for instance Appendix B). One now has a representation of transformation $\widehat{K}^2$, *as a mapping in* CP_2.

Let us also note that, as a consequence of the two simple matricial relations:

$$t(F) = F, \quad t(P) = -P$$

transposition t, can simply be represented as a reflection in the (a_n, b_n)-plane:

$$t: \ (a,b) \ \rightarrow \ (a, \Delta_0 - b) \tag{6.30}$$

From these two representations of t and $\widehat{K}^2$ in the (a_n, b_n)-plane, one gets a representation of $\widehat{I}\, t\, \widehat{I}$, which is actually an involution.

After a change of variables [12]:

$$u_n = \frac{q_0\, q_1}{q_1\, q_2 + q_0\, q_1 + \lambda_2} \cdot \Big(1 + T_0\, a_n\Big),$$
$$v_n = -\frac{q_0\, q_1}{\lambda_2} \cdot \Big(1 + \frac{\mathcal{P}_1}{\det(R_q)}\, a_n + \frac{\mathcal{P}_2}{\det(R_q)}\, b_n\Big) \qquad (6.31)$$

the involutive transformation $\widehat{I}\, t\, \widehat{I}$ takes the remarkably simple form (*independent of any parameter!*):

$$\widehat{I}\, t\, \widehat{I}: \quad (u,v) \;\rightarrow\; (u',v') \;=\; \Big(\frac{u + v - u\,v}{v},\; \frac{u + v - u\,v}{u}\Big) \qquad (6.32)$$

and transformation t is represented as the following (*two parameter*) transformation:

$$t: \quad (u,v) \;\rightarrow\; (u,\; 1 + \epsilon - v + \alpha\, u) \qquad (6.33)$$

where ϵ and α read:

$$\epsilon \;=\; \frac{\lambda_1 - \lambda_2}{\lambda_2},$$
$$\alpha \;=\; -\,\frac{(q_2\,(q_1 + q_3) + \lambda_1)\,(q_1\,(q_0 + q_2) + \lambda_2)}{q_1\, q_2\, \lambda_2} \;=\; -\,\frac{\rho_1\, \rho_2}{\lambda_2} \qquad (6.34)$$

Note that one has the following relation:

$$\frac{u_n}{v_n} \;=\; -\,\frac{\rho_2 \cdot q_{2n+1}}{\lambda_2} \qquad (6.35)$$

Let us now recall again that there does exist an integrable subcase of these mappings associated to class IV. When $\lambda_1 = \lambda_2 = \lambda$ [16], the q_n's actually *satisfy two biquadratic equations* $F(x,y) = 0$ and $F(y,x) = 0$, depending on the parity of n, with:

$$F(x,y) \;=\; (xy - \lambda)(x - \rho_1)(y - \rho_2) - \mu \qquad (6.36)$$

namely:

$$\mu = (\rho_1 - q_{2n+1})\;(\rho_2 - q_{2n})\;(q_{2n}\, q_{2n+1} + \lambda)$$
$$= (\rho_2 - q_{2n+2})\;(\rho_1 - q_{2n+1})\;(q_{2n+1}\, q_{2n+2} + \lambda) \qquad (6.37)$$

It is well known that *biquadratic equations* are associated with elliptic curves [6] [13].

[12] We would like to thank M.P. Bellon for many parallel checks on most of the calculations detailed in this very section.

[13] One should mention the work of several authors who have presented a wide class of mappings of the form $x_{n+1} = f(x_n, x_{n-1})$ which can be seen as discretizations of a second-order ordinary differential equations related to elliptic functions and more precisely biquadratic relations [22–25,28–30].

With these remarkably simple forms for the two involutions generating our group of birational transformations, the integrability condition for the birational mappings of CP_2 simply reads $\epsilon = 0$ (or equivalently $E_0 = E_1$). For this integrable $\epsilon = 0$ subcase the group generated by transformations (6.32) and (6.33) yields a foliation of the (u, v)-plane in terms of curves, which form a *linear pencil of elliptic curves*. This can be seen noticing that, for $\epsilon = 0$, an algebraic expression $\mathcal{I}$ is actually invariant under both transformations $\widehat{I}\,t\,\widehat{I}$ and t:

$$\mathcal{I} \;=\; \frac{(1-u)\,(1-v)\,(v-\alpha\,u)}{u} \tag{6.38}$$

One should also remark that $\epsilon = 0$ is not the only integrability condition for these birational mappings. One can actually verify straightforwardly that $\epsilon = -1$ is also an *integrability condition*: it corresponds to a (rational) degeneracy of the mapping. Condition $\epsilon = -1$ drastically simplifies transformation (6.33) which becomes:

$$t: \;\; (u,v) \;\rightarrow\; (u, \, -v + \alpha\, u) \tag{6.39}$$

One immediately gets an algebraic invariant under transformations (6.32) and (6.33):

$$\mathcal{J} \;=\; \Bigl(\frac{v}{u} - \frac{\alpha}{2}\Bigr)^2 \tag{6.40}$$

This $\epsilon = -1$ case, which corresponds to $\lambda_1 = 0$, yields a *simple rational parameterization* of the iteration.

For heuristic reasons let us consider the $\alpha = 0$ case (which happens to correspond to a *rational parameterization*, when $\epsilon = 0$). From relation (6.38) the variable u can be simply written in terms of the variable v and the invariant $\mathcal{I}$:

$$u \;=\; \frac{v\,(1-v)}{\mathcal{I} + v\,(1-v)} \tag{6.41}$$

and the mappings t and $\widehat{I}\,t\,\widehat{I}$ respectively read:

$$t: \;\; (\mathcal{I},v) \rightarrow \;(\mathcal{I}',v') \;=\; \Bigl(\mathcal{I}\,(1-\frac{\epsilon}{v})\,(1-\frac{\epsilon}{v-1}), \; 1-v+\epsilon\Bigr) \tag{6.42}$$

$$\widehat{I}\,t\,\widehat{I}: \;\; (\mathcal{I},v) \rightarrow \;(\mathcal{I}',v') \;=\; \Bigl(\mathcal{I}, \; 1-\frac{\mathcal{I}}{v-1}\Bigr) \tag{6.43}$$

These very simple representations of the birational transformations of class IV enabled us to perform a large number of numerical calculations which confirm the analysis performed in [16]. The iterations of these transformations often yield orbits which look like curves [16]. The situation, as far as the visualization of *a single orbit* is concerned, is similar to the situation encountered

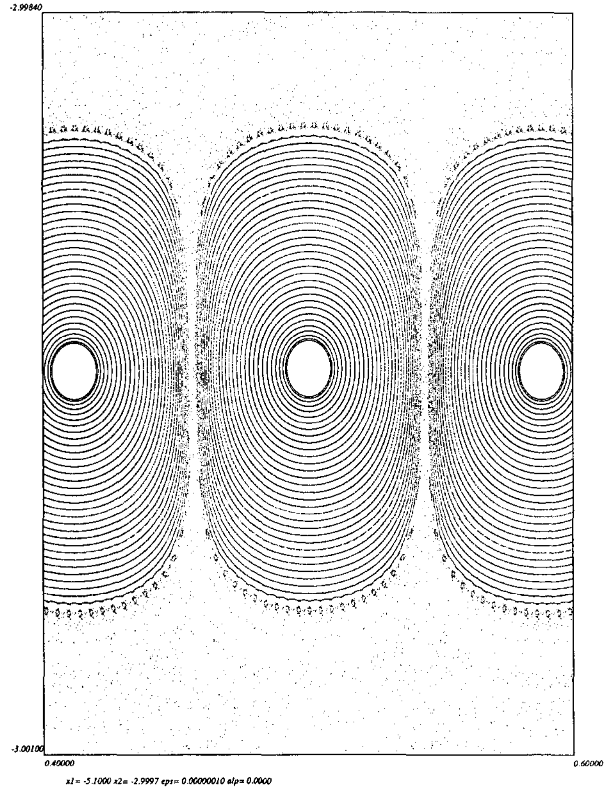

Fig. 4. Hundred orbits corresponding to the iteration of $\widehat{K}^2$, for class IV, in the $(\mathcal{I}, v)$-plane.

in the Henon-Heiles mapping [27]. Let us note however, that *one does not consider perturbations near a fixed point anymore, but in the neighborhood of the whole integrability condition.*

Fig. 4 shows a set of one hundred orbits corresponding to the iteration of $\widehat{K}^2$ in the $(\mathcal{I}, v)$-plane. The orbits are very regular: Fig. 4 gives an illustration of the "almost integrability" described in [16] and also reminiscent of the situation one encounters near elliptic points in dynamical systems [31–34]. At this point it is important to make the following comment: since they are generated by involutions, all our birational transformations are such that transformation $\widehat{K}$ and transformation $\widehat{K}^{-1}$ are conjugated ($\widehat{K} = t \cdot \widehat{I} = t \cdot \widehat{K}^{-1} \cdot t$). When transformation $\widehat{K}$ (or more precisely $\widehat{K}^2$) can be reduced to a mapping on only two variables this means that one has some area preserving properties and one can recover the features of two-dimensional dynamics (elliptic versus hyperbolic points, Arnold's diffusion ... [31–34]), and this explains to a great extend the regularities one encounters here with class IV, even when the mapping is not integrable. However this conjugation properties does not seem sufficient to explain the regularities observed for the other (non-integrable) classes for which the dynamics cannot be reduced to two-dimension any more: volume-(or hypervolume) preserving properties are no longer sufficient to explain our regularities.

6.2. Factorizations for the Integrable subcase of Class IV

The birational transformations for class IV do not generically correspond to integrable mappings [16]. They have been seen to yield an *exponential growth* of the calculations. However, *there exists a subvariety* of $\mathbb{C}P_{q^2-1}$ on which these birational transformations *become integrable*, yielding *algebraic elliptic curves* for arbitrary q. Recalling (6.25), this integrability condition $E_{n+2} = E_{n+1}$ (or equivalently $E_0 = E_1$) can be given, from (6.23), in terms of the entries of the initial matrix $M_0 = R_q$:

$$\frac{m_{22}(m_{33} - m_{31} + m_{11} - m_{13})}{(m_{23} - m_{21})^2} = \text{same expression with } m_{i,j} \to K(M_0)_{i,j} \tag{6.44}$$

An example of a one parameter dependent 4×4 matrix satisfying this integrability condition is given in Appendix C.

In this framework a question naturally pops out: how does the factorizations (4.16), (4.17) and (4.18) modify when one restricts to this integrability condition ? In particular, does the exponential growth of the calculations becomes a polynomial growth when restricted to this integrability condition ?

Restricted to this integrable subcase the factorizations corresponding to the iterations of K for class IV are such that *the first five iterations yield the same factorizations* as for the generic (non-integrable) case:

$$\widetilde{M}_1 = K(\widetilde{M}_0), \quad \widetilde{f}_1 = \det(\widetilde{M}_0), \quad \widetilde{f}_2 = \frac{\det(\widetilde{M}_1)}{\widetilde{f}_1^2}, \quad \widetilde{M}_2 = \frac{K(\widetilde{M}_1)}{\widetilde{f}_1},$$
$$\widetilde{f}_3 = \frac{\det(\widetilde{M}_2)}{\widetilde{f}_1 \cdot \widetilde{f}_2^2}, \quad \widetilde{M}_3 = \frac{K(\widetilde{M}_2)}{\widetilde{f}_2}, \quad \widetilde{f}_4 = \frac{\det(\widetilde{M}_3)}{\widetilde{f}_1^3 \cdot \widetilde{f}_2 \cdot \widetilde{f}_3^2}, \quad \widetilde{M}_4 = \frac{K(\widetilde{M}_3)}{\widetilde{f}_1^2 \cdot \widetilde{f}_3},$$
$$\widetilde{f}_5 = \frac{\det(\widetilde{M}_4)}{\widetilde{f}_1^2 \cdot \widetilde{f}_2^3 \cdot \widetilde{f}_3 \cdot \widetilde{f}_4^2}, \quad \widetilde{M}_5 = \frac{K(\widetilde{M}_4)}{\widetilde{f}_1 \cdot \widetilde{f}_2^2 \cdot \widetilde{f}_4} \quad \cdots \tag{6.45}$$

The difference with the generic factorizations (4.16) comes with the next iterations:

$$\widetilde{f}_6 = \frac{\det(\widetilde{M}_5)}{\widetilde{f}_1^3 \cdot \widetilde{f}_2^2 \cdot \widetilde{f}_3^3 \cdot \widetilde{f}_4 \cdot \widetilde{f}_5^2}, \quad \widetilde{M}_6 = \frac{K(\widetilde{M}_5)}{\widetilde{f}_1^2 \cdot \widetilde{f}_2 \cdot \widetilde{f}_3^2 \cdot \widetilde{f}_5},$$
$$\widetilde{f}_7 = \frac{\det(\widetilde{M}_6)}{\widetilde{f}_2^3 \cdot \widetilde{f}_3^2 \cdot \widetilde{f}_4^3 \cdot \widetilde{f}_5 \cdot \widetilde{f}_6^2}, \quad \widetilde{M}_7 = \frac{K(\widetilde{M}_6)}{\widetilde{f}_2^2 \cdot \widetilde{f}_3 \cdot \widetilde{f}_4^2 \cdot \widetilde{f}_6},$$

$$\widetilde{f}_8 = \frac{\det(\widetilde{M}_7)}{\widetilde{f}_3^3 \cdot \widetilde{f}_4^2 \cdot \widetilde{f}_5^3 \cdot \widetilde{f}_6 \cdot \widetilde{f}_7^2}, \quad \widetilde{M}_8 = \frac{K(\widetilde{M}_7)}{\widetilde{f}_3^2 \cdot \widetilde{f}_4 \cdot \widetilde{f}_5^2 \cdot \widetilde{f}_7}, \cdots \tag{6.46}$$

One remarks that, compared to factorizations (4.16), $\widetilde{f}_1$ *factorizes one more time in* $\det(\widetilde{M}_5)$ and $K(\widetilde{M}_5)$. These additional factorizations propagate and yield for arbitrary n:

$$\det(\widetilde{M}_n) = \widetilde{f}_{n+1} \cdot \widetilde{f}_n^2 \cdot \widetilde{f}_{n-1} \cdot \widetilde{f}_{n-2}^3 \cdot \widetilde{f}_{n-3}^2 \cdot \widetilde{f}_{n-4}^3 \tag{6.47}$$

$$K(\widetilde{M}_n) = \widetilde{M}_{n+1} \cdot \widetilde{f}_n \cdot \widetilde{f}_{n-2}^2 \cdot \widetilde{f}_{n-3} \cdot \widetilde{f}_{n-4}^2 \tag{6.48}$$

which yield:

$$\frac{K(\widetilde{M}_n)}{\det(\widetilde{M}_n)} = \frac{\widetilde{M}_{n+1}}{\widetilde{f}_{n-4} \cdot \widetilde{f}_{n-3} \cdot \widetilde{f}_{n-2} \cdot \widetilde{f}_{n-1}\, \widetilde{f}_n \cdot \widetilde{f}_{n+1}} = \widehat{K}(\widetilde{M}_n) \tag{6.49}$$

The correspondence between the f_n's, associated to the generic (non-integrable) case (see factorizations (4.16)), and these new factorizing polynomials associated to the integrable subcase (see factorizations (6.45)), denoted $\widetilde{f}_n$, read:

$$f_6 = \widetilde{f}_6 \cdot f_1, \quad f_7 = \widetilde{f}_7 \cdot f_2 \cdot f_1, \quad f_8 = \widetilde{f}_8 \cdot f_3 \cdot f_2 \cdot f_1^2, \quad \cdots$$

From Eqs. (6.47), (6.48) and (6.49), one gets linear recurrences on the degrees of the polynomials $\det(\widetilde{M}_n)$ and $\widetilde{f}_n$ (respectively the $\widetilde{\alpha}_n$'s and $\widetilde{\beta}_n$'s):

$$\widetilde{\alpha}_{n+2} = 3\,\widetilde{\beta}_{n-2} + 2\,\widetilde{\beta}_{n-1} + 3\,\widetilde{\beta}_n + \widetilde{\beta}_{n+1} + 2\,\widetilde{\beta}_{n+2} + \widetilde{\beta}_{n+3} \tag{6.50}$$

$$3\,\widetilde{\alpha}_{n+2} = \widetilde{\alpha}_{n+3} + 4\left(2\,\widetilde{\beta}_{n-2} + \widetilde{\beta}_{n-1} + 2\,\widetilde{\beta}_n + \widetilde{\beta}_{n+2}\right) \tag{6.51}$$

and

$$\widetilde{\alpha}_{n+3} + \widetilde{\alpha}_{n+2} = 4\left(\widetilde{\beta}_{n-2} + \widetilde{\beta}_{n-1} + \widetilde{\beta}_n + \widetilde{\beta}_{n+1} + \widetilde{\beta}_{n+2} + \widetilde{\beta}_{n+3}\right) \tag{6.52}$$

giving the following relations on the generating functions:

$$x \cdot \widetilde{\alpha}(x) = \widetilde{\beta}(x) \cdot (1 + 2x + x^2 + 3x^3 + 2x^4 + 3x^5) \tag{6.53}$$

$$(3x - 1) \cdot \widetilde{\alpha}(x) = 4x \cdot \widetilde{\beta}(x) \cdot (1 + 2x^2 + x^3 + 2x^4) - 4 \tag{6.54}$$

and

$$(1 + x) \cdot \widetilde{\alpha}(x) = 4\,(1 + x + x^2 + x^3 + x^4 + x^5) \cdot \widetilde{\beta}(x) + 4 \tag{6.55}$$

The two generating functions $\widetilde{\alpha}(x)$ and $\widetilde{\beta}(x)$ read:

$$\widetilde{\alpha}(x) = \frac{4(1 + x - x^2 + 3x^3)}{(1 + x)(1 - x)^3}, \quad \widetilde{\beta}(x) = \frac{4x}{(1 + x)(1 - x)^3(1 + x + x^2)} \tag{6.56}$$

The linear relations corresponding to the right action of K, namely factorizations (3.10) and (3.11), are *still valid* when one restricts to this integrable case. Therefore the generating functions $\tilde{\mu}(x)$ and $\tilde{\nu}(x)$ still verify relation (3.17) and $\tilde{\mu}(x)$ and $\tilde{\nu}(x)$ read:

$$\tilde{\mu}(x) = \frac{x\,(2 - x + x^3 + x^4 - x^5)}{(1+x)\,(1+x+x^2)\,(1-x)^3}, \quad \tilde{\nu}(x) = \frac{x\,(1 - x + 2x^2)}{(1-x)^3\,(1+x)} \tag{6.57}$$

These relations clearly show that the additional factorizations occurring for this integrable subcase remarkably yield factorizations like (6.49), (6.48), or (6.47) bearing on a *fixed number of polynomials* $\tilde{f}_n$ and even more, to a *polynomial growth* of the calculations instead of the exponential growth previously described (see Section 4.3). One should however note *the occurrence of a third root of unity* in the denominator of the generating functions $\tilde{\beta}(x)$ and $\tilde{\mu}(x)$ (and not for the generating functions $\tilde{\alpha}(x)$ and $\tilde{\nu}(x)$). Of course the occurrence of this new root of unity, different from ± 1, *does not rule out the polynomial growth.*

Finally it is important to note that , in this integrable subcase, the new polynomials $\tilde{f}_n$ defined from the factorization relations (6.47) and (6.48) *do satisfy recurrences bearing on products of a fixed number of polynomials* (instead of "pseudo" recurrences like (4.28)):

$$\frac{\tilde{f}_{n+2}\,\tilde{f}_{n+7}\,\tilde{f}_{n+9} - \tilde{f}_{n+3}\,\tilde{f}_{n+5}\,\tilde{f}_{n+10}}{\tilde{f}_{n+3}\,\tilde{f}_{n+7}\,\tilde{f}_{n+8} - \tilde{f}_{n+4}\,\tilde{f}_{n+5}\,\tilde{f}_{n+9}} = \frac{\tilde{f}_{n+1}\,\tilde{f}_{n+6}\,\tilde{f}_{n+8} - \tilde{f}_{n+2}\,\tilde{f}_{n+4}\,\tilde{f}_{n+9}}{\tilde{f}_{n+2}\,\tilde{f}_{n+6}\,\tilde{f}_{n+7} - \tilde{f}_{n+3}\,\tilde{f}_{n+4}\,\tilde{f}_{n+8}} \tag{6.58}$$

Such a recurrence is very similar to recurrences (3.20) or (3.21) and can be written introducing well-suited variables $\tilde{q}_n$'s defined by:

$$\tilde{q}_{n+4} = \frac{\tilde{f}_n\,\tilde{f}_{n+5}}{\tilde{f}_{n+2}\,\tilde{f}_{n+3}} \tag{6.59}$$

In terms of these variables $\tilde{q}_n$, recurrence (6.58) becomes:

$$\frac{\tilde{q}_{n+2} - \tilde{q}_{n+1}}{\tilde{q}_{n+1}\cdot\tilde{q}_{n+3} - \tilde{q}_n\cdot\tilde{q}_{n+2}} = \frac{\tilde{q}_{n+3} - \tilde{q}_{n+2}}{\tilde{q}_{n+2}\cdot\tilde{q}_{n+4} - \tilde{q}_{n+1}\cdot\tilde{q}_{n+3}} \tag{6.60}$$

which can be integrated as:

$$\tilde{q}_{n+2} - \lambda'\cdot\tilde{q}_{n+1}\cdot\tilde{q}_{n+3} = \rho' \tag{6.61}$$

and in a second step:

$$\frac{1 + \lambda'\cdot\tilde{q}_{n+1}}{\tilde{q}_{n+2}} = \frac{1 + \lambda'\cdot\tilde{q}_{n+4}}{\tilde{q}_{n+3}} \tag{6.62}$$

$$\frac{(1+\lambda'\cdot\widetilde{q}_{n+1})(1+\lambda'\cdot\widetilde{q}_{n+2})(1+\lambda'\cdot\widetilde{q}_{n+3})}{\widetilde{q}_{n+2}} = \frac{(1+\lambda'\cdot\widetilde{q}_{n+2})(1+\lambda'\cdot\widetilde{q}_{n+3})(1+\lambda'\cdot\widetilde{q}_{n+4})}{\widetilde{q}_{n+3}} = \mu' \tag{6.63}$$

which *finally yields a biquadratic relation*:

$$(1+\lambda'\cdot\widetilde{q}_{n+1})(1+\lambda'\cdot\widetilde{q}_{n+2})(\widetilde{q}_{n+1}+\widetilde{q}_{n+2}-\rho') = \mu'\,\widetilde{q}_{n+1}\,\widetilde{q}_{n+2} \tag{6.64}$$

Recalling the two biquadratic relations (6.37) given in Section 6.1, one notices that, taking into account the homogeneity of the q_n's, one can barter these two biquadratics for a single one (changing q_{2n} into q_{2n}/ρ_2 and q_{2n+1} into q_{2n+1}/ρ_1). This last biquadratic is apparently different from (6.64): one would like to see the relation between these two biquadratics bearing respectively on the $\widetilde{q}_n$'s introduced here, and the q_n's introduced in Section 4.3. The relation between the q_n's and the $\widetilde{q}_n$'s reads as follows:

$$\frac{\widetilde{q}_n}{\widetilde{q}_0} = \frac{q_n\,q_{n+1}}{q_0\,q_1} \tag{6.65}$$

After straightforward calculations (corresponding to introduce the product $q_n\,q_{n+1}$ in recurrences (6.13) and (6.14)) one can show in the integrable case, $\lambda_1=\lambda_2=\lambda$, that the biquadratic relation (6.37) yields the biquadratic relation (6.64) with the following correspondence:

$$\lambda' = \frac{1}{\lambda}\,,\quad \rho' = -\frac{\lambda^2+\mu-\lambda\,\rho_1\,\rho_2}{\lambda}\,,\quad \mu' = \frac{\mu}{\lambda^3} \tag{6.66}$$

or conversely:

$$\lambda = \frac{1}{\lambda'}\,,\quad \rho_1\,\rho_2 = \frac{\lambda'+\lambda'^2\,\rho'+\mu'}{\lambda'^2}\,,\quad \mu = \frac{\mu'}{\lambda'^3} \tag{6.67}$$

Since variable $\widetilde{q}_n$ is a homogeneous variable, it is tempting to write recurrence (6.60) in terms of the ratio of two successive $\widetilde{q}_n$'s. In fact this ratio *happens to coincide exactly with the variable* x_n defined by (6.11):

$$x_n = \frac{\widetilde{q}_{n+1}}{\widetilde{q}_n} \tag{6.68}$$

The variable x_n can be written either in terms of the f_n's or in terms of the $\widetilde{f}_n$'s:

$$x_{n+6} = \frac{f_{n+8}\,f_{n+4}\,f_n\,f_{n-4}\,f_{n-8}\cdots}{f_{n+7}\,f_{n+6}\,f_{n+2}\,f_{n-2}\,f_{n-6}\cdots} = \frac{\widetilde{f}_{n+8}\,\widetilde{f}_{n+4}\,\widetilde{f}_{n+3}}{\widetilde{f}_{n+7}\,\widetilde{f}_{n+6}\,\widetilde{f}_{n+2}} \tag{6.69}$$

The relations between the initial values of variables q_n, $\widetilde{q}_n$, f_n, $\widetilde{f}_n$ and of the variables x_n's are given in Appendix D.

With these new variables the integrable recurrence (6.60) reads:

$$\frac{x_{n+2}-1}{x_{n+1}\,x_{n+3}-1} = \frac{x_n\,x_{n+2}}{x_{n+1}} \cdot \frac{x_{n+1}-1}{x_n\,x_{n+2}-1} \tag{6.70}$$

Let us note that, combining (6.70) with itself where n has been shifted by one, one recovers:

$$\frac{x_{n+3}-1}{x_{n+2}\,x_{n+4}-1} = x_n\,x_{n+3} \cdot \frac{x_{n+1}-1}{x_n\,x_{n+2}-1} \tag{6.71}$$

which coincides with relation (4.30). In fact one can actually show, but this will not be performed here, that the x_n's corresponding to class IV do satisfy a *whole hierarchy of recurrences* in the same way it has been proved for class I in [14].

Similarly to [14], one can consider these recurrences for themselves, without referring to our birational transformations acting on $q \times q$ matrices anymore. Again one can see that some of these recurrences *are integrable recurrences* (for instance recurrence (6.70)) and some *are* (generically) *not integrable* (for instance recurrence (6.71)). All the analysis performed in [14] can be applied to the hierarchy emerging from (6.70)), in particular the fact that recurrences like (6.70)) are equivalent to another recurrence of the hierarchy, namely:

$$\frac{x_{n+2}-1}{x_{n+1}\,x_{n+3}-x_{n+2}} = x_n\,x_{n+2} \cdot \frac{x_{n+1}-1}{x_n\,x_{n+2}-x_{n+1}} \tag{6.72}$$

7. Conclusion

The analysis of the factorizations corresponding to the six previously defined classes of transpositions has shed some light on the relations between different properties and structures related to integrable mappings such as the *existence of factorizations* involving a fixed number of polynomials, *the polynomial growth* of the complexity of the iterations of these mappings seen as homogeneous transformations, the existence of *recurrences* bearing on the factorized polynomials f_n's or on other variables such as the x_n's and the *integrability of the mappings.*

A single factorization relation *independent of* q (relation (3.9)) has been shown to be satisfied for classes I, III and V. All these classes *satisfy factorizations where products of a fixed number* of polynomials f_n occur (see (4.31), (4.32)). The polynomial growth of the complexity of the iterations suites quite well with factorizations where products of a fixed number of polynomials occur: this is actually the case for classes I and III which actually verify *the same factorizations* on the f_n's. In fact, surprisingly, it will be shown in [15] (on an example corresponding to the symmetries of a three dimensional vertex model) that a *polynomial growth of the complexity can actually occur even* *with "string-like" factorizations* (like (4.40)). On the other hand, one sees with class V that one can have a factorization of the iteration with a *fixed number of polynomials and, in the same time, an exponential growth of the complexity* of the iterations. Among the two classes satisfying exactly the same factorizations, that is: I and III, *only the first one verifies recurrences on the polynomials* f_n's. For class I, for arbitrary q, these recurrences are *integrable* recurrences and probably are the birational mappings in CP_{q^2-1}. For $q = 4$ the iteration of these transformations associated to classes I, II and III yield algebraic elliptic curves. For $q \geq 5$ class III, yields quite regular orbits which often look like curves, or like a set of curves which seem to lie on higher dimensional varieties. However, the actual status of these higher dimensional varieties is not very clear (abelian varieties ...). Class III is thus, for $q \geq 5$, an interesting class since it would provide *an example of polynomial growth* (and in fact exactly the same factorization as the "nice" class I) *with orbits densifying nice algebraic varieties (abelian surfaces, product of elliptic curves? ...), but not curves anymore.*

A "string-like" factorization relation *independent of* q (relation (4.39)) is satisfied by the two classes IV, VI and also for class II but only for $q \geq 5$. These three sets of factorizations with a "growing" number of polynomials (see relation (4.39)) correspond to an exponential growth of the complexity of the iterations. However, one remarks that class IV *has actually a recurrence on the variables* x_n *but not of course on the variables* f_n's: these variables cannot satisfy simple recurrences like (3.20), but *they satisfy "pseudo-recurrences" like* (4.28). *This recurrence, on the* x_n*'s, is not generically an integrable one* [16]. It however yields orbits which look like *curves* for some domain of the initial conditions [16]. Such a recurrence is an illustration of a transition from integrability to weak chaos, through situations visually similar to the one encountered in the Henon-Heiles mapping [27], or to the situation encountered near elliptic points in the theory of (hamiltonian) dynamical systems of two variables [31–34]. We hope that class II will help understanding the structures of elliptic points for three-dimensional mappings (see relation (4.12)).

Note however that we have been able to prove that the integrable subcase of class IV is actually valid for arbitrary q: *it thus provides an example of integrable mapping in arbitrary dimension even infinite.*

For all these six classes, one has recurrences on a fixed number of variables (see (5.71)). The elimination of these variables may yield quite involved algebraic relations on the remaining variable x_n. The recurrences associated to classes I and IV are the only one which are *recurrences* and not involved algebraic relations between the x_n's. Note that when a recurrence is integrable the corresponding birational transformations are probably also integrable: in fact this appears as the only "handable" way to find the integrable subcase of class IV ...

To sum up the relations between all these various structures and properties are subtle: the only systematic relation being that integrability seems to imply the polynomial growth of the iterations. Let us however recall that *one can have polynomial growth with orbits densifying algebraic surfaces* [15].

The analysis of the factorization of the iteration corresponding to birational transformations such as the one studied here can be seen as a new method to analyze birational mappings and therefore the symmetries of lattice models, in particular models in dimension greater than two. In a forthcoming publication [15] we will consider three-dimensional (and higher dimensional) vertex models and show that the birational transformations associated with these models yield algebraic surfaces *and* polynomial growth.

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Appendix A. A comment on the finite order conditions

In the framework of finite order conditions it is worth mentioning the following very general result (it works with the six classes defined in Section 3.1, and can even be generalized to arbitrary permutations of a $q \times q$ matrix [15]).

For an arbitrary transposition one imposes the two conditions:

$$\Delta_1(R_q) = 0 \tag{A.1}$$

and

$$\Delta_1(t(R_q)) = 0 \tag{A.2}$$

where t denotes a transposition of two entries. The very definition of Δ_1 yields:

$$\Delta_1(R_q) = 0 \Rightarrow \widehat{K}(R_q) = \widehat{I}(R_q) \Rightarrow \widehat{I}(\widehat{K}(R_q)) = R_q \Rightarrow \widehat{K}^2(R_q) = t(R_q) \tag{A.3}$$

Since one assumes $\Delta_1(t(R_q)) = 0$, one similarly gets:

$$\widehat{K}^2(t(R_q)) = t(t(R_q)) = R_q \tag{A.4}$$

Hence for every matrix R_q satisfying (A.1) and (A.2), transformation $\widehat{K}$ is a transformation of *order four*, that is:

$$\widehat{K}^4(R_q) = R_q \tag{A.5}$$

Let us note that the (codimension two) variety given by the two conditions (A.1) and (A.2) contains remarkably simple linear subvarieties. Let us consider for instance a transposition of class VI, namely t exchanging m_{11} and m_{12}, the following conditions yield (A.5):

$$m_{32} = 0, \; m_{33} = 0, \; \ldots, \; m_{3q} = 0 \tag{A.6}$$

These considerations can be generalized for very general permutations of the entries. For instance, let us consider the case where the transposition t is replaced by a 3-cycle C. One can also find remarkably simple conditions on the entries of the matrix such that $\widehat{K}$ is of finite order. Considering the 3-cycle C:

$$m_{11} \rightarrow m_{13}, \quad m_{13} \rightarrow m_{12}, \quad m_{12} \rightarrow m_{11} \tag{A.7}$$

and the following vanishing conditions on the entries:

$$m_{42} = 0, \; m_{43} = 0, \; \ldots, \; m_{4q} = 0 \tag{A.8}$$

Restricted to conditions (A.8) transformation $\widehat{K}$ is such that $\widehat{K}^2 = C$, hence, $\widehat{K}^6 = \mathcal{I}d$, when one restricts to the (linear) subvariety (A.8).

Appendix B

From the parameterization of the (a_n, b_n)-plane (see relation (6.1)), one can actually get the mapping of CP_2 corresponding to $\widehat{K}^2$:

$$\begin{aligned} a_{n+1} &= -a_n + k_4 + \frac{k_5\, b_n + k_6}{k_1\, a_n + k_2\, b_n + k_3} \\ b_{n+1} &= \Delta_0 - b_n + (k_5\, b_n + k_6) \\ &\quad \cdot \left(k_7 + \frac{k_8}{1 + T_0\, a_n} + \frac{k_9\,(k_5\, b_n + k_6)}{(1 + T_0\, a_n)\,(k_1\, a_n + k_2\, b_n + k_3)}\right) \end{aligned} \tag{B.1}$$

where the k_α's denote *rational* expressions depending only on the initial matrix R_q. However the k_α's are not independent, *they do satisfy additional relations*:

$$k_6 = \frac{k_5\,(k_3\, T_0 - k_1)}{k_2\, T_0}, \quad k_7 = \frac{2}{k_5}, \quad k_8 = -\frac{2 + k_4\, T_0}{k_5}, \quad k_9 = -\frac{T_0}{k_5} \tag{B.2}$$

Therefore $\widehat{K}^2$ is represented as a birational transformation in CP_2 depending on *six parameters*. These parameters are of course functions of the q^2 homogeneous entries of the initial matrix R_q. Among the six parameters one can actually drop out four parameters corresponding to (independent) dilatations and translations of variables a_n and b_n, yielding to only two remaining parameters like in Section 6.1.

Appendix C

Let us give an example of one parameter dependent 4×4 matrix satisfying the integrability condition (6.44):

$$\frac{M_0}{3450543} = \begin{bmatrix} -85 & 2636150642/3450543 - u & -78 & -35 \\ 97 & 50 & 100 & 56 \\ 49 & u - 55 & 62 & -59 \\ 45 & -8 & 62 & 92 \end{bmatrix} \tag{C.1}$$

Appendix D

Let us give the expression of the first x_n's in terms of the first f_n's:

$$x_0 = \frac{f_2}{f_1}, \quad x_1 = \frac{f_3}{f_2 f_1}, \quad x_2 = \frac{f_4}{f_3 f_2}, \quad x_3 = \frac{f_5 f_1}{f_4 f_3}, \quad x_4 = \frac{f_6 f_2}{f_5 f_4} \tag{D.1}$$

The expression of the first x_n's in terms of the first $\widetilde{f}_n$ read:

$$x_0 = \frac{\widetilde{f}_2}{\widetilde{f}_1}, \quad x_1 = \frac{\widetilde{f}_3}{\widetilde{f}_2 \widetilde{f}_1}, \quad x_2 = \frac{\widetilde{f}_4}{\widetilde{f}_3 \widetilde{f}_2}, \quad x_3 = \frac{\widetilde{f}_5 \widetilde{f}_1}{\widetilde{f}_4 \widetilde{f}_3}, \quad x_4 = \frac{\widetilde{f}_6 \widetilde{f}_2}{\widetilde{f}_5 \widetilde{f}_4} \tag{D.2}$$

The first homogeneous variables q_n are given in terms of the f_n's as:

$$\begin{aligned} q_2 &= \frac{f_2}{f_1} \cdot q_0, \quad q_3 = \frac{f_3 f_1}{f_2} \cdot q_1, \quad q_4 = \frac{f_4}{f_3 f_1} \cdot q_0, \quad q_5 = \frac{f_5}{f_4 f_2} \cdot q_1, \\ q_6 &= \frac{f_6 f_2}{f_5 f_3 f_1} \cdot q_0, \quad q_7 = \frac{f_7 f_3}{f_6 f_4 f_2 f_1} \cdot q_1, \quad q_8 = \frac{f_8 f_4}{f_7 f_5 f_3 f_1} \cdot q_0, \\ q_9 &= \frac{f_9 f_5}{f_8 f_6 f_4 f_2} \cdot q_1, \quad q_{10} = \frac{f_{10} f_6 f_2}{f_9 f_7 f_5 f_3 f_1} \cdot q_0, \\ q_{11} &= \frac{f_{11} f_7 f_3}{f_{10} f_8 f_6 f_4 f_2} \cdot q_1, \quad q_{12} = \frac{f_{12} f_8 f_4}{f_{11} f_9 f_7 f_5 f_3 f_2} \cdot q_0 \end{aligned} \tag{D.3}$$

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