New Exact Results for the Potts Model

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We present new exact results for the checkerboard q-state Potts model with a magnetic field. An exact expression for the partition function is given on some disorder varieties. We argue that the partition function exhibits, even in the presence of a magnetic field, an unexpected S_4 symmetry. A universal and exact expression for the magnetization discontinuity at the first-order phase-transition point ($q > 4$) is proposed. This expression is shown to depend only on q and to be independent of the coupling constants.

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The Potts model has been the subject of intense research in the last few years.[1] Recently a large number of new exact results have been obtained for this model which appears as an important testing ground in lattice statistics and for the theory of critical phenomena. For instance, one should mention the recent results for the critical exponents of the two-dimensional Potts model, deduced from the conformal covariance assumption,[2] where a whole family of exponents was obtained in a compact elegant form (the Kac determinant formula). The same model has also been considered from the exact integrability point of view.[3] As an extension of this point of view, some exact results have been obtained. For instance, an inversion relation has been found and the critical partition function has been calculated. The critical variety and the critical partition function were recovered from the infinite discrete automorphy group exhibited for this model.[4]

In this Letter, we present a new kind of exact results relative to the checkerboard q-state Potts model with a magnetic field, which is sufficiently general to reduce in some appropriate limits to more known cases. Our purpose is (i) to give the exact expression for the partition function on a new disorder variety; (ii) to present various arguments coming from exact results and different expansions, pointing out the existence of an unexpected S_4 symmetry; and (iii) to give an exact expression for the magnetization discontinuity ($q > 4$). One should note the remarkable simplicity of the results, shedding a new light on the analytical structure of the Potts model, in particular on its dependence on the different coupling constants.

The Potts model Hamiltonian on a checkerboard lattice with N spins is given by

$$-\beta\mathscr{H} = \sum_{\langle ij\rangle} K_{ij}\delta_{\sigma_i,\sigma_j} + H\sum_i \delta_{\sigma_i,0},$$

where σ_i is the spin variable at the site i belonging to Z_q, δ denotes the Kronecker delta symbol, and K_{ij} and H denote the interactions and magnetic field, respectively. The first sum is taken over nearest-neighbor pairs. Four coupling constants K_i ($i = 1$–4) are involved (Fig. 1). For convenience, we use the following notations: $a = e^{K_1}$, $b = e^{K_2}$, $c = e^{K_3}$, $d = e^{K_4}$, $h = e^H$, and $Z = \sum_{\{\sigma\}} \exp(-\beta\mathscr{H}\{\sigma\})$ the partition function.

Let us first recall some of the exact results on this model in the absence of a field. It is well known[1] that Z can be calculated in more or less simple limits: (i) $q \to 0$ which maps into a free-fermion limit of an inhomogeneous six-vertex model,[5] (ii) $q = 2$ which is known as the generalized square Ising model.[6] Furthermore, the critical partition function, for all q, has been deduced either from Z-invariance considerations[7] or from the automorphy group.[8] The critical variety is

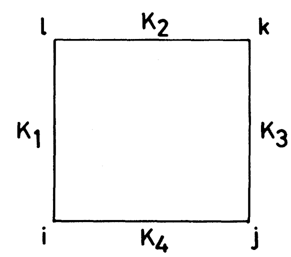

FIG. 1. Elementary cell of the checkerboard Potts model.

given by

$$(q-1)(q-3)=abcd-(q-2)(a+b+c+d)-(ab+ac+ad+bc+bd+cd). \tag{1}$$

On that critical variety, the internal energy as well as the latent heat $(q>4)$ has also been calculated.[9] Besides in the neighborhood of the latter, there are no available exact results, for arbitrary q. In this respect, it was useful to extend the notion of disorder point[10,11] to general models such as the one considered here. The existence of such a solution is relevant from both qualitative (phase diagram, correlation function behavior, etc.) and quantitative (analytical structure of Z, constraints on various expansions, etc.) points of view. The disorder variety of the model without field can be written as

$$\frac{d-1}{(q-1)d+1}+\frac{a-1}{a+q-1}\,\frac{b-1}{b+q-1}\,\frac{c-1}{c+q-1}=0, \tag{2}$$

where the partition function takes the following simple form:

$$Z(a,b,c,d;h=1)=[(a-1)(b-1)(c-1)d/(1-d)]^{1/2}. \tag{3}$$

This solution reduces, as it should, to Rujan's result in the triangular lattice limit[12] $(c\rightarrow\infty)$.

Coming back to the general case of nonzero magnetic field, one now gets two conditions instead of one [Eq. (2)]. Further details will be given elsewhere.[13] Let us just say that these two conditions are also symmetric under the complete permutations of a, b, and c, and that the exact expression for the partition function Z remains given by Eq. (3). Let us also remark that in the limit $c=\infty$ the condition which gives the magnetic field in terms of the coupling constants,

$$h=\frac{a-1}{a+q-1}\,\frac{b-1}{b+q-1}\,\frac{1+(q-1)d}{1-d}, \tag{4}$$

identifies with Eq. (2) (in the $c=\infty$ limit for $h=1$), and that the other condition is a relation between a, b, and d (independent of the field) which is nothing else but the criticality condition for the triangular Potts model in the absence of a magnetic field,

$$abd-(a+b+d)=(q-2). \tag{5}$$

The rational expression for Z on these disorder varieties implies in particular the absence of any analytical singularity on these varieties.

In general two cases must be emphasized according to the relative position of the critical and disorder varieties. A nontrivial intersection set results in strong constraints on critical amplitudes and exponents: vanishing of the first or trivialization of the second (e.g., $-\alpha=$ positive integer). A more intricate situation may occur in the case where multicritical points appear in the problem: In such a case an exact cancelation of singularities may happen. The other possible case is simply the absence of the intersection. Even in such a case the existence of a disorder solution implies some constraints on various expansions (high temperature, large q, etc.). Such is the case of the model studied here in the absence of a field where the varieties have no nontrivial intersection set. This situation is illustrated in Fig. 2 where both varieties are shown in the $K_3=\infty$ limit corresponding to the triangular lattice limit (without field) for $q=3$. A similar situation in the presence of a magnetic field would not be very surprising.

The disorder and critical varieties also present another interesting common property. *A priori* both varieties should only exhibit the symmetry of the square; however, one first remarks that the equation of the critical variety [Eq. (1)] as well as the critical partition function[8] both exhibit the S_4 symmetry; the internal energy and the latent heat also show that symmetry. Then both the expressions of the disorder varieties and the expressions of the partition function

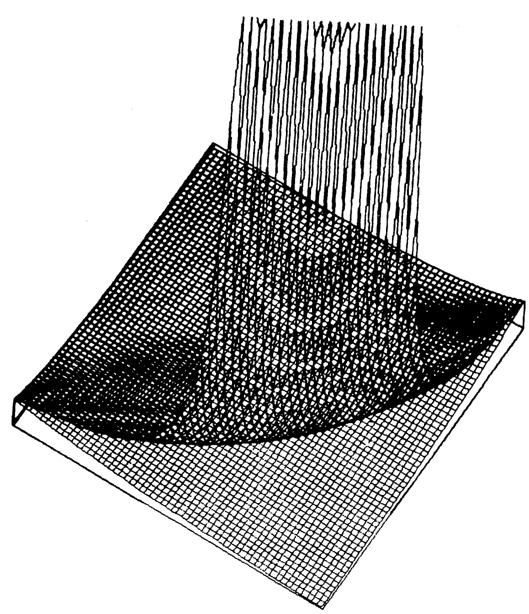

FIG. 2. Critical and disorder varieties for the triangular lattice $(c=\infty)$, without field and for $q=3$. The axis variables are $A=(a-1)/(a+q-1)$, B, and D with $-1/q\leqslant A\leqslant +1$. The upper (lower) surface represents the critical (disorder) variety.

restricted to these varieties are symmetric, either in the presence of or in the absence of the magnetic field, under the permutations of a, b, and c [see Eqs. (2) and (3)]. Taking into account these different results we conjecture the validity of that symmetry in the whole range of parameters: a, b, c, d, h, and q. This conjecture will be supported by different results, which we list in what follows. Further details will be given elsewhere.[14]

(a) Large-q expansion: The large-q expansion of Z has been obtained in the presence of a magnetic field, up to order six in $q^{-1/2}$. On this expression, the S_4 symmetry has been shown to hold.[14] This result is somewhat surprising given the complexity of the expansion in its dependence on various parameters.

(b) The resummed[15] high-temperature (respectively, low-temperature) expansion also shows that symmetry.

(c) $q=2$ limit: The zero-field partition function as well as the spontaneous magnetization are known.[6] A straightforward calculation shows the existence of this symmetry[15] in this Ising limit.

(d) $q \to 0$ limit: Similarly, we have been able[15] to check the S_4 symmetry on Lin and Tang's[5] result for Z.

These various exact results are clearly suppportive of our conjecture. Some of the arguments are directly related to the exact integrability of the model in different limits ($T=T_c$, $q=2$, and $q=0$). In this respect the S_4 symmetry may appear first as a consequence of the star-triangle relation. Indeed, considering these different limits as a solvable inhomogeneous eight-vertex model,[16] it can be shown that Z is S_4 invariant. However, the existence of that symmetry on both the expansions (a) and (b) and the disorder solution suggest the validity of our conjecture.

The occurrence of this intriguing S_4 symmetry sheds a new light on the analytical structure of the model, particularly near criticality. In the following we limit our discussion to the magnetization discontinuity ΔM ($T=T_c$, $q>4$). The exact expression of ΔM has been calculated in different limits[17]: square lattice ($a=c$, $b=d$), triangular lattice ($c=\infty$), and honeycomb lattice ($c=1$). In these cases, ΔM appears as a universal function of q given by

$$\Delta M=\prod_{n=1}^{\infty}(1-x^{2n-1})/(1+x^{2n}), \tag{6}$$

where, for $q>4$, x is defined by $q=x+2+x^{-1}$, $0<x<1$.

One can ask if Eq. (6) holds also for the checkerboard model. This is actually the case as indicated by the following results. Firstly, let us assume that a, b, c, and d are solutions of the following equations:

$$(1-a)(1-b)=q, \quad (1-c)(1-d)=q. \tag{7}$$

These two equations are consistent with the critical condition [Eq. (1)]. A simple generalization of Stephen and Mittag's[18] procedure, based on the star-triangle relation, shows that the diagonal transfer matrices $T(a,b)$ and $T(c,d)$ commute when Eqs. (7) are satisfied. That family of commuting matrices share the same eigenvector ψ_0 (associated to their largest eigenvalue) which must therefore be a function of q only (independent of a, b, c, and d). Hence ΔM, which is related to $\langle\psi_0|\delta_{\sigma_0,0}|\psi_0\rangle$, is a function of q only and thus equal to the expression of Eq. (6). Secondly, using the large q expansion, up to order six in q^{-1}, we have been able[14] to check that ΔM reduces to the expansion of Eq. (6) at small x. Note that the simple result that we obtain comes from a somewhat spectacular cancelation of a large number of terms involving complicated expressions of parameters. This simple expression for ΔM is of course in agreement with the S_4 invariance of the checkerboard model.

In summary, the set of results presented in this Letter shows that the Potts model appears to exhibit unexpected remarkable properties. Clearly, it would be interesting to understand the origin of the S_4 symmetry exhibited here for this particular model. Does it come from a combinatorial theorem specific to the Potts model, or is it a consequence of very general ideas? It is well known that near criticality new symmetries can appear such as the conformal covariance which characterizes in a very precise way the critical behavior. The results presented here emphasize an *a priori* new kind of symmetry, beyond the usual notion of universality: A quantity like ΔM is shown to be independent not only of the lattice but also of all the coupling constants. It is of great interest to see how the hidden symmetry S_4, the conformal covariance, and the integrability structure (Baxter's invariance) are related to each other.

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