Differential Galois groups of high order Fuchsian ODE's.[*]

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Abstract

We present a simple, but efficient, way to calculate connection matrices between sets of independent local solutions, defined at two neighboring singular points, of Fuchsian differential equations of quite large orders, such as those found for the third and fourth contribution ($\chi^{(3)}$ and $\chi^{(4)}$) to the magnetic susceptibility of square lattice Ising model. We use the previous connection matrices to get the exact explicit expressions of all the monodromy matrices of the Fuchsian differential equation for $\chi^{(3)}$ (and $\chi^{(4)}$) expressed in the same basis of solutions. These monodromy matrices are the generators of the differential Galois group of the Fuchsian differential equations for $\chi^{(3)}$ (and $\chi^{(4)}$), whose analysis is just sketched here.

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1 Introduction

Since the work of T.T. Wu, B. M. McCoy, C.A. Tracy and E. Barouch [1], it is known that the expansion in n-particle contributions to the zero field susceptibility of the square lattice Ising model at temperature T can be written as an infinite sum:

$$\chi(T) = \sum_{n=1}^{\infty} \chi^{(n)}(T) \tag{1.1}$$

of $(n-1)$-dimensional integrals [2, 3, 4, 5, 6, 7], the sum being restricted to odd (respectively even) n for the high (respectively low) temperature case.

As far as regular singular points are concerned (physical or non-physical singularities in the complex plane), and besides the known $s = \pm 1$ and $s = \pm i$ singularities, B. Nickel showed [6] that $\chi^{(2n+1)}$ is

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singular for the following finite values of $s = sh(2\,J/kT)$ lying on the $|s \,=\, 1|$ *unit circle* ($m = k = 0$ excluded):

$$2 \cdot \Bigl(s \,+\, {{1} \over {s}}\Bigr) \,\, = \,\,\,\, u^k \,+\, {{1} \over {u^k}} \,+\, u^m \,+\, {{1} \over {u^m}}$$
$$u^{2\,n+1} \,=\, 1, \qquad\qquad -n \,\le\, m, \,\, k \,\,\le\, n \qquad (1.2)$$

In the following we will call these singularities: "Nickel singularities". When n increases, the singularities of the higher-particle components of $\chi(s)$ *accumulate on the unit circle* $|s| \,=\, 1$. The existence of such a *natural boundary* for the total $\chi(s)$, shows that $\chi(s)$ is *not D-finite* (non holonomic[2] as a function of s).

A significant amount of work had been performed to generate *isotropic* series coefficients for $\chi^{(n)}$ (by B. Nickel [6, 7] up to order 116, then to order 257 by A.J. Guttmann and W. Orrick[3]). More recently, W. Orrick *et al.* [8], have generated coefficients[4] of $\chi(s)$ up to order 323 and 646 for high and low temperature series in s, using some *non-linear Painlevé difference equations* for the correlation functions [8, 9, 10, 11, 12]. As a consequence of this non-linear Painlevé difference equation, and the remarkable associated *quadratic double recursion* on the correlation functions, the computer algorithm had a $O(N^6)$ *polynomial growth* of the calculation of the series expansion instead of an exponential growth that one would expect at first sight. However, in such a *non-linear*, non-holonomic, *Painlevé-oriented* approach, one obtains results directly for the total susceptibility $\chi(s)$ which do not satisfy any linear differential equation, and thus prevents the easily disentangling of the contributions of the various holonomic $\chi^{(n)}$'s.

In contrast, we consider here, a *strictly holonomic approach.* This approach [13, 14, 15] enabled us to get 490 coefficients[5] of the series expansion of $\chi^{(3)}$ (resp. 390 coefficients for $\chi^{(4)}$), from which we have deduced [13, 14, 15, 16] the *Fuchsian differential equation* of order *seven* (resp.*ten*) satisfied by $\chi^{(3)}$ (resp. $\chi^{(4)}$). We will focus, here, on the *differential Galois group* of these order seven and ten Fuchsian ODE's.

2 The Fuchsian differential equations satisfied by $\tilde{\chi}^{(3)}(w)$ and $\tilde{\chi}^{(4)}(w)$

Similarly to Nickel's papers [6, 7], we start using the multiple integral form of the $\chi^{(n)}$'s, or more precisely of some normalized expressions $\tilde{\chi}^{(n)}$:

$$\chi^{(n)}(s) \,\,\, = \,\,\, S_{\pm} \,\, \tilde{\chi}^{(n)}(s), \qquad\qquad n = 3, \, 4, \, \cdots \qquad (2.1)$$
$$S_+ \,\,\, = \,\,\, {{(1-s^4)^{1/4}} \over {s}}, \qquad T > T_C \qquad (n \quad odd)$$
$$S_- \,\,\, = \,\,\, (1-s^{-4})^{1/4}, \qquad T < T_C \qquad (n \quad even)$$

where:

$$\tilde{\chi}^{(n)}(w) \,\, = \,\, \int d^nV \quad \Bigl(\prod_{i=1}^{n} \tilde{y}_i\Bigr) \,\, \cdot R^{(n)} \cdot H^{(n)} \qquad (2.2)$$

with (each angle ϕ_i varying from 0 to 2π):

$$d^nV = \quad \prod_{i=1}^{n-1} {{d\phi_i} \over {2\pi}} \quad with \quad \sum_{i=1}^{n} \phi_i = 0, \qquad R^{(n)} \, = \, {{1+\prod_{i=1}^n \tilde{x}_i} \over {1-\prod_{i=1}^n \tilde{x}_i}}$$

[2]The fact this natural boundary may be a "porous" natural frontier allowing some analytical continuation through it is not relevant here: one just need an infinite accumulation of singularities (not necessarily on a curve ...) to rule out the D-finite character of χ.

[3]A.J. Guttmann and W. Orrick private communication.

[4]The short-distance terms were shown to have the form $(T-T_c)^p \cdot (log|T-T_c|)^q$ with $p \ge q^2$.

[5]We thank J. Dethridge for writing an optimized C++ program that confirmed the Fuchsian ODE we found for $\chi^{(3)}$, providing hundred more coefficients all in agreement with our Fuchsian ODE.

$$H^{(n)} = \prod_{i<j} 4 \frac{\tilde{x}_i \tilde{x}_j}{(1-\tilde{x}_i \tilde{x}_j)^2} \cdot \sin^2\left(\frac{\phi_i - \phi_j}{2}\right) \qquad (2.3)$$

Instead of the usual[6, 7] variable s, we found it more suitable to use $w = \frac{1}{2}s/(1+s^2)$ which has, by construction, Kramers-Wannier duality invariance ($s \leftrightarrow 1/s$) and thus allows us to deal with both limits (high and low temperature, small and large s) on an equal footing [13, 14, 15]. The quantities $\tilde{x}_j$ and $\tilde{y}_j$ can be written in the following form [13, 14, 15]:

$$\tilde{x}_j = \frac{2w}{1 - 2w\cos\phi_j + \sqrt{(1-2w\cos\phi_j)^2 - 4w^2}},$$
$$\tilde{y}_j = \frac{2w}{\sqrt{(1-2w\cos\phi_j)^2 - 4w^2}}. \qquad (2.4)$$

It is straightforward to see that $\tilde{\chi}^{(n)}$ is *only a function of the variable* w. From now on, we thus focus on $\tilde{\chi}^{(n)}$ seen as a function of the well-suited variable w instead of s [6, 7]. One may expand the integrand in (2.2) in this variable w, and integrate the angular part.

We do not recall, here, the concepts, tricks and tools that have been necessary to generate very large series expansion for $\tilde{\chi}^{(3)}(w)$ and $\tilde{\chi}^{(4)}(w)$ with a *polynomial growth* of the calculations [13, 14, 15].

Given the expansion of $\tilde{\chi}^{(3)}(w)$ up to w^{490}, the next step amounts to encoding all the numbers in this long series into a *linear* differential equation. Note that such an equation should exist though its order is *unknown*[6]. Let us say that, using a dedicated program for searching[7] for such a finite order linear differential equation with polynomial coefficients in w, we succeeded finally in finding the following linear differential equation of order *seven* satisfied by the 490 terms we have calculated for $\tilde{\chi}^{(3)}$:

$$\sum_{n=0}^{7} a_n \cdot \frac{d^n}{dw^n} F(w) \;=\; 0 \qquad \text{with:} \qquad (2.5)$$
$$a_n \;=\; w^n \cdot (1-4\,w)^{\theta(n-2)}\,(1+4\,w)^{\theta(n-4)}\, P_n(w), \quad n = 6,\, 5, \cdots, 0$$
$$\text{where:} \qquad \theta(m) \;=\; sup(m,\, 0), \qquad \text{and:} \qquad a_7 \;=$$
$$w^7 \cdot (1-w)\;(1+2\,w)\;(1-4\,w)^5\,(1+4\,w)^3\,\left(1+3\,w+4\,w^2\right)\, P_7(w)$$

where $P_7(w), P_6(w) \cdots, P_0(w)$ are polynomials of degree respectively 28, 34, 36, 38, 39, 40, 40 and 36 in w [13].

Furthermore, besides the known singularities (1.2) mentioned above, we remark the occurrence of the roots of the polynomial P_7 of degree 28 in w, and the two *quadratic numbers* roots of $1+3\,w+4\,w^2 = 0$ which are not [17] Nickel singularities (they are not of the form (1.2)). The two quadratic numbers *are not* on the s-unit circle : $|s| = \sqrt{2}$ and $|s| = 1/\sqrt{2}$. These quadratic numbers do not occur in the "physical solution" $\chi^{(3)}$. For P_7, near any of its roots, all the local solutions carry *no logarithmic terms* and *are analytical* since the exponents *are all positive integers*. The roots of P_7 are thus *apparent singularities* [18, 19] of the Fuchsian equation (2.5).

The order seven linear differential operator L_7 associated with the differential equation satisfied by $\tilde{\chi}^{(3)}$ has the following factorization properties [13, 14, 16]:

$$L_7 \;=\; L_1 \oplus L_6, \qquad L_6 \;=\; Y_3 \cdot Z_2 \cdot N_1 \qquad (2.6)$$

where[8] L_1 is a first order differential operator which has the first contribution to the magnetic susceptibility, namely $\tilde{\chi}^{(1)} = 2w/(1-4w)$, as solution.

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[6] A lower bound for the order of this linear differential equation would be extremely useful : such a lower bound *does not* exist at the present moment.

[7] Note that we, first, actually found an order twelve Fuchsian ODE and, then, we reduced it (by factorization of the differential operator) to a seventh order operator. This order twelve differential equation requires much less coefficients in the series expansion to be guessed than the order seven Fuchsian ODE we describe here ! It is *actually easier to find the order twelve* differential equation than the order seven ODE !!

[8] For the notations see [13, 14, 16] for $\tilde{\chi}^{(3)}$, and [15, 16] for $\tilde{\chi}^{(4)}$.

In the same way, we found that the order ten linear differential operator L_{10}, associated with the differential equation satisfied by $\tilde{\chi}^{(4)}$, has the following factorization properties [15, 16]:

$$L_{10} \,=\, N_0 \oplus L_8, \qquad\qquad L_8 \,=\, M_2 \cdot G(L) \tag{2.7}$$

where N_0 is an order two differential operator which has the second contribution to the magnetic susceptibility, $\tilde{\chi}^{(2)}$ as solution and where $G(L)$ is an order four differential operator that can be factorized in a product of four order one differential operators [15].

3 Differential Galois group

A fundamental concept to understand (the symmetries, the solutions of) these exact Fuchsian differential equations is the so-called *differential Galois group* [20, 21, 22, 23, 24], which requires the computation of all the *monodromy matrices* associated with each (non apparent) regular singular point, these matrices being considered *in the same basis*[9]. Differential Galois groups have been calculated for simple enough second order, or even third order, ODE's. However, finding the differential Galois group of such *higher order* Fuchsian differential equations (order seven for $\chi^{(3)}$, order ten for $\chi^{(4)}$) with eight regular singular points (for $\chi^{(3)}$) is not an easy task. Along this side a first step amounts to seeing that the corresponding (order seven, ten) differential operators *do factorize* in smaller order differential operators, as a consequence of some rational and algebraic solutions and other singled out solutions [16]. These factorizations yield a particular block-matrix form of the monodromy matrices [16]. The calculation of local monodromy matrices in some "well-suited" local (Frobenius series solution) bases is easy to perform, however the calculation of the so-called *connection matrices* corresponding to the "matching" of the various well-suited local bases associated with the various regular singularities is a hard *non-local* problem. Of course from the knowledge of all these connection matrices one can immediately write the monodromy matrices in a *unique* basis of solutions[16].

From exact Fuchsian ODE's one can calculate very large series expansions for these (well-suited local Frobenius) solutions, sufficiently large that the evaluation of these series far away from any regular singularity can be performed[10] with a very large accuracy (400, 800, 1000 digits ...). As far as $\chi^{(3)}$ is concerned one can reduce [16] the calculation of these connection and monodromy matrices, to the 6×6 matrices of an order six [16] differential operator L_6 appearing in the decomposition (2.6). Connecting various sets of Frobenius series-solutions well-suited to the various sets of regular singular points amounts to solving a linear system of 36 unknowns (the entries of the connection matrix). We have obtained these entries in floating point form with a very large number of digits (400, 800, 1000, ...). We have, then, been able to actually "recognize" these entries obtained in floating form with a large number of digits [16].

In particular it is shown in [16] that the *connection matrix* between the singularity points 0 and 1/4 (matching the well-suited local series-basis near $w = 0$ and the well-suited local series-basis near $w = 1/4$) is a matrix where the entries are expressions in terms of $\sqrt{3}$, π, $1/\pi$, $1/\pi^2$, ... and a (transcendental) constant I_3^+ introduced in equation (7.12) of [1]:

$$\begin{aligned}
&{1 \over 2\,\pi^2} \cdot \int_1^\infty \int_1^\infty \int_1^\infty dy_1\, dy_2\, dy_3 \Bigl({y_2^2\,-1 \over (y_1^2\,-1)\,(y_3^2\,-1)} \Bigr)^{1/2} \cdot Y^2 \,= \\
&=\, .000814462565662504439391217128562721997861158118508 \cdots \\
&Y \,=\, {y_1\,-y_3 \over (y_1\,+y_2)\,(y_2\,+y_3)\,(y_1\,+y_2\,+y_3)}.
\end{aligned}$$

This transcendental constant can actually be written in term of the *Clausen function* Cl_2 :

$$I_3^+ \,=\, {1 \over 2\pi^2} \cdot \Bigl({\pi^2 \over 3} \,+2\,-3\sqrt{3} \cdot Cl_2({\pi \over 3}) \Bigr) \tag{3.1}$$

[9]These monodromy matrices are the generators of the *monodromy group* which identifies with the differential Galois group when there are no irregular singularities, and, thus, no Stokes matrices [25].

[10]Within the radius of convergence of these series.

where Cl_2 denotes the Clausen function :

$$Cl_2(\theta) \;=\; \sum_{n=1}^{\infty} \frac{\sin(n\,\theta)}{n^2}$$

This constant I_3^+ *can also be written in terms of dilogarithms, polygamma functions or Barnes G-functions* :

$$\begin{aligned}
I_3^+ \;-(1/6\,+\pi^{-2}) \;&=\; -\frac{3\,\sqrt{3}}{2\,\pi^2}\cdot Im\left(dilog\left(1/2-1/2\,i\sqrt{3}\right)\right)\\
&=\; \frac{1}{16\,\pi^2}\cdot\Big(\Psi\,(1,2/3)+\Psi\,(1,5/6)-\Psi\,(1,1/6)-\Psi\,(1,1/3)\Big)\\
&=\; -\frac{\sqrt{3}}{2\,\pi}\cdot\left(\ln\,(2)\;+\ln\,(\pi)\;-6\,\ln\left(\frac{G\,(7/6)}{G\,(5/6)}\right)\right)
\end{aligned}$$

The 6×6 connection matrix $C(0,1/4)$ for the order six differential operator L_6 matching the Frobenius series-solutions around $w=0$ and the ones around $w=1/4$, reads:

$$C(0,\,1/4) \;= \tag{3.2}$$

$$\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & -\frac{9\sqrt{3}}{64\pi} & 0 & 0 & 0\\
0 & -\frac{3\pi\sqrt{3}}{32} & 0 & 0 & 0 & 0\\
5 & \frac{1}{3}-2\cdot I_3^+ & \frac{3\sqrt{3}}{64\pi} & 0 & 0 & \frac{1}{16\pi^2}\\
-\frac{5}{4} & -\frac{3\pi\sqrt{3}}{32} & \frac{45\sqrt{3}}{256\pi} & 0 & \frac{1}{32} & 0\\
\frac{29}{16}-\frac{2\pi^2}{3} & \frac{15\pi\sqrt{3}}{64} & -\frac{225\sqrt{3}}{1024\pi}-\frac{3\pi\sqrt{3}}{64} & \frac{\pi^2}{64} & 0 & 0
\end{array}\right]$$

Not surprisingly[11] a lot of π's "pop out" in the entries of these connection matrices. We will keep track of the π's occurring in the entries of connection matrices through the introduction of the variable $\alpha\,=\,2\,i\,\pi$.

The local monodromy matrices can easily be calculated [16] since they correspond, mostly, to "logarithmic monodromies" and will be deduced from simple calculations using the fact that each logarithm (or power of a logarithm) occurring in a (Frobenius series) solution, is simply changed as follows : $\ln(w)$ $\rightarrow\ \ln(w)\,+\,\Omega$, where Ω will denote in the following $2\,i\,\pi$. From the local monodromy matrix $Loc(\Omega)$, expressed in the $w\,=\,1/4$ well-suited local series-basis, and from the connection matrix (3.2), the monodromy matrix around $w\,=\,1/4$, expressed in terms of the $(w=0)$-well suited basis reads [16]:

$$\begin{aligned}
24\,\alpha^4\cdot M_{w=0}(1/4)(\alpha,\,\Omega)\;&=\;\left[\begin{array}{cc}\mathbf{A} & \mathbf{0}\\ \mathbf{B} & \mathbf{C}\end{array}\right] \qquad (3.3)\\
&=\; C(0,\,1/4)\cdot Loc(\Omega)\cdot C(0,\,1/4)^{-1}
\end{aligned}$$

[11] One can expect the entries of the connection matrices to be *evaluations* of (generalizations of) hypergeometric functions, or solutions of Fuchsian differential equations.

where $\left[\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array}\right]$ and $[\ \mathbf{C}\ ]$ read respectively:

$$\left[\begin{array}{ccc} -24\,\alpha^4 & 0 & 0 \\ -48\,\alpha^4 & 24\,\alpha^4 & -144\,\alpha^2\Omega \\ 0 & 0 & 24\,\alpha^4 \\ -48\,\rho_1 & 32\,\Omega\,\rho_2 & 48\,\Omega\,(9\,\alpha^2+80\,\Omega) \\ 12\,\alpha^2\rho_3 & 4\,(75-4\,\alpha^2)\,\alpha^2\Omega & -300\,\alpha^2\Omega \\ -(87+8\,\alpha^2)\alpha^4 & 0 & 3\,(4\,\alpha^2-75)\,\alpha^2\Omega \end{array}\right],$$

with $\rho_1 = 5\,\alpha^4 + 8\,\Omega^2 + 8\,\Omega^2\alpha^2$, $\rho_2 = 4\,\Omega\,\alpha^2 - 75\,\Omega - 15\,\alpha^2$ and $\rho_3 = 5\,\alpha^2 + 4\,\Omega + 4\,\Omega\,\alpha^2$, and:

$$C = \left[\begin{array}{ccc} 24\,\alpha^4 & -384\,\alpha^2\Omega & 1536\,\Omega^2 \\ 0 & 24\,\alpha^4 & -192\,\alpha^2\Omega \\ 0 & 0 & 24\,\alpha^4 \end{array}\right]$$

Note that the transcendental constant I_3^+ *has disappeared* in the final exact expression of (3.3) *which actually depends only on* α and Ω. This $(\alpha,\,\Omega)$ way of writing the monodromy matrix (3.3) enables to get straightforwardly the $N-$th power of (3.3):

$$M_{w=0}(1/4)(\alpha,\,\Omega)^N \;=\; M_{w=0}(1/4)(\alpha,\,N\cdot\Omega) \qquad (3.4)$$

Let us introduce the following choice of ordering of the eight singularities, namely ∞, 1, 1/4, w_1, $-1/2$, $-1/4$, 0, w_2 ($w_1 = (-3+i\sqrt{(7)})/8$ and $w_2 = w_1^*$ are the two quadratic number roots of $1+3\,w+4\,w^2 = 0$), the first monodromy matrix M_1 is, thus, the monodromy matrix $M_{w=0}(\infty)$ (see (3.3)) at infinity with $\alpha = \Omega = 2i\,\pi$, $\mathcal{M}(\infty)$, the second monodromy M_2 matrix being the monodromy matrix at $w=1$, $\mathcal{M}(1)$, ... This is actually the particular choice of ordering of the eight singularities, such that a product of monodromy matrices *is equal to the identity matrix*[12]:

$$\begin{aligned} M_1\cdot M_2\cdot M_3\cdot M_4\cdot M_4\cdot M_6\cdot M_7\cdot M_8 \;&=\; \mathbf{Id} \\ &=\; \mathcal{M}(\infty)\cdot\mathcal{M}(1)\cdot\mathcal{M}(1/4)\cdot\mathcal{M}(w_1) \\ &\quad\times\mathcal{M}(-1/2)\cdot\mathcal{M}(-1/4)\cdot\mathcal{M}(0)\cdot\mathcal{M}(w_2) \end{aligned} \qquad (3.5)$$

It is important to note that relation (3.5) *is not verified* by the $(\alpha,\,\Omega)$ extension (like (3.3)) of the monodromy matrices M_i. If one considers relation (3.5) for the $(\alpha,\,\Omega)$ extensions of the M_i's, one will find that (3.5) is satisfied *only when* α *is equal to* Ω, but (of course[13]) this $\alpha = \Omega$ matrix identity *is verified for any value of* Ω, not necessarily equal to $2\,i\,\pi$.

3.1 Mutatis mutandis for $\chi^{(4)}$

Similarly to $\chi^{(3)}$ the differential operator for $\chi^{(4)}$ presents remarkable factorizations that yield a particular block-matrix form of the monodromy matrices [16]. Similarly, again, one can consider the (Frobenius series) solutions of the differential operator associated with $\chi^{(4)}$ around $x = 4\,w^2 = 0$ and around the ferromagnetic (and antiferromagnetic) critical point $x=1$ respectively. Again the corresponding *connection matrix* (matching the solutions around the singularity points $x=0$ and the ones around the singularity point $x = 4\,w^2 = 1$) have entries which are expressions in terms of π^2, rational numbers but also of constants like constant I_4^- introduced in [1] which can actually be written in term of the *Riemann zeta function*, as follows :

$$I_4^- \;=\; \frac{1}{16\pi^3}\cdot\Big(\frac{4\,\pi^2}{9} - \frac{1}{6} - \frac{7}{2}\cdot\zeta(3)\Big) \qquad (3.6)$$

[12] Of course, from this relation, one also has seven other relations deduced by cyclical permutations.

[13] A matrix identity like (3.5) yields a set of polynomial (with integer coefficients) relations on $\Omega = 2\,i\,\pi$. The number π being transcendental it is not the solution of a polynomial with integer coefficients. These polynomial relations have, thus, to be *polynomial identities valid for any* Ω.

The derivation of the two results (3.1), (3.6) for the two transcendental constants I_3^+ and I_4^- has never been published[14] but these results appeared in a conference proceedings [26]. We have actually checked that I_3^+ and I_4^- we got in our calculations of connection matrices displayed in [27] as floating numbers with respectively 421 digits and 431 digits accuracy, *are actually in agreement* with the previous two formula. These two results (3.1), (3.6) provide a clear answer to the question of how "complicated and transcendental" some of the constants occurring in the entries of the connection matrices can be. These two remarkable exact formulas (3.1), (3.6) are not totally surprising when one recalls the *deep link between zeta functions, polylogarithms and hypergeometric series* [28, 29, 30, 31]. Along this line, and keeping in mind that we see all our Ising susceptibility calculations as a "laboratory" for other more general problems (Feynman diagrams, ...), we should also recall the various papers of D. J. Broadhurst [32, 33, 34, 35] where $Cl_2(\frac{\pi}{3})$ and $\zeta(3)$ actually occur in a Feynman-diagram-hypergeometric-polylogarithm-zeta framework (see for instance equation (163) in [32]).

Similarly to the previous results for $\tilde{\chi}^{(3)}$ the monodromy matrices written in the *same* basis of solution, deduced from the connection matrices and the local monodromy matrices are such that a product in a certain order of them is the identity matrix. Denoting by $M_{x=0}(0)$, $M_{x=0}(1)$, $M_{x=0}(4)$ and $M_{x=0}(\infty)$ the monodromy matrices expressed in the same $x = 0$ well-suited basis, one obtains:

$$M_{x=0}(\infty) \cdot M_{x=0}(4) \cdot M_{x=0}(1) \cdot M_{x=0}(0) \; = \; \mathbf{Id} \tag{3.7}$$

This matrix identity is valid irrespective of the "not yet guessed" constants [16].

4 Conclusion

The high order Fuchsian equations we have sketched here present many interesting mathematical properties close to the ones of the so-called *rigid local systems* [36], these rigid local systems exhibiting remarkable *geometrical interpretations* [37] as periods of some algebraic varieties. This "rigidity[15]" emerges through the log-singularities of the solutions of these Fuchsian ODE's: the powers of the logarithms of these solutions are "smaller" than one could expect at first sight. It is worth noting that almost all these mathematical structures, or singled-out properties, we sketched here, or in previous publications [13, 14, 15, 16], are far from being specific of the two-dimensional Ising model : they also occur on many problems of lattice statistical mechanics or, even[16], as A. J. Guttmann and I. Jensen saw it recently, on enumerative combinatorics problems like, for instance, the generating function of the three-choice polygon [38].

We have also seen in some of our calculations [14, 15] a clear occurence of hypergeometric functions, hypergeometric series and in some of our calculations (not displayed here) generalizations of hypergeometric functions to *several complex variables*: Appel functions [39], Kampé de Fériet, Lauricella-like functions, polylogarithms [32, 33, 34, 35], Riemann zeta functions, multiple zeta values, ... The occurence of Riemann zeta function or dilogarithms in the two remarkable exact formulas (3.1), (3.6) is not totally surprising when one recalls the *deep link between zeta functions, polylogarithms and hypergeometric series* [28, 29, 30, 31].

We think that such "collisions" of concepts and structures of different domains of mathematics (differential geometry, number theory, ...) are *not* a consequence of the free-fermion character of the Ising model, and that similar "convergence" should also be encountered on more complicated Yang-Baxter integrable models[17], the two-dimensional Ising model first "popping out" as a consequence of its simplicity. In a specific differential framework some of these interesting mathematical properties can clearly be seen in the analyzis of the differential Galois group of these Fuchsian equations.

We have underlined the fact that, beyond a general analyzis of the differential Galois group [20], one can actually find the exact expressions of the *non-local* connection matrices from very simple matching of series calculations, and deduce, even for such *high order* Fuchsian ODE's, *explicit representations* of all

[14] We thank C. A. Tracy for pointing out the existence of these two results (3.1), (3.6) and reference [26].

[15] Let us recall that hypergeometric functions are totally rigid.

[16] The wronskian of the corresponding differential equation in [38] is also rational, the associated differential operator factorizes in a way totally similar to the Fuchsian ODE's for $\chi^{(3)}$ and $\chi^{(4)}$, large polynomial corresponding to apparent singularities also occur, ...

[17] The comparison of the Riemann zeta functions equations obtained for the XXX quantum spin chain [40] with the evaluations of central binomial in [33] provides a strong indication in favor of similar structures on non-free-fermion Yang-Baxter integrable models.

the monodromy matrices in the *same* (non-local) basis of solutions, *providing an effective way* of writing explicit representations of all the elements of the monodromy group. The remarkable form, structures and properties (see (3.2), (3.3), (3.4), (3.5)) of the monodromy matrices in the *same* (non-local) basis of solutions is something one could not suspect at first sight from the general description of the differential Galois group.

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