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More integrable birational mappings

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Abstract

We study birational mappings generated by matrix inversion and permutation of the entries of $q \times q$ matrices. For $q = 3$ we have performed a systematic examination of all the permutations of 3×3 matrices in order to find integrable mappings (of three different kinds) and finite order mappings. This exhaustive analysis gives, among 30 462 classes of mappings, 27 (new) integrable classes of birational mappings and 36 classes yielding finite order recursions associated with these mappings. An exhaustive analysis (with a constraint on the diagonal entries) has also been performed for 4×4 matrices: we have found 8306 new classes of integrable mappings. All these new examples show that integrability can actually correspond to non-involutive permutations. The analysis of the integrable cases specific of a particular size of the matrix and a careful examination of the non-involutive permutations, could shed some light on integrability of such birational mappings. It seems that one has the following result: the non-involutive examples are specific of a given matrix size (3×3 matrix ...) and the permutations which yield integrable mappings for arbitrary matrix size are always involutions.

1. Introduction

In previous papers birational mappings [1–3] having their origin in the theory of exactly solvable models in lattice statistical mechanics [4–9] have been studied. They are generated by *involutive transformations on matrices* corresponding to two kinds of transformations on $q \times q$ matrices: the inversion of the $q \times q$ matrix and an (*involutive*) *permutation of the entries* of the matrix. In these papers, *permutations of two entries* [1–3], as well as permutations corresponding to *discrete symmetries* of lattice models of statistical mechanics [4–9] were first analyzed. Several integrable mappings associated with permutations of $q \times q$ matrices, *for arbitrary* q, have been found this way [1–3]. It has also been shown that the iteration of the associated birational transformations presents some remarkable *factorization properties* [1,2]. These

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factorization properties explain why the complexity of these iterations, instead of having the exponential growth one expects at first sight, may have a *polynomial growth* of the complexity [1,2,10,11]. It has also been shown that the polynomial factors occurring in these factorizations *may satisfy noteworthy non-linear recursion relations* and that some of these recursions were actually *integrable*, yielding elliptic curves [1,2]. In other papers we have also considered birational mappings associated with *non-involutive* permutations of matrix entries and we have also obtained remarkable *factorization properties* [12] but not integrability or even polynomial growth.

We perform here a systematic examination of such birational mappings associated with *all* the permutations of entries of 3×3 matrices and (almost all) permutations of entries of 4×4 matrices as well. This analysis provides a set of new integrable mappings of various number of (homogeneous) variables (9, 16, arbitrary number). Our motivation is not only to accumulate as many new integrable mappings as possible, but also to better understand the structures associated with integrability. For instance, is the integrability of these birational mappings necessarily associated with *involutive* permutations?

Recalls:

Let us first denote $\mathcal{S}_{q\times q}$ the set of permutations of the q^2 (homogeneous) entries of a $q \times q$ matrix and let us also introduce the subset $\mathcal{S}^2_{q\times q}$ (of $\mathcal{S}_{q\times q}$) which consists in a simple interchange of *two* entries. We consider the matrix iteration of an arbitrary $q \times q$ matrix M_0:

$$\mathcal{M}_n = K_t^n(M_0)\,, \qquad (1)$$

where $K_t = t \cdot I$, $I(M) = \det(M) \cdot M^{-1}$ (homogeneous inverse) and $t \in \mathcal{S}^2_{q\times q}$. Transformation K_t is clearly a *birational transformation* on the entries of M_0 since its inverse transformation, $I \cdot t^{-1}$, is also a rational transformation. Let us introduce the permutations of entries $g_{i,j}$ $(0 \leqslant i < j < q)$ which consist in the exchange of column i with column j followed by the exchange of row i with row j. It is straightforward to see (with obvious notations) that

$$K_{g^{-1}\cdot t\cdot g}(M_0) = g^{-1} \cdot K_t(M_0) \cdot g\,. \qquad (2)$$

This means that the equivalence relation, defined by $t' = g^{-1} \cdot t \cdot g$, is compatible with iteration (1). Up to these row and columns relabeling equivalence relation, one can show that $\mathcal{S}^2_{q\times q}$ corresponds to *only six classes* [1]: iteration (1) has to be analyzed for a only one representative in each class. The class denoted class I in [1] can be represented by the interchange of the two entries $M_0[1,2]$ and $M_0[2,1]$. Class I presents remarkable factorization properties at each step of iteration (1). The determinants of the iterated matrices do factorize and all the entries of an iterated matrix also factorize a common (homogeneous) polynomial. Thus, at the nth step of the iteration, one can introduce "reduced matrices" M_n's and introduce homogeneous polynomials denoted f_n's corresponding to these very factorizations. For class I iteration (1) thus yields:

$$M_{n+1} = \frac{K(M_n)}{f_{n-2}^{q-2} f_{n-1}^2 f_n^{q-4}}, \qquad f_{n+1} = \frac{\det(M_n)}{f_{n-2}^{q-1} f_{n-1}^3 f_n^3} \tag{3}$$

with $f_n = 1$ for $n \leqslant 0$. Moreover, the degree of the f_n's grows *polynomially* with n (quadratic growth [1]). Remarkably, these homogeneous polynomials f_n's do satisfy, *independently* of q, a *whole hierarchy of non-linear recursion relations* [1,2]. A simple recursion in this hierarchy is the (integrable) recursion:

$$\frac{f_n f_{n+3}^2 - f_{n+4} f_{n+1}^2}{f_{n-1} f_{n+3} f_{n+4} - f_n f_{n+1} f_{n+5}} = \frac{f_{n-1} f_{n+2}^2 - f_{n+3} f_n^2}{f_{n-2} f_{n+2} f_{n+3} - f_{n-1} f_n f_{n+4}}. \tag{4}$$

This hierarchy is a hierarchy of recursions *integrable* or only compatible with an integrable recursion of the hierarchy [1,2]. Moreover, the orbit of K_t of an arbitrary matrix M_0 is an *elliptic curve* in the parameter space of the q^2 homogeneous entries of M_0: the mapping K_t itself is *integrable* [1] for *arbitrary values of* q. These two integrability properties are of course related [1].

Let us also define $\widehat{K}_t = \widehat{I} \cdot t$, where $\widehat{I}$ is the usual (inhomogeneous) matrix inverse, and $l_n = \det(\widehat{K}_t^n(M_0))$ and $x_n = l_n \cdot l_{n+1}$. The x_n's variables also verify a *whole hierarchy of non-linear recursion relations* [1,2] closely related to the existence of the recursions on the f_n's [1,2]. Let us give the simplest recursion of this hierarchy:

$$\frac{x_{n+2} - 1}{x_{n+1} x_{n+2} x_{n+3} - 1} = \frac{x_{n+1} - 1}{x_n x_{n+1} x_{n+2} - 1} \cdot x_n x_{n+2}. \tag{5}$$

This recursion is *integrable*. In the following, we will use condition (5) as an *integrability criterion* ("class I-integrability").

Among the previously mentioned six classes, class denoted IV in [2,3] also has interesting properties. Class IV can be represented by the interchange of the two entries $M_0[2,1]$ and $M_0[2,3]$. Generically, the iterations of K_t *are not integrable*, however for "many" [3] different initial matrices M_0, the orbits of K_t yield (transcendental [3]) curves. Such a highly regular situation corresponding to (very) weak chaos, has been called "almost integrable" [3]. Moreover there does exist a (codimension-one) algebraic condition, bearing on the entries of the matrix, for which the birational transformations K_t, associated with class IV, actually correspond to *integrable* mappings [2,3]. This integrability condition has been written elsewhere [2]. The factorizations *restricted to this integrable subcase* [1] read for 4×4 matrices:

$$M_{n+1}^{int} = \frac{K(M_n^{int})}{f_n f_{n-2}^2 f_{n-3} f_{n-4}^2}, \qquad f_{n+1} = \frac{\det(M_n^{int})}{f_n^2 f_{n-1} f_{n-2}^3 f_{n-3}^2 f_{n-4}^3} \tag{6}$$

where polynomials f_n's verify:

$$\frac{f_{n+2} f_{n+7} f_{n+9} - f_{n+3} f_{n+5} f_{n+10}}{f_{n+3} f_{n+7} f_{n+8} - f_{n+4} f_{n+5} f_{n+9}} = \frac{f_{n+1} f_{n+6} f_{n+8} - f_{n+2} f_{n+4} f_{n+9}}{f_{n+2} f_{n+6} f_{n+7} - f_{n+3} f_{n+4} f_{n+8}}. \tag{7}$$

[1] The factorizations are more involved in the general case [1].

This recursion is an *integrable* recursion on the f_n's. The birational transformations of class IV yield, for this very integrable case (see (6)), the following (integrable) recursion on the x_n's:

$$\frac{x_{n+2}-1}{x_{n+1}x_{n+3}-1} = \frac{x_{n+1}-1}{x_n x_{n+2}-1} \cdot \frac{x_n x_{n+2}}{x_{n+1}} \,. \tag{8}$$

In general, the birational transformations of class IV only yield:

$$\frac{x_{n+3}-1}{x_{n+2}x_{n+4}-1} = \frac{x_{n+1}-1}{x_n x_{n+2}-1} \cdot x_n x_{n+3} \,. \tag{9}$$

Recursion (9) is *not* an integrable recursion. Of course (8) implies (9).

Condition (8) cannot really be used as an integrability criterion since it is only verified on a (quite involved codimension-one) algebraic variety [1]. Practically, we will use (9) as an integrability criterion: when condition (9) is verified it is easy to find the (codimension-one) algebraic variety on which (8) is actually verified. Thus condition (9) can be seen as an integrability criterion though it is not an integrable recursion. In the following we will call "class IV-integrability" such a situation.

Let us also mention a third example of integrable mapping corresponding to (infinite discrete) symmetries of lattice models of statistical mechanics (vertex models [9, 11, 13]). Let us introduce permutation t_1 which exchanges [2] the two 2×2 off diagonal sub-matrices of a 4×4 R-matrix (note that t_1 does not belong to $\mathcal{S}^2_{4\times 4}$). The mapping $K_{t_1} = t_1 \cdot I$ corresponds to some non trivial non linear symmetry of the (4×4) R matrix of the sixteen vertex [13] model. [3] In that case the factorization scheme reads:

$$M_{n+2} = \frac{K_{t_1}(M_{n+1})}{F_n^2}, \qquad F_{n+2} = \frac{\det(M_{n+1})}{F_n^3},$$

$$\widehat{K}_{t_1}(M_{n+2}) = \frac{K_{t_1}(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{F_{n+1}F_{n+3}} \,. \tag{10}$$

Again one has a hierarchy of recursions integrable, or compatible with integrability [11]. In this last case the simplest recursion on the x_n's reads:

$$\frac{x_{n+2}-1}{x_{n+1}x_{n+2}x_{n+3}-1} = \frac{x_{n+1}-1}{x_n x_{n+1} x_{n+2}-1} \cdot x_n x_{n+1} x_{n+2}^2 \,. \tag{11}$$

This recursion is *integrable*.

Another example of permutation which originates from symmetry analysis of vertex models generalizes the previous permutation t_1 and corresponds to the following action on a $2m \times 2m$ R-matrix [9, 11, 12]:

$$t_1 : R = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \to t_1(R) = \begin{pmatrix} A & C \\ B & D \end{pmatrix}, \tag{12}$$

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[2] Transformation t_1 is a geometrical symmetry of the square lattice [9].

[3] The Baxter model is a Yang–Baxter integrable subcase of this model [9]. One should not confuse the integrability of the *symmetries* of the parameter space of the sixteen vertex model (namely the mappings considered here) and the Yang–Baxter integrability [9]: the sixteen vertex model is not generically Yang–Baxter integrable.

where A, B, C and D are $m \times m$ matrices. This last permutation t_1 corresponds to the analysis of vertex models on a *cubic* (or hypercubic) lattice [11, 12] ($m = 4$, $m = 2^{d-1}$), as well as monodromy matrices of vertex models on a square lattice [11] ($m = 2^N$, N number of sites in the monodromy matrix). The factorization scheme reads (with $q = 2m$):

$$K(M_n) = M_{n+1} \cdot f_n^{q-5} \cdot f_{n-1}^5 \cdot f_{n-2}^{2(q-5)} \cdot f_{n-3}^6 \cdot f_{n-4}^{2(q-5)} \cdot f_{n-5}^6 \cdots ,$$
$$\det(M_n) = f_{n+1} \cdot f_n^{q-4} \cdot f_{n-1}^7 \cdot f_{n-2}^{2(q-4)} \cdot f_{n-3}^8 \cdot f_{n-4}^{2(q-4)} \cdot f_{n-5}^8 \cdot f_{n-6}^{2(q-4)} \cdots . \quad (13)$$

It has been seen [11, 12] that permutation (12) yields a *polynomial growth* of the calculations, however, the f_n's *do not* verify any recursion relation like (5) or (8) (or (9)) or (11). In fact the orbits of the associated mappings K_{t_1} can be seen [11, 12] to (uniformly) densify an (abelian) algebraic variety of dimension g. In principle one can write explicitly the evolution of the points in terms of theta functions of g variables. One could call such a situation a "g-integrability". It would be interesting to find exhaustively, for 3×3 matrices, all the permutations yielding such "g-integrability" (or more simply yielding *polynomial growth* [11, 12]). Unfortunately, we do not have any simple criterion, or any quick and efficient algorithm, to perform such an exhaustive search. Therefore, we restrict in this paper to the "traditional" integrability (foliation of the parameter space in elliptic, or rational, curves).

In the previously mentioned papers [1–3, 11, 12] all the integrable birational mappings have been seen to correspond to the occurrence of one of the three previous hierarchy of recursions on the x_n's represented by (5), (8) (or (9)) and (11). Actually we will, in the following, systematically seek for these three recursions as *integrability criterion*, and verify that the associated hierarchy of recursions [1, 2] are also verified. We will verify if one of the three previous factorization schemes ((3), (6), (10)) is satisfied, or if some new factorization scheme pops out, *only when* a permutation actually verifies one of the three recursions ((5), (8) or (11)). This is our strategy since the analysis of the factorization schemes cannot be easily implemented as an algorithmic procedure: the factorization schemes are not checked directly for each permutation.

Remark. It is possible to extend a permutation of $\mathcal{S}_{q \times q}$ to $\mathcal{S}_{(q+1)\times(q+1)}$ by simply keeping fixed all the new entries.[4] A certain number of results (occurrence of integrable recursion, factorization schemes ...) *do not depend* on the actual size q of the matrix but only on the permutation considered [1, 2]. It is thus tempting to examine exhaustively all the permutations of 3×3 (resp. 4×4) matrices in order to *find new integrable birational transformations independent* of q or, on the contrary, find integrable results *specific* of 3×3 (resp. 4×4) matrices. To some extent this last situation could be of a greater interest to get some hint on the very "nature" of integrability.

[4] These extensions have been called "straight" generalization in [11, 12].

2. Integrability of an arbitrary permutation of $\mathcal{S}_{3\times 3}$

In this section we extend the previous results to an *arbitrary* permutation t of $\mathcal{S}_{3\times 3}$. Let us first notice that since t is not necessarily an involution anymore, the action of the group, generated by t and I, on an initial matrix does *not* only reduce [5] to the iteration of $t \cdot I$. Transformation t^p is just another element of $\mathcal{S}_{3\times 3}$ which is thus considered in a systematic analysis, however many (generically infinite order) transformations like $t \cdot I \cdot t^p \cdot I$ do occur. In this paper we do not analyze this group: we restrict ourself to the analysis and the classification of the mappings $K_t = t \cdot I$, where t is an arbitrary permutation of $\mathcal{S}_{3\times 3}$. The equivalence relation (2) also holds here. The $9! = 362\,880$ elements of $\mathcal{S}_{3\times 3}$ are grouped in 30 462 classes (to be compared to the 6 classes of the previous paragraph). The number of elements in each class is shown in Table 1:

Let us introduce some q-independent encoding of the permutations of entries of $q \times q$ matrices. The entries of a 3×3 (resp. $q \times q$) matrix are labeled as follows:

$$\begin{pmatrix} 0 & 3 & 8 \\ 2 & 1 & 7 \\ 6 & 5 & 4 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 3 & 8 & 15 & \cdots \\ 2 & 1 & 7 & 14 & \cdots \\ 6 & 5 & 4 & 13 & \cdots \\ 12 & 11 & 10 & 9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{14}$$

A 3×3 (resp. $q \times q$) matrix will represent a permutation of the entries of the 3×3 (resp. $q \times q$) matrix. For instance:

$$P_{erm} = \begin{pmatrix} 2 & 0 & 6 \\ 4 & 7 & 8 \\ 5 & 1 & 3 \end{pmatrix} \tag{15}$$

corresponds to the permutation which has the following decomposition in a 4-cycle and a 5-cycle:

$$0 \to 2 \to 4 \to 3 \to 0 \quad \text{and} \quad 8 \to 6 \to 5 \to 1 \to 7 \to 8\,. \tag{16}$$

For numerical purpose it is more convenient to study all permutations and check afterwards which of them are equivalent. This comes from the fact that it is (paradoxically) easier to decide the (5) or (8) or (11)-integrability of a given mapping than to decide if two permutations are equivalent under (2).

Table 1

Number of elements	0	1	2	3	4	5	6	7	8	9	10	11	12
Number of classes	0	2	12	14	34	0	354	0	0	0	0	0	30 046

[5] Up to a semi-direct product by Z_2. A group generated by two involutions, with no relations between them, is isomorphic to the infinite dihedral group [4, 5].

Remark (*Finite order recursions*). It is important to make the following remark. Many (quite involved) permutations are seen to satisfy one of the three previous recursions (5) or (8) or (11) for "pathological" reasons. Suppose, for instance, that a permutation satisfies the following simple recursion on the x_n's:

$$x_n \cdot x_{n+1} \cdot x_{n+2} = 1 \,. \qquad (17)$$

It is straightforward to see that (17) implies that recursion (5) is verified. Recursions (5) or (8) or (11) may thus be verified, not because or a "true" integrability (foliation of the parameter space in elliptic, or rational, curves) but because of a simple recursion on the x_n's like (17). Actually, it is easy to see that (17) yields $x_n = x_{n+3}$. It can be a strong indication that, not only the recursion on the x_n's, but the birational transformation K_t itself, is a *finite order* mapping. Clearly such "trivial" integrability has to be considered separately. As far as such "finite order" integrability is concerned, we have generated *all* the permutations of entries of 3×3 matrices and we have found 36 classes yielding finite order recursions on the x_n's. We have found first 72 permutations organized in 16 classes (having 1 or 2 or 3 or 4 or 6 or 12 elements: all the divisors of 12) yielding $x_n = \pm 1$, and yielding finite order mappings of order 1, 2, 4 or 6. One also has 144 permutations organized in 14 classes (of 6 or 12 elements) corresponding to $x_n x_{n+1} = 1$. Note that they *do not yield finite orbits* (there is no integer p such that $K_t^p(M_0) = \lambda \cdot M_0$). For recursion $x_n x_{n+1} x_{n+2} = 1$ there are 72 permutations organized in 6 classes having 12 elements each, however the examination of the orbits of the corresponding birational transformation K_t shows that one does not have finite orbits in this case either.

Let us remark that these permutations (corresponding to finite order recursions on the x_n's) are quite involved permutations. A list of these ("finite order") permutations is given in Table 2.

The factorization schemes of the corresponding birational transformations K_t of Table 1 are trivial: only a *finite* number of polynomials f_n's is necessary to describe the factorization scheme (for instance f_1, f_2, f_3). An example of such a "trivial" factorization scheme is

$$M_{2n+1} = \frac{K(M_{2n})}{f_1^n}, \qquad M_{2n+2} = \frac{K(M_{2n+1})}{f_2^n},$$
$$f_{2n+1} = \frac{\det(M_{2n})}{f_1^{2n+1} \cdot f_2^n} = 1, \qquad f_{2n} = \frac{\det(M_{2n-1})}{f_1^n \cdot f_2^{2n-1}} = 1 \,. \qquad (18)$$

The expression of the x_n's, in terms of these finite numbers of f_n's, are also very simple, for instance,

$$x_{3p+1} = \frac{f_3}{f_2}, \qquad x_{3p+2} = \frac{f_1}{f_3},$$
$$x_{3p+3} = \frac{f_2}{f_1}, \qquad \text{yielding:} \qquad x_n \cdot x_{n+1} \cdot x_{n+1} = 1 \,. \qquad (19)$$

Table 2

Finite order	038	026	083	062	062	038	172	153
recursion	217 [1]	315 [1]	645 [3]	847 [3]	351 [6]	654 [6]	546 [2]	748 [2]
$x_n = \pm 1$	654	874	271	351	847	217	380	260
	153	127	260	217	271	206	546	531
	260 [12]	564 [12]	153 [4]	654 [4]	083 [6]	784 [6]	380 [2]	487 [2]
	748	308	748	038	645	135	172	602
Finite order	025	026	024	026	074	062	051	047
recursion	316 [6]	314 [12]	316 [12]	814 [12]	326 [12]	347 [6]	347 [12]	351 [12]
$x_n x_{n+1} = 1$	874	875	875	375	815	851	862	862
	026	062	153	206	204	260		
	351 [6]	851 [6]	260 [12]	134 [12]	136 [12]	531 [12]		
	847	347	487	785	785	487		
Finite order	047	136	163	150	260	206		
recursion	851 [12]	204 [12]	250 [12]	263 [12]	758 [12]	478 [12]		
$x_n x_{n+1} x_{n+2} = 1$	362	785	748	748	143	513		

The number in [*bracket*] near each permutation is the cardinality of the corresponding class.

If the birational transformation K_t is a finite order one, one necessarily has such factorization schemes. When the birational transformation K_t is *not* a finite order transformation ($14+6$ classes: $x_n \cdot x_{n+1} = 1$ and $x_n \cdot x_{n+1} \cdot x_{n+2} = 1$) it would be interesting to analyze the orbits of the iteration of K_t.

2.1. *"Genuine" integrability*

Now let us get rid of such "spurious" integrability and concentrate on "true" integrability. It is then straightforward to check exactly, for each selected permutation, if the corresponding recursions (5) or (8) (resp. (9)) or (11) hold and, in a second step, if factorizations (3) or (6) or (10) hold.

We have found the following result: *there are no permutation of entries of* 3×3 *matrices verifying* recursion (11). *This recursion can only be fulfilled for* $q \times q$ matrices for $q \geqslant 4$.

We have then found 108 permutations verifying recursion (5) (and also a whole hierarchy of recursions [1]) and the factorization scheme (3): all these permutations correspond to integrable mappings for $q = 3$ and can be grouped into 23 classes. More specifically three classes verify recursion (5) and the factorization scheme (3) for *arbitrary* q, 21 classes verify recursion (5) and the factorization scheme (3) but for $q = 3$ *only*. Three other classes (second column in Table 3) verify, for $q = 3$, recursion (5) and the factorization scheme (3), however, for $q = 4$, they only verify the factorization scheme (3) but not recursion (5) anymore. Factorization scheme (3) corresponds to a *polynomial growth* of the calculation, which is a strong indication of "g-integrability" [11]. Therefore, for $q = 4$, these last mappings are not integrable anymore but probably "g-integrable".

Table 3

	Integrable for any q, fact. scheme for any q	Integrable for $q = 3$, fact. scheme for $q = 4$	Integrable only for $q = 3$ factorization scheme only for $q = 3$				
Class I	036 217 [3,2] 854	026 317 [3,2] 854	046 315 [6,4] 872	027 318 [3,2] 546	027 315 [6,2] 864	025 854 [3,2] 784	016 325 [6,6] 478
	037 218 [3,2] 564	078 513 [3,2] 624	026 347 [3,2] 851	026 345 [3,2] 871	086 345 [6,4] 271	083 647 [3,2] 251	062 845 [3,2] 371
	038 247 [3,2] 651		081 465 [6,6] 273	147 308 [6,6] 562	146 308 [6,12] 572	206 384 [6,3] 175	047 315 [6,12] 862
	128 307 [3,2] 654		013 625 [6,6] 478	021 465 [6,6] 873			
Class IV	037 218 [6,2] 654		083 245 [6,2] 671	083 654 [12,4] 172	127 645 [12,9] 803		

The two numbers in [*bracket*] near each permutation are the cardinality of the corresponding class and the order of the permutation.

Finally, for "class-IV integrability" (see recursion (8)), we have found 36 permutations grouped into 4 classes: one class (class IV in [3]) is valid for arbitrary q and the three others are specific of $q=3$.

One representant of each of these classes is given in Table 3.

In Table 3 "integrable for Class IV" means that the mapping is integrable only for certain initial matrices M_0 of the iteration (codimension-one algebraic variety [1]).

One remarks that a large number of integrable mappings *specific* of 3×3 matrices *are not involutions* (12 classes among 20 classes).

3. Integrability of permutations of $\mathcal{S}_{q\times q}$ for $q \geqslant 4$

It is possible to extend a permutation of $\mathcal{S}_{q\times q}$ to $\mathcal{S}_{(q+1)\times(q+1)}$ by simply keeping fixed all the new entries ("straight generalization"). Labeling (14) of the entries has the advantage that it can be given without specifying q. The permutation which simply exchanges entry 2 and 3, for example, has been shown to be integrable for arbitrary values of q ("class I-integrability"). It is thus interesting to test, for each integrable permutation of $\mathcal{S}_{3\times 3}$, if it yields an integrable permutation of $\mathcal{S}_{q\times q}$. The results are shown on Table 3, where we have distinguished the different possible cases. Note that the permutations that can be upgraded for arbitrary values of q are *all* involutions (first column of Table 4). Beyond these involutive examples, valid for arbitrary q, it would

Table 4

	Class I	Class IV	Class 16 vertex
Permutations found	3 752	1 567	5 507
Effective number	123 098	43 140	223 008
Number of classes	2 647	945	4 714

be interesting to find integrable mappings *specific* of 4×4 matrices or, on the contrary, that can be generalized to $q \times q$ matrices (with $q \geqslant 5$ but different from these upgraded 3×3 involutive examples). This justifies performing the previous exhaustive analysis but for permutations of entries of 4×4 matrices.

Since the number of permutations of $\mathscr{S}_{4\times 4}$ is very large to explore ($16! = 20\,922\,789\,888\,000$) we have first investigated only $16!/4! = 871\,782\,912\,000$ permutations corresponding to an ordering constraint on the diagonal entries. With the previous encoding of the permutations the diagonal symbols (namely $0, 1, 4, 9$) become, after transformation by a permutation P: $P(0), P(1), P(4), P(9)$. The constraint on the diagonal entries is the following: $P(0) \leqslant P(1) \leqslant P(4) \leqslant P(9)$. This constraint divides by 24 the computer time. It restricts the number of "integrable" permutations (almost by 24) but does not restrict drastically the number of classes since each class is likely to have a representent which satisfies this constraint (this can be easily seen on the previous example of permutations of entries of 3×3 matrices).

When a new permutation is found to verify one of the three previous recursions one checks if this permutation is, or is not, equivalent (up to relabeling (2)) to another one already found. For instance this reduces the 3752 permutations found for "class I-integrability" (see Table 4 above) into only 2647 non-equivalent permutations (classes). For each class one can calculate the number of equivalent permutations (cardinality of the class). Summing these cardinalities for the previous 2647 classes one finds 123 098 permutations associated with "class I-integrability".

The results are summarized in Table 4:

Integrability can be seen to be a quite rare phenomenon: $123\,098 + 43\,140 + 223\,008$ permutations among 16! permutations, that is a ratio of $\simeq 1.86 \times 10^{-8}$. This ratio has to be compared with the ratio emerging from the exhaustive analysis of 3×3 matrices: 27 classes among 30 462 classes that is a ratio of $\simeq 0.886 \times 10^{-4}$.

The exhaustive list of permutations leading to an integrable iteration is available by ftp [15].

Also note that the sixteen-vertex integrability (recursion (11)) *of these* 4×4 *permutations cannot be upgraded to* 5×5 *matrices*: it cannot lead to a 5×5 "sixteen-vertex integrable" permutation using the "straight generalization" previously recalled [11, 12]. However, one can imagine that 5×5 "sixteen-vertex integrable" permutations could exist. An exhaustive examination of permutations of entries of 5×5 matrices would be necessary to get some hint on the specific character of the "sixteen-vertex integrability". Unfortunately, the number of permutations of 5×5 matrices is too large to envisage any exhaustive analysis.

4. Conclusions

We hope that some understanding of the very nature of integrability will emerge from these exhaustive lists of integrable mappings. The case of finite order recursions on the x_n's, associated, or not, with finite orbits of the birational mappings, corresponds to miscellaneous quite involved permutations. Let us concentrate on the "true" integrability: we have seen that, among the permutations of entries of 3×3 matrices yielding (class I-, class IV-, sixteen-vertex-) integrability, the ones which can actually be upgraded for arbitrary q *are all involutions*. Conversely, it is important to note that integrability *does not necessarily require the involutive character of the permutations.* For instance, the two permutations of Table 3 ("class I-integrability" for $q=3$ only):

$$C_1 = \begin{pmatrix} 0 & 4 & 6 \\ 3 & 1 & 5 \\ 8 & 7 & 2 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & 2 & 7 \\ 3 & 1 & 8 \\ 5 & 4 & 6 \end{pmatrix} \tag{20}$$

can be decomposed in a 3-cycle and two 2-cycles. For instance, permutation C_1 decomposes as follows: $3 \to 4 \to 2 \to 3$, $6 \leftrightarrow 8$, $5 \leftrightarrow 7$ and permutation C_2 as: $4 \to 6 \to 5 \to 4$, $7 \leftrightarrow 8$, $2 \leftrightarrow 3$. Another example is provided by the following permutation of entries of 4×4 matrices ("class I-integrability"):

$$C = \begin{pmatrix} 3 & 7 & 15 & 1 \\ 8 & 5 & 11 & 9 \\ 14 & 10 & 6 & 2 \\ 4 & 0 & 13 & 12 \end{pmatrix} \tag{21}$$

which can be decomposed in a 4-cycle, a 5-cycle and a 7-cycle: $0 \to 3 \to 7 \to 11 \to 0$, $4 \to 6 \to 14 \to 9 \to 12 \to 4$, and $1 \to 5 \to 10 \to 13 \to 2 \to 8 \to 15 \to 1$. In fact among the permutations of entries of 3×3 matrices, a quite large number of classes (namely, 12 among 20) specific of 3×3 matrices (see columns "integrable only for $q = 3$, factorization scheme only for $q = 3$ in Table 3) *are not involutions* but permutations of order $3, 4, 6, 9, 12$, respectively. We have a similar situation for permutations of entries of 4×4 matrices. Such examples are precious to understand the very nature of integrability: here we do not have a (birational) representation of the infinite dihedral group anymore [4]. For instance C_1 (and C_2) is a transformation of order six. Thus, besides the iteration of $K = C_1 \cdot I$, there exists a (hyperbolic) Coxeter group generated by a transformation of order six and an involution (namely I) which certainly deserves to be analyzed [14].

One also remarks that the "sixteen-vertex" integrability cannot be represented by an (even involved) permutation of entries of 3×3 matrices. Is it specific of 4×4 matrices or can it be represented using permutations of entries of $q \times q$ matrices $q \geqslant 5$? It seems to be specific of 4×4 matrices. Let us recall that the generalization of the "sixteen vertex integrability" to $2m \times 2m$ matrices (see (13)) corresponds to "g-integrability". When the integrability of a permutation of a $q \times q$ matrix cannot be "upgraded" to a larger value of q it would be interesting to systematically see if it is not changed into a "g-integrability".

We have already started the analysis of permutations of entries of 5×5 matrices: of course this last analysis cannot be exhaustive ($15\,511\,210\,043\,330\,985\,984\,000\,000 \simeq 1.55 \times 10^{25}$ permutations to scan ...). We do hope that the accumulation of results in such very large computer calculations will provide many more examples of *non-involutive "integrable"* permutations that will help to better understand the very structures associated with integrability.

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