From Yang-Baxter equations to dynamical zeta functions for birational transformations*

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Birational transformations have been shown to provide powerful tools for analyzing the Yang-Baxter equations, and, beyond, to perform exact calculations on lattice models of statistical mechanics. In particular the so-called "baxterization problem" can be solved very simply using birational transformations. Beyond, the birational transformations can be studied "per se" in a lattice statistical mechanics framework or in a discrete dynamical system framework. Considering a family of such birational transformations of two variables, depending on two parameters we conjecture here a simple rational expression with integer coefficients for the exact expression of the dynamical zeta function. This yields an algebraic value for the exponential of the topological entropy. Furthermore the generating function for the Arnold complexity is also conjectured to be a rational expression with integer coefficients with the same singularities as for the dynamical zeta function. This leads, at least in this example, to an equality between the Arnold complexity and the (exponential of the) topological entropy. We also give a semi-numerical method to effectively compute the Arnold complexity. We also show that rational generating functions and associated algebraic complexities occur in a much larger framework, namely the iterations of the product of several rational transformations depending on many continuous parameters. Beyond the narrow framework of Yang-Baxter integrable models, these generating function calculations give a way to "classify" non-integrable lattice statistical models and beyond, discrete dynamical systems, providing precise quantitative instruments partitioning between integrable, weakly chaotic and very chaotic systems.

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I. INTRODUCTION

Birational transformations [1–5] naturally pop out as non trivial non-linear symmetries of lattice models of statistical mechanics [6–9] and solid state physics. For example (birational) transformations of the R-matrix of the sixteen-vertex model [10] exist which are non trivial *integrable* symmetries of the parameter space of the model. These transformations find their origin in the so-called *inversion relation* [11] and in the lattice symmetries. They form a (generically infinite discrete) group generated by the composition of such transformations. A worth noticing property

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*"Dedicated to James Mc Guire on the occasion of his 65th birthday"

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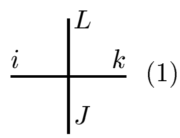
of integrability has been found for some of these transformations, opening the question whether this integrability property is related to an underlying statistical mechanics model or not. To answer this question a wide class of birational mapping has been introduced moving the point of view from statistical mechanics to discrete dynamical system.

These mappings are generated by two kinds of transformations on $q \times q$ matrices: the inversion of the $q \times q$ matrix and a *permutation of the entries* of the matrix. *Permutations of two entries* [7–9], as well as permutations corresponding to *discrete symmetries* of lattice models of statistical mechanics [1–6] were first analysed. Several integrable mappings associated with permutations of $q \times q$ matrices, *for arbitrary* q, have been found [7–9].

These birational symmetries approach provides very powerful tools to solve Yang-Baxter equations or their higher dimensional generalizations[1] (tetrahedron equations ...). They actually provide a fantastic short-cut for these highly overdetermined set equations giving immediately the uniformization of the Yang-Baxter equations whatever it is, whatever complicated it may be [16] (elliptic curves, abelian surfaces, higher dimensional abelian varieties). This approach provides the solution of the so-called *Baxterization problem*[2]. It is also important to underline that these tools can be used *beyond* the "narrow" framework of Yang-Baxter integrability.

A. Birational automorphisms of Yang-Baxter equations

Let us first consider the quite general vertex model where one direction, denoted direction (1), is singled out. Pictorially this can be interpreted as follows :



(1)

where i and k (corresponding to direction (1)) can take q values while J and L take m values. One can define a "partial" transposition on direction (1) denoted t_1. The action of t_1 on the R-matrix is given by [6] :

$$(t_1 R)^{iJ}_{kL} \;=\; R^{kJ}_{iL} \qquad (2)$$

The R-matrix is a $(q\,m) \times (q\,m)$ matrix which can be seen as q^2 blocks which are $m \times m$ matrices :

$$R = \begin{pmatrix} A[1,1] & A[1,2] & A[1,3] & \cdots & A[1,q] \\ A[2,1] & A[2,2] & A[2,3] & \cdots & A[2,q] \\ A[3,1] & A[3,2] & A[3,3] & \cdots & A[3,q] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A[q,1] & A[q,2] & A[q,3] & \cdots & A[q,q] \end{pmatrix} \qquad (3)$$

where $A[1,1]$, $A[1,2]$, ..., $A[q,q]$ are $m \times m$ matrices. With these notations the partial transposition t_1 amounts to permuting all the block matrices $A[\alpha,\beta]$ and $A[\beta,\alpha]$. We use the same notations as in [7–9], that is, we introduce the following transformations, the matrix inverse $\widehat{I}$ and the *homogeneous* matrix inverse I :

$$\widehat{I}: \;\; R \longrightarrow R^{-1}\,, \qquad \text{or :} \qquad I: \;\; R \longrightarrow \det(R) \cdot R^{-1} \qquad (4)$$

The homogeneous inverse I is a homogeneous polynomial transformation on each of the entries of R-matrix, which associates, with each entry, its corresponding cofactor. The two transformations t_1 and $\widehat{I}$ are *involutions* and $I^2 = (\det(R))^{q\,m-2} \cdot \mathcal{I}d$ where $\mathcal{I}d$ denotes the identity transformation. We also introduce the (generically infinite order) transformations :

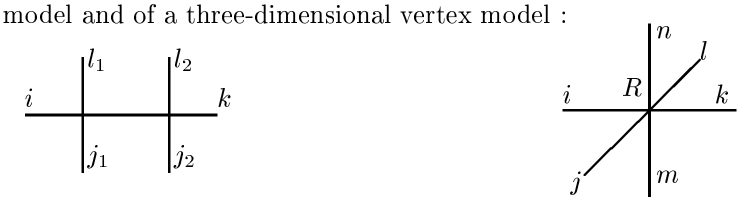
$$K \;=\; t_1 \cdot I \qquad \text{and} \qquad \widehat{K} \;=\; t_1 \cdot \widehat{I} \qquad (5)$$

Transformation $\widehat{K}$ is clearly a *birational transformation* on the entries of the R-matrix, since its inverse transformation, which is $\widehat{I} \cdot t_1$, is obviously a rational transformation. K is a *homogeneous polynomial transformation* on the entries

[1]They are birational automorphisms of the Yang-Baxter equations or of the tetrahedron equations [3,4].

[2]Far beyond the simple linear, or rational, interpolations of knot, or graph, theory.

of the R-matrix. This general framework enables *to take into account the analysis of* N-site monodromy matrices [16] (take $m = q^N$) of two-dimensional models, as well as the analysis of d-dimensional q^d-state *vertex models* (take $m = q^{d-1}$). Let us just give here a pictorial representation of the two sites ($N = 2$) monodromy matrix of a two-dimensional model and of a three-dimensional vertex model :



(6)

For a *three-dimensional cubic vertex model*, the "partial" transposition t_1 associated with one of the three directions of the cubic lattice reads [4,5] :

$$(t_1 R)^{i_1 i_2 i_3}_{j_1 j_2 j_3} \quad = \quad R^{j_1 i_2 i_3}_{i_1 j_2 j_3} \tag{7}$$

Such a situation corresponds to $m = q^2$. Let us restrict to $q = 2$. The analysis of the factorizations [14,16] associated with the iterations of transformation $K = t_1 \cdot I$, acting on an initial R-matrix M_0 corresponding to a general 64-state three-dimensional model (generic 8×8 matrix), gives the following factorizations :

$$M_1 = K(M_0), \qquad f_1 = \det(M_0), \qquad f_2 = \frac{\det(M_1)}{f_1^4}, \qquad M_2 = \frac{K(M_1)}{f_1^3}, \qquad f_3 = \frac{\det(M_2)}{f_1^7 \cdot f_2^4}, \cdots \tag{8}$$

and, for arbitrary n, the following *"string-like" factorizations* :

$$K(M_n) = M_{n+1} \cdot f_n^3 \cdot f_{n-1}^5 \cdot \Big(f_{n-2} \cdot f_{n-3} \cdots f_1\Big)^6 \tag{9}$$

$$det(M_n) = f_{n+1} \cdot f_n^4 \cdot f_{n-1}^7 \cdot \Big(f_{n-2} \cdot f_{n-3} \cdot f_{n-4} \cdots f_1\Big)^8 \tag{10}$$

where the f_n's are homogeneous polynomial expressions of the entries of M_0. Such factorization schemes occur for a large set of birational transformations corresponding to lattice statistical mechanics and even beyond this framework [14,16]. For all these various birational transformations [7–9,14,16] the factorization relations take the following general[3] form at the n-th step of the iterations :

$$\det(M_n) = f_{n+1} \cdot f_n^{\phi_1} \cdot f_{n-1}^{\phi_2} \cdot f_{n-2}^{\phi_3} \cdot f_{n-3}^{\phi_4} \cdot f_{n-4}^{\phi_5} \cdots f_1^{\phi_n} \tag{11}$$

$$K(M_n) = M_{n+1} \cdot f_n^{\eta_0} \cdot f_{n-1}^{\eta_1} \cdot f_{n-2}^{\eta_2} \cdot f_{n-3}^{\eta_3} \cdot f_{n-4}^{\eta_4} \cdots f_1^{\eta_{n-1}} \tag{12}$$

$$\det(M_n) \cdot M_{n+1} = \left(f_{n+1}^{\rho_0} \cdot f_n^{\rho_1} \cdot f_{n-1}^{\rho_2} \cdot f_{n-2}^{\rho_3} \cdot f_{n-3}^{\rho_4} \cdots f_1^{\rho_n}\right) \cdot K(M_n) \tag{13}$$

the exponents η_n's, ϕ_n's and ρ_n's being positive integers. We will denote α_n the degree of the determinant of matrix M_n, and β_n the degree of polynomial f_n and $\alpha(x)$, $\beta(x)$, $\eta(x)$, $\phi(x)$ and $\rho(x)$, the generating functions of the degrees α_n's, β_n's, and of the exponents η_n's, ρ_n's and ϕ_n's in the factorization schemes :

$$\alpha(x) = \sum_{n=0}^{\infty} \alpha_n \cdot x^n, \quad \beta(x) = \sum_{n=0}^{\infty} \beta_n \cdot x^n, \quad \eta(x) = \sum_{n=0}^{\infty} \eta_n \cdot x^n, \quad \phi(x) = \sum_{n=0}^{\infty} \phi_n \cdot x^n, \quad \rho(x) = \sum_{n=0}^{\infty} \rho_n \cdot x^n$$

From factorizations (9), (10), one easily gets the generating functions $\alpha(x)$ and $\beta(x)$:

$$\alpha(x) = \frac{8\,(1+x)^3}{(1-x)^4}, \qquad \beta(x) = \frac{8\,x}{(1-x)^3} \tag{14}$$

This shows that (11) and (12) correspond to a *polynomial growth* of the degrees α_n and β_n. These results can be compared with the ones associated with the analysis of the symmetries of the *sixteen vertex* model [6] for which one gets the simple factorization scheme [6] :

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[3]It should be noticed that slightly more involved, but still stable, factorization scheme may occur where the exponents η_n's and ϕ's depend on the parity of n, or, more generally, on n mod. p : in that case on has p sets of exponents η_n's and ϕ_n's in order to describe these factorization schemes [28,29]. Some examples are given in Appendix A.

$$M_{n+2} = \frac{K(M_{n+1})}{f_n^2}, \qquad f_{n+2} = \frac{\det(M_{n+1})}{f_n^3}, \qquad \frac{K(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{f_{n+1} f_{n+3}} \tag{15}$$

and one has a *hierarchy of integrable recursions* [16] :

$$\frac{f_n\, f_{n+3}^2 - f_{n+4}\, f_{n+1}^2}{f_{n-1}\, f_{n+3}\, f_{n+4} - f_n\, f_{n+1}\, f_{n+5}} = \frac{f_{n+1}\, f_{n+4}^2 - f_{n+5}\, f_{n+2}^2}{f_n\, f_{n+4}\, f_{n+5} - f_{n+1}\, f_{n+2}\, f_{n+6}} \tag{16}$$

The generating functions $\alpha(x)$ and $\beta(x)$ read :

$$\alpha(x) = \frac{4 \cdot (1+\, 3\, x^2)}{(1-x)^3}, \qquad \beta(x) = \frac{4 \cdot x}{(1-x)^3} \tag{17}$$

Again one has a polynomial growth of the calculations, consequence of the integrability of the mapping itself [6]. From these two examples one should not infer that the birational transformations corresponding to lattice statistical mechanics always yield polynomial growth. Vertex models studied by Stroganov or Perk and Schultz corresponding to $q \neq 2$ provide examples of exponential growth of the complexity [16] : this is the generic situation for lattice statistical mechanics. Exponential growth rules out the existence of solutions of the Yang-Baxter equations.

We have used the methods introduced in [7–9] on various examples of vertex models of lattice statistical mechanics. In particular, we have analyzed the factorization properties of discrete symmetries of the parameter space of these lattice models, represented as birational transformations. Different features have emerged from such studies, namely the *polynomial growth of the complexity* of the iterations of these birational transformations [13], the existence of recursion relations bearing on the factorized polynomials f_n. The relation between these properties, or more general structures like the "quasi-integrability" [6], and the integrability of these lattice models of statistical mechanics, has been studied. The analysis of the factorizations corresponding to a specific two-dimensional vertex model has shown how the generic exponential growth of the calculations does reduce to a polynomial growth when the model becomes Yang-Baxter integrable [16]. This gives a first example of the fact that the search for *polynomial growth*[4] of the associated iterations provides a new way to analyse *vertex models* [4,5,15].

B. Birational transformations associated with general permutations of entries of $q \times q$ matrices

These lattice statistical mechanics birational transformations correspond to combining the inversion of a matrix together with various permutations of the entries of the R-matrix representing geometrical symmetries of various euclidean d-dimensional lattice. This is a motivation for considering the following problem [14,16] consisting in analyzing the transformations $K_q = t \circ I$, acting on a $q \times q$ matrices M, for arbitrary permutation t of the entries.

This is a quite large set of transformations : for 3×3 matrices one has 362880 such (birational) transformations to study, and for 4×4 matrices, 20922789888000 transformations have to be studied. A systematic study of these large sets of (birational) transformations is performed elsewhere [28]. Let us first concentrate, in the first part of this paper, on a simple, but very interesting (and tutorial), example of permutation, namely the transposition of the two entries $M_{1,2}$ with $M_{3,2}$ and its associated bi-polynomial transformation K. This transformation has also been analysed in detail in [9]. For $q \times q$ matrices ($q \geq 3$) the factorizations corresponding to the iterations of K read :

$$\begin{aligned} &f_1 = det(M_0), \quad M_1 = K(M_0), \quad f_2 = \frac{det(M_1)}{f_1^{q-2}}, \quad M_2 = \frac{K(M_1)}{f_1^{q-3}}, \quad f_3 = \frac{det(M_2)}{f_1 \cdot f_2^{q-3}}, \quad M_3 = \frac{K(M_2)}{f_2^{q-3}}, \\ &f_4 = \frac{det(M_3)}{f_1^{q-1} \cdot f_2 \cdot f_3^{q-2}}, \qquad M_4 = \frac{K(M_3)}{f_1^{q-2} \cdot f_3^{q-3}}, \qquad f_5 = \frac{det(M_4)}{f_1^2 \cdot f_2^{q-1} \cdot f_3 \cdot f_4^{q-2}}, \quad \cdots \end{aligned} \tag{18}$$

and for arbitrary n :

$$det(M_n) = f_{n+1} \cdot (f_n^{q-2} \cdot f_{n-1} \cdot f_{n-2}^{q-1} \cdot f_{n-3}^2) \cdot (f_{n-4}^{q-2} \cdot f_{n-5} \cdot f_{n-6}^{q-1} \cdot f_{n-7}^2) \cdots f_1^{\delta_n}, \tag{19}$$

$$K(M_n) = M_{n+1} \cdot (f_n^{q-3} \cdot f_{n-2}^{q-2} \cdot f_{n-3}) \cdot (f_{n-4}^{q-3} \cdot f_{n-6}^{q-2} \cdot f_{n-7}) \cdots f_1^{\mu_n} \tag{20}$$

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[4] In fact, the polynomial growth of the calculations [8] correspond to *shift* on an abelian variety C^n/Γ.

where $\mu_n = q-3$ for $n = 1 \pmod 4$, $\mu_n = 0$ for $n = 2 \pmod 4$, $\mu_n = q-2$ for $n = 3 \pmod 4$ and $\mu_n = 1$ for $n = 0 \pmod 4$ and δ_n also depends on the truncation. The exact expressions of the generating functions $\alpha(x)$ and $\beta(x)$ read [9] :

$$\alpha(x) \,=\, \frac{q}{1+x} + \frac{q^2 \cdot x \cdot (1+x^2)}{(1-x)(1+x)(1-x-x^3)}, \qquad \beta(x) \,=\, \frac{q \cdot x \cdot (1+x^2)}{1-x-x^3} \tag{21}$$

It is clear that one has an exponential growth of the degrees α_n's, β_n's : these coefficients grow like λ^n where $\lambda \sim 1.465 \cdots$. This displays the "generic" factorization scheme. However, on various subvarieties (like the codimension one subvariety $\alpha = 0$ see below) the factorization scheme can be modified as a consequence of additional factorizations occurring at each iteration step, thus yielding a *smaller* value for the complexity λ.

This transformation can be seen to restrict to a two-parameter family of mapping of two variables (see (22) below). We now consider this two-parameter family of mapping of two variables, for which much can be said. In particular, we will conjecture an *exact algebraic value* for the (exponential of the) topological entropy and for the Arnold complexity[5]. Furthermore, these two measures of complexity will be found to be equal for all the values of the two parameters, generic or not (the notion of "genericity" is explained below). Note that a fundamental distinction must be made between the various "complexity measures" according to their invariance under certain classes of transformations. One should distinguish, at least, two different sets of complexity measures, the ones which are invariant under the larger classes of variables transformations, like the *topological entropy* or the *Arnold complexity* [22], and the other measures of complexity which also have invariance properties, but under a "less large" set of transformations, and are therefore more sensitive to the details of the mapping (they may depend on the metric like, for instance, the metric entropy [19,20]).

C. A two parameters family of birational transformation

Let us consider K^2 instead of K (which is just a simple change for the complexity λ into λ^2). Transformation K^2 can actually be reduced [9] to a two parameters family of birational transformations $k_{\alpha,\epsilon}$:

$$k_{\alpha,\epsilon} : \qquad (u_{n+1}\,,\; v_{n+1}) \,=\, \Big(1 - u_n + u_n/v_n\,, \;\; \epsilon + v_n - v_n/u_n + \,\alpha \cdot (1 - u_n + u_n/v_n)\Big) \tag{22}$$

which can also be written projectively :

$$\begin{aligned} u_{n+1} &= (v_n t_n - u_n v_n + u_n t_n) \cdot u_n \\ v_{n+1} &= \epsilon \cdot u_n \cdot v_n \cdot t_n + (u_n \,-\, t_n) \cdot v_n^2 + \alpha \cdot (v_n t_n - u_n v_n + u_n t_n) \cdot u_n \\ t_{n+1} &= u_n \cdot v_n \cdot t_n \end{aligned} \tag{23}$$

As far as complexity calculations are concerned, the $\alpha = 0$ case is singled out [26]. In that case, it is convenient to use a change of variables to get the very simple form k_ϵ :

$$k_\epsilon : \qquad (y_{n+1}\,,\; z_{n+1}) \;=\; \Big(z_n + 1 - \epsilon\,, \;\; y_n \cdot \frac{z_n - \epsilon}{z_n + 1}\Big) \tag{24}$$

or on its homogeneous counterpart :

$$(y_{n+1}\,,\, z_{n+1}\,,\, t_{n+1}) \;=\; \Big((z_n + t_n - \epsilon \cdot t_n) \cdot (z_n \,+ t_n)\,, \;\; y_n \cdot (z_n - \epsilon \cdot t_n)\,, \;\; t_n \cdot (z_n \,+ t_n)\Big) \tag{25}$$

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[5] To study the complexity of continuous, or discrete, dynamical systems, a large number of concepts have been introduced [17,18]. A non exhaustive list includes the Kolmogorov-Sinai metric entropy [19,20], the Adler-Konheim-McAndrew topological entropy [21], the Arnold complexity [22], the Lyapounov characteristic exponents, the various fractal dimensions [23,24], $\cdots$.

II. THE COMPLEXITY GROWTH

The correspondence [9] between transformations K_q and $k_{\alpha,\epsilon}$, more specifically between K_q^2 and $k_{\alpha,\epsilon}$, is given in [29]. It is shown below that, beyond this correspondence, K_q^2 and $k_{\alpha,\epsilon}$ share properties concerning the complexity. Transformation K_q is homogeneous and of degree $(q-1)$ in the q^2 homogeneous entries. When performing the $n^{\rm th}$ iterate one expects a growth of the degree of each entries as $(q-1)^n$. It turns out that, at each step of the iteration, some factorization of all the entries occurs. The common factor can be factorized out in each entry leading to a "reduced" matrix M_n, which is taken as the representent of the $n^{\rm th}$ iterate point in the projective space. Due to these factorizations the growth of the calculation is not $(q-1)^n$ but rather λ^n where λ is generically the largest root of $1 + \lambda^2 - \lambda^3 = 0$ (i.e. $1.46557123 < q-1$ [9,13], see also (21)). We call λ the complexity growth, or simply, the complexity. This result is a consequence of a *stable factorization scheme* (see (19), (20)), from which two generating functions[6] $\alpha(x)$ and $\beta(x)$ can be constructed. Generating function $\alpha(x)$ keeps track respectively of the degrees of the determinants of the successive "reduced" matrices and $\beta(x)$ of the degrees of the successive common factors. The actual value of λ is the inverse of the pole of $\beta(x)$ (or $\alpha(x)$) of smallest modulus. The algebraicity of the complexity is, in fact, a straight consequence of the rationality of functions $\alpha(x)$ and $\beta(x)$ with integer coefficients [13]. The same calculations have also been performed on transformations (22) and (23). In that case factorizations also occur, at each step, and generating functions can be calculated. These generating functions are, of course, different from the generating functions for K_q^2 (see [13]) but they have the *same* poles, and consequently the *same complexity growth*. One sees that, remarkably, the complexity λ does not depend on the birational representation considered : K_q^2 for any value of q, $k_{\alpha,\epsilon}$ or the homogeneous transformation (23). It will be useful to define some degree generating functions $G(x)$:

$$G(x) \; = \; \sum_n d_n \cdot x^n \tag{26}$$

where d_n is the degree of some quantities we look at, at each iteration step (numerators or denominators of the two components of k^n, degree of the entries of the "reduced" matrices M_n's, degree of polynomials f_n's extracted in the factorization schemes). The complexity growth λ is the inverse of the pole of smallest modulus (if $G(x)$ is rational) of any of these degree generating functions $G(x)$:

$$\log \lambda \; = \; \lim_{m \rightarrow \infty} {{\log d_m} \over {m}} \tag{27}$$

A. Complexity growth for $\alpha \, = \, 0$

In the $\alpha \, = \, 0$ case, which corresponds to a codimension one variety of the parameter space [26,29], additional factorizations occur reducing further the growth of the complexity. The generating functions are modified and the new complexity is given, for K_q, by equation $1 - \lambda^2 - \lambda^4 \, = \, 0$, i.e. $\lambda \simeq 1.27202 \cdots$. For k_ϵ, which corresponds to K_q^2, the equation reads :

$$1 - \lambda - \lambda^2 \; = \; 0 \tag{28}$$

leading to the complexity $\lambda \simeq 1.61803 \cdots \simeq (1.27202 \cdots)^2$. Not surprisingly, the complexity of the mappings $k_{\alpha,\epsilon}$ for $\alpha = 0$ (see (22)) and the one of mapping k_ϵ (see (24)), are the same: complexity λ corresponds to the asymptotic behavior of the degree of the successive quantities encountered in the iteration (see (27)). Clearly, this behavior remains unchanged under simple changes of variables. Note that this complexity growth analysis can be performed directly on transformation k_ϵ, or on its homogeneous counterpart (25). The number of generating functions in the two cases is not the same, but all these functions lead to the same complexity. In fact complexity λ is nothing but the Arnold complexity [22], known to be invariant under transformations corresponding to a change of variables like the change of variables from (22), for $\alpha = 0$, to (24) (or to (25)). Let us also recall that the Arnold complexity

[6]The generating function $\alpha(x)$ should not be confused with the parameter α.

counts the number of intersection between a fixed line[7] and its $n^{\rm th}$ iterate, which clearly goes as λ^n. Conversely, all these growth calculation evaluations can be seen as a "handy" way of calculating the Arnold complexity.

All these considerations allow us to design a *semi-numerical method* to get the value of the complexity growth λ for any value of the parameter ϵ. The idea is to iterate, with (24) (or (22)), a generic *rational* initial point (y_0, z_0) and to follow the magnitude of the successive numerators and denominators. During the first few steps some accidental simplifications may occur, but, after this transient regime, the integer denominators (for instance) grow like λ^n where n is the number of iterations. Typically a best fit of the logarithm of the numerator as a linear function of n, between $n = 10$ and $n = 20$, gives the value of λ within an accuracy of 0.1%. An *integrable mapping* yields a *polynomial growth* of the calculations [13] : the value of the complexity λ has to be numerically very close to 1. Fig. 1 shows the values of the complexity as a function of the parameter ϵ. The calculations have been performed using an infinite-precision C-library [25].

For most of the values of ϵ we have found $\lambda \simeq 1.618$, in excellent agreement with the value predicted in (28). In [26], it has been shown that the simple rational values $\epsilon = -1, 0, 1/3, 1/2, 1$ yield *integrable* mappings. For these special values one gets $\lambda \approx 1$ corresponding to a *polynomial* growth [26]. In addition, Fig. 1 singles out two sets of values $\{1/4,\, 1/5,\, 1/6, \cdots,\, 1/13\}$ and $\{3/5,\, 2/3\,,\, 5/7\}$, suggesting two infinite sequences $\epsilon = 1/n$ and[8] $\epsilon = (m-1)/(m+3)$ for n and m integers such that $n \ge 4$ and $m \ge 7$ and m odd. We call "non-generic" the values of ϵ of one of the two forms above (together with the integrable values), and "generic" the others. To confirm these suggestions of Fig. 1, we go back to (the matrix) transformation K_q, for $q = 3$, to get a generating function of the degrees of the f_n's extracted at each step of iteration, namely, with the notations of [9,14,16], function $\beta(x)$. From now on, we will give below, instead of $\beta(x)$, the expression of the following complexity generating function defined, for $q \times q$ matrices, as :

$$G_\epsilon^\alpha(q,\, x) \; = \; \frac{\beta(x)}{q \cdot x} \tag{29}$$

In the following the calculations are often displayed for 3×3 matrices and $G_\epsilon^\alpha(q,\, x)$ will simply be denoted $G_\epsilon^\alpha(x)$. Let us recall that the value of the complexity λ is the inverse of the root of smallest modulus of the denominator of this rational function. Examples of these calculations, in order to get the corresponding factorization scheme and deduce the generating function $\beta(x)$, or $G_\epsilon^\alpha(x)$, are given in Appendix A. Choosing an initial $\alpha = 0$ matrix to iterate, we have first obtained the generating function $G_\epsilon(x)$ in the generic case[9] for $\alpha = 0$ (see (A4) in Appendix A) :

$$G_\epsilon(x) \; = \; \frac{1+x+x^3}{1-x^2-x^4} \tag{30}$$

We also got the generating function $G_\epsilon(x)$ for the different "non-generic" cases :

$$G_{1/m}(x) \; = \; \frac{1+x+x^3-x^{2m+1}-x^{2m+3}}{1-x^2-x^4+x^{2m+4}}\,, \qquad \hbox{with} \quad m \ge 4 \tag{31}$$

$$G_{(m-1)/(m+3)}(x) \; = \; \frac{1+x+x^3-x^{2m+6}}{1-x^2-x^4+x^{2m+4}}\,, \qquad \hbox{with} \quad m \ge 7 \quad m \quad \hbox{odd} \tag{32}$$

and :

$$\begin{aligned} G_{\rm int}(x) \; &= \; \frac{1+x+x^3+x^4+x^8+x^{12}}{1-x^2-x^6+x^8-x^{10}+x^{12}+x^{16}-x^{18}} \\ &= \; \frac{1+x\cdot(1+x^2)+x^4\cdot(1+x^4+x^8)}{1-x^2\cdot(1-x^{12})\;-\;x^6\cdot(1-x^2+x^4-x^6+x^8-x^{10}+x^{12})} \end{aligned} \tag{33}$$

for $\epsilon = 1/2$ and $\epsilon = 1/3$. For $\epsilon = 1/m$ ($m \ge 4$) and $\epsilon = (m-1)/(m+3)$ ($m \ge 7$ and m odd), the corresponding complexities are the inverse of the roots of smallest modulus of polynomial :

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[7] Or the intersection of the n-th iterate of *any* fixed algebraic curve together with *any* other possibly different but *fixed* algebraic curve.

[8] Note that $m \rightarrow (m+3)/(m-1)$ is an involution.

[9] It is worth noticing that these results are not specific to 3×3 matrices, for example relation (30) is actually valid simply replacing $G_\epsilon^\alpha(x)$ by $G_\epsilon^\alpha(q,\, x)$.

$$1 - x^2 - x^4 - x^{2m+4} \; = \; 0 \tag{34}$$

in agreement[10] with the values of Fig. 1. This semi-numerical method acts as an 'integrability detector' and, further, provides a simple and efficient way to determine the complexity of an algebraic mapping. Applied to mappings (22), $K_q = t \cdot I$, or (24), it shows that the complexity is, generically, a universal quantity, *independent* of the value of the parameter ϵ, except for the four integrable points, and for two denombrable sets of points.

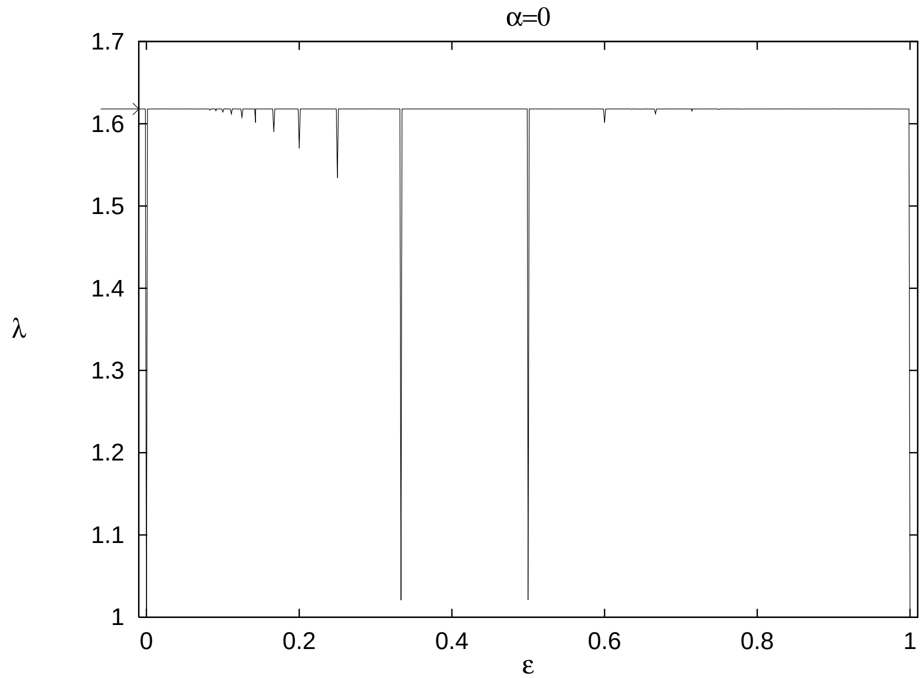

FIG. 1. Complexity for $\alpha = 0$. Complexity λ, for k_ϵ, as a function of ϵ.

B. Complexity growth for $\alpha \neq 0$

These complexity growth calculations can straightforwardly be generalized to $\alpha \neq 0$. As seen in section (I B) (see (21)), the "generic" generating function is :

$$G_\alpha^\epsilon(x) \; = \; \frac{1 + x^2}{1 \, - \, x \, - \, x^3} \tag{35}$$

The pole of smallest modulus of (35) gives $1.46557\cdots$ for the value of the complexity for the matrix transformation K. The complexity for the transformation k_ϵ^α is the square of this value: $\lambda \, = \, 2.14790\cdots$. Fig. 2 shows, for $\alpha \, = \, 1/100$, complexity λ as a function of the parameter ϵ, obtained with the semi-numerical method previously explained. Even with such a "small value" of α the expected drastic change of value of the complexity (namely $1.61803 \, \rightarrow \, 2.14790$) is non-ambiguously seen.

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[10]In this figure the ϵ-axis has been discretized as $M/720$ (M integer) and the extra values 1/7, 1/11, 1/13 and 5/7 have been added.

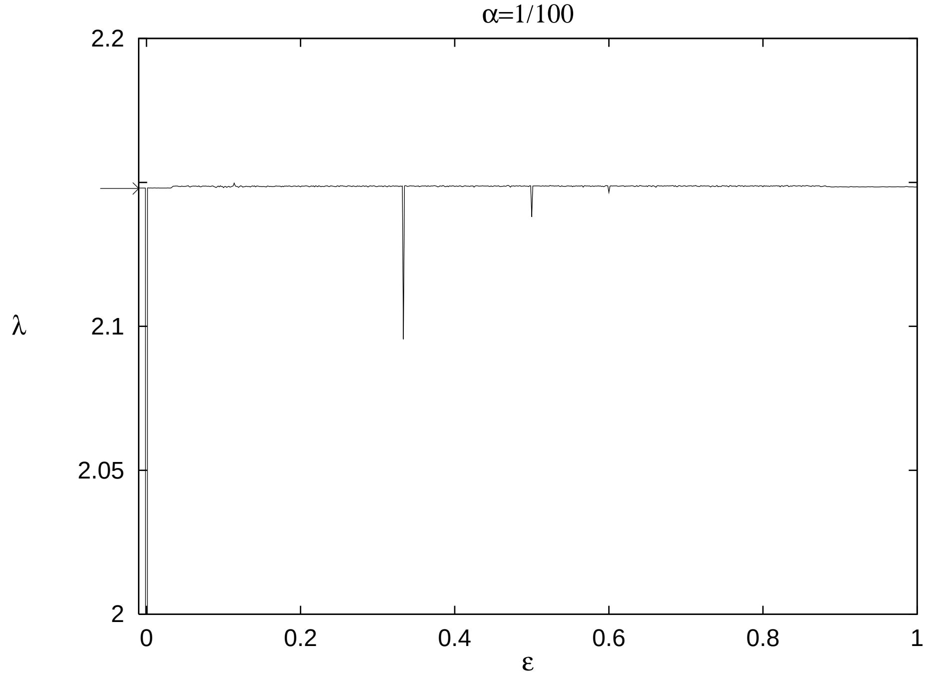

FIG. 2. Complexity λ, for $k_{\alpha,\epsilon}$, as a function of ϵ taken of the form $M/720$ for $\alpha = 1/100$.

Moreover, Fig. 2 clearly shows that, besides the value $\epsilon = 0$ known to be *integrable* whatever α [26], the following values $\epsilon = 1/2$, $\epsilon = 1/3$ and $\epsilon = 3/5$ are associated to a significantly smaller complexity, at least for the discretization in ϵ we have investigated. From these numerical results, and by analogy with $\alpha = 0$, one could figure out that all the $\epsilon = 1/m$ are also *non-generic* values of ϵ. In fact a factorization scheme analysis, like the one depicted in Appendix A), shows that $\epsilon = 1/4$ or $\epsilon = 1/7$ actually correspond to the generic (35). We got similar[11] results for other values of $\alpha \neq 0$. Let us just keep in mind that, besides $\epsilon = 0$ and $\epsilon = -1$, at least $\epsilon = 1/2$, $\epsilon = 1/3$ and $\epsilon = 3/5$ are singled out for $\alpha \neq 0$ in our semi-numerical analysis. The generic expression (for 3×3 matrices) for the generating function $G(x)$, namely (35), is replaced, for the "non-generic" value $\epsilon = 1/2$ (with $\alpha \neq 0$), by :

$$G^{\alpha}_{1/2}(x) = \frac{1 + x + x^3 - x^{16}}{1 - 2x^2 - x^6 + x^8 - x^{10} + x^{12} + x^{16}} = \frac{(1 + x^2) \cdot (1 + x - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12} - x^{14})}{(1 - x^2) \cdot (1 - x^2 - x^4 - 2x^6 - x^8 - 2x^{10} - x^{12} - x^{14})} \tag{36}$$

For the other "non-generic" value of ϵ, $\epsilon = 1/3$, the complexity generating function reads :

$$G^{\alpha}_{1/3}(x) = \frac{1 + x + x^3 - x^{12}}{1 - 2x^2 - x^6 + x^8 + x^{12}} = \frac{(1 + x^2) \cdot (1 + x - x^2 + x^4 - x^6 + x^8 - x^{10})}{(1 - x^2) \cdot (1 - x^2 - x^4 - 2x^6 - x^8 - x^{10})} \tag{37}$$

For the "non-generic" value $\epsilon = 3/5$, the complexity generating function reads :

$$G^{\alpha}_{3/5}(x) = \frac{1 + x + x^3 - x^{20}}{1 - 2x^2 - x^6 + x^8 - x^{10} + x^{12} - x^{14} + x^{16} + x^{20}} \tag{38}$$

The denominator of (38) has a "cyclotomic polynomial" simple form :

$$(1 - x^2) \cdot \Big(1 - x^2 \cdot (1 + x + x^2) \cdot (1 - x + x^2) \cdot (1 + x^4) \cdot (1 + x^8)\Big) \tag{39}$$

[11] However, when varying α and keeping ϵ fixed, new values of the complexity λ occur, λ being some "stair-case" function of α. We will not exhaustively describe the rather involved "stratified" space in the (α, ϵ) plane, corresponding to the various "non generic" complexities.

III. DYNAMICAL ZETA FUNCTION AND TOPOLOGICAL ENTROPY

It is well known that the fixed points of the successive powers of a mapping are extremely important in order to understand the complexity of the phase space. A lot of work has been devoted to study these fixed points (elliptic or saddle fixed points, attractors, basin of attraction, etc), and to analyse related concepts (stable and unstable manifolds, homoclinic points, etc). We will here follow another point of view and study the generating function of the *number* of fixed points. By analogy with the Riemann ζ function, Artin and Mazur [27] introduced a powerful object the so-called *dynamical zeta function* :

$$\zeta(t) \; = \; \exp\left(\sum_{m=1}^{\infty} \#\mathrm{fix}(k^m) \cdot \frac{t^m}{m}\right) \tag{40}$$

where $\#\mathrm{fix}(k^m)$ denotes the number[12] of fixed points of k^m. The generating functions

$$H(t) \; = \; \sum \#\mathrm{fix}(k^m) \cdot t^m \tag{41}$$

can be deduced from the ζ function :

$$H(t) \; = \; t\frac{\mathrm{d}}{\mathrm{dt}}(\log \zeta(t)). \tag{42}$$

If the dynamical ζ function is rational the topological entropy $\log(h)$ is simply related to its pole h :

$$\log h \; = \; \lim_{m\to\infty} \frac{\log\left(\#\mathrm{fix}(k^m)\right)}{m} \tag{43}$$

If the dynamical zeta function can be interpreted as the ratio of two characteristic polynomials of two linear operator A and B, namely $\zeta(t) = \det(1 - t\cdot B)/\det(1 - t\cdot A)$, then the number of fixed points $\#\mathrm{fix}(k^m)$ can be expressed from $\mathrm{Tr}(A^n) - \mathrm{Tr}(B^n)$. For more details on these Perron-Frobenius, or Ruelle-Araki transfer operators, and other shifts on Markov's partition in a symbolic dynamics framework, see for instance [32–35]. In this linear operators framework, the *rationality* of the ζ function, and therefore the algebraicity of the topological entropy, amounts to having a *finite dimensional representation* of the linear operators A and B. In the case of a rational ζ function, h, the exponential of the topological entropy is the inverse of the pole of smallest modulus. Since the number of fixed points remains unchanged under topological conjugaison (see Smale [36] for this notion), the ζ function is also a *topologically invariant function*, invariant under a large set of transformations, and does not depend on a specific choice of variables. Such invariances were also noticed for the complexity growth λ. It is then tempting to make a connection between the *rationality* of the complexity generating function previously given, and a possible *rationality* of the dynamical ζ function. We will also compare the Arnold complexity λ and h, the (exponential of the) topological entropy.

A. Dynamical zeta function for $\alpha = 0$, ϵ generic

We try here to get the expansion of the dynamical zeta function of the mapping k_ϵ (see (24)), for generic[13] values of ϵ. We concentrate on the value $\epsilon = 13/25 = 0.52$. This value is close to the value $1/2$ where the mapping is integrable [26]. One can gain an idea of the number, and localization, of the (real) fixed points looking at the phase portrait of Fig. 3.

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[12] If one of these numbers is infinite the definition breaks down. For instance for integrable mappings there are many algebraic curves such that all their points are fixed points of k^n for some given integer n.

[13] Neither of the form $1/m$, nor of the form $(m-1)/(m+3)$.

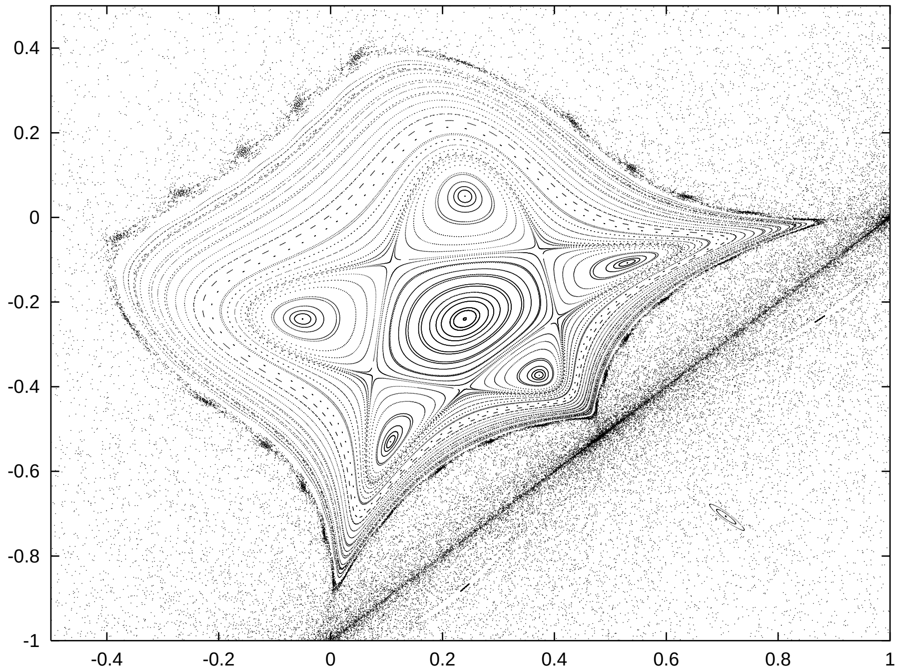

FIG. 3. Phase portrait of k_ϵ for $\alpha = 0$ and $\epsilon = 13/25$. 550 orbits of length 1000 have been generated. 50 orbits start from points randomly chosen near a fixed point of order 5 of $k_\epsilon = k_{13/25}$, and 500 others orbits start from randomly chosen points outside the elliptic region. Only the points inside the frame are shown.

The elliptic fixed point $(y_0, z_0) = (.24\,,\, -.24)$ is well seen, as well as the five elliptic points and the five saddle points of k_ϵ^5. Many points of higher degree are also seen. Transformation k_ϵ has a single fixed point for any ϵ. This fixed point is elliptic for $\epsilon \geq 0$ and localized at $(y_0, z_0) = ((1-\epsilon)/2,\, (\epsilon-1)/2)$. Transformation k_ϵ^2 has only the fixed point inherited from k_ϵ. The new fixed points of k_ϵ^3 are $(2-\epsilon\,, (\epsilon-1)/2)$, $(-1, 1)$ and $((1-\epsilon)/2, \epsilon-2)$. Transformation k_ϵ^4 has four new fixed points. At this point the calculations are a bit too large to be carried out with a literal ϵ, and we particularize $\epsilon = 13/25$. For k_ϵ^5 we have five news elliptic points and five new saddles points. The coordinates z and y of these points are roots of the two polynomials (obtained from resultants) :

$$\begin{aligned} P(z) = \; & z^2\,(25\,z\,-13)\,(1+z)(4375\,z^2+1550\,z-89)\,(175\,z^2\,+106\,z+7) \qquad (44)\\ & \times (25\,z\,+13)^2\,(25\,z^2\,+12\,z\,+1)^2\,(25\,z\,+6)^3 \\ Q(y) = \; & y\,\,(y-1)^2\,\,(25\,y-6)^5\,(25\,y^2-12\,y+1)^3\,(25\,y-12)^2 \qquad (45)\\ & \times (7-106\,y+175\,y^2)^2\,(4375\,y^2-1550\,y-89)^3 \end{aligned}$$

Among the various pairings one can consider, some corresponds to spurious or singular points (components of k_ϵ^5 are of the form $0/0$). For instance $z^2\,(25\,z-13)\,(1+z)\,(25\,z+13)^2 \,=\, 0$ and $y\,\,(y-1)^2\,\,(25\,y-12)^2 \,=\, 0$ correspond to such points to be discarded. After this selection, the five pairings of roots of (44) and (45), giving the five elliptic points, are (0.530283, -0.107335), (-0.050283, -0.24), (0.372665, -0.372665), (0.107335,-0.530283), (0.24, 0.050283) and the five pairings giving the five hyperbolic-saddle points are (0.372665, -0.075431), (0.107335, -0.107335), (0.404568, -0.24), (0.075431, -0.372665), (0.24, -0.404568). This is clearly seen on Fig. 3 where the occurrence of five "petals" corresponding to five elliptic points is obvious, the five hyperbolic points being located between the petals. After discarding the spurious, or singular, points, the fixed points for k_ϵ^5 are $(y\,,\,z)$ points where z and $-y$ are roots of the *same* polynomial $P_5(z)$. For arbitrary value of ϵ, $P_5(z)$ reads :

$$\begin{aligned} P_5(z) \,=\, & \Big((3\,\epsilon-1)\cdot z^2\,+\,(-4\,\epsilon^2\,+14\,\epsilon\,-6)\cdot z\,+\epsilon^3-5\,\epsilon^2+10\,\epsilon-4\Big) \qquad (46)\\ \times & \Big((3\,\epsilon-1)\cdot z^2\,+\,(4\,-6\,\epsilon\,\,-2\,\epsilon^2)\cdot z\,+\,1\,-5\,\epsilon\,+6\,\epsilon^2\Big)\cdot\Big(z^2\,+\,(1\,-\epsilon)\cdot z\,+2\,\epsilon\,-1\Big)\cdot(2\,z+1-\epsilon) \end{aligned}$$

For transformation k_ϵ^6, beyond the fixed points of k_ϵ and k_ϵ^3, one gets two complex saddle fixed points, i.e. transformation k_ϵ has two 6-cycles. For transformation k_ϵ^7, one obtains one elliptic real fixed point, one saddle real fixed point and and two complex saddle fixed points. For transformation k_ϵ^8, one obtains one saddle real fixed point and four complex saddle fixed points. For transformation k_ϵ^9, one obtains one elliptic real fixed point, three saddle real fixed point and and four complex saddle fixed points. For transformation k_ϵ^{10} , one obtains one elliptic real fixed point, one saddle real fixed point and and three complex elliptic fixed points and six saddle complex fixed points. The two elliptic fixed points of k_ϵ^{10} (0.24, -0.874) and (0.874, -0.24) are seen as “ellipse” on Fig. (3). For transformation k_ϵ^{11} , one obtains one elliptic real fixed point, five saddle real fixed point and and twelve complex saddle fixed points. On Fig. (3) a fixed point of k_ϵ^{12} lying on $y + z = 0$ is seen near $y = -13/25$. The polynomials, similar to (44) and (45), as well as the specific pairing of roots, for the successive iterates k_ϵ^N, are available in [37].

It is worth noticing, that among the 53 cycles of k_ϵ of length smaller, or equal, to 11, as much as 44 are on the line $y+z=0$, six are on the line $y+\bar{z}=0$. Two of the three remaining cycles are of length 11, while the last is of length eight. The particular role played by the $y + z = 0$ line can be simply understood. Let us calculate the inverse of the birational transformation (24). It has a very simple form :

$$k_\epsilon^{-1} : \qquad z_{n+1} = y_n - (1-\epsilon) , \qquad y_{n+1} = z_n \cdot \frac{y_n + \epsilon}{y_n - 1} \tag{47}$$

which is nothing but transformation (24) where y_n and $-z_n$ have been permuted. The $y_n \leftrightarrow -z_n$ symmetry just corresponds to the *time-reversal symmetry* $k_\epsilon \leftrightarrow k_\epsilon^{-1}$ transformation. The $y + z = 0$ line is the *time-reversal invariant line.*

Also note that, among these 53 cycles, only one of the 31 complex cycles is of the form $Z_0, Z_1, \cdots Z_p, \bar{Z}_0, \bar{Z}_1, \cdots \bar{Z}_p$ where $Z_i = (y_i, z_i)$ and $\bar{Z}_i$ is the complex conjugate. The 30 remaining complex cycles are actually 15 cycles and their complex conjugates.

For the $\epsilon = 13/25 = 0.52$ example these results are summarized in table Tab. I which gives the number of fixed points, as well as their status :

n	1	2	3	4	5	6	7	8	9	10	11
# fixed points	1	1	4	5	11	16	29	44	76	121	199
# n-cycles	1	0	1	1	2	2	4	5	8	11	18
# elliptic real	1	0	0	0	1	0	1	0	1	1	1
# saddle real	0	0	1	1	1	0	1	1	3	1	5
# elliptic complex	0	0	0	0	0	0	0	0	0	3	0
# saddle complex	0	0	0	0	0	2	2	4	4	6	12
# on $y+z=0$	1	0	1	1	2	2	4	4	6	10	12
# on $y+\bar{z}=0$	0	0	0	0	0	0	0	0	2	0	4

TABLE I. Number and status of the fixed points of $k_{13/25}^n$. n-cycle means cycle of minimum length n

The corresponding phase portrait is very complicated and dominated by the real fixed points [30] which are all saddle or elliptic. We note that the same properties (all points saddle or elliptic) also holds for the complex fixed points.

Local area preserving property : Eventually, one observes an *area preserving* [38] property *in the neighborhood of all the fixed points of* k_ϵ^n : the product of the modulus of the two eigenvalues of the Jacobian (i.e. the determinant) of k_ϵ^n, at all fixed points for $n \le 11$, is equal to 1. This *local* property is rather non trivial : the determinant of the product of the jacobian over an *incomplete* cycle is very complicated and only when one multiplies by the last jacobian does the product of the determinants shrinks to 1.

Dynamical zeta function : The total number of fixed points of k_ϵ^N for N running from 1 to 11, yields the following expansion, up to order eleven, for the generating function $H(t)$ of the number of fixed points :

$$H_\epsilon(t) \; = \; t + t^2 + 4\,t^3 + 5\,t^4 \; + 11\,t^5 + 16\,t^6 + 29\,t^7 + 45\,t^8 + 76\,t^9 + 121\,t^{10} + 199\,t^{11} \; + \cdots \tag{48}$$

This expansion coincides with the one of the *rational* function :

$$H_\epsilon(t) \; = \; \frac{t \cdot (1 \, + \, t^2)}{(1-t^2)\,(1-t-t^2)} \tag{49}$$

which corresponds to a very *simple rational* expression for the dynamical zeta function :

$$\zeta_\epsilon(t) \; = \; \frac{1 \, - t^2}{1 \, - t \, - t^2} \tag{50}$$

Expansion (48) remains unchanged for all the other generic values of ϵ we have also studied.

We conjecture that :

The simple rational expression (50) *is the actual expression of the dynamical zeta function for any generic value of* ϵ.

Comparing expression (28) with (50) one sees that the singularities of the dynamical zeta function happen to coincide with the singularities of the generating functions of the Arnold complexity. In particular the complexity growth λ and h, the exponential of the topological entropy, are *equal*.

In fact, as far as fixed points of k_ϵ^N are concerned, there is also a fixed point at ∞. If one takes into account this fixed point at ∞ as well, the previous definitions are slightly modified :

$$H(t) \quad \longrightarrow \quad H^{(\infty)}(t) \; = \; H(t) \, + \, \frac{t}{1-t}, \qquad \text{and :} \qquad \zeta(t) \quad \longrightarrow \quad \zeta^{(\infty)}(t) \; = \; \frac{\zeta(t)}{1-t} \tag{51}$$

Rational expression (50) becomes :

$$\zeta_\epsilon^{(\infty)}(t) \; = \; \frac{1 \, + \, t}{1 \, - \, t \, - \, t^2} \tag{52}$$

Let us consider the complexity generating function corresponding to the degrees of the *numerators* (or denominators) of the two components of k_ϵ^N. The generating function of the degree of the *numerator* of the z component of k_ϵ^N, we denote $g_z(t)$, *has exactly the same expression, up to* 1, *as* (52) :

$$\begin{aligned} 1 \, + \, g_z(t) \; &= \; \zeta_\epsilon^{(\infty)}(t) \; = \\ &= \; 1 + 2\,t + 3\,t^2 + 5\,t^3 + 8\,t^4 + 13\,t^5 + 21\,t^6 + 34\,t^7 + 55\,t^8 + 89\,t^9 + 144\,t^{10} + 233\,t^{11} \; + \; \cdots \end{aligned} \tag{53}$$

One can also introduce $g_y(t)$ the generating function of the degree of the numerator of the y component of k_ϵ^N, and $h_y(t)$ and $h_z(t)$ the generating functions of the degrees of the denominators of the y and z components of k_ϵ^N :

$$g_y(t) \; = t + 2\,t^2 \; + 3\,t^3 + \cdots, \qquad h_z(t) \; = \; t + 2\,t^2 \; + 4\,t^3 + \cdots, \qquad h_y(t) \; = \; t^2 \; + 2\,t^3 \; + 4\,t^4 + \cdots$$

$$\text{where :} \qquad g_z(t) \; = \; h_z(t) \, + \frac{t}{1-t}\,, \qquad g_y(t) \; = \; h_y(t) \, + \frac{t}{1-t}\,, \qquad g_y(t) \;\; = \;\; t \cdot (1 \, + \, g_z(t)) \tag{54}$$

One has :

$$\zeta_\epsilon^{(\infty)}(t) \;=\; 1 \,+\, g_z(t) \;=\; \frac{g_y(t)}{t} \tag{55}$$

More "canonically" recalling the homogeneous transformation (25), let us denote $g_{hom}(t)$ the generating function of the successive degrees of the y_n, z_n and t_n. For generic ϵ, one has the following relation between $g_{hom}(t)$ and $\zeta_\epsilon^{(\infty)}(t)$:

$$g_{hom}(t) \,+\, \frac{1}{1-t} \;=\; \zeta_\epsilon^{(\infty)}(t) \tag{56}$$

When mentioning zeta functions it is tempting to seek for simple *functional relations* relating $\zeta(t)$ and $\zeta(1/t)$. Let us introduce the following "avatar" of the dynamical zeta function :

$$\widehat{\zeta}(t) \;=\; \frac{\zeta(t)}{\zeta(t) \,-\, 1} \tag{57}$$

Transformation $z \;\rightarrow\; z/(z-1)$ is an involution. One immediately verifies that $\widehat{\zeta}_\epsilon(t)$, corresponding to (50), verifies two extremely simple, and remarkable, functional relations :

$$\widehat{\zeta}_\epsilon(t) \;=\; -\,\widehat{\zeta}_\epsilon(1/t)\,, \qquad \text{and :} \qquad \widehat{\zeta}_\epsilon(t) \;=\; \widehat{\zeta}_\epsilon(-1/t)\,, \tag{58}$$

or on the zeta function $\zeta(t)$:

$$\zeta_\epsilon(1/t) \;=\; \frac{\zeta_\epsilon(t)}{2\cdot\zeta_\epsilon(t) \,-\, 1}\,, \qquad \text{and :} \qquad \zeta_\epsilon(-1/t) \;=\; \zeta_\epsilon(t) \tag{59}$$

The generating function (49) verifies :

$$H_\epsilon(-1/t) \;=\; -\,H_\epsilon(t) \tag{60}$$

Cycle decomposition : An alternative way of writing the dynamical zeta functions relies on the decomposition of the fixed points into *cycles* which corresponds to the Weyl conjectures [39]. Let us introduce N_r the number of irreducible cycles of k_ϵ^r: for instance for N_{12} we count the number of fixed points of k^{12}, that are not fixed points of k_ϵ, k_ϵ^3, k_ϵ^4 or k_ϵ^6, and divide by twelve. One can write the dynamical zeta function as :

$$\zeta_\epsilon(t) \;=\; \frac{1}{(1-t)^{N_1}}\cdot\frac{1}{(1-t^2)^{N_2}}\cdot\frac{1}{(1-t^3)^{N_3}}\;\cdots\;\frac{1}{(1-t^r)^{N_r}}\;\cdots \tag{61}$$

The combination of the N_r's, inherited from the product (61), automatically takes into account the fact that the total number of fixed points of k_ϵ^r can be obtained from fixed points of k_ϵ^p, where p divides r, and from irreducible fixed points of k^r itself (see [39] for more details). A detailed analysis of this cycle decomposition (61) for generic values of ϵ will be detailed elsewhere [30]. The previous exhaustive list of fixed points (up to order twelve) can be revisited in this irreducible cycle decomposition point of view. The results of [37] yield : $N_1 \,=\, 1\,, N_2 \,=\, 0\,, N_3 \,=\, 1\,, N_4 \,=\, 1\,, N_5 \,=\, 2\,, N_6 \,=\, 2\,, N_7 \,=\, 4\,, N_8 \,=\, 5\,, N_9 \,=\, 8\,, N_{10} \,=\, 11\,, N_{11} \,=\, 18$. One actually verifies easily that (50) and (61) have the same expansion up to order twelve with these values of the N_r's. The next N_r's should be $N_{12} \,=\, 25\,, N_{13} \,=\, 40\,, N_{14} \,=\, 58\,,\; N_{15} \,=\, 90\,, \cdots$

Real dynamical functions : Introducing some generating function for the *real* fixed points of k^N, it should be noticed that this generating function has the following expansion up to order eleven for $\epsilon \,=\, .52$:

$$H_\epsilon^{real} \;=\; t + t^2 + 4\,t^3 + 5\,t^4 + 11\,t^5 + 4\,t^6 + 15\,t^7 + 13\,t^8 + 40\,t^9 + 31\,t^{10} + 67\,t^{11} + \cdots \tag{62}$$

This series is a quite "checkered" one. Furthermore, its coefficients depend very much on parameter ϵ. In contrast with generating function (41), the generating function H_ϵ^{real} has *no simple universality property* in ϵ. This series does not take into account the topological invariance in the complex projective space : it just tries to describe the dynamical system in the *real space*. This series H_ϵ^{real} corresponds to the "complexity" *as seen on the phase portrait* of Fig. (3). One sees, here, the quite drastic opposition between the notions well-suited to describe transformations in *complex projective spaces*, and the ones aiming at describing transformations in *real variables*.

B. Dynamical zeta functions for $\alpha = 0$, ϵ non generic

To further investigate the identification of these two notions (Arnold complexity-topological entropy), we now perform similar calculations (of fixed points and associated zeta dynamical functions) for $\epsilon = 1/m$ with $m \geq 4$ and $\epsilon = (m-1)/(m+3)$ with $m \geq 7$ odd (see Appendix B). The calculations are detailed in the Appendix B. All these calculations are compatible with the following single expression of the ζ function :

$$\zeta_{1/m}(t) = \frac{1 - t^2}{1 - t - t^2 + t^{m+2}} \tag{63}$$

We conjecture that this expression is exact, at every order, and for every value of $m \geq 4$. Again this expression coincides with the corresponding expression of the Arnold complexity (see (34) with $t = x^2$).

Similar calculations can also be performed for the second set of *non-generic* values of ϵ, namely $\epsilon = (m-1)/(m+3)$ with $m \geq 7$, m odd (or equivalently $\epsilon = (n-1)/(n+1)$ with $n \geq 4$). Comparing these rational expressions for the dynamical zeta function ((50), (B2), ...), and the rational expressions for the generating functions of the Arnold complexity ((31), (32), (33), ...) for the generic, and non-generic, values of ϵ, one sees that one actually has the *same singularities* in these two sets of generating functions (note that t has to be replaced by x^2 since k_ϵ is associated with transformation K^2 and not K). The identification between the Arnold complexity and the (exponential of the) topological entropy is thus valid, for $\alpha = 0$, *for generic values of* ϵ, and *even for non-generic ones.* It is worth noticing that, due to the topologically invariant character of the dynamical zeta function, these results are of course not specific of the y and z representation of the mapping (24) but are *also valid* for the (u, v) representation (22): in particular the exact expressions of the dynamical zeta functions (namely (50), (B2) in Appendix B), remain unchanged and, of course, the denominators of the complexity generating functions are also the same for generic, or non-generic, values of ϵ.

The *local area preserving* property *in the neighborhood of all the fixed points of* k_ϵ^n previously noticed for $\alpha = 0$, ϵ generic, is also verified for these non generic values of ϵ.

C. Dynamical zeta functions for $\alpha \neq 0$

This $\lambda = h$ identification is not restricted to $\alpha = 0$. One can also consider mapping (22) for arbitrary values of α and ϵ and calculate the successive fixed points. Of course, as a consequence of the higher complexity of the $\alpha \neq 0$ situation (the complexity jumps from $1.61803\cdots$ to $2.14789\cdots$), the number of successive fixed points is drastically increased and the calculations cannot be performed up to order eleven anymore. In the generic case, the expansion of the generating function $H(t)$ of the number of fixed points can be obtained up to order seven :

$$H_\epsilon^\alpha(t) = 2t + 2t^2 + 11t^3 + 18t^4 + 47t^5 + 95t^6 + 212t^7 + \cdots \tag{64}$$

One has two fixed points for $k_{\alpha,\epsilon}$, no new fixed points for $k_{\alpha,\epsilon}^2$, three sets of three new fixed points for $k_{\alpha,\epsilon}^3$ (giving $3 \times 3 + 2 = 11$ fixed points), four sets of four new fixed points for k^4 (giving $4 \times 4 + 2 = 18$ fixed points), nine sets of five new fixed points for k^5 (giving $9 \times 5 + 2 = 47$ fixed points), fourteen sets of six new fixed points for $k_{\alpha,\epsilon}^6$ (giving $14 \times 5 + 3 \times 3 + 2 = 95$ fixed points). This expansion corresponds to the following order seven expansion for the dynamical zeta function :

$$\zeta_\epsilon^\alpha(t) = 1 + 2t + 3t^2 + 7t^3 + 15t^4 + 32t^5 + 69t^6 + 148t^7 + \cdots \tag{65}$$

thus yielding to the following rational expression for the dynamical zeta function :

$$\zeta_\epsilon^\alpha(t) = \frac{(1-t^2)\cdot(1+t)}{1-t-2t^2-t^3} = \frac{(1-x^2)\cdot(1+x^2)^2}{(1-x-x^3)\cdot(1+x+x^3)} \qquad \text{with :} \qquad t = x^2 \tag{66}$$

This expression can also be written :

$$\zeta_\epsilon^\alpha(t) = \frac{(1-t^2)\cdot(1+t)}{1-t\cdot(1+t)^2} \tag{67}$$

If one counts the fixed point at infinity one gets :

$$\zeta^{(\infty)}(t) = \frac{(1+t)^2}{1-t-2t^2-t^3} \tag{68}$$

Let us consider again the complexity generating function corresponding to the degrees of the numerators of the two components of $k_{\alpha,\epsilon}^N$ (see (22)). The generating function $g_v(t)$ for the degrees of the numerators of the v component of $k_{\alpha\epsilon}^N$, *has again exactly the same expression (up to* 1*) as* (68) :

$$1 \,+\, g_v(t) \;=\; \zeta^{(\infty)}(t) \\ \;=\; 1 + 3\,t + 6\,t^2 + 13\,t^3 + 28\,t^4 + 60\,t^5 + 129\,t^6 + 277\,t^7 \,+ \cdots \tag{69}$$

The generating function $g_{hom}(t)$ of the successive degrees of the homogeneous transformation (23) of the u_n, v_n and t_n, reads :

$$g_{hom}(t) \;=\; \frac{t \cdot (3 + t - t^2 - t^3)}{(1-t) \cdot (1 - t - 2\,t^2 - t^3)} \tag{70}$$

Let us recall the "alternative" zeta function (57). It verifies the simple functional relation :

$$t^2 \cdot \widehat{\zeta}_\epsilon^\alpha(t) \cdot \widehat{\zeta}_\epsilon^\alpha(-t) \;=\; -\; \widehat{\zeta}_\epsilon^\alpha(-1/t) \cdot \widehat{\zeta}_\epsilon^\alpha(1/t) \tag{71}$$

The new rational conjecture (66) corresponds to the following expression for $H(t)$:

$$H_\epsilon^\alpha(t) \;=\; \frac{t \cdot (2 \,+ 3\,t^2 \,+ t^3)}{(1 - t^2) \cdot (1 - t \,- 2\,t^2 \,-\, t^3)} \tag{72}$$

Comparing the denominators of (66) and (35), one sees that, like for $\alpha = 0$, there is an identification between the Arnold complexity λ, and h, the exponential of the topological entropy :

$$\lambda \,=\, h \tag{73}$$

The eulerian product Weyl-decomposition (61) of the dynamical zeta function (66) corresponds to the following numbers of r-cycles : $N_1 = 2$, $N_2 = 0$, $N_3 = 3$, $N_4 = 4$, $N_5 = 9$, $N_6 = 14$, $N_7 = 30$, $N_8 = 54$, $N_9 = 107$, $N_{10} = 204$, $N_{11} = 408$, $N_{12} = 25$, $N_{13} = 1593$, $N_{14} = 3162$.

D. Dynamical zeta functions for $\alpha \neq 0$ with ϵ non-generic

For a "non-generic" value of ϵ when $\alpha \neq 0$, namely $\epsilon = 1/2$, the expansions of the generating function $H(t)$ and of the dynamical zeta function suggest the following possible rational expression for the dynamical zeta function :

$$\zeta_{1/2}^\alpha(t) \;=\; \frac{1 + t - t^7}{1 - t - t^2 - 2\,t^3 - t^4 - 2\,t^5 \,- t^6 - t^7} \;=\; \frac{1 \,+ t \cdot \left(1 \,- t^6\right)}{1 \,-\, t \cdot (1 \,- t \,+ t^2) \cdot (1 \,+ t \,+ t^2)^2} \tag{74}$$

This last result has to be compared with (36). The generating function $g_v(t)$ is again in agreement with a relation $1 \,+\, g_v(t) \;=\; \zeta^{(\infty)}(t)$. For another "non-generic" value of ϵ when $\alpha \neq 0$, namely $\epsilon = 1/3$, the expansion of the dynamical zeta function suggests the following possible rational expression :

$$\zeta_{1/3}^\alpha(t) \;=\; \frac{1 + t}{1 - t - t^2 - 2\,t^3 - t^4 - t^5} \;=\; \frac{1 + t}{1 \,-\, t \cdot (1 \,+\, t^2) \cdot (1 \,+ t \,+ t^2)} \tag{75}$$

This last result with has to be compared with (37). These results[14] are again in agreement with an Arnold-complexity-topological-entropy identification (73).

The *local area preserving* property in the neighborhood of all the fixed points of $k_{\alpha,\epsilon}^n$ previously noticed for $\alpha = 0$, is also verified, for $\alpha \neq 0$ for (22), *for generic values of* ϵ *generic, as well as for these non generic values of* ϵ.

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[14]However for the non-generic value of ϵ, $\epsilon \,=\, 3/5$, we do not have enough coefficients in the expansion of the dynamical zeta function to actually compare it with (38).

To sum up : Besides the integrable values, the other non-generic values can be partitioned in two sets : $\{1/m; \quad m > 3\}$ and $\{(m-1)/(m+1); \quad m > 3\}$. In *all* cases the polynomials giving the complexity growth and the topological entropy are the same. These polynomials are listed in Tab. II.

	$\epsilon = 1/3$	$\epsilon = 1/2$	$\epsilon = \frac{1}{m} \quad m > 3$	$\epsilon = \frac{n-1}{n+1} \quad n > 3$
α generic	$1-t-t^2-2t^3-t^4-t^5$	$1-t-t^2-2t^3-t^4-2t^5-t^6-t^7$	*generic see* (66)	(*)
$\alpha = 0$	N-th root of unity	N-th root of unity	$1-t-t^2+t^{m+2}$	$1-t-t^2-t^{2\,n+1}$

TABLE II. The polynomials giving the complexity growth λ and h, the exponential of the topological entropy, in various cases. The symbol(*) means that $\alpha \neq 0$ and $\epsilon = (m-1)/(m+1)$ are not generic, however λ and h are extremely close to the generic value, preventing us to compute them reliably with the semi-numerical method. $\alpha \neq 0$ and $\epsilon = 1/m$ is generic for $m > 3$.

A few comments :

• Heuristically, identification (73) can be understood as follows[15]. The components of k^N, namely y_N and z_N, are of the form $P_N(y,z)/Q_N(y,z)$ and $R_N(y,z)/S_N(y,z)$, where $P_N(y,z)$, $Q_N(y,z)$, $R_N(y,z)$ and $S_N(y,z)$ are polynomials of degree asymptotically growing like λ^N. The Arnold complexity amounts to taking the intersection of the N-th iterate of a line (for instance a simple line like $y = y_0$ where y_0 is a constant) with another simple (fixed) line (for instance $y = y_0$ itself or any other simple line or any *fixed* algebraic curve). For instance, let us consider the N-th iterate of the $y = y_0$ line, which can be parameterized as :

$$y_N \, = \, \frac{P_N(y_0, z)}{Q_N(y_0, z)}\, , \qquad \qquad z_N \, = \, \frac{S_N(y_0, z)}{T_N(y_0, z)}\, , \tag{76}$$

with line $y = y_0$ itself. The number of intersections, which are the solutions of $P_N(y_0, z)/Q_N(y_0, z) = y_0$, grows like the degree of $P_N(y_0, z) \, - \, Q_N(y_0, z) \cdot y_0$: asymptotically it grows like $\simeq \lambda^N$. On the other hand the calculation of the topological entropy corresponds to the number of fixed points of k^N, that is to the number of intersection of the two curves :

$$P_N(y,z) \, - \, Q_N(y,z) \cdot y \, = \, 0\, , \qquad \qquad R_N(y,z) \, - \, S_N(y,z) \cdot z \, = \, 0 \tag{77}$$

which are two curves of degree growing asymptotically like $\simeq \lambda^N$. The number of fixed points is obviously bounded by $\simeq \lambda^{2N}$ but one can figure out that it should (generically) grow like $\simeq \lambda^N$.

• From a general point of view, rational dynamical zeta functions (see for instance [35,40,41]) occur in the literature through theorems where the dynamical systems are asked to be ***hyperbolic***, or through combinatorial proofs using symbolic dynamics arising from Markov partition [42] and even, far beyond these frameworks [43], for the so-called "*isolated expansive sets*"(see [43,44] for a definition of the isolated expansive sets). There also exists an explicit example of a rational zeta dynamical function but only in the case of an *explicit linear* dynamics on the torus R^2/Z^2, deduced from an $SL(2, Z)$ matrix, namely the cat map [18,45] (diffeomorphisms of the torus) :

$$A \, = \, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad B \, = \, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \zeta \, = \, \frac{\det(1 - z \cdot B)}{\det(1 - z \cdot A)} \, = \, \frac{(1-z)^2}{1 - 3 \cdot z + z^2} \tag{78}$$

Note that golden number singularities for complexity growth generating functions have already been encountered (see equation (7.28) in [14] or equation (5) in [46]). In our examples we are not in the context where the known general theorems apply straightforwardly. The question of the demonstration of the rationality of zeta functions we conjectured, remains open.

In the framework of a "diffeomorphisms of the torus" interpretation, the degree of the denominator of a rational dynamical zeta function gives a lower bound of the dimension g of this "hidden" torus C^g/Z^g where the dynamics becomes "linearized". On expression (B2) of Appendix B, valid for $\alpha = 0$ and $\epsilon \, = \, 1/m$, one notes that dimension *g grows linearly* with m. The iteration of some birational transformations which "densify" Abelian surfaces (resp. varieties) has been seen to correspond to polynomial growth of the calculations [16]. Introducing well-suited variables θ_i $(i = 1, \cdots g)$ to uniformize the Abelian varieties, the iteration of these birational transformations just corresponds to a shift[16] $\theta_i \, \rightarrow \, \theta_i \, + \, n \cdot \eta_i$. For such polynomial growth situations, matrix A can be thought as the Jordan matrix associated with this translation, its characteristic polynomial yielding eigenvalues equal to 1.

IV. FROM COMPLEX PROJECTIVE ANALYSIS TO REAL ANALYSIS

The modification of the number of fixed points, from the "generic" values of ϵ to these particular values ($1/m$, $(n-1)/(n+1)$), corresponds to *fusion of some cycles*, or to the *disappearance of other cycles* which become singular points (indeterminations of the form $0/0$). These mechanisms will be detailed in [30]. Let us just mention here that the "non-generic" values of ϵ, like $\epsilon \, = \, 1/m$, correspond to a "disappearance of cycles" mechanism which modifies the denominator of the rational generating functions and thus the topological entropy and the Arnold complexity. In

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[15] We use here the notations of mapping (24) but they can be replaced by the $(u\, , \, v)$ variables of mapping (22).

[16] This "diffeomorphisms of the torus" interpretation is quite obvious on figure 2 of [14].

contrast, there actually exist for k_ϵ, other singled-out values of ϵ , like $\epsilon = 3$ for instance, which correspond to *fusion of cycles*: for instance in the $\epsilon \to 3$ limit, the order three cycle tends to the order one cycle. With the previous cycle notations $N_3 = 1$ becomes $N_3 = 0$, which amounts to multiplying the dynamical zeta function by $1 - t^3$. The dynamical zeta function and function $H(t)$ read :

$$\zeta_3(t) \; = \; \frac{(1-t^2)\cdot(1-t^3)}{1-t-t^2}\, , \qquad\qquad H_3(t) \; = \; t\frac{\mathrm{d}}{\mathrm{d}t}(\log \zeta_3(t)) \; = \tag{79}$$
$$= \; t + t^2 + t^3 + 5\,t^4 + 11\,t^5 + 13\,t^6 + 29\,t^7 + 45\,t^8 + 73\,t^9 + 121\,t^{10} + 199\,t^{11} \; + \; \cdots$$

One notes that such "fusion-cycle" mechanism *does not modify* the denominator of the rational functions, and thus the topological entropy, or the Arnold complexity, *remain unchanged.* However it should be underlined that $\epsilon = 3$ is clearly singled out as far as the *real dynamics* is concerned. The phase portrait, for $\epsilon = 3$, is extremely regular, like the one of an integrable mapping : it really "looks like" a foliation of the $(y\, ,\, z)$ parameter space in elliptic (or rational) curves. Actually, recalling the generating function $H_{real}(t)$ (see (62)), this function and the corresponding zeta function, ζ^{real}, read simple "integrable-like" forms :

$$H_{real}(t) \;\; = \;\; \frac{t}{1-t}\, , \qquad\qquad \text{and :} \qquad\qquad \zeta^{real}(t) \;\; = \;\; \frac{1}{1-t} \tag{80}$$

Of course the orbits in the $\epsilon = 3$ phase portrait are not elliptic curves but are *actually transcendental curves* [30]. The real dynamics "looks like" an integrable one, which is in agreement with the integrable-like form (80), but the mapping, seen as a *complex* (projective) mapping, is actually a chaotic one, with the generic $\alpha = 0$ complexity $\lambda \simeq 1.618033989$.

Other singled out algebraic values of ϵ, besides $\epsilon = 3$, corresponding to the fusion on an N-cycle with the 1-cycle, are for instance for $N = 5$ and $N = 7$:

$$\epsilon^2 - 10\cdot\epsilon + 5 \;\; = \;\; \left(\epsilon - \frac{1-\cos(2\pi/5)}{1+\cos(2\pi/5)}\right)\left(\epsilon - \frac{1-\cos(4\pi/5)}{1+\cos(4\pi/5)}\right) \;\; = \;\; 0$$
$$\epsilon^3 - 21\cdot\epsilon^2 + 35\cdot\epsilon - 7 \;\; = \;\; \left(\epsilon - \frac{1-\cos(2\pi/7)}{1+\cos(2\pi/7)}\right)\left(\epsilon - \frac{1-\cos(4\pi/7)}{1+\cos(4\pi/7)}\right)\left(\epsilon - \frac{1-\cos(6\pi/7)}{1+\cos(6\pi/7)}\right) \;\; = \;\; 0$$

All the (algebraic) ϵ values of the form[17] :

$$\epsilon \;\; = \;\; \frac{1-\cos(2\pi\cdot M/N)}{1+\cos(2\pi\cdot M/N)} \tag{81}$$

for any integer N (with $1 < M < N/2$, M not a divisor of N), do occur in such cycle-fusion mechanism. In fact the number of real fixed points of k_ϵ^N, and thus the phase portrait, depend on parameter ϵ. It is true that these numbers are not universal anymore (independent of ϵ up to a zero measure set of non-generic values of ϵ), however their dependence is not a "wild one". The number of real fixed points of k_ϵ^N depends on ϵ in a "staircase" way. They are constant by interval, the frontiers of the interval corresponding to *algebraic values* like (81). Such a situation can be called "weak universality".

The adaptation of the tools well-suited for topological invariance of dynamical systems seen in complex projective space, for instance the introduction of generating functions of *real fixed points* or "real-dynamical zeta functions" (or simply plots of the number of *real fixed points* for k_ϵ^N, for N fixed, as a function of the parameters of the mapping) shows that the analysis of the real dynamics of our mappings do show some nice algebraic structures and some kind of "weak universality". We thus have a two step procedure for analyzing dynamical systems. A first "universal" step concentrates on the topological entropy (or Arnold complexity) giving a first general classification of the mappings and of the various non-generic subvarieties of the parameters these mappings depend on. For instance, in our example (22), this first analysis shows that it is compulsory to discriminate between the $\alpha \neq 0$ and $\alpha = 0$ situation, and, beyond, between the $\alpha \neq 0$ and $\epsilon = 1/2$ ... on one side, and between the $\alpha = 0$ and $\epsilon = 1/m$ or $\epsilon = (n-1)/(n+1)$ situation on the other side. After this first general classification, the "second step" amounts to considering the algebraic structures corresponding to study the system from the point of view of *real* dynamics. This second step of

[17] Other family of algebraic numbers occur. They will be described elsewhere.

analysis based on "real-dynamical zeta functions" or real-Arnold complexity[18] generating functions, yields, in example (22), to the emergence of a second set of singled-out algebraic values of ϵ which do not modify the Arnold complexity or topological entropy (or equivalently the singularities of the dynamical zeta functions) but do modify the "real Arnold complexity" or the singularities of the "real dynamical zeta function" denoted $1/\lambda_{real}$:

$$\zeta_{real}(t) \;\simeq\; \sum_N \lambda_{real}^N \cdot t^N \tag{82}$$

For instance, for $\epsilon \, = \, 3$ (for $\alpha \, = \, 0$), the real complexity λ_{real} , as seen on the phase portrait, is the "integrable-like" value $\lambda_{real} \, = \, 1$ (see equation (80)). It should be underlined that this $\lambda_{real} \, = \, 1$ situation does not correspond to an integrability (foliation of the space in elliptic or rational curves). The phase portrait "looks like" a foliation of the space in curves. In fact there is no such thing as a "real" integrability, or "transcendental" integrability, in opposition with a "complex" or "algebraic" integrability. The chaotic feature of the mapping reveals through the following fact : the curves one "sees" are actually curves associated with *divergent series* [30]. One has a foliation in terms of curves associated with divergent series which is, at first sight, hard to visually distinguish from a foliation in (integrable) elliptic (or rational) algebraic curves.

V. COMPLEXITY SPECTRUM FOR 3×3 PERMUTATIONS

In view of the previous rational results, and recalling the whole set of rational results obtained for all kinds of birational transformations in [14], a systematic study of the 362880 (birational) transformations K associated with *all* the permutations of entries of 3×3 matrices is tantalizing. This set of transformations is quite large and one would like to reduce it using some symmetries (equivalence classes). One should recall that equivalence classes, corresponding to quite obvious rows and columns relabeling symmetries, had already been introduced [31] and studied. For two permutations in the same "relabeling" class, the complexities of the associated K's are obviously equal. This reduces the 362880 permutations into 30462 "relabeling" equivalence classes in [31]. Fortunately it is possible to go a step further [28] : some "new symmetries" have been discovered[19] which enable to define new equivalence symmetry-classes for the 362880 permutations, reducing a systematic complexity analysis to a careful examination of 2880 representants of 2880 symmetry-classes. Actually one first defines a set of equivalence relations $\mathcal{R}^{(\backslash)}$ such that any two permutations in the same equivalence class of $\mathcal{R}^{(\backslash)}$ automatically have the *same complexity* λ. Heuristically, equivalence relation $\mathcal{R}^{(n)}$ amounts to saying that two equivalent permutations are such that the n-th power of their associated transformations $\widehat{K}$ are conjugated (via particular permutations which can be decomposed into product of row permutations, column permutations and a possible transposition, see [28] for more details). An exhaustive inspection has shown that the equivalence relations $\mathcal{R}^{(n)}$'s "saturate" after $n \, = \, 24$: with obvious notations $\mathcal{R}^{(\infty)} \, = \, \mathcal{R}^{(24)}$. One finds out that the "ultimate" $\mathcal{R}^{(\infty)} \, = \, \mathcal{R}^{(24)}$ classes *can only have* 72 or 144 elements. Among the "ultimate" $\mathcal{R}^{(\infty)} \, = \, \mathcal{R}^{(\in\triangle)}$ classes one wants to separate the classes that were already $\mathcal{R}^{(1)}$ classes, that we will denote from now on $\mathcal{R}^{(1)}_{72}$, or $\mathcal{R}^{(1)}_{144}$, according to their number of elements, and the other ones we will denote $\mathcal{R}^{(\infty)}_{72}$ or $\mathcal{R}^{(\infty)}_{144}$. The 362880 permutations are grouped into 2880 $\mathcal{R}^{(\infty)}$ equivalence classes. We have the prejudice that the $\mathcal{R}^{(\infty)}_{72}$, or $\mathcal{R}^{(\infty)}_{144}$, classes have more "remarkable properties" than the $\mathcal{R}^{(1)}_{72}$ or $\mathcal{R}^{(1)}_{144}$ classes, because $\mathcal{R}^{(\infty)}$ corresponds to quite non-trivial relations. In the table below the respective numbers of $\mathcal{R}^{(1)}_{72}$, $\mathcal{R}^{(1)}_{144}$, $\mathcal{R}^{(\infty)}_{72}$ or $\mathcal{R}^{(\infty)}_{144}$ classes are displayed. Since the complexities do not depend on the chosen representent, we picked a representent in each $\mathcal{R}^{(\infty)}$ class and performed a semi-numerical complexity analysis taking care of these four groups.

A. Semi-numerical approach : numerical growth calculation

A semi-numerical approach to calculate the complexity λ has been detailed in section (II A) and in [29]. These semi-numerical calculations can be applied, mutatis mutandis, to homogeneous transformations K, or $\widehat{K}$, bear-

[18] Let us also recall that the Arnold complexity counts the number of intersections between a fixed (complex projective) line and its $n^{\rm th}$ iterate. One counts here the number of *real* points which are intersections between a *real fixed line* and its $n^{\rm th}$ iterate. With this restriction to *real* points we have lost "most of the universality properties" of the (complex) Arnold complexity.

[19] These symmetries are sketched in [28]. They will not be detailed here. They are related to some "transmutation property" of the matrix inversion $\widehat{I}$ with two permutations of entries.

ing on matrices, iterating an initial matrix with *integer* (or rational) entries chosen in a well-suited way[20]. This semi-numerical method has been applied to 2880 representants representing the 2880 symmetry-classes. For 3×3 matrices the complexities are necessarily such that : $2 \geq \lambda \geq 1$. Remarkably, instead of getting a quite complicated distribution, or spectrum, of values for the complexities, we have obtained values which are always very close (up to the accuracy of the method) to a set of seventeen (besides the $\lambda = 1$ integrability complexity) values : 2, 1.97481, 1.97458, 1.94893, 1.94685, 1.93318, 1.89110, 1.88320, 1.866760, 1.860073, 1.857127, 1.839286, 1.75487, 1.61803, 1.57014, 1.542579, 1.46557 and of course the integrable value $\lambda = 1$.

We got the following results. Among the 2146 classes of the $\mathcal{R}^{(1)}_{144}$ set, we got 2145 classes corresponding to complexities very close to $\lambda \simeq 2$ and a only one class with complexity very close to $\lambda \simeq 1.75487$. Among the 660 classes of the set $\mathcal{R}^{(1)}_{72}$, we got 640 classes corresponding to complexities very close to $\lambda \simeq 2$, and many non trivial complexity values (two classes yield values close to 1.97481, one gives 1.94893, two give 1.94685, ...). Among the set of fourteen $\mathcal{R}^{(\infty)}_{144}$ classes, all classes were seen to correspond to complexities very close to $\lambda \simeq 2$.

The most interesting set (for integrability diggers) is clearly the set $\mathcal{R}^{(\infty)}_{72}$ classes for which, beyond thirty three classes corresponding to the maximal $\lambda = 2$ complexity, and beyond a few non trivial complexity values, one discovers eighteen classes with complexity values numerically very close to one. Actually it is known [31] that some symmetry-classes correspond to situations where the determinantal variables[21] x_n's are periodic (denoted Period. in the table below). This $x_n = x_{n+N}$ situation corresponds to situations where the birational mapping $\widehat{K}$, itself, is of finite order (trivial integrability), but also to polynomial growth situations, that is, $\lambda = 1$ *exactly.* The polynomial growth situations *without any periodicity on the* x_n's are denoted "Pol.gr." in the table below. With our semi-numerical approach it is difficult to discriminate between these two $\lambda = 1$ situations [28] : an examination of the successive x_n's shows that one has nine polynomial growth classes and nine $x_n = x_{n+N}$ periodic classes.

One remarks that most of the classes correspond to complexity values numerically very close to the upper bound $\lambda = 2$. It has also been seen that this upper bound is actually reached for some permutations [14].

These semi-numerical results are revisited and confirmed in the next section (which provides exact factorization scheme calculations), all these results are summarized in the following table :

λ	Associated polynomial	$\mathcal{R}^{(1)}_{144}$	$\mathcal{R}^{(1)}_{72}$	$\mathcal{R}^{(\infty)}_{144}$	$\mathcal{R}^{(\infty)}_{72}$	Total
Total		2146	660	14	60	2880
2	$1-2\cdot x$	2145	640	14	33	2832
1.97481871	$1-2x+x^2-2x^3+x^4-2x^5+x^6$	0	2	0	0	2
1.974584654	$1-x-2x^2-x^3+x^4+2x^5+x^6$	0	1	0	0	1
1.94893574	$1-2x+x^5-x^7$	0	2	0	0	2
1.946856268	$1-x-x^2-x^3-x^4-x^5+x^6$	0	1	0	0	1
1.93318498	$1-2x+x^4-x^5$	0	1	0	0	1
1.891103020	$1-2x+x^2-2x^3+2x^4-2x^5$	0	0	0	1	1
1.88320350	$1-2x+x^2-2x^3+x^4$	0	2	0	6	8
1.866760399	$1-2x+x^3-x^4$	0	1	0	0	1
1.860073051	$1-x-x^2-x^4-2\cdot x^5$	0	1	0	0	1
1.857127516	$1-2x+x^2-x^3-x^5-x^7+x^8-2x^9+x^{10}$	0	1	0	0	1
1.83928675	$1-x-x^2-x^3$	0	2	0	0	2
1.75487766	$1-2x+x^2-x^3$	1	0	0	0	1
1.61803399	$1-x-x^2$	0	3	0	0	3
1.57014731	$1-x-x^3-x^5$	0	1	0	0	1
1.542579599	$1-x-x^3-x^7-x^8$	0	1	0	0	1
1.46557123	$1+x-x^3$	0	0	0	2	2
1 (Pol.gr.)	$1-x, \quad 1-x^N, \cdots$	0	0	0	9	9
1 (Period.)		0	1	0	9	10

Comments : Most of the 362880 birational transformations considered here do correspond to the most chaotic complexity, namely the upper bound $\lambda = 2$: one has 2145 $\mathcal{R}^{(1)}_{144}$ classes, 640 $\mathcal{R}^{(1)}_{72}$ classes, fourteen $\mathcal{R}^{(\infty)}_{144}$ classes and

[20] For integer entries one chooses initial matrices such that their determinants, and the determinants of the first reduced matrices M_n's, are as large as possible, prime numbers.

[21] The variables x_n's are defined by $x_n = \det(\widehat{K}^n(M_0) \cdot \det(\widehat{K}^{n+1}(M_0)$ see [7–9,16].

thirty three $\mathcal{R}_{72}^{(\infty)}$ classes, that is, $2145 \times 144 + 640 \times 72 + 14 \times 144 + 33 \times 72 = 359352$ birational transformations. The ratio of completly chaotic $\lambda = 2$ birational transformations is $r \simeq .99027$. If one is mostly interested by the integrable mappings and, more generally, by the mappings with polynomial growth, one remarks that $\mathcal{R}_{72}^{(\infty)}$ contains *all the integrable*, or *polynomial growth*, mappings and, up to one class in $\mathcal{R}_{72}^{(1)}$, all the mappings such that $x_n = x_{n+N}$, including the situations where mapping $\widehat{K}$, itself, is of finite order (which can be seen as a "trivial" integrability).

B. Revisiting the spectrum though exact factorization scheme

In order to see if this set of seventeen (plus one) values for the complexities really corresponds to a set of eighteen values or if the actual complexity values are just "close" to eighteen values with some "spread", we have revisited all these results and studied the factorization scheme for each of these representents for the various classes, concentrating on the complexities different from the $\lambda = 2$ upper limit. For that purpose we have written a driver which builds automatically the factorization scheme (see (11), (12) and the parity-dependant factorization schemes of Appendix A) for various original matrices[22] till the factorization scheme is stable and can be trusted.

We just display here the two generating functions $\beta(x)$, $\rho(x)$ for only two complexities (see [28] for more details). The other generating functions can be deduced from these two, using linear functional relations between the generating functions [14]. All these factorization scheme calculations confirm the results summarized in the previous table.

Among many symmetry-classes one verifies that one actually obtains :

- $\lambda \simeq 1.570147312$ (corresponding with notations of [31] to permutation 164285073). The expansions of $\rho(x)$ and $\beta(x)$ read :

$$\frac{\beta(x)}{3\,x} = \frac{1 - x^6 - x^{12}}{(1-x^2)\cdot(1-x-x^3-x^5)}, \qquad \rho(x) = \frac{(1+x+x^2)\cdot(1-x+x^2)\cdot(1+x)}{1-x^6-x^{12}} \qquad (83)$$

- $\lambda \simeq 1.839286755$ (corresponding, with notations of [31], to permutation 417063582) :

$$\frac{\beta(x)}{3\,x} = \frac{1 - x^2 - x^3}{(1-x)^2\cdot(1+x)\cdot(1-x-x^2-x^3)}, \qquad \rho(x) = \frac{(x+1)\cdot(1-x+x^4)}{1-x^2-x^3} \qquad (84)$$

It should be noticed that *factorization schemes can be different* from one representent to another one in the same symmetry-class, however, the complexity λ is *independent* of the chosen representent.

Complexity $\lambda \simeq 1.839286755$ can also be obtained with permutation[23] 164273085 for which $\rho(x)$ and $\beta(x)$ read :

$$\frac{\beta(x)}{3\,x} = \frac{1+x+x^2}{1-x-x^2-x^3}, \qquad \rho(x) = \frac{1+x^3+x^4+x^5}{1-x^6} \qquad (85)$$

New singularities : Most of time the *stability of the factorization scheme* and thus, in a second step, the occurrence of *rational* generating functions, corresponds to a simple periodicity of the exponents η_n, ϕ_n or ρ_n in the factorization scheme (11), (12). This periodicity is simply associated to the fact that the "exponent" generating functions have N-th root of unity poles : $1-x^2$, $1-x^8$, $1-x^6, \cdots$. However one sees, on example (84), that one may have a *stability* of the factorization scheme an *exponential growth* of these exponents η_n and ϕ_n. These exponent generating functions, of course, have a growth of their coefficients smaller than λ^n. This growth goes like μ^N where μ is the inverse of the poles of $\rho(x)$, $\phi(x)$ or $\eta(x)$, that is (for (84)), $\mu \simeq 1.324717958 \le \lambda \simeq 1.839286755$. Recalling (85) for which $\mu = 1$ and $\lambda \simeq 1.839286755$ and (84), one sees that one complexity value λ can be associated to *several* values of μ. Conversely permutation 174528603 (with notations of [31]) gives $\lambda \simeq 1.974584654$ (associated with $1 - x - 2x^2 - x^3 + x^4 + 2x^5 + x^6 = 0$) corresponding to :

$$\eta(x) = \frac{x^7}{(1-x^2-x^3)\cdot(1-x+x^2)}, \quad \rho = \frac{1-x+x^7+x^8}{(1-x^2-x^3)\cdot(1-x+x^2)}, \quad \phi = \frac{1-x+x^7+2\cdot x^8}{(1-x^2-x^3)\cdot(1-x+x^2)} \qquad (86)$$

Recalling (84), one sees that *one* "scheme-complexity" μ can actually correspond to *several* complexity growths λ.

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[22] The integer entries in the original matrices are choosen in such a way that the first polynomials f_n's obtained at each iteration step are, as large as possible, prime numbers.
[23] With notations of [31].

VI. COMPLEXITY ALCHEMY

Let us consider eighteen permutations representing the seventeen plus one complexities of the previous table, and the associated birational transformations $K_i = t_i \cdot I$ where $i = 1, \cdots 18$. If one combines one of these birational transformations, namely K_i, with another one, K_j, the complexity corresponding to the "molecule" $\mathcal{K} = K_i \cdot K_j$ obviously coincides with the one of $K_j \cdot K_i$. However it should be noticed that the complexities of these "molecules" $\mathcal{K}$ *do depend* on the represent chosen for each of the eighteen classes. We have systematically performed all the combinations of these eighteen representants with themselves. Among the 18^2 molecules we have obtained many times the maximal complexity $\lambda = 4$, however and remarkably, we got 156 molecules such that $\lambda < 4$, and even 30 molecules such that $\lambda < 3$. The spectrum of (algebraic) complexity values for these 18^2 molecules is extremely rich. When one changes the eighteen complexity representents, the "spectrum" of complexities becomes even richer ...

A. A "molecular" factorization scheme

Let us consider (with notations [31]) permutation 146237058 and its associated $\lambda \simeq 1.9748$ transformation K_1, and permutation 471562380 and its $\lambda \simeq 1.5426$ transformation K_2. From these two "atoms" let us build the "molecule" $\mathcal{K} = K_2 \cdot K_1$ (or molecule $\mathcal{K} = K_1 \cdot K_2$, they obviously have the same complexity). This example is an interesting one since the complexity (obtained from the previous semi-numerical calculations) of $\mathcal{K} = K_2 \cdot K_1$ is smaller than the product of the two complexities of K_1 and K_2 : $\lambda(\mathcal{K}) \simeq 2.897 < 1.9748 \cdot 1.5426 \simeq 3.0463$. The factorization scheme of $\mathcal{K}$ is of the same type as the ones described in [28,29], namely a parity-dependent factorization scheme (which is a straight consequence of the fact that one acts with K_1 and then with K_2 and again ...) :

$$f_1 = det(M_0), \quad M_1 = K_1(M_0), \quad f_2 = det(M_1), \quad M_2 = K_2(M_1), \quad f_3 = \frac{det(M_2)}{f_2}, \quad M_3 = K_1(M_2),$$
$$f_4 = det(M_3), \quad M_4 = K_2(M_3), \quad f_5 = \frac{det(M_4)}{f_2^3 \cdot f_4}, \quad M_5 = \frac{K_1(M_4)}{f_2}, \quad f_6 = \frac{det(M_5)}{f_2^2 \cdot f_4}, \cdots \tag{87}$$

and for arbitrary $n \geq 3$:

$$\begin{aligned} det(M_n) &= f_{n+1} \cdot f_n \cdot f_{n-2}^3 \cdot f_{n-6} \cdot f_{n-8} \cdot f_{n-10} \cdot f_{n-12} \cdot f_{n-14} \cdots \\ K_1(M_n) &= M_{n+1} \cdot f_{n-2} \end{aligned} \tag{88}$$

for n even, and :

$$\begin{aligned} det(M_n) &= f_{n+1} \cdot f_{n-1} \cdot f_{n-3}^2 \cdot f_{n-5} \cdot f_{n-7}^2 \cdot f_{n-9}^2 \cdot f_{n-11}^2 \cdot f_{n-13}^2 \cdots \\ K_2(M_n) &= M_{n+1} \cdot f_{n-3} \cdot f_{n-7} \cdot f_{n-9} \cdot f_{n-11} \cdot f_{n-13} \cdots \end{aligned} \tag{89}$$

for n odd. This yields for the odd and even parts of $\alpha(x)$ and $\beta(x)$ (label "2" for even and "1" for odd) :

$$\beta_2(x) = \frac{6 \cdot x^2}{1 - 3x^2 + x^4 - x^6 - 2x^8}, \qquad \beta_1(x) = \frac{3 \cdot x \cdot (1 - x^2) \cdot (1 - x^4)}{1 - 3x^2 + x^4 - x^6 - 2x^8},$$
$$\alpha_2(x) = \frac{3 \cdot (1 + 4x^4 - 4x^6 + x^8)}{(1 - x^2) \cdot (1 - 3x^2 + x^4 - x^6 - 2x^8)}, \qquad \alpha_1(x) = \frac{6 \cdot x \cdot (1 + x^4 - x^6 + x^8)}{(1 - x^2) \cdot (1 - 3x^2 + x^4 - x^6 - 2x^8)} \tag{90}$$

These generating functions yield a "molecular complexity" : $\lambda \simeq 2.858194057$. These generating functions verify a parity-dependent system of functional relations which generalizes the one described in [13] :

$$\begin{aligned} &x \cdot \alpha_1(x) - \beta_2(x) = F_{2p}(x) \cdot \beta_2(x), \qquad && x \cdot \alpha_2(x) - \beta_1(x) = F_{1m}(x) \cdot \beta_2(x), \\ &\alpha_2(x) - 3 - 2 \cdot x \cdot \alpha_1(x) + 3 \cdot G_{2p} \cdot \beta_2(x) = 0, && \alpha_1(x) - 2 \cdot x \cdot \alpha_2(x) + 3 \cdot G_{1m} \cdot \beta_2(x) = 0 \end{aligned} \tag{91}$$

where :

$$F_{2p}(x) = x^2 + 2x^4 + x^6 + \frac{2 \cdot x^8}{1 - x^2}, \quad F_{1m} = 2x^3 - x^5 + \frac{x}{1 - x^2}, \quad G_{1m}(x) = x^3, \quad G_{2p} = x^4 + \frac{x^8}{1 - x^2}$$

VII. THE "SKY IS THE LIMIT"

It has been seen that, combining two different (bi)rational transformations associated with permutations of the entries, one already gets an extremely rich set of algebraic complexities. Obviously a straight generalization amounts to considering products of three, four .... transformations of the previous table. Not surprisingly all the previous results generalize, mutatis mutandis, yielding again new sets of algebraic complexities. Let us now show that algebraic complexities occur in a much larger framework, corresponding to three quite drastic generalizations. A first generalization will show that there is *nothing specific with permutation* of the entries. The same algebraic results "pop out" for birational transformations which are the combination of a *linear transformation* of the entries of a $q \times q$ matrix and of the matrix inversion. A second generalization will show that there is nothing specific with linear transformations of the entries, and that one *still gets algebraic complexities* replacing *linear* transformations, by *homogeneous polynomial* transformations of the entries. A last generalization will show that a *random* product of birational (or even rational) transformations may yield algebraic complexities.

A. From permutations to linear transformations

Let us show that algebraic complexities occur for (generically) birational transformations, combination of a *linear transformation* of the entries of a $q \times q$ matrix and of the matrix inversion. The previous permutations of entries can actually be "merged" into families of linear transformations depending on r *continuous parameters*. Remarkably we will see that these birational transformations, $K = L \cdot I$, where L is no longer a permutation of the entries but a *linear* transformation on the entries, actually exhibit factorization schemes *exactly similar to the ones* previously described in the case of permutations of entries : how does the factorization scheme (which is a rigid structure) depends on the previous r continuous parameters ? Not surprisingly one can see that these factorization schemes are, generically, *actually constant and independent of the continuous parameters*[24]. Consequently, complexity λ has a universality property : it is actually independent, not only of the initial[25] point M_0, but also of these continuous parameters.

Let us give here a set of generating functions corresponding to factorization schemes associated to bi-polynomial transformations $K = L \cdot I$.

Linear transformations yielding the same complexity as permutations of entries : Let us introduce the *quite general linear transformation* depending on twenty one parameters :

$$L : \quad \left[\begin{array}{ccc} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{array} \right] \qquad \longrightarrow \tag{92}$$

$$\left[\begin{array}{ccc} m_{1,1} & a_{11}\, m_{1,1} + a_{12}\, m_{1,2} + a_{13}\, m_{1,3} + a_{21}\, m_{2,1} + a_{22}\, m_{2,2} + a_{23}\, m_{2,3} + a_{31}\, m_{3,1} + a_{32}\, m_{3,2} + a_{33}\, m_{3,3} & m_{1,3} \\ m_{2,1} & c_{21}\, m_{2,1} + c_{22}\, m_{2,2} + c_{23}\, m_{2,3} & m_{2,3} \\ m_{3,1} & b_{11}\, m_{1,1} + b_{12}\, m_{1,2} + b_{13}\, m_{1,3} + b_{21}\, m_{2,1} + b_{22}\, m_{2,2} + b_{23}\, m_{2,3} + b_{31}\, m_{3,1} + b_{32}\, m_{3,2} + b_{33}\, m_{3,3} & m_{3,3} \end{array} \right]$$

and let us consider the iterations of the homogeneous transformation $K = L \cdot I$. They read a stable factorization scheme identical to one factorization scheme already obtained for one representent of the previous table, corresponding to the complexity $\lambda \simeq 1.618033$. The factorization scheme, up to f_4 and M_4, is the same as the generic factorization scheme (18) (for $q = 3$) but gets modified with f_5, becoming for arbitrary n, instead of (19) and (20) :

$$\begin{aligned} det(M_n) &= f_{n+1} \cdot f_n \cdot (f_{n-1} \cdot f_{n-2}^2) \cdot (f_{n-3} \cdot f_{n-4}^2) \cdot (f_{n-5} \cdot f_{n-6}^2) \cdot (f_{n-7}^2 \cdot f_{n-8}^2) \cdots , \\ K(M_n) &= M_{n+1} \cdot f_{n-2} \cdot f_{n-4} \cdot f_{n-6} \cdot f_{n-8} \cdot f_{n-10} \cdots \end{aligned} \tag{93}$$

This yields the following generating functions :

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[24]Except on some submanifold (probably subvariety) of this r-dimensional parameter space, where the factorization scheme actually becomes different, associated with a *smaller* complexity : on these subvarieties one can only expect *more factorizations* than in the generic r-dimensional parameter space.

[25]Generically of course : on some codimension-one, or codimension-two, algebraic varieties of the space of entries of M_0 the factorization scheme may be modified yielding another (smaller of course) value of λ.

$$\alpha(x) \, = \, \frac{3 \cdot (1+x+x^3)}{(1-x^2)\cdot(1-x-x^2)}\, , \quad \frac{\beta(x)}{3\,x} \, = \, \frac{1}{1-x-x^2}\, , \quad \rho(x) \, = \, \frac{1}{1-x}\, , \quad \eta(x) \, = \, \frac{x^2}{1-x^2}\, , \quad \phi(x) \, = \, \frac{1+x+x^3}{1-x^2}$$

For a codimension-one subvariety $\mathcal{V}$ of these twenty one parameters, the linear transformation is not invertible anymore [28]. It is worth noticing that, even restricted to $\mathcal{V}$ where transformation K is not birational anymore, but just rational, the complexity λ remains unchanged, that is equal to $1.618\cdots$ Let us note that we have also found [28] linear transformations L, yielding non trivial algebraic complexities, which are not deformations of any permutation of entries.

B. From linear transformations to homogeneous polynomial transformations

There is nothing specific with linear transformations. Let us consider the following homogeneous polynomial transformation Q_r of degree r :

$$Q_r : \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{bmatrix} \qquad \longrightarrow \qquad \begin{bmatrix} m_{1,1}^r & m_{1,2}^r & m_{1,3}^r \\ m_{2,1}^r & m_{2,2}^r & m_{2,3}^r \\ m_{3,1}^r & m_{3,2}^r & m_{3,3}^r \end{bmatrix} \tag{94}$$

and the associated homogeneous transformation $K_r \, = \, Q_r \cdot I$. The iteration of transformation K_r yields a stable factorization scheme which gives, for arbitrary $r \geq 2$ the following generating functions :

$$\alpha(x) \, = \, \frac{3 \cdot (1+2\,x)}{1+2\cdot(1-r)\cdot x - r\cdot x^2}\, , \qquad \frac{\beta(x)}{3\,x} \, = \, \frac{1}{1\,+\,2\cdot(1-r)\cdot x\,-\,r\cdot x^2}\, , \quad \eta(x) \, = \, r\, , \quad \phi(x) \, = \, 1+2\cdot x$$

C. From periodic products of (bi)rational transformations to random products

It has previously been shown (see section (II)), that the two-dimensional mapping :

$$k_\epsilon : \qquad (y\,,\,z) \quad \longrightarrow \quad \Bigl(z\,+\,1\,-\,\epsilon\,, \ \ y\cdot\frac{z\,-\,\epsilon}{z+1}\Bigr) \tag{95}$$

yields (generically) a complexity $\lambda \, \simeq \, 1.61803$ associated with polynomial $1-t-t^2$. The *same* complexity can be obtained with the (generic) eight-parameters two-dimensional mapping :

$$k : \qquad (y\,,\,z) \quad \longrightarrow \quad \Bigl(a_1\cdot z\,+\,a_2\,, \ \ (a_3\cdot y\,+\,a_4)\cdot\frac{a_5\cdot z\,+\,a_6}{a_7\cdot z\,+\,a_8}\Bigr) \tag{96}$$

or even the nine parameters mapping :

$$k : \qquad (y\,,\,z) \quad \longrightarrow \quad \Bigl(a_1\cdot z\,+\,a_2\,, \ \ \frac{a_3\cdot y\cdot z\,+\,a_4\cdot y\,+\,a_5\cdot z\,+\,a_6}{a_7\cdot y\,+\,a_8\cdot z\,+a_9}\Bigr) \tag{97}$$

This last example can even be generalized to :

$$k : \qquad (y\,,\,z) \quad \longrightarrow \quad \Bigl(a_1\cdot z\,+\,a_2\,, \ \ \frac{a_3\cdot y^n\cdot z^n\,+P(y,z)}{Q(y,z)}\Bigr) \tag{98}$$

where $P(y,z)$ and $Q(y,z)$ are some polynomial expressions. The generating function of the successive degree of the numerator of the z component of the N-th iterate of these various families of transformations (96), (97), (98) seems to identify (as far as we have been able to check it) with a dynamical zeta function $\zeta(t)$ and reads :

$$1\,+\,g_z(t) \, = \, \frac{1+t}{1-t-t^2} \, = \, \zeta(t) \tag{99}$$

This generating function *remains unchanged* if one considers the sequence product of two (generic) transformations (95) : $k_{\epsilon_1}\cdot k_{\epsilon_2}\cdot k_{\epsilon_1}\cdot k_{\epsilon_2}\cdot k_{\epsilon_1}\cdots$, or even the sequence associated with the iteration of a "molecule" $\mathcal{K}_M$ product of

M different (generic) transformations k_ϵ. Of course if one prefers to consider directly $\mathcal{G}_M$, the degree generating function of $\mathcal{K}_M$, one gets for $M = 2, 3, \cdots$:

$$1 + \mathcal{G}_2(T) = \frac{1}{1 - 3\cdot T + T^2}, \quad 1 + \mathcal{G}_3(T) = \frac{1+T}{1 - 4\cdot T - T^2}, \quad 1 + \mathcal{G}_4(T) = \frac{1+T}{1 - 7\cdot T + T^2}, \tag{100}$$
$$1 + \mathcal{G}_5(T) = \frac{1 + 2\cdot T}{1 - 11\cdot T - T^2}, \quad 1 + \mathcal{G}_6(T) = \frac{1 + 3\cdot T}{1 - 18\cdot T + T^2}, \quad 1 + \mathcal{G}_7(T) = \frac{1 + 5\cdot T}{1 - 29\cdot T - T^2}, \quad \cdots$$

and for arbitrary M :

$$1 + \mathcal{G}_M(T) = \frac{1 + G(M)\cdot T}{1 - F(M)\cdot T + (-1)^M \cdot T^2}, \qquad where: \quad F(M) = F(M-1) + F(M-2) \quad and:$$
$$G(M) = G(M-1) + G(M-2) \quad with: \quad F(2) = 0, \quad F(3) = 4, \quad G(2) = 0, \quad G(3) = 1 \tag{101}$$

When comparing (99) and (100) or (101) the variable T must be seen as $T = t^M$. Since these results are valid for any product of M transformations k_ϵ, they are, in particular, valid in the limit where the k_ϵ's are all equal, which amounts to replacing k_ϵ into $\mathcal{K} = k_\epsilon^M$. The generating function amounts to "extracting", in the series expansion of (99), the coefficients of every M-th power of t . For instance the denominators of (100) are just the resultant (in t) of $1 - t - t^2$ and $t^M - T$. One has for any integer M :

$$1 + \mathcal{G}_M(t^M) = 1 + \frac{1}{M}\cdot\sum_{n=0}^{M-1} g(\omega^N\cdot t), \qquad \text{where}: \quad \omega^M = 1 \tag{102}$$

More generally, generating function $1 + g_z(t)$ in (99) *remains unchanged* if one considers a *random product of transformations in the family* (96) or even (97). Furthermore relations (54) and (55) are still valid. It seems that one even has a relation $1 + g_z(t) = \zeta_{rand}(t)$ for some dynamical zeta function suitably defined for random products.

All these mappings can also be seen as recursions. For instance (96) becomes :

$$z_{n+2} = \Big(a_3\cdot(a_1\cdot z_n + a_2) + a_4\Big)\cdot\frac{a_5\cdot z_{n+1} + a_6}{a_7\cdot z_{n+1} + a_8} \tag{103}$$

A "stochastic evolution" corresponding to *random* recursions (103) is thus seen to actually yield an *algebraic complexity* namely $\lambda \simeq 1.618033...$

VIII. COMMENTS AND SPECULATIONS

In practice it is numerically easier to get the Arnold complexity generating functions than getting the dynamical zeta functions. If one assumes the rationality of the dynamical zeta function and the identification between Arnold complexity and (exponential of the) topological entropy, getting the Arnold complexity generating functions may be seen as a simpler way to "guess" the denominator of the dynamical zeta functions.

Among the various complexity growth generating functions some seem to be more "canonical" and to identify exactly with the ζ function (see (55)).

The denominators of all the rational zeta functions encountered here are of the form : $1 - t\cdot Y(t)$ where $Y(t)$ is product of *cyclotomic polynomials* [47,48]. This is particularly obvious on expressions (33) and also on expressions (32), or (C1), or even (B2). We do not have any l-adic cohomology interpretation (see for instance [39] page 453) of this cyclotomic polynomial "encoding" of the zeta functions or of the complexity functions $G(q, x)$. Most of these rational expressions for zeta functions satisfy very simple functional relations and one also expects, for (C1) or (C2) for instance, more involved but, still simple, functional relations, may be similar to the ones obtained by Voros in [49]. Many of the generating functions $G(q, x)$ can also be seen to satisfy simple functional relations relating $G(q, x)$ and $G(q, 1/x)$. This will be detailed elsewhere[26].

The analysis developed here can be applied to a very large set of birational transformations of an *arbitrary* number of variables, yielding again *rational* generating functions [14,28]. Moreover, these generating functions are always simple rational expressions with integer coefficients (thus yielding *algebraic numbers* for the Arnold complexity). Most have

[26]For instance the generating function of the degrees $g(x)$ given by equation (5) in [46] verifies $g(x) + g(1/x) = 1$.

the previously mentioned "cyclotomic encoding" [14]. At this point the question can be raised[27] to see if the iteration of *any* birational transformation of *an arbitrary number* of variables always yields rational generating functions for the Arnold complexity.

It has also been shown that same results hold, mutatis mutandis, for *rational* transformations *which are not birational* anymore (also see (7.7) and (7.28) in [14]) or which are combination of homogeneous polynomial transformations of the entries of a $q \times q$ matrix, together with the matrix inversion, yielding again new algebraic spectrum independent of a large number of *continuous* deformation parameters : any proof of the rationalities of these generating functions should not depend to heavily on a simple *reversibility* of the mapping [50], or on the fact that the transformations should be rational transformations with *integer* coefficients.

One thus gets rational generating functions for quite arbitrary products of *rational* transformations which are not *invertible anymore*, and *may depend on many continuous parameters*. The set of birational transformations is a "huge" one, and the set of rational transformations is even "larger". One can imagine (if one believes in "some" universality of dynamical systems) that "most" of the dynamical systems could be very closely "approximated" by such transformations having algebraic complexity values. It will be important to try to define, for a given discrete dynamical system, what is the "best" approximation of this system in terms of birational, or even rational, transformations.

IX. CONCLUSION

Transformations generated by the composition of permutations of the entries and matrix inverse, naturally emerge in the analysis of lattice statistical mechanics symmetries [16], and provide a set of efficient new tools to study lattice statistical models (and beyond discrete dynamical systems).

The Yang-Baxter equations have been seen as a "laboratory" to elaborate these concepts. In return, these tools provide a systematic way to find integrable symmetries of the parameter space of the lattice model, which is a first (and compulsory) step to find Yang-Baxter integrable models. Beyond the narrow framework of Yang-Baxter integrable models, the "birational approach" gives a way to "classify" non-integrable lattice statistical models and beyond, discrete dynamical systems, providing a classification of these systems according to their more or less chaotic character.

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APPENDIX A: FACTORIZATION SCHEME FOR $\alpha = 0$, ϵ GENERIC

For matrix

$$M_0 \, = \, \left[\begin{array}{ccc} 4785 & 1305 & -2221 \\ 2175 & 9570 & 1305 \\ -18270 & 3480 & -16054 \end{array} \right] \tag{A1}$$

which corresponds to $\alpha \, = \, 0$ and $\epsilon \, = \, .52$, the generic ($\alpha \neq 0$) factorization scheme (18) becomes[28] :

$$f_1 \, = \, det(M_0) \, , \quad M_1 \, = \, K(M_0) \, , \quad f_2 \, = \, \frac{det(M_1)}{f_1} \, , \quad M_2 \, = \, K(M_1) \, , \quad f_3 \, = \, \frac{det(M_2)}{f_1^2 \cdot f_2} \, , \quad M_3 \, = \, \frac{K(M_2)}{f_1} \, ,$$
$$f_4 \, = \, \frac{det(M_3)}{f_1 \cdot f_2 \cdot f_3} \, , \qquad M_4 \, = \, K(M_3) \, , \qquad f_5 \, = \, \frac{det(M_4)}{f_1^2 \cdot f_2^2 \cdot f_3^2 \cdot f_4} \, , \qquad M_5 \, = \, \frac{K(M_4)}{f_1 \cdot f_2 \cdot f_3} \, ,$$

[27] After [14].

[28] These results can straightforwardly be generalized to $q \times q$ matrices, they are just a bit more involved.

$$f_6 = \frac{det(M_5)}{f_1 \cdot f_2^2 \cdot f_3 \cdot f_4 \cdot f_5}, \qquad M_6 = \frac{K(M_5)}{f_2}, \qquad f_7 = \frac{det(M_6)}{f_1^2 \cdot f_2 \cdot f_3^2 \cdot f_4^2 \cdot f_5^2 \cdot f_6}, \qquad M_7 = \frac{K(M_6)}{f_1 \cdot f_3 \cdot f_4 \cdot f_5}$$
$$f_8 = \frac{det(M_7)}{f_1 \cdot f_2 \cdot f_3 \cdot f_4^2 \cdot f_5 \cdot f_6 \cdot f_7}, \qquad M_8 = \frac{K(M_7)}{f_4}, \qquad f_9 = \frac{det(M_8)}{f_1^2 \cdot f_2^2 \cdot f_3^2 \cdot f_4 \cdot f_5^2 \cdot f_6^2 \cdot f_7^2 \cdot f_8}, \cdots$$

and for arbitrary n :

$$\begin{aligned} det(M_n) &= f_{n+1} \cdot (f_n \cdot f_{n-1}^2 \cdot f_{n-2}^2 \cdot f_{n-3}^2) \cdot (f_{n-4} \cdot f_{n-5}^2 \cdot f_{n-6}^2 \cdot f_{n-7}^2) \cdots \\ K(M_n) &= M_{n+1} \cdot (f_{n-1} \cdot f_{n-2} \cdot f_{n-3}) \cdot (f_{n-5} \cdot f_{n-6} \cdot f_{n-7}) \cdots \end{aligned} \qquad (A2)$$

for n even and :

$$\begin{aligned} det(M_n) &= f_{n+1} \cdot (f_n \cdot f_{n-1} \cdot f_{n-2} \cdot f_{n-3}^2) \cdot (f_{n-4} \cdot f_{n-5} \cdot f_{n-6} \cdot f_{n-7}^2) \cdots \\ K(M_n) &= M_{n+1} \cdot f_{n-3} \cdot f_{n-7} \cdot f_{n-11} \cdot f_{n-15} \cdot f_{n-19} \cdots \end{aligned} \qquad (A3)$$

for n odd.

The exact expressions of the generating functions $\alpha(x)$ and $\beta(x)$ read[29] :

$$\alpha(x) = \frac{3}{1+x} + \frac{3 \cdot \beta(x)}{1-x^2}, \qquad \text{where :} \qquad \beta(x) = 3 \cdot \frac{x \cdot (1+x+x^3)}{1 - x^2 - x^4} = -3 + 3 \cdot (1+x)/(1-x^2-x^4) \qquad (A4)$$

These generating functions give a complexity $\lambda \simeq 1.272019649$. It is important to note that factorization scheme (A2), (A3) is *actually stable, but of a slightly more general form,* compared to (18), or the ones described in [14] : recalling the generating functions $\eta(x)$ and $\phi(x)$ of the exponents that occur in the factorization scheme (see section (I A) or equations (8.6) and (8.10) in [14]), one must now introduce *two sets* of such exponents generating functions, η_1, ϕ_1, η_2, ϕ_2, in order to keep track of the parity of n, and even split these four functions into their odd and even parts :

$$\eta_{i2} = (\eta_i(x) + \eta_i(-x))/2, \qquad \eta_{i1} = (\eta_i(x) - \eta_i(-x))/2, \qquad \phi_{i2} = \cdots \qquad i = 1, 2$$

We must also decompose $\alpha(x)$ and $\beta(x)$ in odd and even parts: $\alpha_1(x) = (\alpha(x) - \alpha(-x))/2$, $\alpha_2(x) = (\alpha(x) + \alpha(-x))/2$, $\beta_1(x) = (\beta(x) - \beta(-x))/2$, $\beta_2(x) = (\beta(x) + \beta(-x))/2$, namely :

$$\beta_2(x) = \frac{3 \cdot x^2 \cdot (x^2+1)}{1-x^2-x^4}, \qquad \beta_1(x) = \frac{3 \cdot x}{1-x^2-x^4},$$

$$\alpha_2(x) = \frac{3 \cdot (1 + 2x^2 + 2x^4)}{(1-x^2)(1-x^2-x^4)}, \qquad \alpha_1(x) = \frac{3 \cdot x \cdot (2 + x^2 + x^4)}{(1-x^2)(1-x^2-x^4)} \qquad (A5)$$

Instead of functional relations (8.6) and (8.10) in [14], one now has the following relations :

$$\begin{aligned} &\alpha_1(x) - 2 \cdot x \cdot \alpha_2(x) + 3 \cdot x \cdot (\eta_{12}(x) \cdot \beta_2(x) + \eta_{11}(x) \cdot \beta_1(x)) = 0, \\ &\alpha_2(x) - 2 \cdot x \cdot \alpha_1(x) - 3 + 3 \cdot x \cdot (\eta_{22}(x) \cdot \beta_1(x) + \eta_{21}(x) \cdot \beta_2(x)) = 0, \\ &x \cdot \alpha_1(x) - \beta_2(x) - (\phi_{21}(x) \cdot \beta_1(x) + \phi_{22}(x) \cdot \beta_2(x)) = 0, \\ &x \cdot \alpha_2(x) - \beta_1(x) - (\phi_{11}(x) \cdot \beta_2(x) + \phi_{12}(x) \cdot \beta_1(x)) = 0 \end{aligned} \qquad (A6)$$

where the odd and even part of the exponents generating functions $\eta_1(x)$, $\phi_1(x)$, $\eta_2(x)$, $\phi_2(x)$, read :

$$\eta_{12}(x) = \frac{x^2}{1-x^4}, \qquad \eta_{11}(x) = \frac{x}{1-x^2}, \qquad \eta_{22}(x) = 0, \qquad \eta_{21}(x) = \frac{x^3}{1-x^4},$$
$$\phi_{11}(x) = \frac{x \cdot (2x^2+1)}{1-x^4}, \qquad \phi_{12}(x) = 2\frac{x^2}{1-x^2}, \qquad \phi_{21}(x) = \frac{x}{1-x^2}, \qquad \phi_{22}(x) = \frac{x^2 \cdot (2x^2+1)}{1-x^4},$$

[29] Result (A4) corresponds to a very simple expression for $\rho(x)$ (see for instance equation (8.12) in [14]).

Period four in the factorization scheme (A2), (A3) corresponds to the occurrence of a $1 - x^4 = 0$ singularity for these exponents generating functions.

The "stability" of factorization scheme (18) corresponds to the following $(n \to n+1)$-property : the exponents of the f_n's occurring at the m-th step of iteration are also the one's at $(m+1)$-th step of iteration the f_n's being changed into f_{n+1} : at each new iteration step one only needs to find the exponent of f_1 (if any). The "stability" of factorization scheme (A2), (A3) is a straight generalization mod.2. of the previous property : the exponents of the f_n's occurring at the m-th step of iteration are also the one's at $(m+2)$-th step of iteration the f_n's being changed into f_{n+2}. Let us now note that the initial matrix :

$$M_0 = \begin{bmatrix} 1 & 3 & x \\ 5 & 2 & 3 \\ -4 & 8 & -x-3 \end{bmatrix} \tag{A7}$$

which corresponds to $\alpha = 0$ and $\epsilon = -22/25$ *for any* x, do not yield the same factorization scheme as (A2), (A3), but still the same singularity associated with polynomial $1 - x^2 - x^4$. One actually gets the following generating functions:

$$\alpha(x) = \frac{3 \cdot (1 + 2x + 2x^2 + 4x^3 + 2x^4 + x^5)}{(1 - x^2 - x^4)(1 - x^2)}, \qquad \beta(x) = \frac{3 \cdot x \cdot (1 + x) \cdot (1 + x^2)}{1 - x^2 - x^4} \tag{A8}$$

Note that the "even" generating functions $\alpha_2(x)$ and $\beta_2(x)$ are the same as in (A5). The "odd" generating functions $\alpha_1(x)$ and $\beta_1(x)$ read :

$$\alpha_1(x) = \frac{3 \cdot x \cdot (2 + 4x^2 + x^4)}{(1 - x^2 - x^4) \cdot (1 - x^2)}, \qquad \beta_1(x) = \frac{3 \cdot x \cdot (1 + x^2)}{1 - x^2 - x^4} \tag{A9}$$

It is corresponds to the factorization scheme :

$$\begin{aligned} det(M_n) &= f_{n+1} \cdot (f_n \cdot f_{n-1} \cdot f_{n-2} \cdot f_{n-3}^2) \cdot (f_{n-4} \cdot f_{n-5} \cdot f_{n-6} \cdot f_{n-7}^2) \cdots \\ K(M_n) &= M_{n+1} \cdot f_{n-3} \cdot f_{n-7} \cdot f_{n-11} \cdot f_{n-15} \cdot f_{n-19} \cdots \end{aligned} \tag{A10}$$

for n even and :

$$\begin{aligned} det(M_n) &= f_{n+1} \cdot (f_n \cdot f_{n-1}^2 \cdot f_{n-2}^2 \cdot f_{n-3}^2) \cdot (f_{n-4} \cdot f_{n-5}^2 \cdot f_{n-6}^2 \cdot f_{n-7}^2) \cdots \\ K(M_n) &= M_{n+1} \cdot (f_{n-1} \cdot f_{n-2} \cdot f_{n-3}) \cdot (f_{n-5} \cdot f_{n-6} \cdot f_{n-7}) \cdots \end{aligned} \tag{A11}$$

for n odd. This factorization scheme is the same as (A2), (A3) where odd and even parity are permuted. The generating functions verify the functional equations :

$$\begin{aligned} &\alpha_1(x) - 2 \cdot x \cdot \alpha_2(x) + 3 \cdot \frac{x^4}{1 - x^4} \cdot \beta_1(x) = 0, \\ &\alpha_2(x) - 2 \cdot x \cdot \alpha_1(x) + 3 \cdot \Big(\frac{x^3}{1 - x^4} \cdot \beta_1(x) + \frac{x^2}{1 - x^2} \cdot \beta_2(x)\Big) - 3 = 0, \\ &x \cdot \alpha_1(x) - \beta_2(x) - \Big(\frac{x\,(2x^2 + 1)}{1 - x^4} \cdot \beta_1(x) + 2\,\frac{x^2}{1 - x^2} \cdot \beta_2(x)\Big) = 0, \\ &x \cdot \alpha_2(x) - \beta_1(x) - \Big(\frac{x^2\,(2x^2 + 1)}{1 - x^4} \cdot \beta_1(x) + \frac{x}{1 - x^2} \cdot \beta_2(x)\Big) = 0 \end{aligned} \tag{A12}$$

This factorization scheme is a slight modification of the previous one (the η_{ij}'s and ϕ_{ij} are just permuted : $\phi_{2j} \leftrightarrow \phi_{1j}$ and $\eta_{2j} \leftrightarrow \eta_{1j}$). In fact condition $\alpha = 0$ factorizes into several codimension-one varieties [29]. These subvarieties yield the same complexity *but not the same factorization schemes.*

1. Factorization scheme for $\alpha \neq 0$, ϵ non generic

Let us come back to $\alpha \neq 0$ with the non-generic value $\epsilon = 1/2$. We consider here $\alpha = 396/6095 \simeq .06497128$. Up to the thirteenth iteration one has the previously described $(n \to n+1)$-property, but this property is broken with f_{15} in favor of the $(n \rightarrow n+2)$-property previously encountered. The previously introduced odd-even-parity dependent exponents generating functions $\eta_{ij}(x)$ and $\phi_{ij}(x)$ now read :

$$\eta_{12}(x) \, = \, x^2 + x^6 + x^{10} + x^{12}\,, \qquad \eta_{11}(x) \, = \, x^3 + x^7 + x^{11} + \frac{x^{15}}{1-x^4}\,,$$

$$\eta_{22}(x) \, = \, x^2 + x^6 + x^{10} + \frac{x^{14}}{1-x^4}\,, \qquad \eta_{21}(x) \, = \, x^3 + x^7 + x^{11}\,,$$

$$\phi_{11}(x) \, = \, x + 2\,x^3 + 2\,x^7 + x^9 + 2\,x^{11} + x^5 + 2\,x^{13}\,, \quad \phi_{12}(x) \, = \, \frac{(1+2\,x^2)\cdot x^{14}}{1-x^4} + x^2 + 2\,x^4 + x^6 + 2\,x^8 + x^{10} + 2\,x^{12}\,,$$

$$\phi_{21}(x) \, = \, x + 2\,x^3 + 2\,x^7 + x^9 + 2\,x^{11} + x^5 + \frac{(1+2\,x^2)\cdot x^{13}}{1-x^4}\,, \quad \phi_{22}(x) \, = \, x^2 + 2\,x^4 + x^6 + 2\,x^8 + x^{10} + 2\,x^{12}$$

from which one deduces, from relations (A6), the rational expressions of the α_i's and β_i's :

$$\begin{aligned}
\beta_2(x) \, &= \, \frac{3\cdot x^2\cdot(1\,+\,x^2)}{(1-x^2)\cdot(1-\,x^2\,-\,x^4\,-2\,x^6\,-\,x^8\,-2\,x^{10}\,-\,x^{12}\,-\,x^{14})}\,, \\
\beta_1(x) \, &= \, \frac{3\cdot x\cdot(1\,+\,x^2)\cdot(1\,+\,x^4)\cdot(1\,+\,x^8)}{1\,-x^2\,-x^4\,-2\,x^6\,-x^8\,-2\,x^{10}\,-x^{12}\,-x^{14}}\,, \\
\alpha_2(x) \, &= \, 3\cdot\frac{1\,+2\,x^2\,+5\,x^4\,+4\,x^6\,+5\,x^8\,+4\,x^{10}\,+5\,x^{12}\,+5\,x^{14}\,+3\,x^{16}}{(1-x^2)\cdot(1-x^2\,-x^4\,-2\,x^6\,-x^8\,-2\,x^{10}\,-x^{12}\,-\,x^{14})}\,, \\
\alpha_1(x) \, &= \, 3\cdot x\cdot\frac{(2+4\,x^2+4\,x^4\,+\,5\,x^6\,+\,4\,x^8\,+5\,x^{10}\,+4\,x^{12}+\,4\,x^{14})}{(1\,-\,x^2)\cdot(1-x^2\,-x^4\,-2\,x^6\,-x^8\,-2\,x^{10}\,-x^{12}\,-x^{14})}
\end{aligned} \qquad \text{(A13)}$$

yielding the rational expressions (36) for $\beta(x)$.

These results have also been checked, using the previously depicted semi-numerical complexity growth evaluation method, for $\epsilon \, = \, 1/2$ and $\alpha \, = \, 396/6095 \, \simeq \, .06497\cdots$. The following value for the complexity has been obtained : $\lambda \, \simeq \, 1.46199$, in good agreement with the exact algebraic value deduced from (A13), namely : $\lambda \, \simeq \, 1.46188\cdots$ (to be compared with the generic algebraic value of λ, $\lambda \, \simeq \, 1.4655\cdots$ associated with $1-x-x^3 \, = \, 0$).

The singularities of (A13) are in agreement with the dynamical zeta function calculated for these values of α and ϵ:

$$\zeta(t) \, = \, \frac{1+t-t^7}{1-t-t^2-2\,t^3-t^4-2\,t^5-t^6-t^7} \, = \, \frac{1+t\cdot\left(1\,-\,t^6\right)}{1\,-\,t\cdot(1\,-t\,+t^2)\cdot(1\,+t\,+t^2)^2}$$

These calculations can also be performed, for $\alpha \neq 0$, for the other non-generic value of ϵ : $\epsilon \, = \, 1/3$. As far as the factorization scheme is concerned one gets exactly the same scenario as the one for $\epsilon \, = \, 1/2$, the breaking of the $(n \, \rightarrow \, n+1)$-property and the occurrence of a $(n \, \rightarrow \, n+2)$-property taking place with f_{11} instead of f_{15} previously. For $\epsilon \, = \, 1/3$ and, for instance, for $\alpha \, = \, 237/6095 \, \simeq \, .038884\cdots$, one gets expressions (37) for $\beta(x)$:

$$\beta(x) \, = \, \frac{3\cdot x\cdot(1\,+x^2)\cdot(1\,+x\,-\,x^2\,+\,x^4\,-\,x^6\,+\,x^8\,-\,x^{10})}{(1-x^2)\cdot(1-x^2-x^4-2\,x^6-x^8-x^{10})} \qquad \text{(A14)}$$

Again these results have been compared with the complexity growth deduced from the semi-numerical method, for $\epsilon \, = \, 1/3$ and $\alpha \, = \, 237/6095 \, \simeq \, .038884\cdots$. We have obtained the following value for the complexity : $\lambda \, \simeq \, 1.44865$ in good agreement with the exact algebraic value deduced from (A14), namely : $\lambda \, \simeq \, 1.44717\cdots$.

The singularities of (A14) are in agreement with the dynamical zeta function calculated for these values of α and ϵ:

$$\zeta(t) \, = \, \frac{1+t}{1-t-t^2-2\,t^3-t^4-t^5} \, = \, \frac{1+t}{1\,-t\cdot(1\,+\,t^2)\cdot(1\,+t\,+t^2)}$$

APPENDIX B: DYNAMICAL ZETA FUNCTIONS FOR $\alpha \, = \, 0$ WITH ϵ NON-GENERIC

To further investigate the identification of these two notions (Arnold complexity-topological entropy), we now perform similar calculations (of fixed points and associated zeta dynamical functions) for $\epsilon \, = \, 1/m$ with $m \, \geq \, 4$ and $\epsilon \, = \, (m-1)/(m+3)$ with $m \, \geq 7$ odd. The calculations have been performed for $\epsilon \, = \, 1/m$ for $m \, = \, 4\,,\,5\,,7$ and 9, giving the expansion of $H_\epsilon(t)$ up to order eleven :

$$\begin{aligned}
H_{1/4}(t) &= t+t^2+4\,t^3+5\,t^4+11\,t^5+10\,t^6+22\,t^7+29\,t^8+49\,t^9+71\,t^{10}+111\,t^{11}+\cdots\\
H_{1/5}(t) &= t+t^2+4\,t^3+5\,t^4+11\,t^5+16\,t^6+22\,t^7+37\,t^8+58\,t^9+91\,t^{10}+144\,t^{11}+\cdots\\
H_{1/7}(t) &= t+t^2+4\,t^3+5\,t^4+11\,t^5+16\,t^6+29\,t^7+45\,t^8+67\,t^9+111\,t^{10}+177\,t^{11}+\cdots\\
H_{1/9}(t) &= t+t^2+4\,t^3+5\,t^4+11\,t^5+16\,t^6+29\,t^7+45\,t^8+76\,t^9+121\,t^{10}+188\,t^{11}+\cdots
\end{aligned} \tag{B1}$$

All these expressions are compatible with this single expression of the ζ function :

$$\zeta_{1/m}(t) \; = \; \frac{1\,-\,t^2}{1\,-\,t\,-\,t^2\,+\,t^{m+2}} \tag{B2}$$

We conjecture that this expression is exact, at every order, and for every value of $m \ge 4$. Again this expression is in agreement with the polynomial expression giving the Arnold complexity (see (34)). If one counts the point at infinity the zeta function (51) becomes :

$$\zeta^{(\infty)}_{1/m}(t) \; = \; \frac{1\,+\,t}{1\,-\,t\,-\,t^2\,+\,t^{m+2}} \tag{B3}$$

Let us consider again the complexity generating function corresponding to the degrees of the numerators of the two components of k^N_ϵ . The generating function $g_z(t)$ for the degrees of the numerators of the z component of k^N_ϵ , for $\epsilon \, = \, 1/m$, *has again exactly the same expression (up to* 1*) as* (B3) :

$$1\,+\,g_z(t) \;\; = \;\; \zeta^{(\infty)}_{1/m}(t)$$

Note that relations (54) are still valid.

The generating function $g_{hom}(t)$ of the successive degrees of the homogeneous transformation (25) of the y_n, z_n and t_n, reads :

$$g_{hom}(t) \; = \; \frac{1-t^{m+3}}{(1-t)\cdot(1-t-t^2+t^{m+2})}$$

As far as functional relations relating $\zeta(t)$ and $\zeta(\pm 1/t)$ are concerned, recalling (57), one immediately verifies that $\widehat{\zeta}(t)$, corresponding to (B2), verifies the simple functional relation :

$$t^{m+1}\cdot\widehat{\zeta}_{1/m}(t) \;\; = \;\; \widehat{\zeta}_{1/m}(1/t)\,, \qquad \text{or :} \qquad \zeta_{1/m}(1/t) \; = \; \frac{t^{m+1}\cdot\zeta_{1/m}(t)}{t^{m+1}\cdot\zeta_{1/m}(t)\,-\,\zeta_{1/m}(t)\,+\,1}$$

Actually $\widehat{\zeta}_{1/m}(t)$ has a very simple n-th root of unity form :

$$\widehat{\zeta}_{1/m}(t) \;\; = \;\; \frac{1-t^2}{t\cdot(1-t^{m+1})}$$

Also note that when m is odd, and only in that case, $\widehat{\zeta}_{1/m}(t)$ *also satisfies* the functional relation :

$$t^{m+1}\cdot\widehat{\zeta}_{1/m}(t) \;\; = \;\; -\,\widehat{\zeta}_{1/m}(-1/t)$$

No simple functional relation, similar to (60), can be deduced on $H_{1/m}(t)$.

Similar calculations can also be performed for the second set of non-generic values of ϵ , namely $\epsilon = (m-1)/(m+3)$ with $m \ge 7$, m odd. For $m = 7$, that is $\epsilon = 3/5$, one gets, up to order eleven, the *same expansion* as the one for $\epsilon \, = \, 1/7$:

$$H_{3/5}(t) \; = \; t+t^2+4\,t^3+5\,t^4+11\,t^5+16\,t^6+29\,t^7+45\,t^8+67\,t^9+111\,t^{10}+177\,t^{11}+\cdots$$

yielding again, if this equality of expansions is still true at higher orders, the dynamical zeta function :

$$\zeta_{3/5}(t) \; = \; \frac{1\,-\,t^2}{1\,-\,t\,-\,t^2\,+\,t^9} \qquad \text{or :} \qquad \zeta^{(\infty)}_{3/5}(t) \; = \; \frac{1\,+\,t}{1\,-\,t\,-\,t^2\,+\,t^9}$$

Again the generating function of the numerator of the z component of k^N_ϵ , $g_z(t)$, *has exactly the same expression, up to* 1*, as* $\zeta^{(\infty)}_{3/5}(t)$:

$$1 + g_z(t) = \zeta^{(\infty)}_{3/5}(t) = \frac{g_y(t)}{t} =$$
$$= 1 + 2t + 3t^2 + 5t^3 + 8t^4 + 13t^5 + 21t^6 + 34t^7 + 55t^8 + 88t^9 + 141t^{10} + 226t^{11} + \cdots$$

For $m = 9$, that is $\epsilon = 2/3$, one gets :

$$H_{2/3}(t) = t + t^2 + 4t^3 + 5t^4 + 11t^5 + 16t^6 + 29t^7 + 45t^8 + 76t^9 + 121t^{10} + 177t^{11} + \cdots$$

A compatible zeta function could be[30] :

$$\zeta_{2/3}(t) = \frac{1 - t^2 - t^{11} - t^{12} - t^{13}}{1 - t - t^2 + t^{11}} \tag{B4}$$

Rational expression (B4) is not the same as (B2), however it has the same pole. Note that relations (54) are still valid for $\epsilon = 2/3$ and $\epsilon = 3/5$. At the order where the iterations have been performed, a relation like $1 + g_z(t) = \zeta^{(\infty)}_{2/3}(t)$ is not ruled out. One gets, however, a very simple expression for $g_y(t)/t$:

$$\frac{g_y(t)}{t} = \frac{1 + t}{1 - t - t^2 + t^{11}}$$

which rules out a simple $\zeta_{2/3}(t)^\infty = g_y(t)/t$ relation (see (55)).

APPENDIX C: DYNAMICAL ZETA FUNCTIONS FOR $\alpha \neq 0$ WITH ϵ NON-GENERIC

For a "non-generic" value of ϵ when $\alpha \neq 0$, namely $\epsilon = 1/2$, the expansion of the generating function $H(t)$ and of the dynamical zeta function read respectively :

$$H^{\alpha}_{1/2}(t) = 2t + 2t^2 + 11t^3 + 18t^4 + 47t^5 + 95t^6 + 198t^7 + \cdots$$
$$\zeta^{\alpha}_{1/2}(t) = 1 + 2t + 3t^2 + 7t^3 + 15t^4 + 32t^5 + 69t^6 + 146t^7 + \cdots$$

A possible rational expression for the dynamical zeta function is for instance :

$$\zeta^{\alpha}_{1/2} = \frac{1 + t - t^7}{1 - t - t^2 - 2t^3 - t^4 - 2t^5 - t^6 - t^7} = \frac{1 + t \cdot (1 - t^6)}{1 - t \cdot (1 - t + t^2) \cdot (1 + t + t^2)^2} \tag{C1}$$

This last result has to be compared with (36).

The generating function $g_v(t)$ corresponding to the degrees of the numerators of the v component of $k^N_{\alpha,1/2}$ reads :

$$1 + g_v(t) = \frac{1 + t - t^7}{(1 - t) \cdot (1 - t - t^2 - 2t^3 - t^4 - 2t^5 - t^6 - t^7)}$$
$$= 1 + 3t + 6t^2 + 13t^3 + 28t^4 + 60t^5 + 129t^6 + 275t^7 + \cdots$$

This expression is again in agreement with a relation $1 + g_v(t) = \zeta^{(\infty)}(t)$.

For another "non-generic" value of ϵ when $\alpha \neq 0$, namely $\epsilon = 1/3$ the expansion of the generating function $H(t)$ and of the dynamical zeta function read respectively :

$$H^{\alpha}_{1/3}(t) = 2t + 2t^2 + 11t^3 + 18t^4 + 42t^5 + 83t^6 + 177t^7 + \cdots$$
$$\zeta^{\alpha}_{1/3}(t)(t) = 1 + 2t + 3t^2 + 7t^3 + 15t^4 + 31t^5 + 65t^6 + 136t^7 + \cdots$$

A possible rational expression for the dynamical zeta function is for instance :

$$\zeta^{\alpha}_{1/3}(t) = \frac{1 + t}{1 - t - t^2 - 2t^3 - t^4 - t^5} = \frac{1 + t}{1 - t \cdot (1 + t^2) \cdot (1 + t + t^2)} \tag{C2}$$

---

[30] The series are not large enough to confirm this form. A set of simple and quick calculations seem to give for the next coefficients $\cdots + 296t^{12} + 469t^{13} + 785t^{14} + \cdots$ in agreement with (B4).

A possible generating function $g_v(t)$ corresponding to the degrees of the numerators of the v component of $k^N_{\alpha,1/3}$ reads :

$$
\begin{aligned}
1 + g_v(t) \;&=\; \frac{1+t+t^5-t^6}{(1-t)\cdot(1-t-t^2-2\cdot t^3-t^4-t^5)} \;= \\
&=\; 1+3\,t+6\,t^2+13\,t^3+28\,t^4+60\,t^5+125\,t^6+262\,t^7+\cdots
\end{aligned}
$$

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