Physics Letters A 159 (1991) 221–232
North-Holland

PHYSICS LETTERS A

Integrable Coxeter groups ☆

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Received 25 February 1991; accepted for publication 29 July 1991
Communicated by A.R. Bishop

We describe the construction of a class of mappings in projective space $\mathbb{C}P_N$ for any N. These mappings are non-linear representations of Coxeter groups by birational and therefore almost everywhere defined and invertible transformations. We give specific examples of the construction and exhibit algebraic invariants. The class of mappings we consider has a variety of behaviours according to the number of independent invariants. We introduce the notion of integrability of a group of mappings. The concept is related to the notion of integrability in the realm of statistical physics and field theory as will appear elsewhere. There is a natural set of deformation parameters of our mappings, allowing for a study of their stability. We comment on the algebraic structures we are handling.

1. Introduction

The purpose of this Letter is to present the construction and initiate the study of a class of rational mappings in projective space $\mathbb{C}P_N$. This construction takes its roots in the study of lattice spin models and vertex models, and will eventually serve the construction of integrable models in any dimensions. We will actually show in another publication that the transformations we consider are symmetries of basic equations for integrable models, i.e. star–triangle equations, Yang–Baxter equations and their higher dimensional generalizations (as the tetrahedron equations) [1,2].

We consider a Coxeter group [3] generated by a finite number of involutions, $I, J, K, \ldots$, with no other relations than $I^2=J^2=K^2= \ldots =1$. The number of involutions is 2^{d-1} for vertex models if d is the lattice dimension.

It is usual to represent Coxeter groups by reflections around hyperplanes in vector spaces, the paradigm being Weyl reflections in the root space of Lie algebras.

What we do is construct explicit non-linear representations of this group in terms of (bi)rational involutive transformations in projective space. These transformations are obtained from elementary considerations on matrix algebra. They are actually inversions of matrices linked with the so-called inversion relations of lattice statistical mechanics.

We denote by Γ the representation thus obtained of the original Coxeter group. Γ acts as an automorphy group (and may be considered as a group of symmetries) of various quantities of interest in statistical mechanics (partition function, critical manifolds, phase diagram, ...), and are of great help for calculating them.

Our construction provides with explicit examples of birational automorphisms of $\mathbb{C}P_N$ for any N, especially (but not only) *with integer coefficients*. These mappings have non-trivial properties which we exemplify: we exhibit orbits which are dense in algebraic subvarieties of $\mathbb{C}P_N$. In other words Γ may have a number of algebraic invariants, and thus reduce to automorphisms of non-trivial subvarieties of $\mathbb{C}P_N$. As a byproduct, this produces infinitely many rational points on some of these varieties.

For the sake of simplicity we describe here the construction related to two-dimensional lattice spin models, with nearest neighbour chiral interactions.

☆ Work supported by CNRS.

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We thus construct a group Γ of transformations, a priori isomorphic [#1] to the semi-direct product $\mathbb{Z}_2 \ltimes \mathbb{Z}$ (alias infinite dihedral group) [#2]. *Our method is not limited to this case. The construction extends to vertex models and to higher lattice dimensions [6,2].*

In section 2 we describe the construction of our mappings from two inversion operations on matrices. In the case of spin and vertex models in more than two lattice dimensions, the construction is similar, but from more than two involutions.

We then study in section 3 the orbits of the action of this group Γ in the parameter space $\mathbb{CP}_N$.

We describe in section 4 some invariants of the action of Γ. Guided by the origin of the construction (and its relation to symmetries of the star-triangle equations [1,2]), we also consider specific trajectories in $\mathbb{CP}_N \times \mathbb{CP}_N$, under the action of Γ, acting as $\mathrm{diag}(\Gamma \times \Gamma^{-1})$, and exhibit more invariants of this action. The number of invariants depends on the model, i.e., on the representation of the Coxeter group we consider.

The quest for invariants is deeply related to the resolution of the star-triangle equations. The fundamental reason is that these invariants define algebraic varieties where the possible spectral parameters – appearing in known solutions [7] – lives [8,1].

It is a crucial issue in the resolution of the star-triangle (respectively Yang–Baxter, tetrahedron, etc.) equations, to decide on which type of varieties the spectral parameters live. The most probable answer is just: *algebraic groups*, i.e. products of subgroups of $GL(n)$ by Abelian varieties. The $GL(n)$ part is related to certain similarity transformations of the matrices our construction is based upon (gauge transformations for vertex models, alias weak graph duality).

Another important issue is to understand the structure of the group Γ when the number of generating involutions is greater than two, i.e. for lattice dimensions larger than two.

We will say that the action of Γ is *quasi-integrable* if the orbits of Γ lie in a non-trivial algebraic subvariety of the parameter space $\mathbb{CP}_N$. This terminology is motivated by two facts. The first one is that Γ enters the construction of the symmetry group of the star-triangle relation [1]. The property of quasi-integrability is apparently a necessary but not sufficient condition for the existence of non-trivial solutions of the star-triangle relations. The second is that the setting of our construction is exactly the one used in the analysis of stability of dynamical systems à la Poincaré [9–11]. The property we call quasi-integrability is nothing but the existence of closed orbits in the Poincaré section of phase space. We shall call image varieties the (algebraic) varieties in which the orbits lie. Our mappings are not restricted to any interval, as is sometimes the case in the literature on iterated mappings.

Our construction has a priori a certain rigidity, showing in fact that our mappings have *integer* coefficients. There is however a natural set of continuous deformation parameters of our mappings. This allows for a study of their stability, and of the appearance of chaos as was already largely explored for lower dimensional systems. One feature of our mappings is that they are – by construction – almost everywhere invertible. Their natural deformations enjoy the same property.

Among the orbits of Γ, the ones where the group has a *finite order* representation pop out immediately: they appear in the search for invariants, in the resolution of the star-triangle equations, as well as in the numerical analysis of our mappings, ... This is no surprise, since for instance in the context of the study of Hamiltonian systems, they are just the resonant tori, and in the context of integrable models, they account for the existence of higher genus solutions [12–14] (they signal the existence of an increased symmetry [15,16]). This situation has numerous avatars in various domains of mathematical physics as Tutte–Behara numbers for chromatic polynomials, representations of quantum groups appearing in rational conformal field theories, rationality of critical exponents in statistical mechanics, representations of Hecke algebras, knot theory and the list is far from being exhaustive [17,18]. All this

[#1] The representation of Γ is faithful in the examples we describe here. It is an open question to decide in the general case whether or not the representation of this Coxeter group introduces any additional relations.

[#2] If $N=1$ the transformations are linear and we recover the Hecke group and other subgroups of $PSL(2, \mathbb{C})$ [4]. This is nothing but the well suited rational parametrization of the standard scalar Potts model [5].

legitimates a detailed study of the finite order orbitals.

In section 5, we describe some features of the algebraic structures we are handling. We note in particular the existence of collineations intertwining the different inverses in some cases of spin models and generically for vertex models in any dimension. This intertwining [#3] reduces in definite cases to an isomorphism between the matrix product and the dyadic product (element by element), thus generalizing the Kramers–Wannier duality transformation.

2. Construction

We start from two-dimensional q-state spin models with nearest neighbour interactions. The matrix of Boltzmann weights is a $q\times q$ matrix M with complex homogeneous entries m_{ij}. Choosing a specific model means fixing q and constraining the Boltzmann weights. We only want to retain the constraints between weights preserved by the two inversions:

$$I:\quad M\to M^{-1}\,, \tag{1}$$

$$J:\quad m_{ij}\to 1/m_{ij}\,, \tag{2}$$

I is the matrix inverse and J is the element by element or dyadic inverse. This allows us to keep track of the abovementioned inversion symmetries. In the inversion relations of statistical mechanics [7,20–23], one acts on the Boltzmann weights for vertical bonds with I (J) and on the horizontal bonds with J (I).

We restrict ourselves to constraints on the matrix of Boltzmann weights of the form $m_{ij}=m_{kl}$ for a number of pairs of indices. Such constraints are automatically preserved by J and are in fact up to signs the only invariant linear constraints. This amounts to giving a partition of the set of the entries of M such that all elements of a given part are set equal. Clearly only a limited number of partitions give a pattern that the matrix inverse I will preserve. We shall call these patterns *admissible* patterns [24].

The number of remaining homogeneous parameters in an admissible pattern is lower than q^2. It equals the number of parts in the pattern. It will be denoted by N in the following. On these remaining parameters $x_0, x_1, ..., x_{N-1}$ (that is to say an element of $\mathbb{C}\mathrm{P}_{N-1}$), the action of the dyadic inverse J is simply $x_k\to 1/x_k$. The action of the matrix inverse I is $x_k\to i_k(x_0, x_1, ..., x_{N-1})$, where the i_k are polynomials with integer coefficients in the x_j.

An extensive study shows that the number of admissible patterns is indeed extremely small compared to the number of all patterns, that is to say the number $\mathscr{P}(q^2)$ of partitions of q^2 elements. To give an idea of their scarcity we have to compare

$$\mathscr{P}(q^2)=\sum_{s=1}^{q^2}\sum_{k=0}^{s-1}(-1)^k\frac{(s-k)^{q^2-1}}{k!(s-1-k)!} \tag{3}$$

with the number of admissible patterns. The evaluation of the latter may be obtained by an exhaustive inspection and is 17 for the $q=3$ case to be compared to $\mathscr{P}(9)=21147$, and 187 for the $q=4$ case to be compared to $\mathscr{P}(16)=10480142147\approx 10^{10}$. We must furthermore say that most of them are related by trivial permutations of the row and columns. The investigation and results for the $q=5$ and $q=6$ models (recalling that $\mathscr{P}(25)\approx 4.6\times 10^{18}$ and $\mathscr{P}(36)\approx 3.8\times 10^{30}$) will be detailed in ref. [24].

We shall use in the following a number of specific examples of admissible patterns:

P1. The general cyclic 4×4 matrix

$$M=\begin{pmatrix} x_0 & x_1 & x_2 & x_3\\ x_3 & x_0 & x_1 & x_2\\ x_2 & x_3 & x_0 & x_1\\ x_1 & x_2 & x_3 & x_0\end{pmatrix}, \tag{4}$$

i.e., $q=4$, $N=4$. It is the matrix of Boltzmann weights of the four-state chiral Potts model. We shall also consider its subpattern SP1 obtained by taking an additional symmetry condition $x_1=x_3$, leading to $q=4$, and $N=3$. This subcase is equivalent to the symmetric Ashkin–Teller model.

P2. The general cyclic 5×5 matrix, i.e., coefficients $x_0, ..., x_4$, $q=5$, $N=5$, and its symmetric subpattern SP2 obtained with $x_1=x_4$ and $x_2=x_3$, or equivalently by quotienting the algebra of the group $\mathbb{Z}_5$ by the $\mathbb{Z}_5$-automorphism $x\to x^{-1}$ (we get $N=3$).

[#3] This intertwining between two products in our matrix algebras is not of the same nature as the compatibility conditions between Lie-algebra structures encountered in r-matrix formalism of the classical Yang–Baxter equations (difference between the approaches of Drin'feld and Semenov-Tjan-Shanksy). This intertwining is not straightforwardly related to the intertwining between two coproducts encountered in quantum groups [19].

We thus have an Abelian three-dimensional algebra generated by $\{\mathbf{1}, A=\sigma+\sigma^4, B=\sigma^2+\sigma^3\}$ where σ is the generator of $\mathbb{Z}_5$ (shift). The product law is

$$A^2=2+B\,,\quad B^2=2+A\,,\quad AB=BA=A+B\,. \qquad (5)$$

P3. A $q=6$ pattern, which is not cyclic, nor symmetric and with $N=3$. The matrix form is

$$\begin{pmatrix} x & y & z & y & z & z \\ z & x & y & z & y & z \\ y & z & x & z & z & y \\ y & z & z & x & z & y \\ z & y & z & y & x & z \\ z & z & y & z & y & x \end{pmatrix}. \qquad (6)$$

This is the general element $x\mathbf{1}+yA+zB$ of an Abelian subalgebra of the algebra of the (non-Abelian) permutation group of three objects [24,25]. We have the following product law in this subalgebra,

$$A^2=\mathbf{1}+B\,,\quad B^2=2+2A+B\,,$$

$$AB=BA=\mathbf{1}+A+B\,. \qquad (7)$$

Introducing for this model the variables $u=y/x$, $v=z/x$, the explicit formulae for the inversion I are

$$u\rightarrow\frac{-u^2-u+2v^2}{1+u+2v-u^2-2uv-v^2}\,, \qquad (8)$$

$$v\rightarrow\frac{u^2+vu-v^2-v}{1+u+2v-u^2-2uv-v^2}\,. \qquad (9)$$

P4. A reduced cyclic 7×7 matrix, with $q=7$, $N=3$. The matrix may be written as the generic element of a three-dimensional subalgebra of the algebra of $\mathbb{Z}_7$. If σ is the generator of $\mathbb{Z}_7$, the subalgebra is generated by $\{\mathbf{1}, A=\sigma+\sigma^2+\sigma^4, B=\sigma^3+\sigma^5+\sigma^6\}$. This is the subalgebra of the algebra of $\mathbb{Z}_7$ obtained by quotienting by the $\mathbb{Z}_7$-automorphism $x\rightarrow x^2$. The product law in this subalgebra is

$$A^2=A+2B\,,\quad B^2=B+2A\,,$$

$$AB=BA=3+A+B\,. \qquad (10)$$

Deformations. It is straightforward to construct deformations of our mappings. It suffices for example to replace the involution J (sending x_i into $1/x_i$), by $\tilde{J}$ sending x_i into a_i/x_i, where the a_i are the deformation parameters [6]. These deformations do not spoil the invertibility of the mappings. One should analyse both the perturbations around quasi-integrable mappings and the large deformations, especially the ones with integer a_i. For vertex models the situation is even simpler since the various involutions we consider are related by permutations of the parameters. These permutations are similarity transformations on the matrices and one may deform them straightforwardly. These deformations are studied in a parallel publication [6,26].

3. Orbits

A very efficient way to analyse the trajectories of a point under the action of Γ is to look at them. Figs. 1–4 are pictures of the trajectories of points under the iteration of IJ (that is to say the $\mathbb{Z}$ part of Γ given in the $(N-1)$-dimensional space of the variables $u=x_1/x_0$, $v=x_2/x_0$, $w=x_3/x_0$, ...).

Fig. 5 is of a different nature: we consider a point in a double copy of the parameter space $\mathbb{C}\mathrm{P}_{N-1}$. We act with IJ on the first copy, and with its inverse JI on the second one. We take the same starting point in both copies. We take P3, where $N=3$, as example. The points in the first (second) copy have coordinates u, v ($\bar{u}$, $\bar{v}$). We draw the projection of the trajectories on the coordinate plane u, $\bar{u}$. This analysis is justified by the following considerations from exactly solvable model theory in statistical mechanics: the star–triangle contains three pairs of copies of the parameter space. The compatibility relations of this overdetermined system implies in particular that the two members of each pair belong to an algebraic variety which is stable with the abovementioned action of IJ [1]. We are drawing this variety, keeping in mind that the existence of this variety is not a sufficient condition for the existence of solutions of the star–triangle equations.

We have experimentally four different types of orbits:

- the quasi-integrable case;
- the resonant (finite) case;
- the Hoffstadter-type orbits;
- the contracting (expanding) case.

By quasi-integrable we mean that the orbits will remain inside algebraic subvarieties of the parameter space. By contracting we mean that the orbits will contract towards some fixed stable submanifolds, independently of the initial point of the orbit. Notice

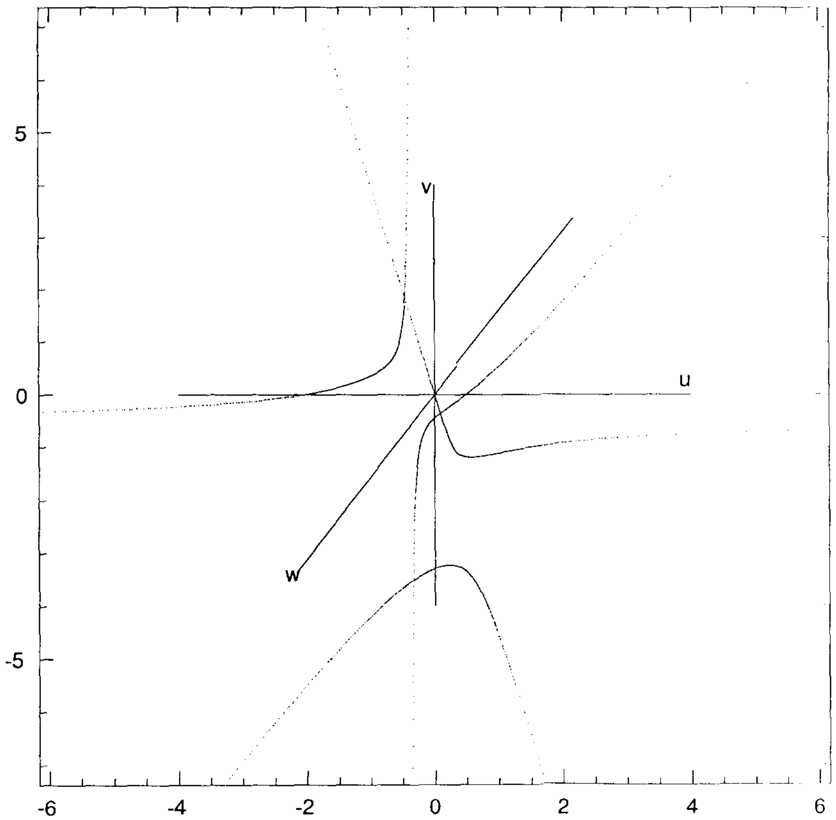

Fig. 1. 3d perspective of the orbit of (8, −7, 8.5) for P1.

that the two do not exclude, as exemplifies the Ashkin–Teller model SP1, according to the value of some algebraic invariant Δ for the model [27,6]. By resonant we mean that the orbit is finite.

Figs. 1, 2 and 3 illustrate the quasi-integrable cases P1, SP2, P3. What is remarkable is that in the quasi-integrable case, the orbits look like the image subvarieties, especially when these are curves. This makes the graphic representation of the iteration a very good detector of quasi integrability. As we will see in refs. [28,25], we have for P1, SP2, and P3, an elliptic uniformization [29], making IJ a mere translation $\theta \to \theta+\lambda$ of the uniformizing parameter (spectral parameter θ). The situation is that we have non-resonant tori in the terminology of Hamiltonian systems, or equivalently translations on the circle S^1 with a shift not commensurate to the circumference (irrational "rotation number"). Notice that going to the action–angle like coordinates (invariants, spectral parameter) linearizes the action of I and J. If the image varieties are Abelian varieties, that is to say quotients of $\mathbb{C}^m$ by some lattice, the action of Γ must be compatible with the lattice. It is more than conceivable that generic orbits are dense in the image curves. In this case, we can recover properties on the whole curve from those on the infinite discrete set of points of the orbit.

We have not given any picture of contracting cases here. In this situation, one could construct a fundamental domain for the action of the group Γ, taking advantage of the fact that Γ is generated by involutions [6].

In the example P4 (fig. 4), there is clearly no image curve. Beware that there is an extremely fast accumulation of numerical errors in the iteration. The qualitative features of the orbit are nevertheless correct. It is worth noticing that although the examples SP2 and P4 are both obtained in a similar way from

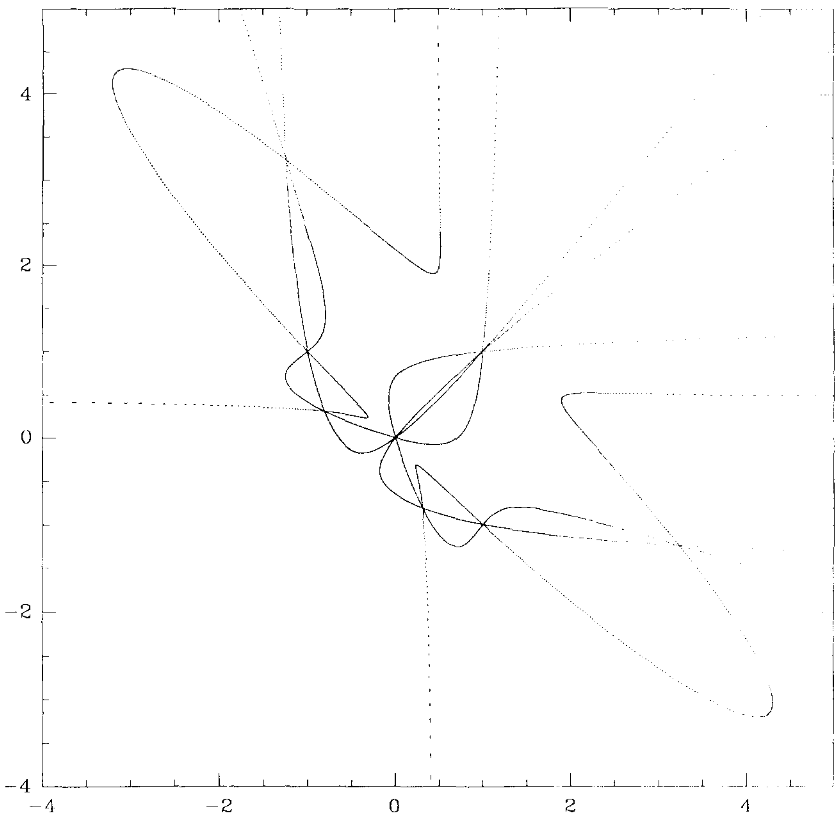

Fig. 2. A few curves in the pencil for SP2.

cyclic groups [30,31], they behave quite differently. Notice also that the Hoffstadter butterfly-like orbit of P4 has a remarkable structure: the role of the infinite set of special points $u=1$, $v=\sin[(n+1)\alpha]/\sin(n\alpha)$, with $\tan\alpha=\bar{7}$, and of the symmetric points obtained by exchanging u and v, will be detailed elsewhere [28].

Fig. 5 shows the existence of curves in the double copy we have considered. This is a direct consequence of the existence of the above invariant, and the ergodicity of the mapping IJ on generic image curves. The equation of the curve is $\theta+\bar{\theta}=\text{const}$ with evident notations. This equation is meaningful only if the two starting points, in the two copies of the parameter space belong to the same curve (i.e., have the same invariant). By choosing identical starting points, we have ensured this condition. However any pair of points on the same curve would do. We would then get another curve in the $(u, \bar{u})$ coordinate plane, loosing the $u \leftrightarrow \bar{u}$ symmetry which appears in fig. 5.

4. Invariants

The localization of the trajectories on curves, or more generally smooth submanifolds of the space of parameters, rather than clouds (see P4), is the sign of the existence of *algebraic* invariants of the group Γ. The number of invariants depends on the representation of the Coxeter group. Equating these invariants to some constant gives the equations of the image subvarieties.

We indeed have such invariants. They are

For P1:

$$\Delta_1 = \frac{(x_0^2+x_2^2)x_1x_3+(x_1^2+x_3^2)x_0x_2-2(x_0^2x_2^2+x_1^2x_3^2)}{(x_1^2-x_3^2)(x_0^2-x_2^2)}, \tag{11}$$

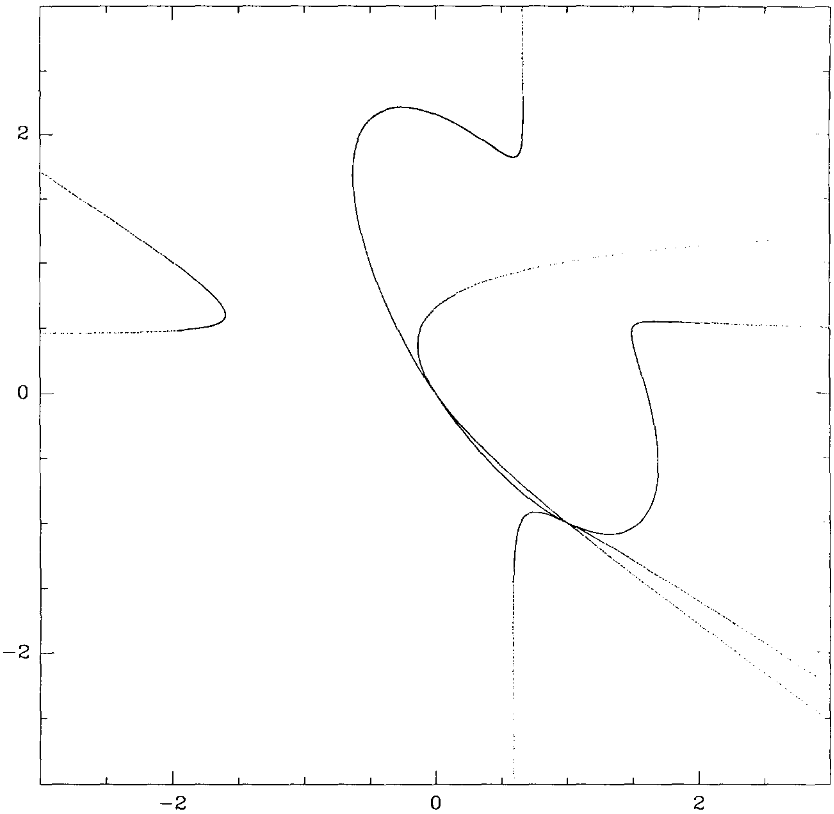

Fig. 3. Orbit of (1.5, 0.5) for P3.

(11)

$$\Delta_2 = \frac{(x_1 - x_3)(x_0 + x_2)}{(x_1 + x_3)(x_0 - x_2)} . \tag{12}$$

We have a set of curves left invariant by Γ. This is shown by fig. 1, and verified explicitly by a direct calculation. Note that the vanishing of the numerator in eq. (11) (i.e., $\Delta_1 = 0$) is the equation of the algebraic variety of Au-Yang et al. for which the four-state chiral Potts model is integrable [12,13,32]. Points in this variety have a finite orbit of order 8. The value $\Delta_1 = \infty$ (i.e., the denominator in eq. (11) vanishes) corresponds to the symmetric Ashkin–Teller model SP1. In these limits, Δ_2 trivializes. Notice also that there is a large arbitrariness in the choice of Δ_1 and Δ_2. Any choice of two algebraically independent invariants is good. We may barter Δ_1 for a simpler invariant

$$\Delta'_1 = \frac{(x_0^2 - x_1 x_3)(x_2^2 - x_1 x_3)}{(x_0 - x_2)(x_0 + x_2)(x_1 - x_3)(x_1 + x_3)} . \tag{13}$$

For SP2 (with the notations u and v) the invariant is

$$\Delta_{\mathrm{SP2}} = \frac{(u^2 + v^2 + 3uv)(u-1)(v-1)}{2u^2v^2 + 2uv - (u^3 + v^3) - uv(u+v)} . \tag{14}$$

This is shown by fig. 2, and verified explicitly by a direct calculation. We have a linear pencil of curves, with intersection points at which the invariant is undetermined. *This is possible only because the transformations we consider are singular at some points.* This means that our mappings are defined in $\mathbb{C}\mathrm{P}_{N-1}$ minus some subvarieties. Notice that the points of the curve $\Delta_{\mathrm{SP2}} = \infty$ have a finite orbit of order 6 under Γ.

For P3 (with the notations u and v) the invariant is

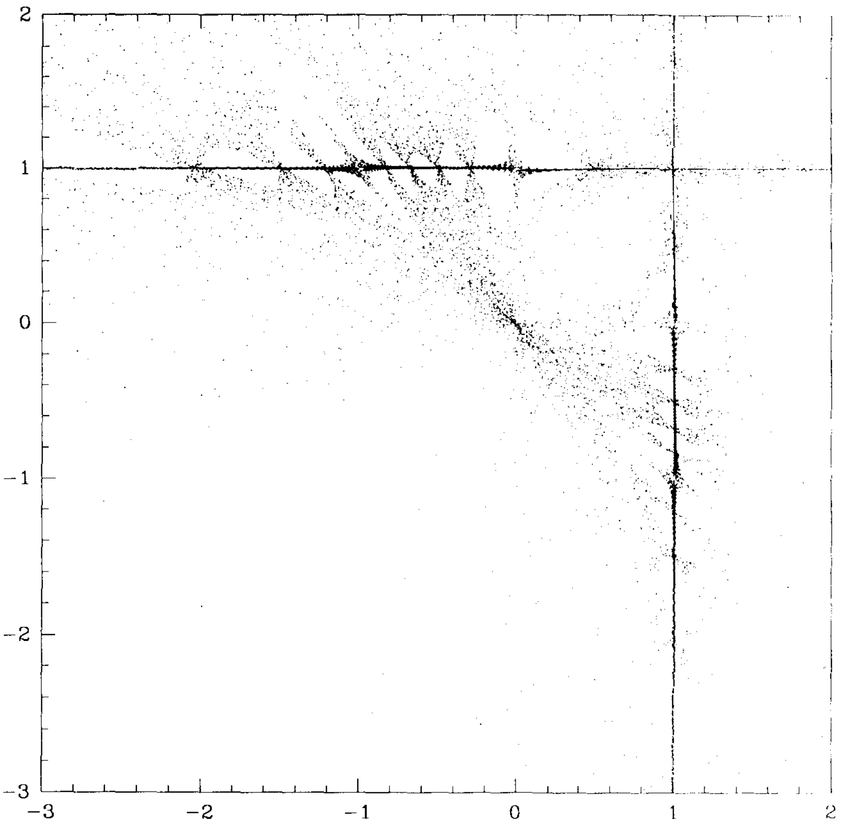

Fig. 4. A generic orbit for P4.

$$\Delta_{P3} = \frac{(2v^2+2vu-u^2-2u^3-2vu^2+v^2u)(u-v^2)^2}{(v+u)^4(1-u)(1-v)^2} . \quad (15)$$

The non-existence of invariants does not impeach the existence of conserved subvarieties. For instance in P4, the two lines $u=1$ and $v=1$ are exchanged by IJ and are stable by J, although they clearly *do not* belong to any underlying linear pencil of curves. Similarly, the standard scalar Potts line $u=v$ is globally invariant.

Moreover subvarieties made out of points having finite orbits are automatically stable. For example the integrability variety of Au-Yang et al. is such a variety [14,23]. More examples are easy to find explicitly by writing the condition $(IJ)^r=\mathrm{Id}$, for arbitrary integer r. Of course if r is divisible by r', we recover as particular case the points satisfying the condition with r'. When we have a pencil of image curves we guess that all these subvarieties belong to the pencil.

As an explicit example we may give the points of order r for $r=1, ..., 11$ in the case P3. For $r=1$, 2 or 4, we have only isolated points for the orbits of order r. Remarkably, for the other values of r, the points of order r actually lie on curves of the pencil. $r=3$ corresponds to the component $u-v^2=0$ of the curve $\Delta_{P3}=0$; $r=6$ corresponds to the whole curve $\Delta_{P3}=0$; $r=5$ corresponds to the curve $\Delta_{P3}=-1$. For higher values of r, there will be different curves of points of order r corresponding to fixed Δ_{P3}. For $r=7$, there are two curves with $\Delta_{P3}=\frac{1}{2}(7\pm3\sqrt{5})$. For $r=8$, we have two values $\Delta_{P3}=\frac{1}{2}(3\pm\sqrt{7})$. For $r=9$, the values of Δ are the three roots of the polynomial x^3+3x^2-6x+1 which have the numerical values 1.226, −4.411, 0.1847. For $r=10$, we have naturally the solution of $r=5$ and in addition the values of Δ corresponding to the three roots of x^3-9x^2+7x-1

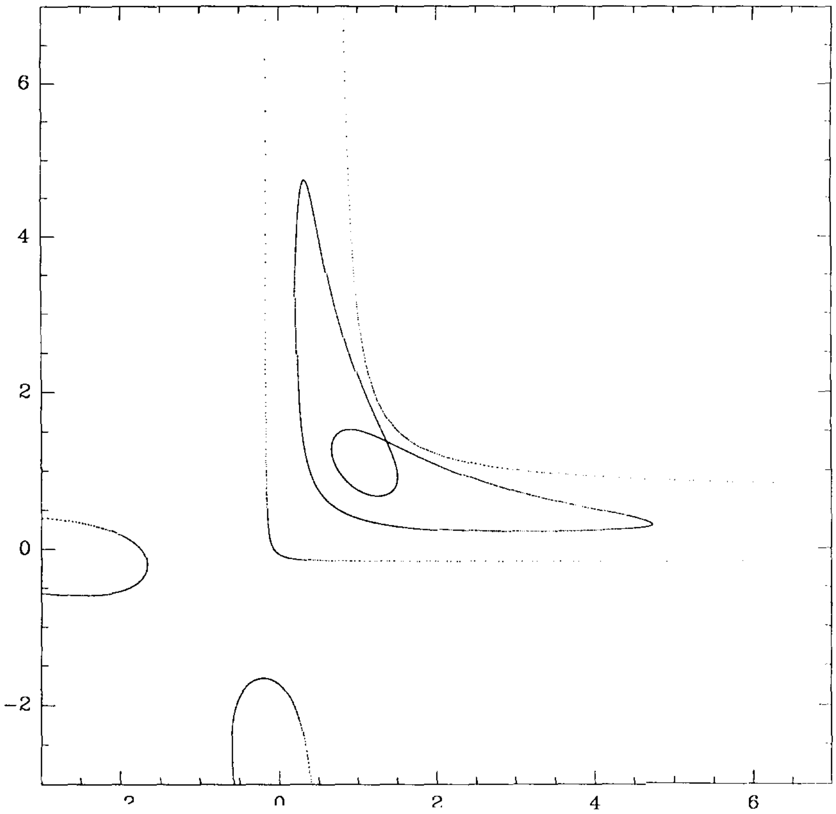

Fig. 5. Orbit of $[(1.5, 0.8), (1.5, 0.8)]$ in the $(u, \bar{u})$ plane for P3.

which are 8.156, 0.1867, 0.6563. Finally the points of order $r=11$ lie on the curves with Δ_{P3} solution of the fifth degree equation $x^5-13x^4+55x^3-17x^2-4x+1$ which has three real roots: -0.2516, 0.1873 and 0.4193.

Remark 1. If the action of Γ is quasi-integrable we see that Γ *is an infinite set of automorphisms of the image subvarieties.* Therefore, if the image subvarieties are curves, they are necessarily of genus 0 or 1 [29]. A detailed analysis to be found in parallel publications [28,25] shows that we have pencils of curves which are generically of genus 1. For a finite set of values of Δ, the curve factorizes in a number of genus 0 components. It is clearly the case for the line $u=v$ associated with the standard scalar Potts limit for which one can introduce a rational parametrization to represent Γ [5]. For instance, in the model SP2, this line $u=v$ comes together with a hyperbola of equation:

$$2uv+3(u+v)+2=0 .$$

Numerically, the line $u=v$ is not stable and the iterations of IJ escape to the hyperbola. Remarkably enough the existence of an invariant stabilizes the numerics. This turns out to be an extremely fruitful aspect of the numerical iteration of our mappings.

We have genus zero image curves *only* for the following values of Δ_{SP2}: $\frac{5}{2}$, $\frac{1}{2}$, 0, and $\frac{1}{2}(3\pm\sqrt{5})$. For the following values of Δ_{P3}, we will also have product of genus zero curves: 0, ∞, $\frac{3}{16}$, 8. In all of these cases, the curve will in fact decompose in different parts which may or may not be invariant under the group Γ by themselves.

Remark 2. When there exist image curves, we may write down differential equations for the running point of these curves. For example for P3, they are

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\phi(u,v)(9u^2v^2-2v^2u-9uv^3+3u^3v+9u^2v^3$$
$$-9u^3v^2+3uv^4-6u^4v-2u^3v^3+6vu^2-4v^3-6v^4$$
$$+3u^4+3v^5-4u^4v^2+6u^2v^4)(v-1)\,, \qquad (16)$$

$$\frac{\mathrm{d}v}{\mathrm{d}t}=2\phi(u,v)(-2v^2u-u^2v^2-3uv^3+4u^3v+4u^2v^3$$
$$-u^3v^2+2uv^4-3u^4v-u^3v^3-vu^2+3u^3-2v^4$$
$$+2u^4-2u^5-2u^4v^2+3u^2v^4)(u-1)\,, \qquad (17)$$

where $\phi(u,v)$ is an arbitrary function.

Remark 3. Remember that all points in the orbit of a rational point are themselves rational, since the transformation IJ has integer coefficients. We have thus given examples of algebraic varieties *with an infinite number of rational points.*

Remark 4. For all our examples, the standard scalar Potts limit is always compatible with Γ. This limit corresponds to $\Delta_{\mathrm{SP2}}=\frac{5}{2}$ for SP2, $\Delta_{\mathrm{P3}}=\frac{3}{16}$ for P3, and $\Delta_1=\infty$ and $\Delta_2=0$ for P1.

5. Some comments on the algebraic structures

From the point of view of algebra, all our examples P1, SP1, SP2, P2, P3, P4 present a common feature. There exists a collineation C intertwining the two involutions I and J.

This is particularly remarkable for the model P3, where this collineation reads

$$u\to\frac{1-v}{1+2u+3v}\,,\quad v\to\frac{1-u}{1+2u+3v}\,. \qquad (18)$$

This collineation not only intertwines between the inverses corresponding to the two products (matrix product and element by element, i.e. dyadic product) but is actually an *isomorphism between the two products.* If we denote $M\cdot N$ the matrix product and $M*N$ the dyadic product. By construction all admissible patterns close under the dyadic product. For our examples they also close under the matrix product, and we have

$$C(M\cdot N)=C(M)*C(N)\,. \qquad (19)$$

This isomorphism may exist only because the algebra of matrices given by the pattern we consider is Abelian. It is the simultaneous diagonalization of the three basis matrices $\{\mathbf{1}, A, B\}$,

$$C(\mathbf{1})=\mathbf{1}+A+B\,, \qquad (20)$$

$$C(A)=2-B\,, \qquad (21)$$

$$C(B)=3-A\,. \qquad (22)$$

Notice that there are admissible patterns without this algebra structure, and with such a collineation between the inverses. There are also admissible patterns without this collineation between the two inverses for mappings with more than two variables.

Beware that C is not unique, and *is not of finite order.* However, in the standard scalar Potts limit $(u=v)$, *C is nothing but the duality transformation,* and it verifies $C^2=1$.

Notice also that the invariant given for P3 in section 4 is not invariant by C, while the invariants given for P1, and SP2 are invariant by the corresponding collineations (which in these cases are nothing but the Kramers–Wannier duality transformations, that is to say the Fourier transform in $\mathbb{Z}_n$ [33,34]).

6. Conclusion

The construction we have presented opens the way to much more work in various directions. A first set of questions is associated to exactly solvable models in mathematical physics. A second set is linked with mapping theory and dynamical systems. In the first set we have for instance:

– An exhaustive classification of admissible patterns of two-dimensional lattice spin or vertex models. It will lead to interesting results in elementary matrix algebra, algebraic geometry, Diophantine problems, and statistical mechanics [24]. We may even introduce exclusion rules very straightforwardly by setting some weights to zero and perform the same analysis.

– The description of the locus where representations of Γ have finite order is a crucial step in this classification [28,25]. We do think they are a lode for the integrable-model-digger.

– The higher-dimensional equivalent problem, in addition to its interest per se, is the key to the uncovering of higher dimensional integrable models. The situation is qualitatively different as far as the nature of the group generated by the involutions is concerned, since the number of generating involutions increases [2].

All the results thus obtained will enlighten the role of the various algebraic structures we have described.

If we consider the implications for dynamical systems and also mappings, several questions could be addressed:

– The deformations of quasi-integrable mappings, for spin as well as for vertex models [6].

– How important is the invertibility? Is it equivalent to the possibility to be written as a product of involutions? (It seems to be the case in $\mathbb{C}P_2$ where reversible mappings can be written as the product of two involutions.) Moreover, if we modify our mappings in such a way that they become non-invertible, the nature of the orbits changes completely and one stumbles on a strange attractor [6].

– Does there exist an area (volume) preserved by our mappings? In the quasi-integrable case with image curves, the answer is yes and is related to the existence of a uniformizing parameter on the curves and to the fact that IJ is just a translation of that parameter [25]. In the same spirit, one should find the Poisson structure on the space of parameters yielding a motion on the image subvarieties.

– The description of a dynamical system (if any), of which our mappings are a discretization. This is to be linked with recent results on the discrete time presentation of Hamiltonian integrable systems [35–38]. The invariants will be conserved quantities of the motion.

– The analysis of ergodicity of these mappings, within the algebraic image varieties determined by the invariants of our transformations, in order to study the orbits of Γ. The questions to be answered are: How do the orbits fill the image varieties, or equivalently: Is the orbit a dense subset of the variety? Does the orbit have accumulation points? Does Γ have a fundamental domain? Is there a Lyapunov function?

Acknowledgement

We would like to thank J. Avan, O. Babelon, A. Douady, M. Dubois-Violette, J.F. Mestre, M. Talon, and B. Teissier, for very stimulating discussions and comments. Two of us (JMM and CMV) would like to thank the Research Institute for Theoretical Physics (TFT) of the University of Helsinki for hospitality and support.

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