# Hyperbolic Coxeter groups for triangular Potts models*

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**Abstract.** The symmetry groups, generated by the inversion relations of lattice models of statistical mechanics on triangular lattices, are analysed for vertex models and for the standard scalar Potts model with two- and three-site interactions. These groups are generated by three inversion relations and are seen to be generically very large ones: hyperbolic groups. Two situations for which the representations of these groups degenerate into smaller ones, hopefully compatible with integrability, are considered. The first reduction for the vertex triangular model corresponds to the situation where the vertex of the triangular model coincides with the left- or right-hand side of a Yang–Baxter relation. In this case the representation of the group is isomorphic, up to a semi-direct product by a finite group, to $\mathbb{Z} \times \mathbb{Z}$. The second reduction for $q$-state Potts models occurs for particular values of $q$, the so-called Tutte–Beraha numbers. For this model, algebraic varieties, including the known ferromagnetic critical variety, happen to be invariant under such large groups of symmetries.

As a byproduct, this analysis provides nice birational representations of hyperbolic Coxeter groups.

## 1. Introduction

In previous papers [1, 2] it has been shown that there exist non-trivial, nonlinear symmetries acting on the parameter space of lattice models of statistical mechanics generated by the so-called *inversion relations* [3–6]. These nonlinear groups of symmetries appeared as powerful tools to study integrable models in lattice statistical mechanics, for instance to find the critical varieties of their phase diagrams [7]. These symmetry groups can also be seen as symmetries of the Yang–Baxter equations (or star–triangle equations, when dealing with spin models) and their higher-dimensional generalizations. It is important to note that these groups exist as symmetry groups of lattice models *even when one is no longer restricted to an integrable framework* [8, 9].

In this point of view, the straight, but tedious, analysis of a three-dimensional model through transfer-matrix formalism, or any other classical method of lattice statistical mechanics is replaced by an analysis of the transformations corresponding to the symmetries, acting *in the parameter space* and therefore, at first sight, less sensitive to the lattice space and of course even to the dimension of the lattice.

However, in both cases, integrable or not, known in the literature, a drastic difference *seems* to appear between two- and three-dimensional models, suggesting a way to understand the obstruction for three-dimensional integrability associated with generic

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three-dimensional symmetry groups, and also suggesting giving an algebraic definition to the notion of the dimension of the model. In this framework the dimension of the lattice re-emerges through the 'size' of the symmetry group. As far as two-dimensional models are concerned, the symmetry groups known in the literature are either finite groups [10, 11] or groups isomorphic to products of $\mathbb{Z}$ up to a semi-direct product by a finite group [8, 11, 12]. Whereas for lattice models of dimension three, these symmetry groups are much larger: generically they are free groups with three generators. With such symmetry groups, the *very existence of solutions of the tetrahedron equations*† having a '*generic* three-dimensional symmetry' *seems problematic* [12]: the only possibility for solutions of the tetrahedron equations are probably cases where the representations of such 'large' groups *degenerate into products of* $\mathbb{Z}$ or even into *finite groups* [16, 17].

It will be shown here that the analysis of the symmetry group of models on triangular lattices weakens this opposition between dimensions two and three. More precisely, this study suggests that the coordination number of the lattice could be a parameter more relevant for the structure of the symmetry group than the lattice dimension. In the following, we will analyse a vertex model on a triangular lattice, and the standard scalar $q$-state Potts model with two- and three-spin interaction [18]. Generically, their symmetry groups are free groups with two generators. One recovers a situation similar to the one encountered in dimension three; these models on a triangular lattice thus provide examples giving hints for the analysis of such large symmetry groups in dimension three.

Moreover, these hyperbolic Coxeter groups of symmetries can actually degenerate into more 'reasonable' groups leaving room for integrability in two cases: one corresponding to a $\mathcal{Z}$-invariant-like situation [19] and another one, reminiscent of the occurrence of *Tutte–Beraha numbers* [20].

Finally we will consider the consequences of these symmetries, with a special emphasis on criticality conditions. Actually, many criticality conditions have been conjectured in the literature of lattice models of statistical mechanics, and of course all these conjectures were algebraic [21, 9, 22]. However, when exactly proved, they were *always related to some integrability of the model, the algebraicity thus being a consequence of the integrability* [23]; with a noticeable exception: the (self-dual) critical variety given by Wu [18, 24], on the two- and three-site Potts model on the triangular lattice, which we revisit here.

Here we will analyse this algebraic variety, which is a critical condition in some 'ferromagnetic region' [24], but is *not related to any simple*‡ '*Yang–Baxter-like integrability*'. Moreover, we will discuss the status of other remarkable varieties emerging from this algebraic study.

## 2. Symmetries of lattice models

Let us recall the symmetry group generated by the *inversion relations* for lattices of coordination number six, first on the cubic three-dimensional vertex model [16, 17] and then on the triangular lattice.

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† Generalizations of the Yang–Baxter equations in dimension three [8, 13–15].

‡ If a Yang–Baxter integrability exists for this model it cannot be simply parametrized in terms of the elliptic or rational curves one is used to in exactly solvable models, or even surface products of curves (see the solution of the star–triangle for the chiral Potts model [10]). One would have a much more involved parametrization as far as effective algebraic geometry is concerned.

## 2.1. Inversion relations and the group $\Gamma_{3D}$

Let us consider a vertex model on a three-dimensional cubic lattice of size $M \times M$. With each bond is associated a variable with $q$ possible states. A Boltzmann weight $w(i, j, k, l, m, n)$ is assigned to each vertex configuration [25], and can be represented pictorially by
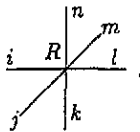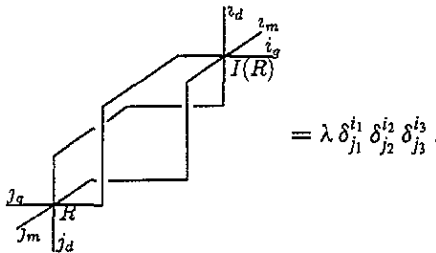


The $q^6$ homogeneous weights $w(i, j, k, l, m, n)$ are first arranged in a $q^3 \times q^3$ matrix $R$ of entries:

$$R^{ijk}_{lmn} = w(i, j, k, l, m, n).$$

One may [17] introduce an involution $I$ which transforms $R$ into $IR$ according to

$$\sum_{\alpha_1, \alpha_2, \alpha_3} (IR)^{i_1 i_2 i_3}_{\alpha_1 \alpha_2 \alpha_3} R^{\alpha_1 \alpha_2 \alpha_3}_{j_1 j_2 j_3} = \lambda \, \delta^{i_1}_{j_1} \, \delta^{i_2}_{j_2} \, \delta^{i_3}_{j_3}$$

where $\lambda$ is an arbitrary multiplicative factor. This relation can be represented pictorially:



$$= \lambda \, \delta^{i_1}_{j_1} \, \delta^{i_2}_{j_2} \, \delta^{i_3}_{j_3}.$$

The inversion transformation $I$ amounts to taking the inverse of the $q^3 \times q^3$ matrix $R$. One also introduces the partial transpositions $t_1$, $t_2$ and $t_3$ with

$$(t_1 R)^{i_1 i_2 i_3}_{j_1 j_2 j_3} = R^{j_1 i_2 i_3}_{i_1 j_2 j_3}$$

and similar definitions for $t_2$ and $t_3$.

For three-dimensional vertex models, one has *four* such involutions acting as symmetries of the $R$-matrix [17]:

$$I_2 = I \qquad I_3 = t_1 I t_2 t_3 \qquad I_4 = t_2 I t_3 t_1 \qquad I_1 = t_3 I t_1 t_2. \qquad (2.1)$$

These four involutions generate an infinite discrete group $\Gamma_{3D}$ [17]. Let us note that the full transposition is nothing but the product $t = t_1 t_2 t_3$.

Considering the parameter space as a projective space (the entries of the $R$-matrix are homogeneous parameters), the elements of the group $\Gamma_{3D}$ have a *nonlinear* representation in terms of *birational transformations*. This group of symmetry of the parameter space of the model is very large. This is in fact a *hyperbolic* Coxeter group [26–31].

*Remark.* coming back to integrability, it has been shown that the tetrahedron equations (generalization in three dimensions of the Yang–Baxter equations [9, 13, 14, 32]) do have an infinite group of symmetry generated by *four* involutions $K_1$, $K_2$, $K_3$, $K_4$ [17]. They satisfy various relations, for instance $(K_1 K_2 K_3 K_4)^2 = \mathcal{I}d$, where $\mathcal{I}d$ denotes the identity transformation. This group of symmetry of the tetrahedron equation is quite 'monstrous' since the number of elements of length smaller than $l$ is of exponential growth with respect to $l$, unlike the symmetry group of the Yang–Baxter equations which identifies with the affine Coxeter group $A_2^{(1)}$ [16, 17, 27].

## 2.2. *Inversion relations on the triangular lattice*

For the triangular lattice the vertex Boltzmann weight [18] also reads $w(i, j, k, l, m, n)$ and can be represented by
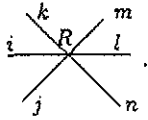


Similarly to the cubic vertex model [16, 17], the weights may be arranged in a $q^3 \times q^3$ matrix. However, for the triangular model *there are only three inversion transformations*, $I_1, I_2, I_3$, which actually coincide with three of the four of the cubic lattice (2.1). The fourth transformation $I_4$ corresponds to a non-planar picture, which is meaningless for the triangular lattice. Let us denote $\Gamma_{\text{triang}}$ the symmetry group generated by $I_1, I_2, I_3$. As will be shown in the following, using the equivalence between vertex and spin representation for this model [18], *this group also has generically an exponential growth*.

Let us recall the results obtained by Baxter, Temperley and Ashley on the triangular vertex and spin models [18]. They noticed that the integrable case discovered by Kelland for a triangular vertex model (a 20-vertex model) [33], actually corresponds to the following situation: the vertex Boltzmann weight can alternatively be seen as either a left- or right-hand side of a Yang–Baxter equation (more generally this refers to the $\mathcal{Z}$-invariance concept) [19]:



$$\tag{2.2}$$

In the framework of this very model, they brought out the correspondence between such a vertex model and the *standard scalar q-state Potts model for anisotropic triangular lattices with two- and three-site interaction* (only on up-pointing triangles) *through the Lieb–Temperley algebra* [18, 34]. In terms of the two- and three-site interaction spin model, these integrability conditions correspond to having *no three-spin interaction* and also to being *at the transition temperature* [18].

In the following, the symmetry group of both vertex and spin models on triangular lattice will be analysed.
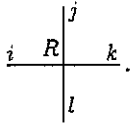
## 3. Triangular vertex model

As far as the triangular vertex model is concerned, an interesting subcase pops out, for which the group *no longer has exponential growth*. It occurs when the vertex of the triangular model spreads out into three square vertices $(A, B, C)$ (i.e. the left- or right-hand sides of a Yang–Baxter relation):



$$\tag{3.1}$$

This model is a generalization of model (2.2), *without assuming any Yang–Baxter integrability condition.*

In order to write the three inverse transformations $I_1, I_2, I_3$ restricted to this subcase (3.1), let us introduce $I$ and $J$, the two inverse transformations on the square lattice vertex model [17]. A Boltzmann weight $w(i, j, k, l)$ is assigned to each square vertex configuration [25]:

$$
\begin{array}{c}
i \;\; \underline{\phantom{xx} R \phantom{xx}} \;\; k \\
\end{array}
$$

(with $j$ above and $l$ below the vertex $R$).

The $q^4$ homogeneous weights $w(i, j, k, l)$ are first arranged in a $q^2 \times q^2$ matrix $R$:

$$
R_{kl}^{ij} = w(i, j, k, l).
$$

We introduce (see [1, 16, 17]) the inverse $I$ by

$$
\sum_{\alpha,\beta} R_{\alpha\beta}^{ij} \, (IR)_{uv}^{\alpha\beta} = \delta_u^i \, \delta_v^j = \sum_{\alpha,\beta} (IR)_{\alpha\beta}^{ij} \, R_{uv}^{\alpha\beta}
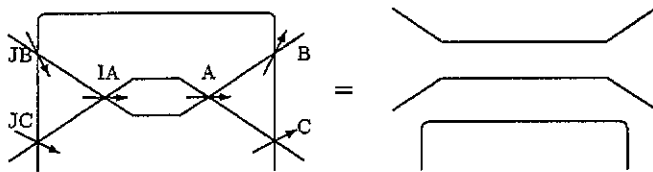$$

and the other inverse $J$ by

$$
\sum_{\alpha,\beta} R_{v\beta}^{\alpha u} \, (JR)_{j\beta}^{\alpha i} = \delta_u^i \, \delta_v^j = \sum_{\alpha,\beta} (JR)_{\alpha j}^{i\beta} \, R_{\alpha v}^{u\beta}.
$$

Similarly to the situation occurring for the cubic lattice, $I$ and $J$ are two involutions related by a partial transposition (denoted $t_1$ in [35]) of the indices: $J = t_1 I t_1$. Namely, $t_1$ reads $(t_1 R_{kl}^{ij}) = R_{il}^{kj}$.

The three inverse transformations $I_i$'s read

$$
I_1 (A, B, C) = (IA, JC, JB)
$$
$$
I_2 (A, B, C) = (JC, IB, JA)
$$
$$
I_3 (A, B, C) = (JB, JA, IC)
$$

as shown in the following picture:



Let us now introduce the following transformations $i_1 = I_2 I_3$, $i_2 = I_3 I_1$, $i_3 = I_1 I_2$, which are generically of *infinite order*:

$$
i_1 (A, B, C) = \left(K^{-1}C, KA, B\right)
$$
$$
i_2 (A, B, C) = \left(C, K^{-1}A, KB\right) \qquad \text{where} \quad K = I\,J.
$$
$$
i_3 (A, B, C) = \left(KC, A, K^{-1}B\right)
$$

Obviously group $\Gamma_{\text{triang}}$ is also generated by $i_1, i_2, i_3$, up to a semi-direct product by a finite group. These new generators do not commute, but they only differ from the generators

$\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$, which *commute* two by two, by the 3-cycle, $r(A, B, C) = (C, A, B)$. These three generators $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ act on a triplet $(A, B, C)$ as follows:

$$\mathcal{I}_1(A, B, C) = (A, \mathbf{K}^{-1}B, \mathbf{K}C)$$
$$\mathcal{I}_2(A, B, C) = (\mathbf{K}A, B, \mathbf{K}^{-1}C)$$
$$\mathcal{I}_3(A, B, C) = (\mathbf{K}^{-1}A, \mathbf{K}B, C).$$

One easily notes the following relations:

$$\mathcal{I}_1 r = \mathbf{i}_2 \qquad r\,\mathcal{I}_1 = \mathbf{i}_3 \qquad \mathcal{I}_2 r = \mathbf{i}_3 \qquad r\,\mathcal{I}_2 = \mathbf{i}_1 \qquad \mathcal{I}_3 r = \mathbf{i}_1 \qquad r\,\mathcal{I}_3 = \mathbf{i}_2.$$

Because of the commutation of its generators, the group generated by $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ is, at first sight, the following group:

$$\{\mathcal{I}_1^{n_1} \mathcal{I}_2^{n_2} \mathcal{I}_3^{n_3}, (n_1, n_2, n_3) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\}.$$

Since $\mathcal{I}_1 \mathcal{I}_2 \mathcal{I}_3 = $ identity, one has $n_1 + n_2 + n_3 = 0$. Thus the group generated by $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. As a direct consequence $\Gamma_{\text{triang}}$ is *isomorphic to $\mathbb{Z} \times \mathbb{Z}$ upto the semi-direct product by a finite group* [7, 36].

Heuristically, one can understand this subcase as follows: the symmetry group $\Gamma_{\text{triang}}$ of the 'six-legs' Boltzmann weight, $R$, becomes quite similar to the symmetry group of the Yang–Baxter equations, which is known to be isomorphic to $A_2^{(1)}$ [17].

## 4. Triangular spin model

### 4.1. Notations for the spin model

Let us now consider the standard scalar $q$-state Potts model on a triangular lattice with nearest-neighbour interaction and three-spin interaction only on the up-pointing triangles:
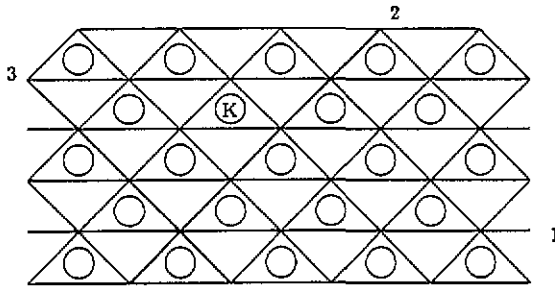


The partition function of the models reads:

$$\mathcal{Z} = \sum_{\{\sigma_i\}} \prod_{\langle i,j \rangle} e^{K_1 \delta_{\sigma_i,\sigma_j}} \prod_{\langle j,k \rangle} e^{K_2 \delta_{\sigma_j,\sigma_k}} \prod_{\langle k,l \rangle} e^{K_3 \delta_{\sigma_k,\sigma_l}} \prod_{\langle upt \rangle} e^{K \delta_{\sigma_i,\sigma_j} \delta_{\sigma_j,\sigma_k}}. \tag{4.1}$$

The first three products denote the product over the edge two-site interaction Boltzmann weights along the three directions of the triangular model, and the last product denotes the product of all up-pointing triangles of the three-site interaction Boltzmann weights. The sum is taken over all spin configurations.

In this framework one can now introduce the following notations:

$$x_i = e^{K_i} \qquad i = 1, 2, 3$$
$$x = e^K \tag{4.2}$$
$$y = x\,x_1\,x_2\,x_3 - (x_1 + x_2 + x_3) + 2.$$

Of course for $q = 2$, the model degenerates into the nearest-neighbour interaction triangular Ising model since the three-site interaction becomes irrelevant. Therefore one will not consider this $q = 2$ case in the following (even if most of the results one will get are also valid in this very case).

### 4.2. Duality transformation

Let us recall that on this model a *duality transformation* does exist [18, 24]. With notations (4.2) this duality, denoted $D$, reads

$$D: \quad \begin{cases} x_i \longrightarrow x_i^* = 1 + q \, \dfrac{x_i - 1}{y} \\[2mm] x \longrightarrow x^* = \dfrac{x_1^* + x_2^* + x_3^* - 2 + q^2/y}{x_1^* x_2^* x_3^*} \\[2mm] y \longrightarrow y^* = \dfrac{q^2}{y}. \end{cases} \tag{4.3}$$

This duality is associated with a rotation of 180° of the corresponding vertex model on a triangular lattice through the correspondence detailed in [18]. $D$ is an involution.

    Introducing well suited homogeneous variables, the duality transformation $D$ can be represented as a linear transformation $D_h$ (see section (4.6)), which satisfies the relation $D_h^2 = q^2 \, \mathcal{I}d$, where $\mathcal{I}d$ denotes the identity transformation. The hyperplanes stable by $D_h$ correspond to eigenforms associated with eigenvalues $\pm q$. The eigenspace corresponding to $q$ is of dimension four, the associated eigenplanes reading

$$x \, x_1 x_2 x_3 - (x_1 + x_2 + x_3) + q + 2 = 0 \tag{4.4}$$

$$x \, x_1 x_2 x_3 - x_i + x_j + x_k + q - 2 = 0 \qquad \text{with} \quad \{i, j, k\} = \{1, 2, 3\}. \tag{4.5}$$

The eigenform associated with eigenvalue $-q$ reads

$$x \, x_1 x_2 x_3 - (x_1 + x_2 + x_3) + 2 - q = 0. \tag{4.6}$$

The two self-dual varieties symmetric under permutations of $1, 2$ and $3$ ((4.4) and (4.6)) have already been introduced in [18, 24]. They can respectively be written as follows:

$$y = -q \qquad \text{and} \qquad y = q.$$

Hyperplane (4.6) is a subvariety of the critical manifold in some ferromagnetic region [24], whereas (4.4) has no such property. Let us notice that hyperplane (4.6) is *the only variety stable point by point by duality* $D$.

    Note that *the well known case, of no three-site interaction*, (that is, $x = 1$) *is not stable*† *under* $D$. Namely, variety $x = 1$ becomes

$$(x_1 x_2 + x_2 x_3 + x_3 x_1 - x_1 - x_2 - x_3 - x \, x_1 x_2 x_3 + 1) \, y$$
$$+ q \, (x_1 - 1) (x_2 - 1) (x_3 - 1) = 0. \tag{4.7}$$

### 4.3. Disorder solutions

*Disorder varieties* are algebraic varieties for which dimensional reductions occur for vertex or spin models, thus enabling us to calculate exactly physical quantities such as partition

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† In particular, one does not recover for $x = 1$ the Kramers–Wannier duality for Potts models [22]: Kramers–Wannier duality maps the triangular lattice onto the honeycomb lattice.

functions, correlation functions ... [36, 37]. A straightforward calculation, using a 'disorder criterion' explained in [38], yields the following disorder conditions:

$$x\, x_1\, x_2\, x_3 - (x_1 + x_2 + x_3) + 2 - q + q\, x_i = 0 \qquad i = 1, 2, 3. \qquad (4.8)$$

When there is no three-site interaction ($x = 1$) one recovers the known disorder conditions of the two-site nearest-neighbour triangular Potts model [38, 39].

One directly sees that these disorder conditions are nothing but the vanishing conditions of the $x_i^*$'s.

As should be the case [36], these three disorder varieties have no intersection with the ferromagnetic critical variety (4.6).

### 4.4. Inversion relations

The inversion relations [7, 36] for the two- and three-site interaction spin model can be represented pictorially as follows:
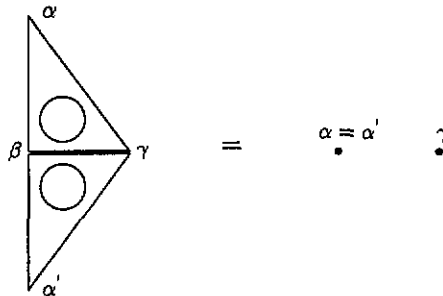


which means analytically that

$$\sum_\beta w(\alpha, \beta, \gamma) \qquad I(w)(\beta, \alpha', \gamma) = \lambda\, \delta_{\alpha, \alpha'}. \qquad (4.9)$$

The Boltzmann weight $w(\alpha, \beta, \gamma)$ of model (4.1) is invariant under a common shift of each spin $\alpha, \beta$ and $\gamma$. Therefore, $\gamma$ can be fixed in a particular colour, namely zero. Thus the Boltzmann weight can be represented by a $q \times q$ matrix ($\alpha$ being the column index, and $\beta$ the row index), with entries $w(\alpha, \beta, 0)$. Equation (4.9) thus becomes the following matricial relation:

$$\mathcal{W} I(\mathcal{W}) = \lambda\, \mathcal{I}d_q$$

where $\mathcal{I}d_q$ denotes the $q \times q$ identity matrix, and the $q \times q$ matrix Boltzmann weight $\mathcal{W}$ reads

$$\mathcal{W} = \begin{pmatrix} x\, x_1\, x_2\, x_3 & x_2 & x_2 & \cdots & \cdots & x_2 \\ x_3 & x_1 & 1 & 1 & \cdots & 1 \\ x_3 & 1 & x_1 & 1 & \cdots & 1 \\ \vdots & & 1 & 1 & x_1 & \ddots & \vdots \\ \vdots & & \vdots & \vdots & \ddots & \ddots & 1 \\ x_3 & 1 & 1 & \cdots & 1 & x_1 \end{pmatrix}.$$

Using a '$\mathbb{Z}_{q-1}$ Fourier transformation' [40, 41], this $q \times q$ matrix can be block-diagonalized into one $2 \times 2$ block and a $(q - 2) \times (q - 2)$ matrix proportional to the identity matrix, $(x_1 - 1) \times \mathcal{I}d_{q-2}$. Then one can easily obtain the matrix inverse $I(\mathcal{W})$. Note that $I(\mathcal{W})$ is

of the same form as $\mathcal{W}$, $x$, $x_1$, $x_2$, $x_3$, being changed according to the following *birational transformation* $I$:

$$
I : \begin{cases}
x \longrightarrow \dfrac{(x\,x_1 - 1)^2\,(x_1 + q - 2)}{\left(x\,x_1^2 + x\,x_1\,(q-3) - q + 2\right)(x_1 - 1)} \\[2ex]
x_1 \longrightarrow -\dfrac{x\,x_1^2 + x\,x_1\,(q-3) - q + 2}{x\,x_1 - 1} = 2 - q - x_1 + \dfrac{x_1\,(x-1)}{x_1\,x - 1} \\[2ex]
x_2 \longrightarrow \dfrac{x_1 - 1}{x_3\,(x\,x_1 - 1)} \\[2ex]
x_3 \longrightarrow \dfrac{x_1 - 1}{x_2\,(x\,x_1 - 1)}\,.
\end{cases}
\tag{4.10}
$$

Obviously, permutations of indices 1, 2 and 3 are also symmetries of the model. Introducing $p_{23}$ the permutation of $x_2$ and $x_3$, and similarly $p_{31}$ and $p_{12}$, one can define the three following transformations:

$$
\begin{aligned}
I_1 &= p_{23}\,I = I\,p_{23} \\
I_2 &= p_{31}\,p_{12}\,I\,p_{12} = p_{12}\,I\,p_{12}\,p_{31} \\
I_3 &= p_{12}\,p_{31}\,I\,p_{31} = p_{31}\,I\,p_{31}\,p_{12}
\end{aligned}
$$

corresponding to the three[†] inversion transformations of the model [42].


*4.5. The symmetry group*

Inversion $I$, permutations of $x_1$, $x_2$, $x_3$, and duality relation $D$ (defined by (4.3)) generate a symmetry group of the parameter space of the model, denoted $\Gamma_{\mathrm{upt}}$ in the following.

At this point it is worth noticing that duality transformation $D$, *does actually commute* with $I$, and also with the group of permutations $S_3$. This commutation property enables us to see $\Gamma_{\mathrm{upt}}$ as a *hyperbolic Coxeter group* generated by two infinite-order transformations, up to the semi-direct product by a finite group. These generically *infinite-order* transformations read

$$
J_1 = I_3\,I_2 \qquad J_2 = I_1\,I_3 \qquad J_3 = I_2\,I_1\,.
\tag{4.11}
$$

By definition, the $J_i$'s satisfy relation

$$
J_3\,J_2\,J_1 = \mathcal{I}dentity\,.
\tag{4.12}
$$

Two of these $J_i$'s generate $\Gamma_{\mathrm{upt}}$, up to the semi-direct product by a finite group.

Let us recall that for generic values of $q$, when $x = 1$, $\Gamma_{\mathrm{upt}}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ up to a semi-direct product by a finite group and degenerates into a *finite group* for *Tutte–Beraha numbers* [20] $(q = 2 - 2\cos(k\pi/N))$. In fact for $x = 1$, the $J_i$'s do commute and the elements of group $\Gamma_{\mathrm{upt}}$ read

$$
\gamma = J_1^{n_1}\,J_2^{n_2}\,I_1^{\alpha_1} \qquad \text{where} \quad \alpha_1 = 0, 1\,.
$$

Generically $n_1$ and $n_2$ are relative integers. For $q$ a Tutte–Beraha number associated with $N$, $n_1$ and $n_2$ run into $\{0, \ldots, N-1\}$, the group $\Gamma_{\mathrm{upt}}$ being therefore isomorphic to the product $\mathbb{Z}_N \times \mathbb{Z}_N \ltimes \mathbb{Z}_2$.

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† This existence of three involutions singles out the triangular lattice among the bi-dimensional models, from the symmetry-group analysis point of view. Let us notice that even for involved models, like the checkerboard Potts model with multispin interactions, one still gets two involutions and therefore a representation of the infinite dihedral group.

In order to analyse the general case ($x \neq 1$), let us introduce the 3-cycle $c = p_{31}\, p_{12}$, and let us write the $J_i$'s in term of $c$ and a only one (generically) infinite-order transformation, namely $(c\,I)^2$:

$$J_1 = c\,(c\,I)^2\,c^2 \qquad J_2 = c^2\,(c\,I)^2\,c \qquad J_3 = (c\,I)^2\,.\tag{4.13}$$

*4.5.1. Transformation $(c\,I)^2$.* For the sake of simplicity, one will consider transformation $(c\,I)^2$ as a *homogeneous* transformation, introducing $x_0 = x\,x_1\,x_2\,x_3$ and a fifth homogenization variable $t$. One can then define a homogeneous inverse $I_h$ (corresponding to (4.10)):

$$I_h : \quad \begin{cases} x_0 \to -x_1 - (q-2)\,t \\[4pt] x_1 \to -(q-2)\,\dfrac{x_0\,t - x_2\,x_3}{x_1 - t} - x_0 \\[8pt] x_2 \to x_2 \\[4pt] x_3 \to x_3 \\[4pt] t \to \dfrac{x_0\,t - x_2\,x_3}{x_1 - t}\,. \end{cases}$$

Transformation $c\,I_h$ then reads

$$c\,I_h : \quad \begin{cases} x_0 \to -x_1 - (q-2)\,t \\[4pt] x_1 \to x_3 \\[4pt] x_2 \to -(q-2)\,\dfrac{x_0\,t - x_2\,x_3}{x_1 - t} - x_0 \\[8pt] x_3 \to x_2 \\[4pt] t \to \dfrac{x_0\,t - x_2\,x_3}{x_1 - t}\,. \end{cases}$$

One notices that $u_3 = x_1 + x_2 + (q-2)\,t$ and $v_3 = x_3 - x_0$ are permuted by transformation $c\,I_h$: $u_3 \leftrightarrow v_3$. With these new variables, one also has:

$$c\,I_h : \quad \begin{cases} x_1 \to x_3 \\[4pt] x_3 \to u_3 - x_1 - (q-2)\,t \\[4pt] t \to F_0\,t \end{cases} \qquad \text{where} \quad F_0 = \dfrac{x_0\,t - x_2\,x_3}{t\,(x_1 - t)}\,.$$

Transformation $(c\,I_h)^2$ then reads

$$(c\,I_h)^2 : \quad \begin{cases} u_3 \to u_3 \\[4pt] v_3 \to v_3 \\[4pt] x_1 \to u_3 - x_1 - (q-2)\,t \\[4pt] x_3 \to v_3 - x_3 - (q-2)\,F_0\,t \\[4pt] t \to F_0\,F_1\,t \end{cases}$$

where $F_1 = F_0(c\,I_h)$ is the same expression as $F_0$, where the $x_i$'s have been replaced by their images by $c\,I_h$.

One can now define the successive iterates of $F_0$ by transformation $c\,I_h$, which will be called $F_n$ in the following: $F_{n+1} = F_n(c\,I_h)$. One can also introduce new variables $A_n$ defined as the successive products of the $F_n$'s:

$$A_n = F_0\,F_1\,F_2\,\ldots\,F_{2n-1}\,.$$

Taking $A_0 = 1$, one gets, for any integer $n$, the explicit expression of $(c\, I_h)^{2n}$:

$$(c\, I_h)^{2n} : \begin{cases} u_3 \to u_3 \\ v_3 \to v_3 \\ x_1 \to (-1)^n x_1 + \dfrac{1 - (-1)^n}{2} u_3 + (-1)^n (q-2)\, t \displaystyle\sum_{k=0}^{n-1} (-1)^k A_k \\ x_3 \to (-1)^n x_3 + \dfrac{1 - (-1)^n}{2} v_3 + (-1)^n (q-2)\, t \displaystyle\sum_{k=0}^{n-1} (-1)^k A_k\, F_{2k} \\ t \to A_n\, t\,. \end{cases}$$

One can then show recursively that

$$A_n - 1 = a_n\, (A_1 - 1) + b_n\, (q-2)\, \frac{x_2 - x_1}{t} \tag{4.14}$$

the $a_n$'s and $b_n$'s satisfying the following recurrences:

$$\begin{aligned} a_{n+1} &= \left((q-2)^2 - 1\right) a_n - (q-2)^2\, b_n + 1 \\ b_{n+1} &= a_n - b_n\,. \end{aligned} \tag{4.15}$$

One can initiate these recurrences with $a_0 = b_0 = 0$ (that is, $A_0 = 1$). One then gets

$$\begin{aligned} a_n &= \frac{1}{q\,(q-4)}\, (\lambda_+^n + \lambda_-^n - 2) \\ b_n &= \frac{-1}{q\,(q-4)\,(q-2)}\, \left(\lambda_+^{n-1/2} + \lambda_-^{n-1/2} + (q-2)\right) \end{aligned} \tag{4.16}$$

where $\lambda_\pm$ are the roots of the quadratic polynomial $z^2 + \left(2 - (q-2)^2\right) z + 1$.

*Remark.* One has to consider $q = 0$ and $q = 4$ separately†. Solutions of recurrences (4.15) now read

$$a_n = n^2 \qquad b_n = \tfrac{1}{2} n\,(n-1)\,.$$

Equation $A_N = 1$ has no other solution than $n = 0$, therefore transformation $(c\, I)^2$ is of infinite order. Moreover, recalling the $x = 1$ limit, one gets that this transformation is equivalent to some translation in the variables $1/(1 + x_i)$.

### 4.5.2. Tutte–Beraha numbers.

Let us recall that, when there is no three-site interaction (that is, $x = 1$), there does exist particular values of $q$, the so-called Tutte–Beraha numbers [20, 43], for which transformations $J_i$'s, or equivalently transformation $(c\, Ih)^2$, become finite-order transformations. Introducing $q_\pm$, the roots of the second-order equation $z^2 + (q-2)\, z + 1$, $q$ corresponds to a Tutte–Beraha number when $q_\pm^2$ are $N$th-*root of unity*. In the $x = 1$ case, it has been shown that for these values‡ of $q$ the $J_i$'s *are transformations of order* $N$ [7, 42]. Amazingly, this situation still holds for the generic case (with $x \neq 1$).

In fact, one notices that $\lambda_\pm = q_\pm^2$, so one has to calculate transformation $(c\, I_h)^2$ when $\lambda_\pm^N = 1$. In this case, relations (4.14) and (4.16) yield

$$a_N = b_N = 0 \qquad \text{and} \qquad A_N = 1\,. \tag{4.17}$$

† At first sight, one should also consider $q = 2$ as a particular case. In fact only $a_n$ is relevant and it is actually given by relation (4.16) with $q = 2$.

‡ One considers only $N \geqslant 2$, since for $N = 1$ (that is $q = 0$ or $q = 4$) transformation $(c\, I)^2$ is of infinite order (see remark in the previous section).

Straightforward calculations yield

$$\sum_{k=0}^{N-1}(-1)^k a_k = 0 \qquad \sum_{k=0}^{N-1}(-1)^k b_k = \frac{1-(-1)^N}{2(q-2)^2} \qquad (4.18)$$

and these relations (4.18) enable us to get

$$\sum_{k=0}^{n-1}(-1)^k A_k = \frac{1-(-1)^N}{2}\left(1+\frac{x_2-x_1}{t(q-2)}\right).$$

Finally, one notes that $x_1$ is invariant by $(c\,I_h)^{2N}$, as well as variable $t$ (see equation (4.17)). At first sight, one should also verify that $x_3$ is preserved by transformation $(c\,I_h)^{2N}$, in fact in the following (see section (4.6)) one will see that there exists a rational invariant under the whole group $\Gamma_{\text{upt}}$. Since this invariant involves variable $x_3$, it is not necessary to perform this last calculation: the invariance of $x_3$ under $(c\,I_h)^{2N}$ is a straightforward consequence of the invariance of $u_3, v_3, x_1$ and $t$.

One has thus established for $q = 2 - 2\cos(k\,\pi/N)$ (a Tutte–Beraha number) that transformation $(c\,I_h)^{2N}$ reduces to identity, that is equivalently:

$$J_i^N = \mathcal{I}d \qquad \text{with} \qquad i = 1, 2, 3. \qquad (4.19)$$

*Remark.* Such Coxeter groups can be seen as the *fundamental group* of a surface of genus $g$ minus $k$ points [31]. Here one has a genus-zero Riemann surface minus three points. At this step the Coxeter group one has to deal with is reminiscent of the Schwarz's triangular groups†. Considering a geodesic triangle of angles $\pi/n_1$, $\pi/n_2$, $\pi/n_3$, and considering $S_1$, $S_2$, $S_3$ the symmetries with respect to the edges of the triangle, and defining the 'rotations'

$$R_1 = S_2\,S_3 \qquad R_2 = S_3\,S_1 \qquad R_3 = S_1\,S_2$$

the $R_i$'s verify

$$R_i^{n_i} = \mathcal{I}d \qquad \text{with} \quad i = 1, 2, 3$$

and

$$R_1\,R_2\,R_3 = \mathcal{I}d.$$

In the study of these triangular groups, three different cases have to be distinguished: depending on $1/n_1 + 1/n_2 + 1/n_3$ greater, lower or equal to 1.

Because of symmetry of our triangular Potts model we have here $n_1 = n_2 = n_3 = N$. *The only Euclidean case is $N = 3$*, the other values of $N$ yielding hyperbolic triangles and hyperbolic geometries.

$N = 2$ corresponds to $q = 2$, that is, the Ising subcase of the model (for which the three-site interaction becomes irrelevant).

Thus, the first interesting case is $N = 3$, that is, $q = 1$ or $q = 3$.

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† Such groups have been obtained from the analysis of the ratios of solutions of second-order differential equations ramified in three points.
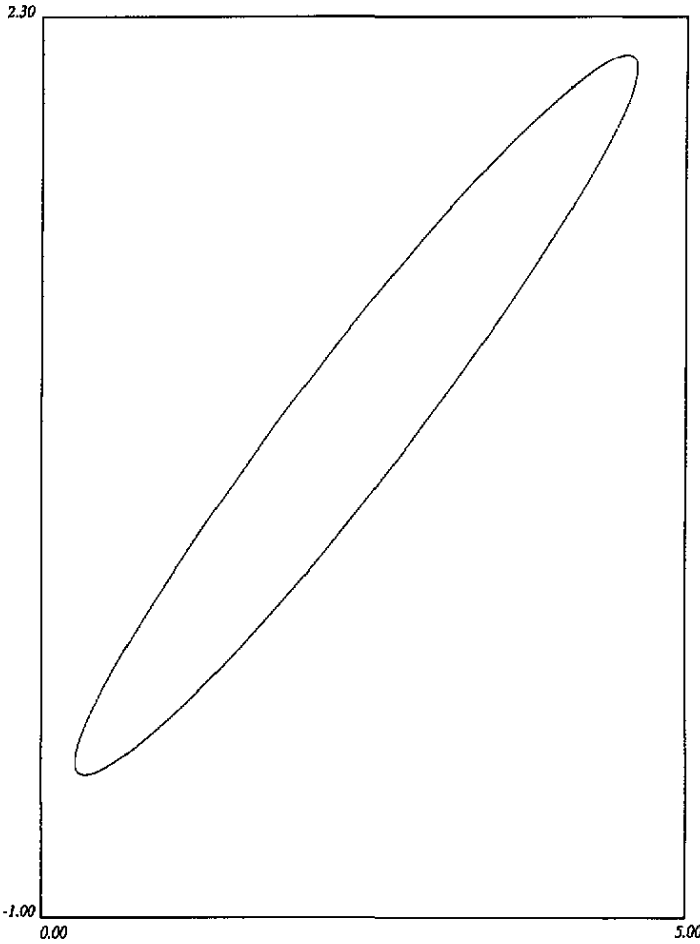
**Figure 1.** Two-dimensional projection of an orbit of transformation $(c\,I)^2$, for $q = 3.5$.

*4.5.3. The Euclidean case: $q = 1$ or $q = 3$.* In this section we will restrict $N$ to $N = 3$.
Introducing the well suited transformations

$$G_1 = p_{12}\, J_1\, p_{31} \qquad G_2 = p_{23}\, J_2\, p_{12} \qquad G_3 = p_{31}\, J_3\, p_{23} \qquad (4.20)$$

we will show that for $N = 3$, $\Gamma_{\mathrm{upt}}$ is not a group with an exponential growth anymore but
reduces down to $\mathbb{Z} \times \mathbb{Z}$ up to a semi-direct product by a finite group (like the Affine Coxeter
group $A_2^{(1)}$ [16]).

First, one notices that the $G_i$'s do satisfy a relation similar to relation (4.12):

$$G_3\, G_2\, G_1 = \mathcal{I}d. \qquad (4.21)$$

Let us first study the group $\mathcal{G}$, generated by $G_1$, $G_2$ and $G_3$. With relations (4.11) and
(4.11) the $G_i$'s can be written in terms of transformation $I$ and of the 3-cycle $c$:

$$G_1 = c^2\, I\, c^2\, I\, c^2 \qquad G_2 = I\, c^2\, I\, c \qquad G_3 = c\, I\, c^2\, I. \qquad (4.22)$$

Using $(c\,I)^6 = \mathcal{I}dentity$, $G_1\, G_2$ reads

$$G_1\, G_2 = c^2\, I\, c^2\, I\, c^2\, I\, c^2\, I\, c = c^2\, (c\,I)^{-4}\, c^2 = c^2\, (c\,I)^2\, c^2 = I\, c\, I\, c^2$$
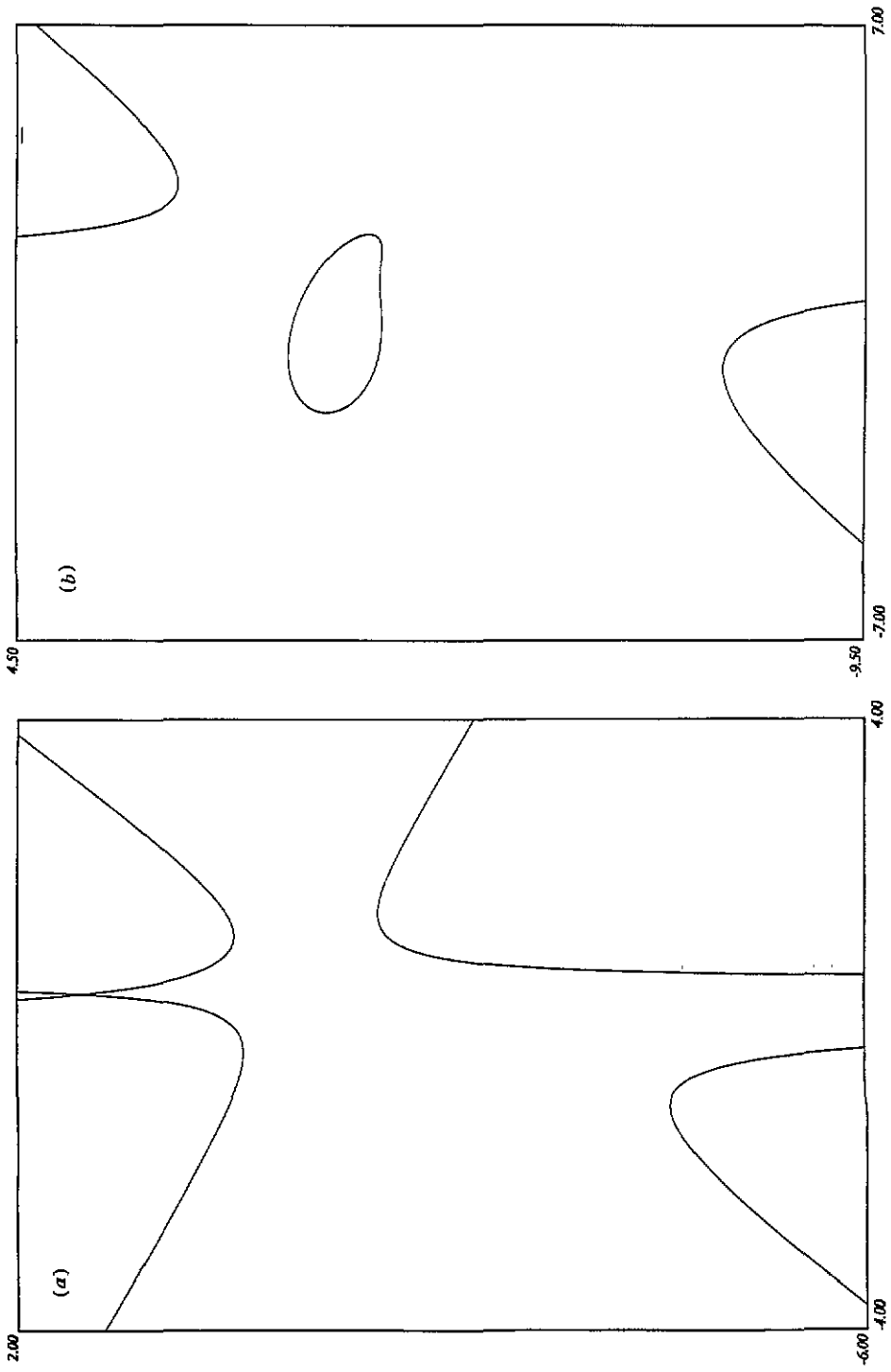
$$= G_2\, G_1. \qquad (4.23)$$

**Figure 2.** (*a*), (*b*)Two-dimensional projections of two orbits corresponding respectively to the iterations of transformations $G_2$ and $G_3$, for $q = 3$. (*c*) Orbit generated by transformations $G_2$ and $G_3$, for $q = 3$: this surface is the product of the two curves of figures (*a*) and (*b*). (*d*) Surface generated by transformations $G_2$ and $G_3$, for $q = 1$.
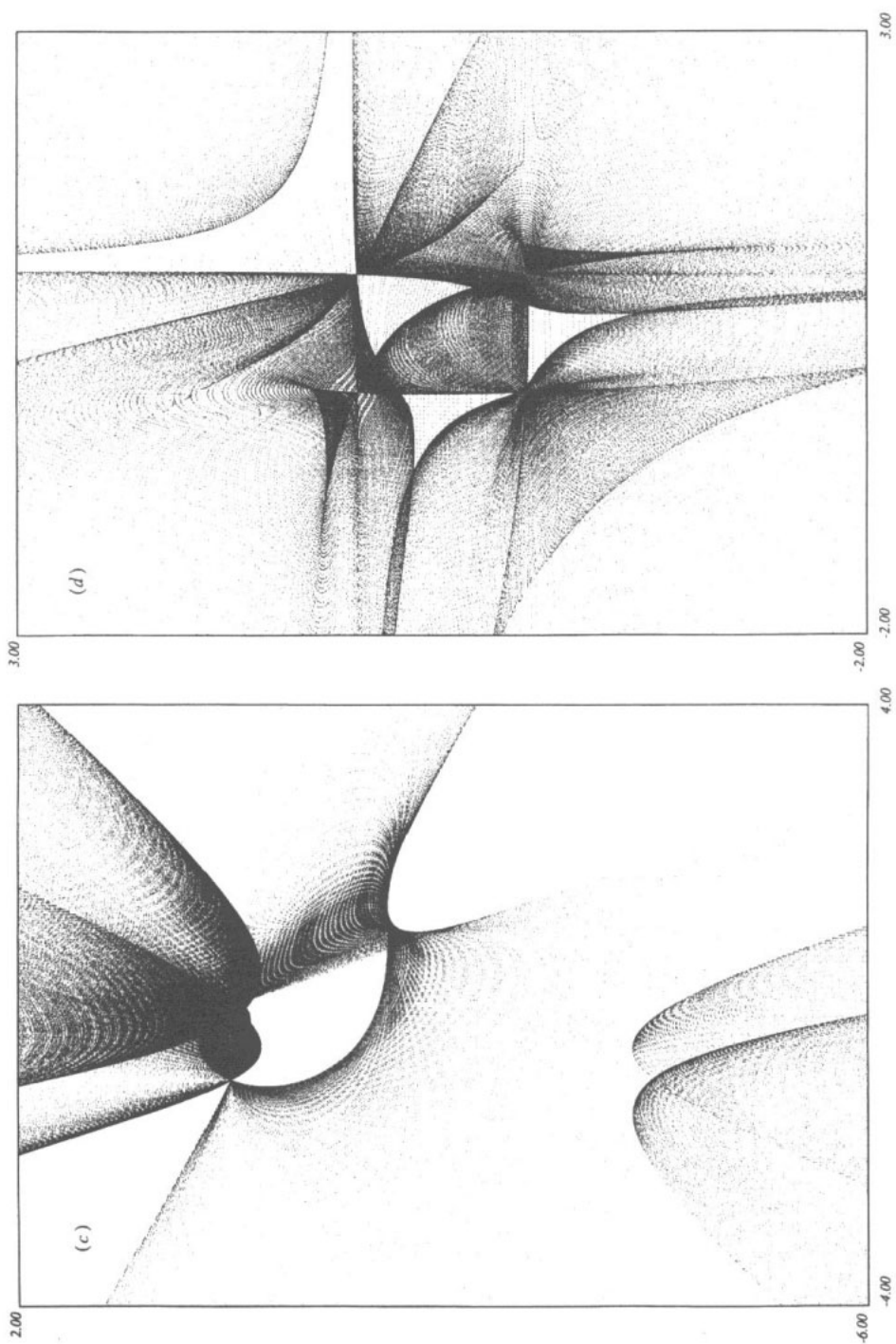
**Figure 2.** Continued.

Thus, the $G_i$'s *actually commute*. From relations (4.21) and (4.23), it is clear that a generic element of $G$ reads:

$$g = G_1^{n_1} G_2^{n_2}$$

where $n_1$ and $n_2$ are relative integers, which explicitly means that $G$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Let us now show that $\Gamma_{upt}$ is isomorphic to $G$ up to a semi-direct product by a finite group.

$\Gamma_{upt}$ can be seen to be generated by $I$ and $c$, up to some semi-direct product by a finite group. From relation (4.22), one gets at once

$$c\,I\,c\,I = G_1^{-1}\,c^2 \qquad c\,I\,c^2\,I = G_3 \qquad c^2\,I\,c\,I = G_2^{-1} \qquad c^2\,I\,c^2\,I = G_1\,c$$
$$I\,c\,I\,c = G_3^{-1}\,c^2 \qquad I\,c\,I\,c^2 = G_3^{-1} \qquad I\,c^2\,I\,c = G_2 \qquad I\,c^2\,I\,c^2 = G_2\,c\,.$$

Thus $\Gamma_{upt}$ is isomorphic to $G$, up to a semi-direct product by a finite group, that is, isomorphic to $\mathbb{Z} \times \mathbb{Z}$ up to a semi-direct product by a finite group.

*4.5.4. Numerical analysis.*   For $0 < q < 4$ the infinite set of points of the orbits of the automorphy group is dense in an algebraic curve while, in the other case, they accumulate to fixed points. This situation has already been noticed [44] in the $x = 1$ subcase. Therefore *in this section, we will restrict our study to* $0 < q < 4$.

To complete the analysis of the symmetry group, one has to study its generically infinite-order generators (the $J_i$'s). We will draw here their orbits in the four-dimensional parameter space ($\mathbb{C}P_4$) of the model. From relation (4.13), it is clear that the iterations of the $J_i$'s amount to performing the iteration of transformation $(c\,I)^2$. For generic values of $q$ (of course different from Tutte–Beraha numbers, see section (4.5.2)), the iteration of $(c\,I)^2$ *yields curves*. Figure 1 shows such a curve obtained for $q = 3.5$ (which is not a Tutte–Beraha number). (In all figures, $x_0 = 2$, $x_1 = 3$, $x_2 = 4$, $x_3 = 1.5$, $N = 100\,000$.)

For Tutte–Beraha numbers, since the $J_i$'s are finite-order transformations, one has to *consider other elements of the group*. As far as the Euclidean case is concerned ($q = 1$ or $q = 3$), let us recall that the $G_i$'s are the commuting generators of the symmetry group isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Figure 2(*a*) illustrates the iteration of $G_2$ for $q = 3$. *Remarkably one again gets curves*. Of course, iterating $G_3$ for $q = 3$ also yields curves, as can be seen in figure 2(*b*). Considering one orbit of the symmetry group generated by the $G_i$'s, we get, as we should, a *surface* which can clearly be seen in figure 2(*c*) as the *product of curves* like 2(*a*) and (*b*). This last figure gives a nice illustration of the $\mathbb{Z} \times \mathbb{Z}$ structure of the group. One gets similar results for the other Euclidean case $q = 1$: figure 2(*d*) shows the surface corresponding to one orbit of the whole symmetry group generated by the $G_i$'s.

Amazingly, the $G_i$'s which no longer commute when $q$ is no longer equal to 1 or 3, *do yield curves*, as can be seen on figure 3, which represents the iteration of $G_2$ for $q = 0.5$ (which is not a Tutte–Beraha number).

All these examples are remarkable: if one consider the iteration of more involved elements of the group, one generically gets quite chaotic figures (except for $q = 1$ or $q = 3$). Figure 4(*a*) shows such a 'chaotic' orbit for a Tutte–Beraha number ($q = 2 + \sqrt{3}$) and figure 4(*b*) for $q = 3.5$ (which is not a Tutte–Beraha number). Both figures 4(*a*) and (*b*) correspond to the iteration of $J_1\,J_2^2$.

These last figures and the study of many other orbits not given here give a good hint of the complexity of these infinite Coxeter groups. They are generically of exponential growth, *even when additional relations occur* (see relation (4.19)).

This numerical study indicates that for generic values of $q$, the generators of the symmetry group (the $J_i$'s) seem integrable since their iterations yield curves apparently in the
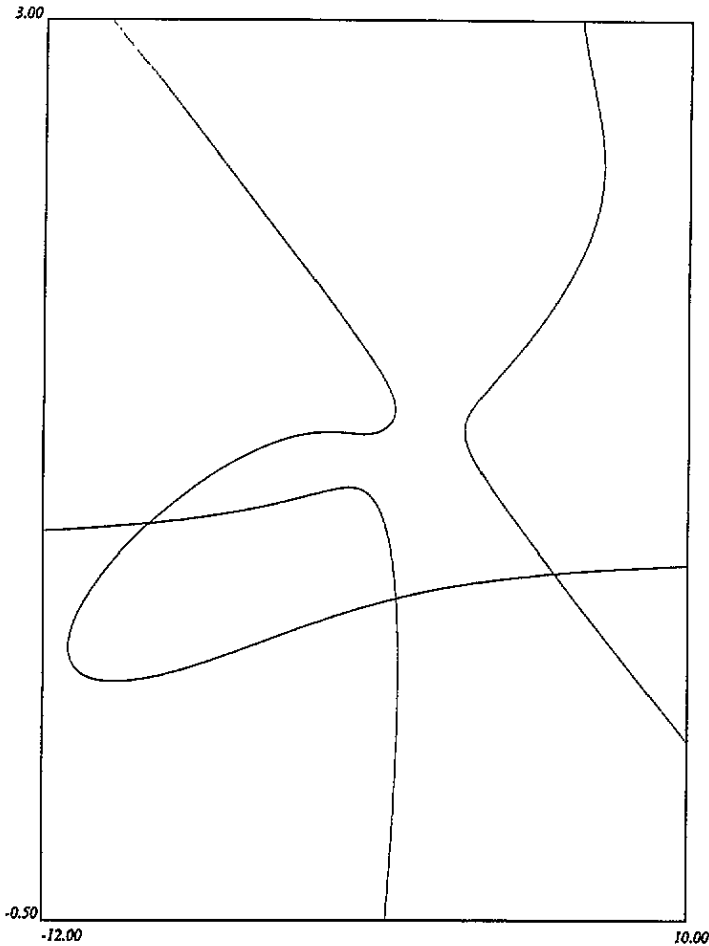
**Figure 3.** Two-dimensional projection of an orbit of transformation $G_2$ for $q = 0.5$.

whole parameter space. Moreover, the $G_i$'s seem to satisfy the same property for any value of $q$, though they emerged from the analysis of the Euclidean case ($q = 1$ or $q = 3$). A way to verify this assumption is to give the algebraic equations of these curves. For this purpose, in the next section we will seek algebraic varieties invariant under the $J_i$'s and the $G_i$'s.

### 4.6. Group invariants

Let us first remark that there exist three (homogeneous) polynomials, of degree $1, 2$ and $3$ respectively, invariant under permutation of $x_1, x_2$ and $x_3$, and *covariant* under transformation $I$. These three polynomials read

$$D_1 = x_1 + x_2 + x_3 - x_0 + (q - 2)t$$
$$D_2 = t(x_1 + x_2 + x_3 + x_0 - t) - x_1 x_2 - x_2 x_3 - x_3 x_1$$
$$D_3 = t^2 x_0 - x_1 x_2 x_3.$$

Let us note that the cofactor (under the action of $I$) of $D_3$ is the product of the respective cofactors (under the action of $I$) of $D_1$ and $D_2$.
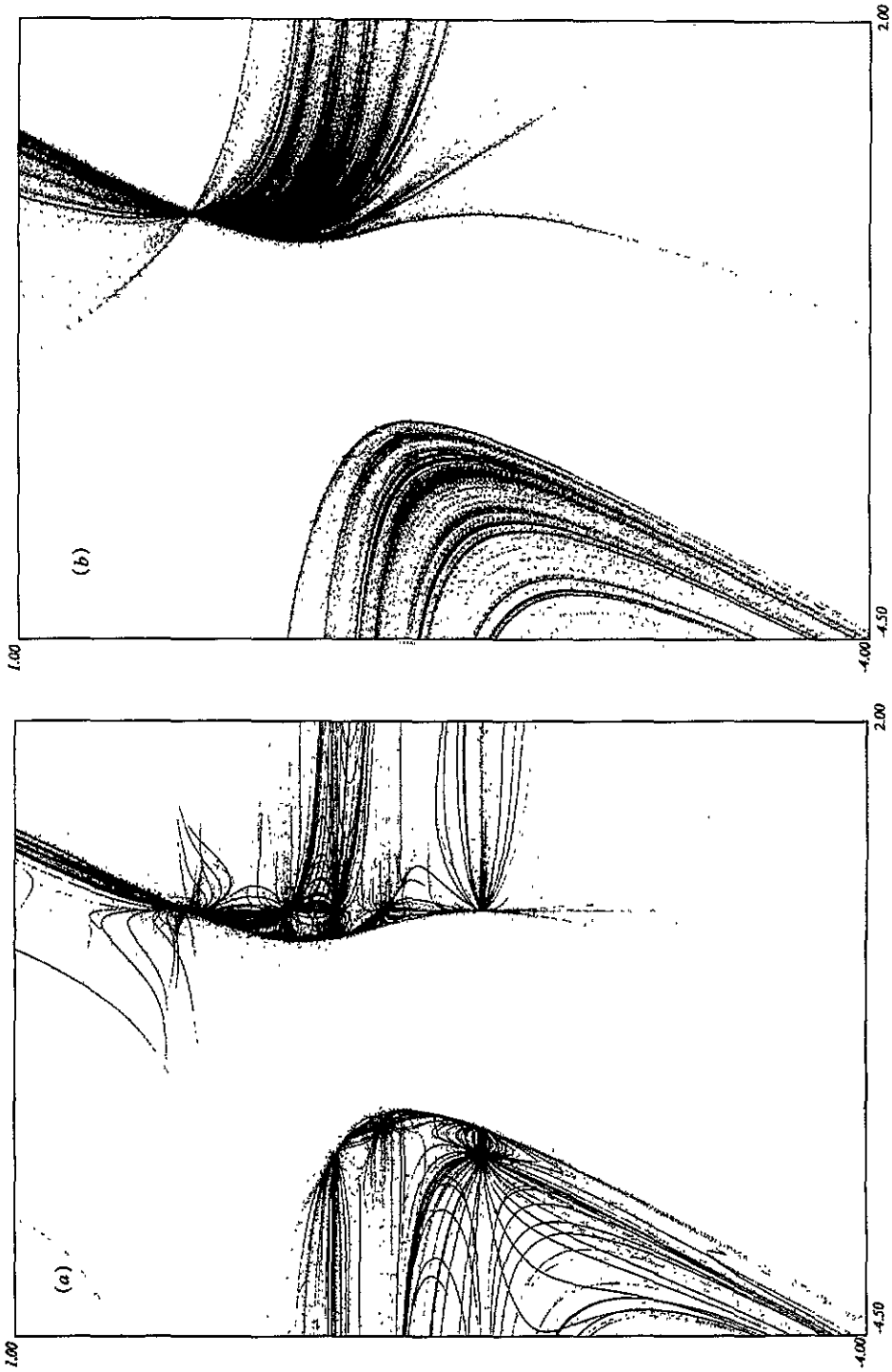
**Figure 4.** Two-dimensional projections of two orbits corresponding respectively to the iterations of transformation $J_1 J_2^2$ for a Tutte–Beraha number ($q = 2 + \sqrt{3}$) and for a generic value of $q$ ($q = 3.7$).

As a consequence, one directly gets an *invariant* under the group generated by $I$ and the permutations of $1, 2$ and $3$:

$$\Delta = \frac{D_1 \, D_2}{D_3} . \tag{4.24}$$

This provides, for arbitrary $q$, a *canonical foliation of the parameter space* ($\mathbb{C}P_4$) by co-dimension-one algebraic varieties.

It has been seen in the previous section that the iterations of the $J_i$'s yield curves in the whole parameter space. In order to prove that these curves are actually algebraic, one has to exhibit two other algebraic invariants for these transformations. From relations (4.13), it is clear that one can restrict the study to transformation $(cI)^2$. One can show that the two polynomials

$$E_1 = x_1 + x_2 - x_3 + x_0 + (q - 2)\,t$$
$$E_2 = t\,(x_1 + x_2 - x_3 - x_0 - t) - x_1\,x_2 + x_2\,x_3 + x_3\,x_1$$

are actually covariant under the action of $(cI)^2$. These expressions happen to have respectively the same cofactors (under transformation $(cI)^2$) as $D_1$ and $D_2$. This provides immediately two additional *algebraic invariants* under $(cI)^2$:

$$\Delta_1 = \frac{D_1}{E_1} \qquad \Delta_2 = \frac{D_2}{E_2} .$$

Curves like figure 1 are thus given as intersections of cubics, quadrics and hyperplanes, namely

$$\Delta = \delta \qquad \Delta_1 = \delta_1 \qquad \Delta_2 = \delta_2 \tag{4.25}$$

where the $\delta$'s denote arbitrary constants.

Such algebraic curves, with an infinite number of automorphisms, are either elliptic or rational curves [23]. Amazingly, eliminating $x_0$ and $x_3$ from relations (4.25) one gets (coming back to inhomogeneous variables)

$$(\delta_1 + 1)\,(\delta_2 + 1)\,(x_1\,x_2 - 1) =$$
$$(4\,\delta_1\,\delta_2\,\delta\,(x_1 + x_2 - 2) + (\delta_1 - 1)\,(\delta_2 - 1))\,(x_1 + x_2 - 2 + q)$$

which proves that these curves are actually *rational curves*.

The previous numerical analysis indicated remarkable occurrence of curves, when iterating the $G_i$'s for any value of $q$. Let us for instance consider $G_3$. One notices that polynomials

$$F_1 = x_3$$
$$F_4 = (x_1\,x_3 + x_2\,x_3 - x_3\,t - x_0\,t)\,(x_1\,x_2 + (q - 3)\,x_0\,t - (q - 2)\,x_0\,x_3)$$

are actually covariant under the action of $G_3$. The values of the cofactors of these $F_i$'s enable us to get two $G_3$ invariants

$$\Delta_1' = \frac{D_1}{F_1} \qquad \Delta_4' = \frac{D_1\,D_3}{F_4} .$$

Figures like 2(*a*), 2(*b*) or 3 are thus algebraic (elliptic) curves given by intersections of cubics, hyperplanes and quartics.

Let us recall that for $q = 1$ or $q = 3$, these $G_i$'s do commute and that: $G_3\,G_2\,G_1 = \mathcal{I}d$. We have just seen that each of the $G_i$'s generates algebraic elliptic curves. Therefore for $q = 1$ and $q = 3$, the orbits of the group generated by the $G_i$'s yield algebraic surfaces which are *products of two elliptic curves*, as clearly seen on figure 2(*c*). Since this surface is

stable under permutations of $x_1$, $x_2$ and $x_3$, it is natural to give its equation without referring to two of the $G_i$'s, that is, without having any direction singled out. For $q = 3$ an additional polynomial

$$D_5 = -x_1 x_2 x_3 \left(x_0^2 + x_1^2 + x_2^2 + x_3^2 - t^2\right) + x_0 \left(x_1^2 x_2^2 + x_2^2 x_3^2 + x_3^2 x_1^2\right)$$

is symmetric under permutations of 1, 2 and 3 and covariant under $I$. Symmetric invariant

$$\Delta_5' = \frac{D_1^3 D_2}{D_5}$$

together with invariant (4.24) thus give symmetric equations of these algebraic surfaces. Similarly for $q = 1$, polynomial

$$D_3' = x_0 D_2$$

is a symmetric covariant under the action of $I$, yielding the following symmetric invariant:

$$\Delta_3' = \frac{D_1^3}{D_3'} .$$

Let us now recall that duality transformation $D$, defined in section (4.2), is also a symmetry of the model, which commutes with transformation $I$ and with permutations of 1, 2 and 3. Let us notice that $D$ is actually a *linear* transformation when written in terms of homogeneous variables:

$$D_h : \quad \begin{cases} x_0 \to x_0 + (q - 1)(x_1 + x_2 + x_3 + (q - 2)t) \\ x_1 \to (q - 1)x_1 + x_0 - x_2 - x_3 - (q - 2)t \\ x_2 \to (q - 1)x_2 + x_0 - x_1 - x_3 - (q - 2)t \\ x_3 \to (q - 1)x_3 + x_0 - x_2 - x_1 - (q - 2)t \\ t \to x_0 - x_1 - x_2 - x_3 + 2t . \end{cases}$$

Considering the previous covariant polynomials, one notices that five of them are 'eigen-polynomials' of the duality transformation:

$$\begin{cases} D_1 \to -q\, D_1 \\ D_2 \to q^2 D_2 \\ E_1 \to q\, E_1 \\ E_2 \to q^2 E_2 \\ D_3' \to q^2 D_3' . \end{cases}$$

As far as the other covariant polynomials are concerned, one has to barter them for new homogeneous polynomials, namely

$$D_{3d} = 2q\, D_3 - D_1 D_2$$
$$F_{1d} = 2q\, x_3 - D_1$$
$$F_{4d} = 2q\, F_4 - (q^2 - 3q + 1) D_1 (D_3 - x_3 D_2)$$
$$D_{5d} = 2q^3 D_5 - D_1^3 D_2$$

to get the self-dual covariants.

Algebraic varieties $D_1$, $D_2$ and $D_{3d}$ are actually remarkable since they *do have covariance properties with respect to the whole group* $\Gamma_{\text{upt}}$, which is (generically) a hyperbolic group. From the point of view of effective algebraic geometry, it provides examples of algebraic varieties *with very large groups of automorphisms*. Moreover these varieties also provide examples of algebraic varieties with an infinite number of rational points (when $q$ is rational itself). This is a direct consequence of the representation of the hyperbolic group $\Gamma_{\text{upt}}$ in terms of *birational transformations with integer coefficients* with

respect to the $x_i$'s and $q$. This situation can straightforwardly be generalized to algebraic numbers.

In the next section, we discuss the actual status of these algebraic varieties.

### 4.7. Remarkable algebraic varieties

Let us first recall that critical manifolds have to be compatible with all the symmetries of the lattice model. The self-dual variety (4.6) is already known as the critical variety of the Potts model for arbitrary $q$ in some ferromagnetic region [24]. It is also known that this variety is, as it should be, stable under the whole hyperbolic Coxeter group [42]. One recovers this result, noticing that this symmetric self-dual variety (4.6) is nothing but variety $D_1 = 0$. Conversely, the same argument confirms that the self-dual variety (4.4) cannot be critical since it is not stable under $\Gamma_{\text{upt}}$.

Moreover in the previous section, other algebraic varieties have been shown to be stable under the whole infinite Coxeter group, namely the vanishing conditions of the expressions: $D_2$, $D_{3d}$ and also $D_{5d}$ when $q = 3$. We do not consider here $D_3' = 0$ (for $q = 1$) since it reduces to the previous $D_2 = 0$ case and to condition $x_0 = 0$ for which the analysis of the model becomes of a more 'combinatorial' nature and deserves a specific study.

Variety $D_2 = 0$ was not previously known in the literature: it is a good candidate for being a critical variety. Unfortunately, the $x = 1$ limit of this variety is singular for $\Gamma_{\text{upt}}$ and is thus of no help to decide the status of this variety. New Monte Carlo calculations will be performed to answer this question.

Moreover, in the $x = 1$ limit another algebraic variety has already been introduced in [45]. Monte Carlo calculations of the $q = 3$ isotropic limit of the model have been performed on this subvariety [46]. These studies confirmed the existence of an antiferromagnetic critical point (in addition to the well known ferromagnetic one) probably corresponding to a first-order transition. Though very close to this variety in the $x = 1, q = 3$ isotropic region, this antiferromagnetic critical point is definitely different [46]. This negative result does not rule out the existence of an extension of this variety to some $x \neq 1$ domain as a critical variety in some region of the parameter space which could depend on the value of $q$. This question will be addressed in further publications.

Variety $D_{3d} = 0$ is also a good candidate for criticity. Unfortunately, no partial results are available in the literature. If one comes back to $D_3 = 0$, which is not self-dual, this condition corresponds to the vanishing of the three-spin interaction ($x = 1$). Variety $x = 1$ is well known [36] and plays a special role: the symmetry group $\Gamma_{\text{upt}}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ (up to some semi-direct product by a finite group). Since duality $D$ commutes with $\Gamma_{\text{upt}}$, the *dual variety of $x = 1$ also corresponds to the degeneracy of* $\Gamma_{\text{upt}}$ *into a group isomorphic to* $\mathbb{Z} \times \mathbb{Z}$ (up to some semi-direct product by a finite group). This remarkable variety (4.7) also reads

$$D_3^* = q^2 (q\, D_3 - D_1\, D_2) = 0 \qquad \text{or equivalently} \quad \Delta = q\,. \tag{4.26}$$

Finally for $q = 3$, the vanishing condition of the self-dual expression $D_{5d}$, as well as condition $D_5 = 0$ and its dual variety, also requires further studies.

Clearly one needs further Monte Carlo simulations, with a particular emphasis on the Euclidean case $q = 3$. One will try to see if, besides the known variety $D_1 = 0$, others of the above-mentioned algebraic varieties are actually critical. As far as phase diagrams are concerned, the intersections of these algebraic varieties could play a special role (multicritical points ?...). Note that $D_1 = 0$ and $D_2 = 0$ do have intersection points

in the ferromagnetic domain.

Let us recall that other remarkable varieties have been mentioned in section (4.3): the disorder varieties (4.8). The analysis of the action of the hyperbolic group on these varieties has also to be performed, in order to generalize the analysis already achieved in the $x = 1$ limit [38].

## 5. Conclusion

The symmetry group generated by inversion relations has been analysed for vertex triangular lattice models and for the standard scalar Potts model with two- and three-site interactions on the triangular lattice. The group generated by three involutions is seen to be generically a very large one, like a free group. Two situations for which the representations of this group degenerate into smaller ones, hopefully compatible with integrability, have been considered. The first reduction for the vertex triangular model corresponds to the situation where the vertex of the triangular model coincides with the left- or right-hand sides of a Yang–Baxter relation. The representation of the group is isomorphic, up to a semi-direct product by a finite group, to $\mathbb{Z} \times \mathbb{Z}$. The second reduction for $q$-state Potts models occurs for particular values of $q$, the so-called Tutte–Beraha numbers [20, 43]. For these values of $q$, some of the (generically infinite-order) generators are of finite· order. However, even with such additional relations on the generators, one still gets groups an with exponential growth, except for $q = 1$ or 3. Nevertheless such additional relations on the generators occur on particular algebraic varieties, yielding a degeneracy of the group into products of $\mathbb{Z}$. We have seen in this paper that $x = 1$ and its dual variety (4.26) are such varieties. It would be interesting to search systematically for these varieties.

As far as this Potts model is concerned, a set of algebraic varieties stable under hyperbolic Coxeter groups has emerged. In particular, one recovers the self-dual algebraic variety, known as critical in some ferromagnetic region. These new results strongly suggest further Monte Carlo calculations to clarify the phase diagram of the model.

As a byproduct, this analysis provides nice birational representations of hyperbolic Coxeter groups and also algebraic varieties having such large groups of automorphisms.

This first analysis of hyperbolic Coxeter symmetry groups for lattice models, including degeneracy subcases, should help us to better understand the symmetries of three-dimensional models and the occurrence of true three-dimensional integrability.

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