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Growth-complexity spectrum of some discrete dynamical systems

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Abstract

We first study the iteration of birational mappings generated by the composition of the matrix inversion and of a permutation of the entries of 3×3 matrices, and consider the degree $d(n)$ of the numerators (or denominators) of the corresponding successive rational expressions for the nth iterate. The growth of this degree is (generically) exponential with n: $d(n) \simeq \lambda^n$. λ is called the growth complexity. We introduce a semi-numerical analysis which enables to compute these growth complexities λ for all the 9! possible birational transformations. These growth complexities correspond to a spectrum of eighteen algebraic values. We then drastically generalize these results, replacing permutations of the entries by homogeneous polynomial transformations of the entries possibly depending on many parameters. Again it is shown that the associated birational, or even rational, transformations yield algebraic values for their growth complexities.

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1. Introduction and recalls

Birational transformations have been seen to be a powerful tool to analyze the symmetries of the parameter space of lattice models of statistical mechanics [1,2] and to seek for some possible Yang–Baxter integrability [3,4]. Beyond the lattice statistical mechanics framework, birational transformations are worthy to be studied *per se*, as discrete dynamical systems. In particular, one should mention many works on birational plane mappings (Cremona transformations) [5] and more recently results on the integrability of these birational plane mappings [6,7]. Discrete dynamical systems have been intensively studied (see for example [8–11]). Among them polynomial examples, like the Henon map [12], have been precious to understand some features of chaos. Beyond, rational mappings

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are of special interest since they allow some analytical calculations. Furthermore, the rational transformations also allow numerical calculations which can be performed with any wanted precision : the existence of singularities in the rational transformations one iterates, and their possible 'proliferation' is not in fact a numerical obstruction. We will first consider mappings generated by the composition of the matrix inverse and some arbitrary, but fixed, permutation of the entries of $q \times q$ matrices. The results, displayed in this paper, are given for $q = 3$, but are actually valid, *mutatis mutandis* [2], for arbitrary q values.

1.1. *Recalling a previous* 3 × 3 *analysis*

Integrability of a mapping amounts to saying that all the orbits of the iteration correspond to elliptic, or rational, algebraic curves [13,14]. From the point of view of the growth [1] of the complexity of the successive iterations [13,14], such integrability in curves always yields a *polynomial growth* of the calculations [1,2], instead of the exponential growth one generically expects. Conversely, polynomial growth is not restricted to integrability in curves but may correspond to orbits 'densifying' Abelian varieties [1,2].

A first exhaustive analysis of all the 9! birational transformations generated by the composition of the matrix inversion and of a (fixed) permutation of the entries of 3 × 3 matrices has already been performed concentrating on the extraction of integrable mappings [19]. This analysis was exhaustive, but restricted to particular integrability criteria [2]. Even from this 'integrability-digger' point of view some integrable mappings are missing (for example, the so-called 'Class III' mappings of [20], as well as some polynomial growth situations). In the first part of this paper we will revisit these 9! = 362880 birational mappings without any *a priori* integrability criterion and with the help of a new equivalence relation among permutations (symmetry). This analysis exactly yields *all* the polynomial growth situations, and, far beyond, *classifies* the exponential growth situations. The classification relies on the value of the *Arnold complexity* [21] of the mapping. This complexity can be obtained [22] from generating functions associated with factorization schemes [2] detailed below.

1.2. *Factorization scheme and generating functions*

We use the same notations as in [13,14,20], that is, we introduce the following two transformations, the usual matrix inverse $\hat{I}$ and the *homogeneous* matrix inverse I :

$$\hat{I}: \quad M_0 \to M_0^{-1}, \quad \text{and} \quad I: \quad M_0 \to \det(M_0) \cdot M_0^{-1} \tag{1}$$

The homogeneous inverse I is a homogeneous polynomial transformation, which associates, with each entry of M_0, its corresponding cofactor. Transformation t is any (fixed) permutation of the entries of the 3 × 3 matrix. We also introduce the (generically infinite order) transformations:

$$K = t \cdot I \quad \text{and} \quad \hat{K} = t \cdot \hat{I} \tag{2}$$

Transformation $\hat{K}$ is clearly a *birational transformation* [13,14].

For all the various birational transformations associated with permutations of the entries of 3 × 3 matrices, the following factorization relations happen to occur *at each* iteration step [2] :

$$f_1 = \det(M_0), \quad M_1 = K(M_0), \quad f_2 = \frac{\det(M_1)}{f_1^{\phi_1}}, \quad M_2 = \frac{K(M_1)}{f_1^{\eta_0}}, \quad f_3 = \frac{\det(M_2)}{f_1^{\phi_2} \cdot f_2^{\phi_1}}, \quad M_3 = \frac{K(M_2)}{f_1^{\eta_1} \cdot f_2^{\eta_0}}$$

[1] When one iterates a rational transformations the 'size' of the successive rational expressions, corresponding to the Nth iterate, grows, in general, exponentially. In particular, the *degree* of these successive rational expressions has, generically, an exponential growth [1,15]. Growth of the calculations related with factorizations were also introduced by Veselov for some particular Cremona transformations [16–18].

[2] Associated with particular recursions [19] on some 'determinantal' variables [13,14] x_n's, also introduced here in Section 1.2.

and for arbitrary n:

$$\det(M_n) = \det(M_n) = \prod_{k=0}^{n} f_{n+1-k}^{\phi_k} \quad \text{with} \quad \phi_0 \equiv 1 \tag{3}$$

$$K(M_n) = \left(\prod_{k=0}^{n-1} f_{n-k}^{\eta_k}\right) \cdot M_{n+1} \tag{4}$$

$$\det(M_n)\, M_{n+1} = \left(\prod_{k=0}^{n} f_{n+1-k}^{\rho_k}\right) \cdot K(M_n) \quad \text{with} \quad \rho_0 \equiv 1 \tag{5}$$

defining the positive integer exponents η_n, ϕ_n and ρ_n such that $\rho_k = \phi_k - \eta_{k-1}$, $n \geq k > 0$, $\eta_{-1} \equiv 0$. The f_n's are homogeneous polynomials of the entries of M_0. These factorizations allow to define, at each iteration step, the successive f_n's one can 'factor out', and the 'reduced matrices' M_n's, such that their entries are homogeneous polynomial expressions of the initial entries, and have no further factorization. One finds out, looking at the first 30 iterations, that one recovers the *same* exponents (η_n, ϕ_n, ρ_n) at each iteration step (up to the last emerging coefficient for f_1). We assume that this regularity property holds for arbitrary n. This regularity [3] property assumption is crucial in our analysis.

We will denote by α_n, the degree of the determinant of matrix M_n, and by β_n the degree of polynomial f_n and $\alpha(x)$, $\beta(x)$, $\eta(x)$, $\phi(x)$ and $\rho(x)$, the generating functions of the degrees α_n's, β_n's, and of the exponents η_n's, ρ_n's and ϕ_n's in the factorization schemes:

$$\alpha(x) = \sum_{n=0}^{\infty} \alpha_n x^n, \qquad \beta(x) = \sum_{n=1}^{\infty} \beta_n x^n, \qquad \eta(x) = \sum_{n=0}^{\infty} \eta_n x^n, \qquad \phi(x) = \sum_{n=0}^{\infty} \phi_n x^n, \qquad \rho(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

where $\alpha_0 = 3$ and $\beta_1 = 3$. It is straightforward to show [2] that the existence of the *stable* factorization schemes (3) and (4) yields the following simple linear relations between these various 'exponents generating functions', for instance $x\,\eta(x) = \phi(x) - \rho(x)$, and the 'degree generating functions':

$$\alpha(x) + 3x\,\eta(x)\,\beta(x) = 3 + 2\,x\,\alpha(x) \tag{6}$$

$$x\,\alpha(x) = \phi(x)\,\beta(x) \tag{7}$$

$$3 + 3\,\rho(x)\,\beta(x) = (1+x)\,\alpha(x) \tag{8}$$

When analytically iterating an arbitrary transformation K, the degree of the successive polynomial expressions one encounters, grow exponentially: α_n or $\beta_n \simeq \lambda^n$, where λ measures the growth of the calculations and is closely related, for mappings of two variables, with the notion of Arnold [4] complexity [21,22] (more precisely with the asymptotic behaviour of the Arnold complexity). From now on λ will be called the 'growth *complexity*' or more simply, G-complexity. When the degree generating functions $\alpha(x)$ or $\beta(x)$ happen to be rational functions, the

[3] In fact it is shown in [22] that other slightly more general factorizations scheme can occur on some $\hat{K}$-invariant subvarieties (yielding smaller Arnold complexity values). Such slightly more general factorizations scheme will also be detailed below (see Appendix B). However, for the transformations K associated with permutations of $q \times q$ matrices, for a generic initial matrix, one gets factorization schemes like Eqs. (3) and (4), also depicted in [2].

[4] More precisely the Arnold complexity $C_A(n)$ is proportional (for plane maps) to $d(n)$, the degree of the nth iteration of the birational mapping which behaves like $d(n) \simeq \lambda^n$. This 'degree notion' was introduced by A.P. Veselov in exact correspondence with the general Arnold definition [21]. Note that the concept of Arnold complexity *is not* restricted to two-dimensional maps.

G-complexity λ is obviously the inverse of the pole of smallest modulus. Recalling the 'determinantal' variables [13,14,20] x_n's defined by:

$$x_n(M_0) = \det(\hat{K}^{n+1}(M_0))\ \ \det(\hat{K}^n(M_0)) \tag{9}$$

one finds out that these determinantal variables happen to decompose on a product of the homogeneous polynomials f_n's *only*:

$$x_n(M_0) = f_{n+1}^{w_0}\, f_n^{w_1}\, f_{n-1}^{w_2}\, f_{n-2}^{w_3}\, f_{n-3}^{w_4}\, f_{n-4}^{w_5} \cdots \quad \text{with} \quad w_0 = 1,\, w_i \in \mathbf{Z} \tag{10}$$

which defines some, at first sight, 'new' exponents w_n's and consequently a, at first sight, 'new' generating function $\mathcal{W}(x)$:

$$\mathcal{W}(x) = \sum_{n=0}^{\infty} w_n\, x^n \tag{11}$$

It is worth noting that the determinantal variables x_n's induce the homogeneous polynomials f_n's emerging from the factorization schemes (3) and (4) and no other homogeneous polynomials. The variables x_n's are well-suited since they are invariant under a multiplication of M_0 by a constant: $M_0 \to Cst \cdot M_0$. In other words, the x_n's are homogeneous expressions of degree zero. Concentrating on the degrees of the left-hand side, and right-hand side, of Eq. (10)), one gets the following 'degree equation':

$$0 = \beta_{n+1} w_0 + \beta_n w_1 + \cdots + \beta_{n-p} w_{p+1} + \cdots + \beta_1 w_n \tag{12}$$

from which one immediately deduces the simple functional equation:

$$\mathcal{W}(x)\, \beta(x) = \beta_1\, w_0\, x = 3x \tag{13}$$

This result is immediately generalized to $q \times q$ matrices. Relation (13) becomes $\mathcal{W}(x)\, \beta(x) = q\, x$. From Eq. (13) one actually sees that $\mathcal{W}(x)$ is *not* a new generating function: it is simply related to the degree generating function $\beta(x)$. The G-complexity λ is associated to the *zeroes* of $\mathcal{W}(x)$.

From relations (6)–(8), one easily gets the 'degree generating functions' $\alpha(x)$ and $\beta(x)$ from two of the 'exponent generating functions' (for instance $\phi(x)$ and $\eta(x)$ or $\eta(x)$ and $\rho(x)$). As a matter of fact, for most of the permutations, the factorization schemes are *periodic* ($\eta_n = \eta_{n+N}$, $\phi_n = \phi_{n+N}$ and $\rho_n = \rho_{n+N}$ for some integer N). Consequently, the exponent generating functions $\phi(x)$ and $\eta(x)$, or $\rho(x)$, are *rational* functions with Nth root of unity poles [2] (see, for instance, the exponent generating function $\rho(x)$ in Eq. (14) or (16)). In a second step one deduces, from relation (6), or (7), rational expressions for the degree generating functions $\alpha(x)$ and $\beta(x)$. However, it will be seen below that, for some permutations, the factorization schemes *are still regular*, but not with periodic exponents (see the exponent generating function $\rho(x)$ in Eq. (15) or (17)): the exponents η_n, ϕ_n and ρ_n *grow exponentially*, but one remarks that the associated generating functions $\eta(x), \phi(x), \rho(x)$ are *still rational*, and, thus, $\alpha(x)$ and $\beta(x)$ *are also rational*. The exponent generating functions can be seen as an 'encoding' of the degree generating functions, and, thus of the G-complexity λ. Remark that all these rational expressions involve *integer* coefficients, yielding *algebraic* values for their poles and for the growth of the calculations: the degrees of the successive rational expressions, namely α_n's and β_n's grow like λ^n, where λ is an *algebraic number*, and for regular factorization schemes (like (15) or (17), see below) the exponents η_n and ϕ_n grow like μ^n, where μ is the 'scheme complexity'. Note that μ is obviously such that $\mu \le \lambda$. Exponent μ is also the inverse of the pole of smallest modulus of the exponent generating functions. The G-complexity λ allows all kinds of handy, efficient, and formal, or semi-numerical, calculations. We will present below such a semi-numerical method and apply it to *all the* permutations of entries of 3×3 matrices [5].

2. G-complexity spectrum analysis for permutations

2.1. A semi-numerical method

All these considerations allow us to design a *semi-numerical method* to get the value of the G-complexity λ for the iteration of *rational* transformations. The idea is to iterate, with $\hat{K}$, a generic initial matrix with *integer* entries. After one iteration step the entries become rational and we follow the magnitude of the successive numerators and denominators. During the first few iteration steps some ‘accidental’ simplifications may occur, but, after this transient regime, the integer denominators (for instance) grow like λ^n, where n is the number of iterations. One can systematically improve the method as follows: the initial matrix is chosen in such a way that it avoids, as much as possible, any ‘accidental’ additional factorization in comparison with the factorization scheme associated with a generic matrix. For instance, in a factorization scheme framework like Eqs. (3) and (4), one chooses the initial matrix M_0 with integer entries such that the determinant, and most of its cofactors, are prime numbers as large as possible. One may impose further constraints on the initial matrices, for instance, that the first homogeneous polynomials f_2 and f_3 are also prime numbers as large as possible. These conditions down-size the probability that all the entries of the reduced matrices M_n, or the polynomials f_n’s, could be divisible by some accidental additional f_1, f_2 or f_3. Such initial matrices, well-suited for the iteration of the homogeneous transformation K, are also well-suited for the iteration of the (bi)rational transformation $\hat{K}$. In practice, we start with a set of initial matrices and keep only the one for which the *less* factorizations occur (non-generic factorization can only correspond to *additional* factorizations).

The computations are done using an infinite precision C-library [6] [23]. We perform as many iterations as possible during a given CPU time T. This number of iterations, n, is such that $T \simeq \lambda^n$. For λ close to 2 and $T = 60\,\mathrm{s}$, n is of the order of 20 and a best fit of the logarithm of the numerator as a linear function of n, between $n = 10$ and $n = 20$, gives the value of λ within an accuracy of 0.1%. For smaller values of λ (typically $\lambda < 1.5$) the number of iterations is larger, but the accuracy, for a given CPU time, is smaller. In such ‘difficult’ cases one analytically finds the factorizations up to $n = 7$ and implement the first steps of these factorization schemes in the semi-numerical method. We are then almost guaranteed that no accidental factorizations will occur for $n > 7$, and therefore, we can average over many initial matrices. Even so it remains difficult to discriminate between a truly polynomial growth [1,2] ($\lambda = 1$) and an exponential growth with $\lambda \simeq 1$. The G-complexity values close to one clearly need to be revisited by other methods we present below.

2.2. Equivalence relations between permutations

Even if this semi-numerical algorithm is efficient it is quite time consuming to use it directly on the 9! permutations. To classify the G-complexities associated to a large set of (birational) transformations like the one associated to the

[5] However, one should keep in mind that there is nothing specific with 3×3 matrices. These results simply generalize to $q \times q$ matrices (see for instance [2]).

[6] The multi-precision library gmp (GNU MP) is part of the GNU project. It is a library for arbitrary precision arithmetic, operating on signed integers, rational numbers and floating points numbers. It is designed to be as fast as possible, both for small and huge operands. The current version is: 2.0.2. Targeted platforms and Software/Hardware requirements are any Unix machine, DOS and others, with an operating system including files and a C compiler.

Table 1

λ	Polynomial	$\mathcal{R}^{(1)}_{144}$	$\mathcal{R}^{(1)}_{72}$	$\mathcal{R}^{(\infty)}_{144}$	$\mathcal{R}^{(\infty)}_{72}$	Total
2	$1-2x$	2145	640	14	33	2832
1.97481871	$1-2x+x^2-2x^3+x^4-2x^5+x^6$	0	2	0	0	2
1.97458465	$1-x-2x^2-x^3+x^4+2x^5+x^6$	0	1	0	0	1
1.94893574	$1-2x+x^5-x^7$	0	2	0	0	2
1.94685627	$1-x-x^2-x^3-x^4-x^5+x^6$	0	1	0	0	1
1.93318498	$1-2x+x^4-x^5$	0	1	0	0	1
1.89110302	$1-2x+x^2-2x^3+2x^4-2x^5$	0	0	0	1	1
1.88320350	$1-2x+x^2-2x^3+x^4$	0	2	0	6	8
1.86676040	$1-2x+x^3-x^4$	0	1	0	0	1
1.86007305	$1-x-x^2-x^4-2\cdot x^5$	0	1	0	0	1
1.85712752	$1-2x+x^2-x^3-x^5-x^7+x^8-2x^9+x^{10}$	0	1	0	0	1
1.83928675	$1-x-x^2-x^3$	0	2	0	0	2
1.75487766	$1-2x+x^2-x^3$	1	0	0	0	1
1.61803399	$1-x-x^2$	0	3	0	0	3
1.57014731	$1-x-x^3-x^5$	0	1	0	0	1
1.54257960	$1-x-x^3-x^7-x^8$	0	1	0	0	1
1.46557123	$1-x-x^3$	0	0	0	2	2
1 (Pol.gr.)	$1-x,\ 1-x^N,\ \ldots$	0	0	0	9	9
1 (Period.)		0	1	0	9	10
Total		2146	660	14	60	2880

9! permutations of 3×3 matrices, one certainly needs to reduce this set as much as possible. For instance, one can try to find symmetries such that two permutations, related by the symmetry, yield the same G-complexity λ. These symmetries allow to build *equivalence classes* and, thus, to restrict the exhaustive analysis to only one representative in each class. Furthermore, one may have the prejudice that any non-trivial symmetry could enable to explain a possible integrability structure of the mappings and beyond, the structures associated with the classification of the G-complexity of these mappings.

There actually exist quite trivial symmetries, corresponding to relabeling of rows and columns [19], for which the G-complexities of the associated K's are obviously equal. It is possible to go a step further and define a set of equivalence relations $\mathcal{R}^{(n)}$ between the permutations, yielding new equivalence classes such that any two permutations in the same 'new' equivalence class, $\mathcal{R}^{(n)}$, automatically have the *same G-complexity* λ. Equivalence relation $\mathcal{R}^{(n)}$ amounts to saying that two equivalent permutations are such that the nth power of their associated transformations $\hat{K}$ are conjugated (via particular permutations, product of row permutations, column permutations and possibly the transposition, see Appendix A for more details). An exhaustive inspection has shown that the equivalence relations $\mathcal{R}^{(n)}$'s 'saturate' after $n=24$: with obvious notations $\mathcal{R}^{(\infty)}=\mathcal{R}^{(24)}$. One finds out that the 'ultimate' $\mathcal{R}^{(\infty)}$ equivalence classes *can only have* 72, or 144, elements. Among the 'ultimate' $\mathcal{R}^{(\infty)}$ classes, one wants to distinguish between the classes that were already $\mathcal{R}^{(1)}$ classes, that we will denote from now on by $\mathcal{R}^{(1)}_{72}$, or $\mathcal{R}^{(1)}_{144}$, according to their number of elements , and the other ones we denote $\mathcal{R}^{(\infty)}_{72}$ or $\mathcal{R}^{(\infty)}_{144}$. Being an $\mathcal{R}^{(\infty)}$ equivalence class which does not reduce to a $\mathcal{R}^{(1)}$ equivalence class, means the existence of several non-trivial relations between the permutations in the $\mathcal{R}^{(\infty)}$ equivalence class (see Eq. (A.3) in Appendix A). This implies strong constraints on the respective orbits. One, thus, expects more properties, and structures, inherited from this fact. The 362880 permutations are grouped into 2880 equivalence classes (instead of 30462 'relabeling' equivalence classes in [19]). In Table 1 the number of the respective $\mathcal{R}^{(1)}_{72}$, $\mathcal{R}^{(1)}_{144}$, $\mathcal{R}^{(\infty)}_{72}$ and $\mathcal{R}^{(\infty)}_{144}$ classes is displayed. Since the complexities do not depend on the chosen representative, we picked a representative in each $\mathcal{R}^{(\infty)}$ class and performed, for it, the semi-numerical method previously explained.

For 3×3 matrices, the G-complexities are necessarily such that: $2 \geq \lambda \geq 1$. Remarkably, instead of getting a quite complicated distribution, or spectrum, of values for the G-complexities, we have obtained values which are always very close, up to the accuracy of the method, to a set of 17 values given in the left column of Table 1 (see below) and, of course, the integrable value $\lambda = 1$. To test the accuracy of the method we got G-complexities for two representatives of the *same* class (that should, as we know, have exactly the same G-complexity value). We always obtained an equality of the corresponding G-complexities, up to an error of 10^{-3}. This accuracy is, however, not always sufficient enough to discriminate between some G-complexities displayed in the left column of Table 1. In order to fix our mind it is necessary to obtain the exact expressions of these G-complexity values, for instance by getting the factorization schemes (3) and (4), and, thus, the generating functions $\alpha(x)$ and $\beta(x)$.

2.3. Revisiting the G-complexity spectrum via exact factorization schemes

For most of the $\mathcal{R}^{(\infty)}$ equivalence classes (2832 out of 2880), the G-complexity values, obtained with our semi-numerical method, are extremely close to the upper limit $\lambda = 2$. In fact, one can figure out that these G-complexity values are actually exactly equal to 2. Therefore, we can focus on the analysis of the remaining 48 classes, finding systematically their factorization schemes and associated generating functions. We actually found these factorization schemes and the associated generating functions, and were actually able to see that the previous numerical spectrum *exactly* corresponds to 18 algebraic values listed in Table 1. Among these eighteen algebraic values, let us take four illustrative examples. We give for each example, the permutation representing the $\mathcal{R}^{(\infty)}$ equivalence class, the value of λ, and μ, defined in Section 1.2, the expressions of $\beta(x)$ and $\rho(x)$, since they respectively correspond to the simplest 'degree generating function' and 'exponent generating function'. The other generating functions can be deduced from these two, using linear functional relations (6)–(8) between the generating functions [2]. Furthermore, relation (10) remains valid for all the factorization schemes associated with all the various permutations studied here. We first give the permutation t itself, using the notation, already used in [19], where $p_0 p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8$ means that $(t\tilde{M})_i = \tilde{M}_{p_i}$, the entries of the matrix being enumerated consecutively (i.e., $M_{11} = \tilde{M}_0, M_{12} = \tilde{M}_1, M_{13} = \tilde{M}_2,\ M_{21} = \tilde{M}_3, \ldots, M_{33} = \tilde{M}_8$).

- *First example.* Permutation 407326518 yields $\lambda \simeq 1.61803\cdots$ and $\mu = 1$ and:

$$\frac{\beta(x)}{3x} = \frac{1-x^2}{1-x-x^2}, \qquad \rho(x) = \frac{1}{(1-x)^2(1+x)} \tag{14}$$

- *Second example.* Permutation 417063582 yields $\lambda \simeq 1.83928\cdots$ and $\mu \simeq 1.32471\cdots$ and:

$$\frac{\beta(x)}{3x} = \frac{1-x^2-x^3}{(1-x)^2(1+x)(1-x-x^2-x^3)}, \qquad \rho(x) = \frac{(1+x)(1-x+x^4)}{1-x^2-x^3} \tag{15}$$

- *Third example.* Permutation 164273085 yields $\lambda \simeq 1.83928\cdots$ and $\mu = 1$ and:

$$\frac{\beta(x)}{3x} = \frac{1+x+x^2}{1-x-x^2-x^3}, \qquad \rho(x) = \frac{1+x^3+x^4+x^5}{1-x^6} \tag{16}$$

- *Fourth example.* Permutation 174528603 yields $\lambda \simeq 1.97458\cdots$ and $\mu \simeq 1.32471\cdots$ and:

$$\frac{\beta(x)}{3x} = \frac{1-x^2-x^3}{(1-x)(1-x-2x^2-x^3+x^4+2x^5+x^6)}, \quad \rho(x) = \frac{1-x+x^7+x^8}{(1-x+x^2)(1-x^2-x^3)} \tag{17}$$

The exhaustive analysis of the factorization schemes, and the associated degree, and exponent, generating functions ($\alpha(x), \beta(x), \eta(x), \phi(x)$ and $\rho(x)$), confirms that the G-complexities are actually independent of the representative in the equivalence class. On the contrary, the factorization schemes and the associated degree, and exponent, generating functions may depend [7] on the chosen representative in the equivalence class. In other words, to two permutations in the same class of equivalence correspond the same (up to $1-x$ or $1+x$, or Nth root of unity factors) denominators for the degree generating functions $\alpha(x)$, $\beta(x)$. By contrast the numerators, as well as the exponents generating functions are *representative dependent* (see the previous four examples). Most of time the *stability of the factorization scheme* and thus, in a second step, the occurrence of *rational* generating functions, corresponds to a simple periodicity of the exponents η_n, ϕ_n or ρ_n in the factorization scheme (3) and (4). This periodicity is simply associated to the fact that the exponent generating functions have Nth root of unity poles: $1-x^2$, $1-x^8$, $1-x^6, \cdots$ (see $\rho(x)$ in Eq. (16)). However, one sees, in examples (15) and (17), that one may have a *stability* of the factorization scheme with an *exponential growth* of these exponents η_n and ϕ_n. These exponent generating functions, of course, have a 'scheme complexity' μ smaller that the G-complexity λ. This 'scheme complexity' μ is the inverse of the poles of $\rho(x)$, $\phi(x)$ or $\eta(x)$, that is (for Eq. (17)), $\mu \simeq 1.32471\cdots \leq \lambda \simeq 1.83928\cdots$. Recalling Eq. (16), for which $\mu = 1$ and $\lambda \simeq 1.8392\cdots$, and Eq. (15), one sees that the *same* G-complexity λ can be associated to *several* 'scheme-complexity' μ. Conversely, comparing the fourth example (17) and the second example (15), one sees that one 'scheme-complexity' μ can actually yield *several* G-complexities λ.

2.4. To sum up

All these factorization scheme calculations confirm the results of the semi-numerical method and are summarized in Table 1. Most of the 362880 birational transformations considered here do correspond to the most 'chaotic G-complexity', namely the upper bound $\lambda = 2$: one has 359568 such $\lambda = 2$ birational transformations, that is 99.0873% of all the birational transformations. It is known [19], that some symmetry-classes correspond to situations where the determinantal variables x_n's, defined by (9), are periodic (denoted by 'Period.' in Table 1). This $x_n = x_{n+N}$ situation may correspond to situations where mapping $\hat{K}$, itself, is of finite order (trivial integrability), but also to polynomial growth situations, that is, $\lambda = 1$ *exactly*. One remarks that $\mathcal{R}_{72}^{(\infty)}$ contains *all the integrable*, or *polynomial growth*, mappings and, up to one class in $\mathcal{R}_{72}^{(1)}$, all the mappings such that $x_n = x_{n+N}$, including the situations where mapping $\hat{K}$, itself, is of finite order.

The actual value of the upper bound $\lambda = 2$ comes from the fact that the homogeneous transformation K is a *quadratic* transformation of the nine homogeneous entries of the initial 3×3 matrix.

Let us remark that all these birational (bipolynomial for K) transformations have *other* (birational) representations than the 'straight' representation as a mapping bearing on eight (nine homogeneous) variables inherited from Eqs. (1) and (2). For instance, for the mappings corresponding to the G-complexity $\lambda \simeq 1.46557123\cdots$, it can be shown that a point in the plane spanned by M_0, $K^2(M_0)$ and $K^4(M_0)$ is transformed, by K^2, into another point of the *same* plane:

$$K^2(M) = b_0 M_0 + b_1 K^2(M_0) + b_2 K^4(M_0) \qquad \text{where} \qquad M = a_0 M_0 + a_1 K^2(M_0) + a_2 K^4(M_0)$$

and, thus, K^2 can be represented as a birational transformation of only *two* variables (see Appendix A in [22]). Similar relations occur for all the birational transformations associated with 3×3 matrices and classified in Table 1, the previous (projective) plane being replaced by a d-dimensional space spanned by M_0, $K^2(M_0), \cdots, K^{2d}(M_0)$, where $d \leq 8$ for 3×3 matrices [8]. The previous semi-numerical calculations can be applied to these other representations

[7] Considering one $\mathcal{R}^{(\infty)}$ equivalence class, one does not get as many factorization schemes as the number of elements in the equivalence class. It seems, inspecting directly all the 9! factorization schemes (but only up to 12 iteration steps), that, most of the time, one gets, at most, two possible factorization schemes for a given $\mathcal{R}^{(\infty)}$ equivalence class, and that the set of all the possible factorization schemes would be 21 (besides the polynomial growth situations which can be quite 'rich').

[8] Examples with $d = 3$ or $d = 6$ are given in [19] for 4×4 matrices.

as birational transformations on d variables and yield the same values for the G-complexities (see for instance [22] for $\lambda \simeq 1.46557123\cdots$).

3. Various generalizations

We now show that all these results also apply for a much larger set of rational transformations. The number of permutations of entries of 3×3 matrices being finite it has been possible to perform an exhaustive analysis. For more general transformations, depending on continuous parameters, it is not anymore possible and we will proceed just with chosen examples. These examples always combine homogeneous transformations of the entries of a matrix together with the matrix inversion. Therefore, the transmutation relations, detailed in Appendix A, still apply, yielding again non-trivial symmetries for these new set of transformations.

3.1. Combining different K's

Let us first consider permutation 146237058, and its associated $\lambda \simeq 1.97481\cdots$ transformation K_1, and permutation 471562380 and its $\lambda \simeq 1.54258\cdots$ transformation K_2. Let us compose the two previous transformations. From these two 'atoms' we build the 'molecule' $\mathcal{K} = K_2 \cdot K_1$. Note that $\mathcal{K} = K_1 \cdot K_2$, obviously has the same G-complexity.

This example is an interesting one since the G-complexity (obtained from the previous semi-numerical calculations) of the 'molecule' $\mathcal{K} = K_2 \cdot K_1$ is smaller than the product of the two G-complexities of K_1 and K_2: $\lambda(\mathcal{K}) \simeq 2.897 < 1.9748 \times 1.5426 \simeq 3.0463$. In general the combination of two G-complexities λ_1 and λ_2 gives a G-complexity for the 'molecules' larger than the product $\lambda_1 \cdot \lambda_2$, often equal to the upper bound (here $\lambda_{\text{upper}} = 4$).

The factorization scheme of $\mathcal{K}$ is of the same type as the one described in [22], namely a '*parity-dependent*' factorization scheme. It is detailed in Appendix B and yields a degree generating function $\beta(x)$:

$$\frac{\beta(x)}{3x} = \frac{1 + 2x - x^2 - x^4 + x^6}{1 - 3x^2 + x^4 - x^6 - 2x^8} \tag{18}$$

The G-complexity of the molecule $\mathcal{K}$ does not identify with the G-complexity of K_1, or the one of K_2. It is a true *new* algebraic number. This algebraic expression for the G-complexity of the molecule is in good agreement with the semi-numerical value obtained above. We have systematically studied such 'molecules' for a choice of 18 representatives of the 18 G-complexities of Table 1 combined with themselves, and beyond, with *other* representatives. If one barters the permutation t_2 for another representative $t_2^{(2)}$ in the same $\mathcal{R}^{(\infty)}$ class, transformation K_2 being modified accordingly ($K_2 \to K_2^{(2)}$), the new 'molecule' $\mathcal{K}^{(2)} = K_2^{(2)} \cdot K_1$ yields, in general, *another* algebraic value for the G-complexity λ: the equivalence relation $\mathcal{R}^{(\infty)}$ is no longer compatible with the 'molecular structure'.

For all these 'molecules' the *parity-dependent* factorization scheme, yields algebraic numbers for the G-complexities of these molecules in agreement with the values obtained from our semi-numerical now applied for the 'molecules'. Combining among themselves all the permutations yields a large number of different algebraic G-complexities, much larger than the number of G-complexities obtained combining only representatives of the $\mathcal{R}^{(\infty)}$ classes among themselves.

3.2. From permutations to linear transformations

We got algebraic results on birational mappings associated with permutation of the entries. We now address the following question: are these structures (existence of a stable factorization scheme) dependent of the fact that we are dealing with permutations? In other terms, does one loose these *algebraic* properties when deforming the permutations in most general transformations? The most simple, and natural, generalization amounts to replacing the permutation of the entries by *linear* combination on the entries.

Let us now consider a first example, namely the *quite general linear transformation* depending on 21 parameters:

$$L:\ \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{bmatrix} \to \begin{bmatrix} m_{1,1} & a_{11}m_{1,1}+a_{12}m_{1,2}+a_{13}m_{1,3}+a_{21}m_{2,1}+a_{22}m_{2,2}+a_{23}m_{2,3}+a_{31}m_{3,1}+a_{32}m_{3,2}+a_{33}m_{3,3} & m_{1,3} \\ m_{2,1} & c_{21}m_{2,1}+c_{22}m_{2,2}+c_{23}m_{2,3} & m_{2,3} \\ m_{3,1} & b_{11}m_{1,1}+b_{12}m_{1,2}+b_{13}m_{1,3}+b_{21}m_{2,1}+b_{22}m_{2,2}+b_{23}m_{2,3}+b_{31}m_{3,1}+b_{32}m_{3,2}+b_{33}m_{3,3} & m_{3,3} \end{bmatrix} \tag{19}$$

This particular form singles out the rows of the 3×3 matrix (and, thus, can be understood as an RCT-compatible form, see Appendix A). Similarly to the previous paragraphs let us introduce the homogeneous transformation $K = L\cdot I$. Factorizations again occur at each iteration step. These factorizations correspond to a *stable factorization scheme* giving a growth like λ^N, where $\lambda \simeq 1.61803\cdots$. It is of the general type described in Eqs. (3) and (4). This yields the following generating functions:

$$\frac{\beta(x)}{3x} = \frac{1}{1-x-x^2}, \qquad \rho(x) = \frac{1}{1-x} \tag{20}$$

These results are actually valid for any ‘sufficiently generic’ choice of the 21 parameters. One thus has a first ‘universality’ property: the G-complexity λ is ‘generically’ not dependent of the previous 21 parameters. Furthermore, relation (10) (and consequently relation (13) *remains also valid* for all the factorization schemes associated with all the linear transformations studied in this section. The G-complexity $\lambda \simeq 1.61803\cdots$ (corresponding to polynomial $1-x-x^2$) is a G-complexity value already found in Table 1, in the 16th row. It is noteworthy that no choice of the 21 parameters leads to a permutation of any of the three classes corresponding to $\lambda \simeq 1.61803\cdots$. Besides the identity, the only choice of parameters, leading to a permutation, is $b_{12} = a_{32} = 1$, all others being zero. The permutation is then the transposition $M_{1,2} \leftrightarrow M_{3,2}$ which corresponds to the G-complexity $\lambda \simeq 1.46557\cdots$. This transposition (denoted by class IV in [20]) is not isolated in the 21 parameters set of transformations. Actually, if the parameters verify the conditions $b_{12}a_{32} - a_{12}b_{32} = c_{22} = 1$ and $a_{12}+b_{32} = 0$, all the other parameters being zero, one then gets additional factorizations, the modified factorization scheme yielding:

$$\rho(x) = \frac{1}{1-x}, \qquad \frac{\beta(x)}{3x} = \frac{1+x^2}{1-x-x^3} \tag{21}$$

and again $\lambda \simeq 1.46557\cdots$. There is also *polynomial growth* subcases, for instance $a_{12} = c_{22} = b_{32} = 1$, a_{11} and a_{22} arbitrary non-zero, all the other ones being zero. Generally speaking, having a G-complexity generically independent of r parameters (here twenty one), one can only expect *more* factorizations on some subvariety of the r-dimensional space, and consequently a *smaller* G-complexity value λ on this very subvariety.

We now give another 11 parameter example associated with the following linear transformation:

$$L:\ \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{bmatrix} \to \begin{bmatrix} m_{1,1} & m_{1,2} + b_{21}m_{2,1} + b_{22}m_{2,2} + b_{23}m_{2,3} & m_{1,3} \\ m_{2,1} & a_{12}m_{1,2} + a_{21}m_{2,1} + a_{22}m_{2,2} + a_{23}m_{2,3} + a_{32}m_{3,2} & m_{2,3} \\ m_{3,1} & m_{3,2} + c_{21}m_{2,1} + c_{22}m_{2,2} + c_{23}m_{2,3} & m_{3,3} \end{bmatrix} \tag{22}$$

For $K = L \cdot I$ the corresponding generating functions are:

$$\rho(x) = \frac{1}{1-x}, \qquad \frac{\beta(x)}{3x} = \frac{1-x^3}{1-x-x^2-x^3+x^4} \tag{23}$$

The numerator of $\beta(x)$ does not appear in Table 1: this mapping has a *new value* for the G-complexity $\lambda \simeq 1.72088\cdots$, *not previously obtained* for any of the 9! permutations.

Family (22), depending on 11 continuous parameters, also enables to address the following problem: is the growth complexity crucially dependent on the *reversible character* [24] of the transformations? In fact one may lose the birational character of K when, for instance, the linear transformation L becomes singular. This is very easy to realize for some condition on the 11 parameters (codimension one subvariety). For instance, taking $b_{22} = 2$, $a_{22} = 87$, $a_{12} = 5$, $a_{32} = 7$, $c_{22} = 11$, all the other parameters being zero, leads to a *non-invertible* mapping $K = L \cdot I$. One easily verifies that the factorization scheme, the associated generating functions and, thus, the G-complexity λ, are *unchanged* in this case and, more generally, on such singular subvarieties. With this first rational, non invertible, example one sees that the rational character of the generating functions is not a consequence of a 'simple' invertibility of the mapping (see also [2]).

3.3. From linear transformations to homogeneous polynomial transformations

There is nothing specific with linear transformations. For instance, let us consider the following quadratic transformation depending on 21 parameters (which is reminiscent of Eq. (19)):

$$Q: \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{bmatrix} \to$$

$$\begin{bmatrix} m_{1,1}^2 & a_{11}m_{1,1}^2+a_{12}m_{1,2}^2+a_{13}m_{1,3}^2+a_{21}m_{2,1}^2+a_{22}m_{2,2}^2+a_{23}m_{2,3}^2+a_{31}m_{3,1}^2+a_{32}m_{3,2}^2+a_{33}m_{3,3}^2 & m_{1,3}^2 \\ m_{2,1}^2 & c_{21}m_{2,1}^2+c_{22}m_{2,2}^2+c_{23}m_{2,3}^2 & m_{2,3}^2 \\ m_{3,1}^2 & b_{11}m_{1,1}^2+b_{12}m_{1,2}^2+b_{13}m_{1,3}^2+b_{21}m_{2,1}^2+b_{22}m_{2,2}^2+b_{23}m_{2,3}^2+b_{31}m_{3,1}^2+b_{32}m_{3,2}^2+b_{33}m_{3,3}^2 & m_{3,3}^2 \end{bmatrix} \tag{24}$$

The homogeneous transformation $K = Q \cdot I$ gives again *a stable factorization scheme*. In this case, where Q is no longer a linear transformation, but a homogeneous polynomial transformation of degree r (here $r = 2$), the factorization scheme remains of the general forms (3) and (4). As far as generating functions are concerned some modifications have to be done. Firstly, the ρ_n's, and associated $\rho(x)$, should be replaced by the γ_n's defined by:

$$\hat{K}(M_n) = \frac{K(M_n)}{\det(M_n)^r} = \frac{M_{n+1}}{f_{n+1}^{\gamma_0} \cdot f_n^{\gamma_1} \cdot f_{n-1}^{\gamma_2} \cdots} \tag{25}$$

and the corresponding generating function $\gamma(x)$. The linear relations between $\eta(x)$, $\phi(x)$ and $\gamma(x)$ are slightly modified (see Eqs. (C.1) and (C.4) in Appendix C). Secondly, relation (10) is no longer valid here. A new relation has to be introduced playing the same role. Transformation $\hat{K} = Q \cdot \hat{I}$ is a homogeneous transformation of degree $-r$. Instead of introducing the determinantal variables x_n through Eq. (9), let us introduce $\tilde{x}_n$ by:

$$\tilde{x}_n(M_0) = \det(\hat{K}^{n+1}(M_0)) \left(\det(\hat{K}^n(M_0))\right)^r \tag{26}$$

These new determinantal variables $\tilde{x}_n$ are well-suited ones since they are *invariant under a rescaling of* M_0: $\tilde{x}_n(Cst \cdot M_0) = \tilde{x}_n(M_0)$. Relation (10) becomes:

$$\tilde{x}_n(M_0) = f_{n+1}^{W_0} f_n^{W_1} f_{n-1}^{W_2} f_{n-2}^{W_3} \cdots f_0^{W_{n+1}} \tag{27}$$

Again, one can introduce the generating function of these exponents W_n and see that relation (13) still holds. From the stable factorization scheme of $K = Q \cdot I$ one now gets:

$$\frac{\beta(x)}{3x} = \frac{1}{1-3x-2x^2}, \qquad \gamma(x) = 2\frac{1-x}{1-2x} \tag{28}$$

This gives a G-complexity value $\lambda \simeq 3.5615\cdots$. Let us consider the expression of $\alpha(x)$:

$$\alpha(x) = \frac{3(1+x-2x^2+4x^3)}{(1+2x)(1-2x)(1-3x-2x^2)} \tag{29}$$

In this expression one sees that other poles occur. The inverse of these additional poles, namely ± 2, are actually smaller that the complexity value $3.56155\cdots$. The existence of ‘subdominant’ poles already occurred with permutations of entries, or linear transformations (see $\beta(x)$ in Eq. (15)): we often had $1-x$, or $1+x$, additional factors in the expressions of the degree generating functions. With expression (29), one sees the occurrence of a $1-2x$ factor instead of $1-x$ factors.

There is also nothing specific with quadratic transformations. Let us introduce the simple homogeneous polynomial of degree r:

$$Q_r: \quad \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{bmatrix} \to \begin{bmatrix} m_{1,1}^r & m_{1,2}^r & m_{1,3}^r \\ m_{3,2}^r & m_{2,2}^r & m_{3,1}^r \\ m_{2,3}^r & m_{2,1}^r & m_{3,3}^r \end{bmatrix} \tag{30}$$

and its associated homogeneous transformation $K = Q_r \cdot I$. Its factorization scheme is very simple, it reads for $r \geq 2$ (for $r = 1$ transformation (30), and $K = Q_r \cdot I$, become trivial):

$$M_n = \frac{K(M_{n-1})}{f_{n-1}^r}, \qquad \det(M_n) = f_{n+1} f_n^2 \tag{31}$$

which yields the following linear relations on the α_n’s and β_n’s (see also Appendix C):

$$\alpha_n = 2r\alpha_{n-1} - 3r\beta_{n-1}, \qquad \alpha_n = \beta_{n+1} + 2\beta_n \tag{32}$$

It gives the following generating functions for arbitrary $r \geq 2$:

$$\frac{\beta(x)}{3x} = \frac{1}{1+2(1-r)x-rx^2}, \qquad \eta(x) = r, \qquad \phi(x) = 1+2x, \qquad \gamma(x) = r\cdot(1+x) \tag{33}$$

For homogeneous polynomials of degree r one can show that subdominant poles, like $1-rx$, may occur instead of the previous $1-x$ and $1-2x$ factors.

For $r = 2$, one remarks that one gets a degree generating function:

$$\frac{\beta(x)}{3x} = \frac{1}{1-2x-2x^2} \tag{34}$$

which is not the limit of Eq. (28). The generic G-complexity corresponding to Eq. (28), namely $\lambda \simeq 3.56155\cdots$ is changed, for Eq. (30) taken for $r = 2$, into $\lambda \simeq 2.73205\cdots$. There actually exist many subvarieties of the 21 parameter space of transformation (24) on which the generic G-complexity $\lambda \simeq 3.56155\cdots$ is modified into another (smaller) algebraic value. One remarks that the subvarieties of the 21 parameter space of transformation (19) (for instance, $a_{12} = c_{22} = b_{32} = 1$, a_{11} and a_{22} arbitrary non-zero, all the other ones being zero, previously mentioned as a polynomial growth subcase) also yield ‘non-generic’ G-complexities for Eq. (24).

4. Conclusion

In previous papers [22,28], it has been shown that the (exponential of the) topological entropy, and the growth complexity λ (related to the asymptotic of the Arnold complexity for two-dimensional mappings), actually identify on various simple two-dimensional birational examples, and that these quantities are actually algebraic numbers. The generating functions corresponding to these two 'complexity measures', namely the *dynamical zeta function* [25–28] and the various 'degree' generating functions (like $\beta(x)$) were shown to be simple *rational expressions* with integer coefficients [22], the dominant poles in these two sets of generating functions being the same. When one analyzes birational transformations depending on more than two variables, it becomes very difficult to calculate even the first coefficients of the expansion of the dynamical zeta function. On the contrary, the calculations on the degree generating functions can be quite easily performed, even for birational transformations of many variables (the q^2 entries of a matrix [2]).

Analyzing exhaustively a first finite set of 362880 birational transformations (associated with all the permutations of 3×3 matrices), we have obtained non-trivial, but still simple, 'spectrum' of 18 algebraic G-complexities for the corresponding dynamical systems. In a second step it has been shown that these results can be drastically generalized along three different lines *preserving the algebraic character of the G-complexities*. Firstly, one can combine these birational transformations together, and gets extremely rich sets of *algebraic* G-complexities. Secondly, one can consider (generically birational) transformations, associated with linear transformations of the entries of 3×3 matrices, and still gets sets of algebraic G-complexities. Remarkably, one has another *universality* property here: these algebraic G-complexities do not depend on many of the continuous parameters associated with the linear transformations. Thirdly, one still gets sets of algebraic G-complexities with *rational* transformations (associated with homogeneous polynomial transformations on the entries) which again can depend on many continuous parameters. With this last generalization we have completely lost any invertible character of the transformations. On the top of that these 3×3 matrix calculations can be simply generalized to $q \times q$ matrices *for arbitrary* [9] q. Combining several of these rational transformations depending on several continuous parameters together, one certainly gets again rich sets of *algebraic* G-complexities.

Appendix A. A transmutation property of the matrix inversion

Let us sketch here some non-trivial symmetries between the permutations. The transformations, considered in Sections 1.2 and 2.3, are products of matrix inversion and permutations of the entries. Any such non-trivial symmetry of the birational transformations $\hat{K}$ should correspond to a non-trivial relation between matrix inversion and permutations of the entries of the matrix. Such relations actually exist. They correspond to a 'transmutation' property between the inversion and permutations P and Q. There actually exist two permutations P and Q such that:

$$P \cdot \hat{I} = \hat{I} \cdot Q \tag{A.1}$$

Permutations, such that a 'transmutation' relation (A.1) is satisfied, do exist: one can easily build examples by combining product of permutations that permutes *only rows* of a $q \times q$ matrix (that we will denote by 'R'), permutations that permutes *only columns* of a $q \times q$ matrix (that we will denote by 'C') and, possibly, the matrix transposition we denote by 'T'. Examples of permutations P and Q, such that (A.1) is satisfied, read:

$$P = R \cdot C \cdot T^{\epsilon} \qquad \text{where} \quad \epsilon = 0, \ \text{or} \ 1 \tag{A.2}$$

[9] The 'spectrum' of values of the G-complexity λ depends on q, see for instance [2].

and similarly for permutation Q. A permutation P having such a decomposition (A.2) will be called an 'RCT' permutation.

Let us consider two permutations t_1 and t_2, yielding the two birational transformations $\hat{K}_1 = t_1 \cdot \hat{I}$ and $\hat{K}_2 = t_2 \cdot \hat{I}$, respectively. Let us introduce the following relation of equivalence between two permutations t_1 and t_2: t_1 and t_2 will be related if they are such that there exists an 'RCT' permutation, b_0, such that:

$$\hat{K}_1^n = b_0 \cdot \hat{K}_2^n \cdot b_0^{-1} \tag{A.3}$$

Relation (A.3) can easily be seen to define a relation of equivalence between t_1 and t_2, we will denote $\mathcal{R}^{(n)}$:

$$t_1 \mathcal{R}^{(n)} t_2 \tag{A.4}$$

Note that this $\mathcal{R}^{(n)}$ equivalence relation is compatible with the inverse in the permutation group $t \quad \rightarrow \quad t^{-1}$. Also, note that the equivalence of two permutations, up to simple rows and columns relabeling, is an $\mathcal{R}^{(1)}$ equivalence, however, conversely, the $\mathcal{R}^{(1)}$ equivalence does not reduce to the simple, and quite trivial, equivalence of two permutations up to simple rows and columns relabeling. Obviously, rows and columns relabeling of the matrices do not modify their integrability properties [19], as well as the growth of the calculations.

It is obvious that if $t_1 \mathcal{R}^{(n)} t_2$ then $t_1 \mathcal{R}^{(n \times p)} t_2$ for any natural integer p. This is a consequence of the fact that:

$$\hat{K}_1^n = b_0 \cdot \hat{K}_2^n \cdot b_0^{-1} \quad \text{yields} \quad \hat{K}_1^{np} = b_0 \cdot \hat{K}_2^{np} \cdot b_0^{-1} \tag{A.5}$$

If two permutations, t_1 and t_2, are in the same equivalence class with respect to $\mathcal{R}^{(m)}$, and if t_2 and t_3 are in the same equivalence class with respect to $\mathcal{R}^{(n)}$ where $n \neq m$, t_1 and t_3 are in the same equivalence class with respect to $\mathcal{R}^{(n \times m)}$, or with respect to $\mathcal{R}^{(N)}$ for some 'large enough' integer N. In fact, it can be shown, on the example of the equivalence classification of the permutations of 3×3 matrices, that this value of N corresponding to the ('asymptotic' equivalence) relation is actually equal to $N = 24$.

If two permutations t_1 and t_2 are in the same equivalence class, with respect to $\mathcal{R}^{(m)}$, the G-complexities (which are real positive numbers), associated with their respective birational transformations $\hat{K}_1$ and $\hat{K}_2$, we denote by λ_1 and λ_2 are, as a straight consequence of Eq. (A.3), related by:

$$\lambda_1^m = \lambda_2^m \tag{A.6}$$

Therefore, one sees that their G-complexities are equal: $\lambda_1 = \lambda_2$. In particular, if one considers the (largest) equivalence classes corresponding, for 3×3 matrices, to $\mathcal{R}^{(24)}$, all the representatives in one of these $\mathcal{R}^{(24)}$ equivalence classes will have the *same growth complexity* λ.

Appendix B. A molecular factorization scheme

The factorization scheme of $\mathcal{K} = t_1 \cdot I \cdot t_2 \cdot I$, corresponding to permutation 146237058 and permutation 471562380 (see Section 3.1), is of the same type as the one described in [22], namely a *parity-dependent* factorization scheme (which is a straight consequence of the fact that one acts with K_1, and then with K_2, and again ...):

$$f_1 = \det(M_0) \quad M_1 = K_1(M_0), \quad f_2 = \det(M_1), \quad M_2 = K_2(M_1), \quad f_3 = \frac{\det(M_2)}{f_2}, \quad M_3 = K_1(M_2),$$
$$f_4 = \det(M_3), \quad M_4 = K_2(M_3), \quad f_5 = \frac{\det(M_4)}{f_2^3 f_4}, \quad M_5 = \frac{K_1(M_4)}{f_2}, \quad f_6 = \frac{\det(M_5)}{f_2^2 f_4}, \quad \cdots \tag{B.1}$$

and for arbitrary $n \geq 3$:

$$\begin{aligned} \det(M_n) &= f_{n+1}\, f_n\, f_{n-2}^3\, f_{n-6}\, f_{n-8}\, f_{n-10}\, f_{n-12}\, f_{n-14} \cdots \\ K_1(M_n) &= M_{n+1}\, f_{n-2} \end{aligned} \tag{B.2}$$

for n even and:

$$\begin{aligned} &\det(M_n) = f_{n+1}\, f_{n-1}\, f_{n-3}^2\, f_{n-5}\, f_{n-7}^2\, f_{n-9}^2\, f_{n-11}^2\, f_{n-13}^2 \cdots \\ &K_2(M_n) = M_{n+1}\, f_{n-3}\, f_{n-7}\, f_{n-9}\, f_{n-11}\, f_{n-13} \cdots \end{aligned} \tag{B.3}$$

for n odd. This yields the following expressions for the odd and even parts of $\alpha(x)$ and $\beta(x)$ ('2' for even and '1' for odd):

$$\begin{aligned} &\beta_2(x) = \frac{6x^2}{1-3x^2+x^4-x^6-2x^8}, \qquad \beta_1(x) = \frac{3x(1+x^2)\,(-1+x)^2\,(x+1)^2}{1-3x^2+x^4-x^6-2x^8} \\ &\alpha_2(x) = \frac{3(1+4x^4-4x^6+x^8)}{(1-x^2)(1-3x^2+x^4-x^6-2x^8)}, \qquad \alpha_1(x) = \frac{6x(1+x^4-x^6+x^8)}{(1-x^2)(1-3x^2+x^4-x^6-2x^8)} \end{aligned} \tag{B.4}$$

These generating functions yield a 'molecular G-complexity': $\lambda \simeq 2.8581\cdots$. These generating functions verify a parity dependent system of functional relations which generalizes the ones described in [2]:

$$\begin{aligned} &x\,\alpha_1(x) - \beta_2(x) = F_{2p}(x)\beta_2(x), \qquad x\,\alpha_2(x) - \beta_1(x) = F_{1m}(x)\beta_2(x) \\ &\alpha_2(x) - 3 - 2x\alpha_1(x) + 3G_{2p}\beta_2(x) = 0, \qquad \alpha_1(x) - 2x\alpha_2(x) + 3G_{1m}\beta_2(x) = 0 \end{aligned} \tag{B.5}$$

where:

$$F_{2p}(x) = x^2 + 2x^4 + x^6 + \frac{2x^8}{1-x^2}, \quad F_{1m} = 2x^3 - x^5 + \frac{x}{1-x^2}, \quad G_{1m}(x) = x^3, \quad G_{2p} = x^4 + \frac{x^8}{1-x^2}$$

Appendix C. Exponent generating functions for homogeneous polynomial transformations of degree r

Let us consider a homogeneous transformation Q_r of degree r (like Eq. (30), or like Eq. (24) for $r = 2$) and its associated homogeneous transformation $K = Q_r \cdot I$. Relations (3) and (4) are still valid but yield a slight modification of the linear functional relations (6) and (7), namely:

$$((q-1)rx - 1)\cdot\alpha(x) + q - qx\eta(x)\beta(x) = 0 \tag{C.1}$$

$$x\alpha(x) = \phi(x)\beta(x) \tag{C.2}$$

Let us recall that, for homogeneous transformations of degree r, one must introduce, instead of $\rho(x)$, the generating function $\gamma(x)$ (see Section 3.3) defined by:

$$\hat{K}(M_n) = \frac{K(M_n)}{\det(\mathrm{M_n})^{\mathrm{r}}} = \frac{M_{n+1}}{f_{n+1}^{\gamma_0}\, f_n^{\gamma_1}\, f_{n-1}^{\gamma_2}\cdots} \tag{C.3}$$

This last relation yields a new relation:

$$q + q\gamma(x)\beta(x) = (1+rx)\alpha(x) \tag{C.4}$$

which has to be compatible with the previous two Eqs. (C.1) and (C.2):

$$r\phi(x) = \gamma(x) + x\eta(x) \tag{C.5}$$

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