J. Phys. A: Math. Gen. 16 (1983) 2257–2265. Printed in Great Britain

Study of a two-dimensional fully frustrated model

T Garel† and J M Maillard‡

† Laboratoire de Physique des Solides§, Bâtiment 510, Université Paris-Sud, 91405 Orsay, France
‡ DPh/SRM, CEN Saclay, BP 2, 91191 Gif-sur-Yvette, France

Received 2 December 1982

Abstract. We consider a fully frustrated Ising model on a square lattice, depending on four parameters. The partition function is shown to be equivalent to the two-parameter (ferromagnetic) Onsager partition function. This result generalises a relation established by Southern *et al*, and can be checked in the particular case of the (one-parameter) Villain model. In particular, the residual entropy of the Villain model is linked to the free energy of the Onsager model at criticality. The reduction from four to two parameters, occurring in this mapping, is studied in the ligh' of the inverse relation satisfied by the partition function of the frustrated model.

1. Introduction

It is generally believed that frustration and disorder are responsible for the unusual properties of spin glasses (Fischer 1982, Toulouse 1982). In order to disentangle these two effects, periodic frustrated models have been studied. Various questions arise such as:

(i) the existence of a phase transition and the nature of the low-temperature phase (Wannier 1950, Villain 1977, Derrida *et al* 1978, André *et al* 1979, Williams 1982);

(ii) the behaviour of correlation functions (Stephenson 1970, Gabay 1980, Forgacs 1981);

(iii) the possible influence of the frustration network periodicity (Bryskin *et al* 1980, Longa and Olès 1980).

In this work, we will follow a path suggested first by Jüngling (1975) and then by Southern *et al* (1980), who were able to map the isotropic fully frustrated Ising model on a square lattice (Villain model) onto a symmetric eight-vertex model (Baxter 1972). Here, we generalise this result to a fully frustrated model depending on four coupling constants. The partition function of this enlarged model, which can be mapped onto a free-fermion case of the Baxter model, is shown to be equivalent to the partition function of an anisotropic ferromagnetic model (Onsager 1944). At this particular level, we are, therefore, able to 'defrustrate' the model‖. One should note, however, that the temperature interval $(0, \infty)$ of the fully frustrated case corresponds to the interval (T_c, ∞) of the ferromagnetic case. From a physical point of view, it is interesting to see the precise correspondence between the low-temperature properties of the fully

§ Laboratoire associé au CNRS.
‖ This mapping does not *a priori* hold for more sophisticated objects, such as correlation functions (Gabay 1980).

frustrated model (ground-state entropy, etc) and some (finite) critical quantities of the Onsager model. From a more mathematical point of view (Jaekel and Maillard 1982a, b), the problem is to know whether the partition function of a given model can be completely determined by

(i) a functional equation, called the inverse relation (Stroganov 1979);

(ii) the geometrical symmetries of the model.

This question can be studied in the light of the reduction of the number of parameters in the 'defrustration' mapping (two in the Onsager case, four in the frustrated model). Such a reduction is certainly a non-trivial property; a similar phenomenon occurs in other contexts (Baxter and Enting 1978). We will see that the four-parameter frustrated model satisfies an inverse relation; this fact will be used to try to find the partition function.

The paper is organised as follows: the model is defined in § 2, where we show its relation to a constrained Baxter model and to the Onsager ferromagnetic solution. Section 3 deals with some particular examples (Villain model). The inverse relation using a diagrammatic expansion is presented in § 4. Finally, some comments are given on the link between frustrated and non-frustrated models.

2. The model and its relation to ferromagnetic models

2.1. Mapping onto a Baxter model

The model is defined in figure 1, and its reduced Hamiltonian reads

$$-\mathscr{H} = -H/k_{\mathrm{B}}T = \sum_{\langle ij\rangle} \{K\}_{ij}\mu_i\mu_j. \tag{1}$$

In equation (1), $\langle ij\rangle$ denotes nearest-neighbour pairing, $\mu_i = \pm 1$, and the reduced coupling constants $\{K\}_{ij}$ may take the values K_1, $\pm K_2$, K_3, $\pm K_4$ (see figure 1). The isotropic case ($K_1 = K_2 = K_3 = K_4$) corresponds to Villain's model and has been considered by Southern *et al* (1980). Related problems with two or three coupling constants have been considered by André *et al* (1979) and Villain *et al* (1980). If N is the number of sites, the partition function per site is given in the thermodynamic limit by

$$Z^N(K_1, K_2, K_3, K_4) = \operatorname*{Tr}_{\{\mu_i = \pm 1\}} \mathrm{e}^{-\mathscr{H}}. \tag{2}$$

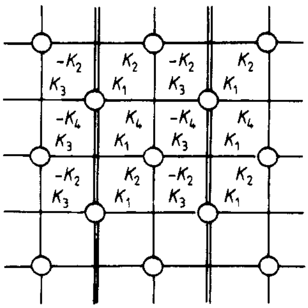


Figure 1. The four-parameter fully frustrated model. Double lines denote negative coupling constants ($-K_2$ or $-K_4$). Circled spins are to be integrated out.

In order to calculate (2), it is convenient to integrate out the circled spins of figure 1. Two non-equivalent kinds of site occur (figure 2). Let us denote their spins by σ (figure 2(*a*)) and τ (figure 2(*b*)). It is shown in the appendix that the partial trace over spins σ generates nearest-neighbour interactions $\{K_{12}, K_{23}, K_{34}, K_{41}\}$, next-nearest-neighbour interactions $\{K_{13}, K_{24}\}$ and a four-spin interaction $\{K_{1234}\}$. The same operation on spins τ yields the same types of interaction, except that one gets $\{-K_{12}, -K_{23}, -K_{34}, -K_{41}\}$ as nearest-neighbour couplings. As a consequence, the partition function does not depend on these terms. We therefore obtain an Ising representation $\{K_{13}, K_{24}, K_{1234}\}$ of the Baxter model (Kadanoff and Wegner 1971).

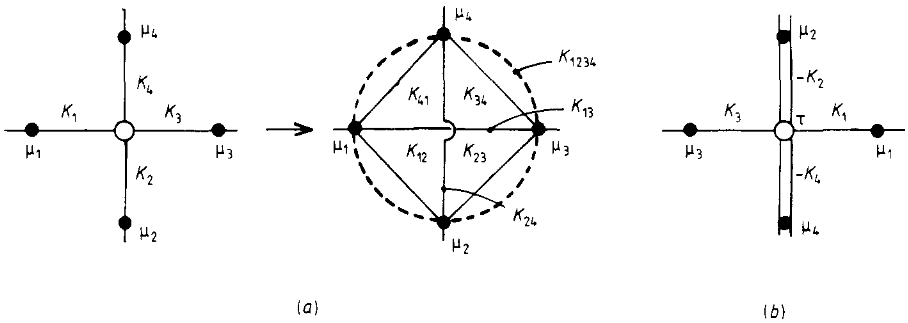


Figure 2. Two kinds of site: (*a*) sum over spin σ, surrounded by four ferromagnetic bonds and (*b*) sum over spin τ, surrounded by two ferro- and two antiferromagnetic bonds. The results are the same as (*a*) except that $(K_{12}, K_{23}, K_{34}, K_{41})$ change sign.

If one sets $\exp K_{13} = A$, $\exp K_{24} = B$, $\exp K_{1234} = C$, the vertex parameters (a, b, c, d) of the Baxter model are then defined as

$$a = \lambda ABC \tag{3a}$$

$$b = \lambda C/AB \tag{3b}$$

$$c = \lambda A/BC \tag{3c}$$

$$d = \lambda B/AC \tag{3d}$$

where λ is a multiplicative constant which does not enter critical properties (Kadanoff and Wegner 1971). Using the appendix, one can express (a, b, c, d) as functions of the original parameters (K_1, K_2, K_3, K_4):

$$a^2 = \cosh(K_1+K_2+K_3+K_4)\cosh(K_1-K_2+K_3-K_4) \tag{4a}$$

$$b^2 = \cosh(K_1+K_2-K_3-K_4)\cosh(K_1-K_2-K_3+K_4) \tag{4b}$$

$$c^2 = \cosh(K_1-K_2+K_3+K_4)\cosh(K_1+K_2+K_3-K_4) \tag{4c}$$

$$d^2 = \cosh(-K_1+K_2+K_3+K_4)\cosh(K_1+K_2-K_3+K_4). \tag{4d}$$

Equations (4) in turn show that this Baxter model obeys the free-fermion condition (Fan and Wu 1970)

$$a^2+b^2=c^2+d^2. \tag{5}$$

2.2. Mapping onto a ferromagnetic Ising model

If $Z(a, b, c, d)$ is the partition function of the Baxter model, one can show that (Baxter 1978):

(i) there is a duality operation D (see equation (3.7) of Baxter (1978))

$$(a, b, c, d) \xrightarrow{D} (a^*, b^*, c^*, d^*)$$

such that $Z(a, b, c, d) = Z(a^*, b^*, c^*, d^*)$ and

(ii) there exists an involution I

$$(a, b, c, d) \xrightarrow{I} (a, b, c, -d)$$

such that $Z(a, b, c, d) = Z(a, b, c, -d)$.

Combining both operations, it is easy to show that

$$(a, b, c, d) \xrightarrow{D_0 I_0 D} (a', b', c', d') \tag{6}$$

implies $Z(a, b, c, d) = Z(a', b', c', d)$. Equations (6) explicitly read

$$2a' = a - b + c + d \tag{7a}$$

$$2b' = -a + b + c + d \tag{7b}$$

$$2c' = a + b + c - d \tag{7c}$$

$$2d' = a + b - c + d. \tag{7d}$$

The free-fermion condition (equation (5)) now reads

$$a'b' = c'd' \tag{8}$$

which shows that this Baxter model is a ferromagnetic Ising model (Fan and Wu (1970). Following the Kadanoff–Wegner (1971) correspondence, one finds that $(a'b'c', d')$ can be mapped onto a set $(\lambda', A', B', C' = 1)$ with

$$a' = \lambda' A' B' \tag{9a}$$

$$b' = \lambda' / A' B' \tag{9b}$$

$$c' = \lambda' A' / B' \tag{9c}$$

$$d' = \lambda' B' / A'. \tag{9d}$$

Setting $A' = \exp(K'_{13})$ and $B' = \exp(K'_{24})$, we finally get the (anisotropic) coupling constants of the Ising case:

$$\exp(2K'_{13}) = \frac{a'}{d'} = \frac{a - b + c + d}{a + b - c + d} \tag{10a}$$

$$\exp(2K'_{24}) = \frac{a'}{c'} = \frac{a - b + c + d}{a + b + c - d}. \tag{10b}$$

Equations (10) can be checked easily in one-dimensional limiting cases (one of the four K_i going to zero or infinity).

Let us summarise this section. The original frustrated model described by (K_1, K_2, K_3, K_4) has been mapped onto a Baxter model (a, b, c, d) constrained by the free-fermion condition. This model can in turn be mapped onto an anisotropic Ising model (K'_{13}, K'_{24}). The relation between these models is contained in equations (4) and (10).

2.3. *Some properties of the model*

2.3.1. Isotropy. In order to get an isotropic Ising model, one must have

$$K'_{13} = K'_{24} \qquad (\text{or } c' = d'),$$

that is

$$c = d.$$

Using equations (4c) and (4d), the isotropy condition can be rewritten as

$$\sinh 2K_1 \sinh 2K_3 = \sinh 2K_2 \sinh 2K_4. \tag{11}$$

This condition is obviously satisfied in the Villain model ($K_i = K$).

2.3.2. Correspondence between temperatures. It is easy to check that the relation between temperatures in the frustrated and ferromagnetic cases is monotonic, at least for the Villain model ($K_i = K$) and the model ($K_1 = K_3$, $K_2 = K_4$). In the general case, we just point out that T_c of the Onsager model corresponds to $T = 0$ of the frustrated model. Starting from the critical condition (McCoy and Wu 1973)

$$\sinh 2K'_{13} \sinh 2K'_{24} = 1 \tag{12}$$

we have, since $a'b' = c'd'$,

$$\sinh 2K'_{13} = \tfrac{1}{2}\left(\frac{a'}{d'} - \frac{b'}{c'}\right) \qquad \sinh 2K'_{24} = \tfrac{1}{2}\left(\frac{a'}{c'} - \frac{b'}{d'}\right).$$

Using equations (7), one sees that (12) implies

$$ab/cd = \infty, \tag{13}$$

that is, $T = 0$ in the frustrated model (equations (4)).

3. Application to the Villain model

Let us consider the Villain (1977) model where $K_i = K$, as an example. Some of the results have been obtained by Southern *et al* (1980). Equations (4a)–(4d) read

$$a^2 = \cosh 4K \qquad b^2 = 1 \qquad c^2 = d^2 = \cosh^2 2K.$$

The correspondence with the (isotropic) Ising model, $K'_{13} = K'_{24} = K'$, is given by (10):

$$\exp(2K') = \frac{2\cosh 2K - 1 + (\cosh 4K)^{1/2}}{1 + (\cosh 4K)^{1/2}}. \tag{14}$$

One easily checks that ($K \to \infty$) implies ($\exp(2K') = 1 + \sqrt{2}$) and, more generally, that equation (13) and the Onsager module (McCoy and Wu 1973)

$$k_0 = 2 \sinh 2K'/\cosh^2 2K' \tag{15}$$

yield the correct Villain (1977) module

$$k_0 = k_V = \tanh^2 2K. \tag{16}$$

In short, we have

$$\sinh 2K' = \frac{1}{2}\left(\frac{a-b+2c}{a+b} - \frac{2c+b-a}{a+b}\right) = \frac{a-b}{a+b},$$

that is

$$k_0 = \frac{a^2-b^2}{a^2+b^2}$$

with equation (16) as a result. This shows that the angular integrals (McCoy and Wu 1973) of the two partition functions are the same for any temperature. This allows us to link the residual entropy S_V of the Villain model with the Onsager free energy at criticality F_c:

$$S_V = \tfrac{1}{2}(-\beta_c F_c - \tfrac{1}{2}\ln 2) = G/\pi$$

where G is Catalan's constant (Fisher 1961, Kasteleyn 1963) and $\beta_c = T_c^{-1}$.

4. Study of the inverse functional relation

The notion of an inverse relation, leading to a functional equation on the partition function of some lattice models, was first introduced in statistical mechanics by Stroganov (1979), and intensively used by many authors, among them Baxter (1980, 1982). It has been noticed by one of us that many models satisfy an inverse relation (Jaekel and Maillard 1983). Within the scope of this paper, the inverse functional relation can be written as

$$Z(K_1, K_2, K_3, K_4)Z(K_1 + \tfrac{1}{2}i\pi, -K_2, K_3 + \tfrac{1}{2}i\pi, -K_4) = ((2i\sinh 2K_1)(2i\sinh 2K_2))^{1/2} \tag{17}$$

where Z is defined in equation (2). Of course, $Z(K_1, K_2, K_3, K_4)$ is symmetric under the symmetry group of the square (C_{4v}).

Let us first recall Baxter's result on the two-dimensional anisotropic ferromagnetic Ising model (Baxter 1980); the partition function of this model can be completely determined by using

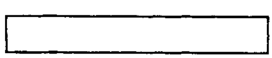
(i) the inverse relation on the partition function and the symmetry between the two coupling constants of this model, say K_0 and K_0', and

(ii) a resummed high-temperature expansion (see e.g. Jaekel and Maillard 1982b), and the fact that in this expansion $\tanh^2 K_0 = 1$ is the only singularity.

It is rather tempting to see whether $Z(K_1, K_2, K_3, K_4)$ can be similarly determined, by using the inverse relation (equation (17)) and the geometrical symmetry (C_{4v}). Let us introduce the following notations:

$$t_1 = \tanh K_1, \qquad t_2 = \tanh K_2, \qquad t_3 = \tanh K_3, \qquad t_4 = \tanh K_4.$$

At the lowest order in t_2 and t_4, we have the following diagram (see Jaekel and Maillard 1982b for further details)



$$\left(\frac{-t_1^2 - t_3^2 + 2t_1^2t_3^2}{1-t_1^2t_3^2}\right)\left(\frac{t_2^2+t_4^2}{2}\right) \tag{18}$$

At this order, the (resummed) high-temperature expansion of $Z(K_1, K_2, K_3, K_4)$ clearly satisfies the inverse relation (17). At the next order, let us consider the coefficient of $t_2^2 t_4^2$. The highest-order singularities occur in the diagram

One can convince oneself that the full $t_2^2 t_4^2$ coefficient can be written in the form

$$\frac{a(t_1^2+t_3^2)+bt_1^2t_3^2+c(t_3^2t_1^4+t_1^2t_3^4)+dt_1^4t_3^4+e(t_1^6t_3^4+t_1^4t_3^6)+ft_1^6t_3^6}{(1-t_1^2t_3^2)^3}. \qquad (19)$$

Due to the C_{4v} symmetry, coefficient a can be derived from the above mentioned coefficient (18). From the inverse relation (17), coefficients f, $a+e$, $b+d$, c are known. Therefore, to determine completely the $t_2^2 t_4^2$ coefficient, some extra information is needed (for instance the $t_1^2 t_2^2 t_3^2 t_4^2$ coefficient). At higher orders, it will be necessary to add more and more extra information to determine the partition function. These difficulties† are clearly due to the great number of parameters (four in the (K_1, K_2, K_3, K_4) model). (It should be noticed that in the restricted model $(K_1 = K_3, K_2 = K_4)$, the partition function satisfies exactly the same inverse and symmetry relations as in the (anisotropic) Onsager model. The only singularities which occur, seem to be $\tanh^2 K_1 = \pm 1$. It may well be that these inverse and symmetry relations determine the partition function as in the Onsager case.)

As a matter of fact, § 2 sheds some light on the extra information needed; the four-parameter frustrated model was shown to be equivalent to the two-parameter (ferromagnetic) Ising model, for which the inverse and symmetry relations determine completely the partition function (Baxter 1980). Therefore the inverse relation and the geometrical symmetry C_{4v}, when combined with the equivalence developed in § 2, determine completely the partition function of the frustrated model. In this particular case, the missing information is seen to be some rather sophisticated property, namely the reduction of a four-parameter model to a two-parameter model.

A similar property can be found in the literature, where some two-point correlation function on the honeycomb lattice depends only on two parameters and not on the three coupling constants of the model (Baxter and Enting 1978).

5. Conclusion

We have shown how a fully frustrated Ising model depending on four parameters can be mapped onto a ferromagnetic Ising model, which depends only on two. In some sense, one can say that fluctuations in two dimensions are important enough to 'defrustrate' the model, with the caveat that T_c of the Onsager case corresponds to $T = 0$ in the frustrated model. We would like to call attention to the non-trivial reduction of the number of parameters in this mapping. Such a reduction is seen to be very useful when one tries to determine the partition function of the fully frustrated model via the inverse relation.

† We have already encountered similar problems with the two-dimensional Potts and three-dimensional Ising models, where we tried to characterise the missing information (Jaekel and Maillard 1982a, b).

Appendix. Relation to a (constrained) Baxter model

Let us perform a partial trace over spins (figure 2(*a*)):

$$\operatorname*{Tr}_{\sigma=\pm 1} \exp[\sigma(K_1\mu_1+K_2\mu_2+K_3\mu_3+K_4\mu_4)] = 2\cosh\left(\sum_{i=1}^{4} K_i\mu_i\right). \tag{A1}$$

Equation (A1) can be rewritten as

$$2\cosh\left(\sum_{i=1}^{4} K_1\mu_1\right) = 2\lambda \exp(K_{12}\mu_1\mu_2+K_{23}\mu_2\mu_3+K_{34}\mu_3\mu_4+K_{41}\mu_4\mu_1) \times \exp(K_{13}\mu_1\mu_3+K_{24}\mu_2\mu_4)\exp(K_{1234}\mu_1\mu_2\mu_3\mu_4). \tag{A2}$$

This is turn yields

$$\begin{aligned} \cosh(K_1+K_2+K_3+K_4) &= \lambda \exp(K_{12}+K_{23}+K_{34}+K_{41})\exp(K_{13}+K_{24})\exp(K_{1234}) \\ \cosh(K_1-K_2+K_3-K_4) &= \lambda \exp(-K_{12}-K_{23}-K_{34}-K_{41})\exp(K_{13}+K_{24})\exp(K_{1234}). \end{aligned} \tag{A3}$$

Equation (4*a*) is then easily obtained. Equations (4*b*)–(4*d*) can be similarly derived.

The partial trace over spin τ (figure 2(*b*)) replaces (K_2, K_4) by ($-K_2$, $-K_4$) in equation (A1). This amounts to changing (μ_2, μ_4) to ($-\mu_2$, $-\mu_4$) in (A2), which in turn can be considered as changing the sign of ($K_{12}, K_{23}, K_{34}, K_{41}$). Combining partial traces on σ and τ (figure 1), we obtain the cancellation of these terms.

References

André G, Bidaux R, Carton J P, Conte R and de Sèze L 1979 *J. Physique* **40** 479
Baxter R J 1972 *Ann. Phys., NY* **70** 193
—— 1978 *Phil. Trans. R. Soc.* A **289** 315
—— 1980 *Fundamental Problems in Statistical mechanics V* (*Proc. Enschede Summer School*) ed E G D Cohen (Amsterdam: North-Holland)
—— 1982 *J. Stat. Phys.* **28** 1
Baxter R J and Enting I G 1978 *J. Phys. A: Math. Gen.* **11** 2463
Bryskin V V, Goltsev A Yu and Kudinov E E 1980 *J. Phys. C: Solid State Phys.* **13** 5999
Derrida B, Maillard J M, Vannimenus J and Kirkpatrick S 1978 *J. Physique Lett.* **39** 465
Fan C and Wu F Y 1970 *Phys. Rev.* B**2** 723
Fischer K H 1982 Spin glasses (*Jülich Preprint*)
Fisher M E 1961 *Phys. Rev.* **124** 1664
Forgacs G 1981 *Phys. Rev.* B**22** 4473
Gabay M 1980 *J. Physique Lett.* **41** 427
Jaekel M T and Maillard J M 1982a *J. Phys. A: Math. Gen.* **15** 1309
—— 1982b *J. Phys. A: Math. Gen.* **15** 2241
—— 1983 *J. Phys. A: Math. Gen.* **16** 1975
Jüngling K 1975 *J. Phys. C: Solid State Phys.* **8** L169
Kadanoff L P and Wegner F J 1971 *Phys. Rev.* B **4** 3989
Kasteleyn P W 1963 *J. Math. Phys.* **4** 287
Longa L and Olès A M 1980 *J. Phys. A: Math. Gen.* **13** 1031
McCoy B and Wu T T 1973 *The Two Dimensional Ising Model* (Cambridge, Mass.: Harvard University Press)
Onsager L 1944 *Phys. Rev.* **65** 117
Southern B W, Chui S T and Forgacs G 1980 *J. Phys. C: Solid State Phys.* **13** L827
Stephenson J 1970 *J. Math. Phys.* **11** 413

Stroganov Y G 1979 *Phys. Lett.* **74A** 116
Toulouse G 1982 Frustrations et désordres (*ENS Preprint*)
Villain J 1977 *J. Phys. C: Solid State Phys.* **10** 1717
Villain J, Bidaux R, Carton J P and Conte R 1980 *J. Physique* **41** 1263
Wannier G 1950 *Phys. Rev.* **79** 357
Williams J K 1982 *J. Phys. C: Solid State Phys.* **15** 275