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Symmetries of lattice models in statistical mechanics and effective algebraic geometry

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Résumé. — Nous rééxaminons la paramétrisation du modèle à huit vertex symétrique et ses généralisations (modèles à seize-vertex), en soulignant le rôle joué par une "pre-Ansatz de Bethe" reliée aux relations quadratiques de Frobenius sur les fonctions thêtas. Cette relation correspond à une relation d'entrelacement de deux courbes elliptiques identiques, $y_2 = P_3(z) = 4z^3 - g_2 z - g_3$. Diverses expressions explicites de quantités associées aux fonctions elliptiques (g_2, g_3, modules des fonctions elliptiques,...) sont données. Nous considérons, tout particulièrement, les sous-cas du modèle à seize-vertex pour lesquels les trois racines de $P_3(z)$ peuvent être données explicitement par une expression simple. De plus, deux sous-variétés algébriques pour lesquelles apparaît une multiplication complexe sont données explicitement sur l'exemple du modèle de Baxter. Les diverses symétries de ces modèles sont examinées à la lumière de la géométrie algébrique effective. Nous montrons qu'il existe une relation étroite entre la physique des modèles et les symétries et transformations agissant sur les variétés algébriques les paramétrisant.

Abstract. — The elliptic parametrization of the symmetric eight-vertex model and its generalizations (sixteen-vertex models) is revisited, underlying the role played by a "pre-Bethe Ansatz" condition closely related to the quadratic Frobenius relation on theta functions. This relation corresponds to an intertwining of two identical elliptic curves $y^2 = P_3(z) = 4z^3 - g_2 z - g_3$. Explicit expressions for various quantities associated to the elliptic functions (g_2, g_3, modulus of the elliptic functions,...) are given. One concentrates on subcases of the sixteen-vertex model for which the three roots of $P_3(z)$ can be given explicitly in a simple form. Moreover, two algebraic subvarieties of the Baxter model, for which complex multiplication occurs, are given explicitly. The various symmetries occurring in these models are understood in the light of effective algebraic geometry. We show that there is a close relation between the physics of the problems and the symmetries and transformations acting on the algebraic varieties parametrizing the models.

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1. Introduction.

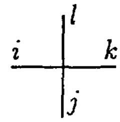
It has been shown [1] that integrable models in lattice statistical mechanics are necessarily associated with algebraic varieties. Up to now, only genus zero, or one, algebraic curves (up to the noticeable exception of the chiral Potts model [2, 3]) have occurred as a recurrent feature of exactely solvable models in statistical mechanics [4-6]. This leads to elliptic or rational parametrizations, which are relevant for the description of the solutions of the Yang-Baxter equations [7, 4], for the explicit construction of the Bethe Ansatz [8, 9], or the exact calculation of the partition function using the so-called "inversion trick" [7, 10, 11]. It has also been shown that these curves have a set of automorphisms corresponding to an infinite group (denoted Γ in the sequel) generated by the inversion relations of the model [11]. This explains the occurrence of genus zero, or one, algebraic curves in so many integrable models of statistical mechanics [11]. More recently, it has been shown that these (elliptic or rational) curves can be generated as orbits of the group Γ [12-14]. This group happens to be closely related to a symmetry group of the Yang-Baxter equations (isomorphic to the affine Weyl group $A_2^{(1)}$, see [15, 16]), however, it not only acts as a symmetry of integrability, but is a symmetry group of the *whole phase diagram whatever the model* [17].

In the parameter space of the sixteen-vertex model, the orbits of the group Γ stay on *elliptic curves* [18]. The whole parameter space of the general sixteen-vertex model is thus parametrized in terms of these elliptic curves. An analysis of a "pre-Bethe Ansatz" equation leads to an algebraic *modular invariant* which gives a canonical foliation of the parameter space [18]. This algebraic expression is actually invariant by *all the symmetries of the model*, in particular the weak-graph transformations [19] (linear transformations on the R-matrix), and the group Γ (non-linear transformations on the R-matrix). This gives therefore the most appropriate parametrization to describe the physics of the model and the best candidates for criticality and disorder conditions [4, 20, 18].

In the following, we revisit the elliptic parametrization of the sixteen-vertex model [18] and its submodels, underlying the role played by a "pre-Bethe Ansatz" equation closely related to the quadratic Frobenius relation on theta functions [21-23]. This equation corresponds to an intertwinning of two (identical) elliptic curves: $y^2 = P_3(z) = 4z^3 - g_2 z - g_3$. Various quantities associated to the elliptic functions (g_2, g_3, modulus of the elliptic functions,..) are given explicitly. One underlines subcases of the sixteen-vertex model for which the three roots of $P_3(z)$ can be given in a simple form. Some algebraic subvarieties for which *complex multiplication* occurs [24-26], are also emphasized. The symmetries of the sixteen-vertex models are understood in the light of (effective) algebraic geometry. In particular, one underlines a subgroup of symmetry isomorphic to the permutation group of three elements $\mathcal{S}_3$. This algebraic geometry point of view leads to discriminate the various symmetries of the model and understand their (often subtle) relations. For instance, one has to distinguish the weak-graph transformations (irrelevant parameters to be "gauged-away"), the group Γ (a key symmetry to understand the integrability of the models), a modular group [27] and, more generally, the *transformations of elliptic functions* (isogenies): Landen transformations [28],.. For some sixteen-vertex models, these last symmetries (which amount to multiplying the ratio of the period of elliptic functions) *correspond to the renormalization group*. This approach, and this point of view, are not restricted to the sixteen-vertex model, or more generally to two-dimensional models.

2. General case: the sixteen-vertex model.

Let us consider the sixteen-vertex model on a square lattice [29]. The Boltzmann weights are arranged in a 4×4 matrix R of entries r^{ij}_{kl} corresponding to the configuration:



We use the notation for R :

$$R = \begin{pmatrix} a & e & f' & d' \\ g & b & c' & h' \\ h & c & b' & g' \\ d & f & e' & a' \end{pmatrix} \tag{2.1}$$

The model is insensitive to a rescaling of all entries by a common factor. The (complexified) parameter space is thus the projective space $\mathbb{CP}_{15}$.

A number of transformations act on R and correspond to symmetries of the parameter space: the group G of "gauge-like" transformations, which are linear transformations on the matrix R (weak-graph transformations [19, 30], see also p. 456 of [29] in the particular case of the sixteen-vertex model) and the group Γ of symmetries generated by the inversion relations of the model [12, 13]. The group Γ can be represented in terms of *birational* transformations in the parameter space $\mathbb{CP}_{15}$ [12]. For this model, a detailed analyzis of the compatibility between the (linear) group G and the (non-linear) group Γ has been performed in [18]: this compatibility *can be typified in a modular invariant* which foliates the parameter space. This modular invariant gives the best candidates for the critical (and disorder) varieties [1, 4, 18, 31, 32]. It has to be compared with the invariants of the linear group G only (the "Hilbert's syzygies" [33, 34]), introduced by Perk and Wu [35] or Gwa and Wu [36]. Unfortunately, for the most general sixteen-vertex model, the algebraic expressions associated with this modular invariant are homogeneous polynomials of degree 24 in the sixteen homogeneous parameters of the model, which are the sum of several millions of monomials [18],... Any exact calculation with such expressions is hopeless, and it is necessary to concentrate on sixteen-vertex models for which factorizations of the modular invariant occur. One illustrates here, on subcases of the sixteen-vertex model for which such factorizations occur, the analyzis of these models from the point of view of algebraic geometry. The key point of this study is grounded on the analysis of biquadratic equations associated to the model (more precisely associated to the construction of a "pre-Bethe Ansatz" for the model) [18].

Since Γ is a symmetry group of the model, one seeks for subcases of the sixteen-vertex model compatible with the group Γ. The study of the subcases of the sixteen-vertex model (defined by equalities between the entries of the associated 4×4 R-matrix) has been performed elsewhere, leading to a restricted list of such "admissible patterns" [12, 13, 37, 38]. Actually, one only has 62 admissible patterns which can be classified into eight classes [38]:

- the most general sixteen-vertex model depending on sixteen homogeneous parameters
- four admissible classes depending on ten homogeneous parameters
- three admissible classes depending on eight homogeneous parameters. For instance, one

of these three classes corresponds to the following 4×4 R-matrix:

$$R = \begin{pmatrix} a & e & f & d \\ g & b & c & h \\ h & c & b & g \\ d & f & e & a \end{pmatrix} \tag{2.2}$$

Admissible patterns of less than eight homogeneous parameters are subcases of the previous admissible classes, obtained by imposing more equalities between the entries.

The well-known symmetric eight-vertex Baxter model [39, 8], can be seen as a subcase of most of these admissible patterns.

Note that, among these parameters, some of them may be irrelevant and "gauged-away" (using the weak-graph duality transformation [19]). In this framework, one can, for instance, recall a (ten parameter-dependent) subcase of the sixteen-vertex model defined by equalities between the entries *up to a sign* given by the following 4×4 R-matrix:

$$R = \begin{pmatrix} a & e & f' & d' \\ g & b & c' & -f' \\ h & c & b & -e \\ d & -h & -g & a \end{pmatrix}$$

which can be reduced to the (four parameter-dependent) symmetric eight-vertex model by weak-graph duality transformations (see p. 170 of [19]).

3. Biquadratic equations: towards Bethe Ansatz.

3.1 GENERALITIES. — It is not necessary to recall the relevance of the Bethe Ansatz to solve two-dimensional lattice models in statistical mechanics, or one-dimensional quantum Hamiltonians [21, 7, 4, 29, 9, 39]. The method due to Bethe [40, 21] aims at a direct determination of the eigenvectors. The relation between the Bethe Ansatz and other features of integrability (infinite number of conserved quantities, family of commuting transfer matrices, Yang-Baxter equations,...) will not be detailed here (see [21]).

The explicit construction of the Bethe Ansatz on the symmetric eight-vertex Baxter model [8] is explained simply in [9]. The details of this very construction will not be given here. Let us just recall that it amounts to build eigenvectors made up of linear combinations of product vectors. One of the keys to the Bethe-Ansatz [8, 9] is *the occurrence* (see Eqs. (B.10), (B.11a)) in [8] *of vectors which are pure tensor products* of the form $(v \otimes w)$ and *which R maps onto other pure tensor products* $(v' \otimes w')$. In the case of elliptic parametrization, such a property is related to the so-called *quadratic Frobenius relations on theta functions* [22, 21, 23], which may give a representation of a Zamolodchikov algebra [41]. It is worth recalling that the Zamolodchikov algebra is an "almost" [42] sufficient condition for the Yang-Baxter equation to be satisfied.

Let us write that R maps a pure tensor product onto a pure tensor product in the general case of the sixteen-vertex model. Denoting:

$$v = \binom{1}{p}, \quad w = \binom{1}{q}, \quad v' = \binom{1}{p'}, \quad w' = \binom{1}{q'}$$

this "pre-Bethe Ansatz" equation reads:

$$R(v \otimes w) = \mu \cdot v' \otimes w' \tag{3.1}$$

where μ is a multiplicative factor, yielding the two biquadratic relations:

$$e_9 + e_8\ p + e_7\ p' + e_6\ pp' + e_5\ p^2 + e_4\ p'^2 + e_3\ p^2p' + e_2\ \ pp'^2 + e_1\ p^2p'^2 = 0 \tag{3.2}$$
$$f_9 + f_8\ q + f_7\ q' + f_6\ qq' + f_5\ q^2 + f_4\ q'^2 + f_3\ q^2q' + f_2\ qq'^2 + f_1\ q^2q'^2 = 0 \tag{3.3}$$

where the e_i's and f_i's are quadratic polynomials of the homogeneous parameters $a_1, .. , d_4$ of the R-matrix [18].

These two biquadratic equations yield the *same elliptic curve* [18, 43, 38, 9] (see next subsection). The transformation $p \rightarrow p'$ (or $q \rightarrow q'$) actually corresponds to a *shift* of some parameter describing the elliptic curve (let us see for instance Eq. (4.100) in [9]). In the case of the Baxter model, this enables to build eigenvectors made up of linear combinations of product vectors [9]. This can be seen as the first step to build the Bethe Ansatz.

3.2 SYMMETRIES AND INVARIANTS OF THE BIQUADRATIC RELATIONS. — The two biquadratic relation (3.2) and (3.3) actually differ for the above mentioned patterns [38], except for a few ones for which they remarkably identify. A group $\mathrm{G_{Bethe}}$ acts naturally on the biquadratic equations (3.2) and (3.3). $\mathrm{G_{Bethe}}$ is isomorphic to $\mathrm{sl_2 \times sl_2 \times sl_2 \times sl_2}$ [18]. The four copies of $\mathrm{sl_2}$ act respectively on v, w, v', w'. $\mathrm{G_{Bethe}}$ has a linear action on the 4×4 matrix R :

$$R \rightarrow g_{1L}^{-1} \cdot g_{2L}^{-1} \cdot R \cdot g_{1R} \cdot g_{2R} \tag{3.4}$$

where $g_{1L}, g_{2L}, g_{1R}, g_{2R}$ are $\mathrm{sl_2}$ matrices, i.e. homographic transformations on p, p', q, q', or equivalently, linear transformations on the e_i's and f_i's. $\mathrm{G_{Bethe}}$ generalizes the weak-graph duality transformations acting on R [19, 30]. The group G of weak-graph transformations is a subgroup of $\mathrm{G_{Bethe}}$ with $g_{1L} = g_{1R}$ and $g_{2L} = g_{2R}$:

$$R \rightarrow g_1^{-1}\ \ g_2^{-1}\ \ R \cdot g_1 \cdot g_2 \tag{3.5}$$

The elliptic parametrization of (3.2) (or (3.3)) is obtained as follows: the discriminant D of (3.2), considered as a quadratic polynomial in p, is a polynomial in p' of the form:

$$D = A_0\ p'^4 + 4\ A_1\ p'^3 + 6\, A_2\ p'^2 + 4\, A_3\ p' + A_4 \tag{3.6}$$

The transformations of D under $\mathrm{sl_2}$ transformations acting on the vector v' (homographic transformations of p') have two fundamental invariants g_2, g_3 and the modular invariant $J = g_3^2 / \left(g_2^3 - 27 g_3^2\right)$ [44, 45]:

$$g_2 = A_0A_4 - 4A_1A_3 + 3A_2^2 \tag{3.7}$$
$$g_3 = A_0A_2A_4 + 2A_1A_2A_3 - A_0A_3^2 - A_4A_1^2 - A_2^3 \tag{3.8}$$

g_2 and g_3 are also invariant (since D is) under $\mathrm{sl_2}$ transformations acting on the vector v (homographic transformations of p). Let us denote Δ the discriminant of the degree four polynomial D. Δ reads:

$$\Delta = g_2^3 - 27 g_3^2 \tag{3.9}$$

Similar calculations can be performed exchanging the role of p and p' in equation (3.2): they lead to the *same* g_2 and g_3. *More remarkably, the same analysis on equation* (3.3), *seen alternatively as a quadratic polynomial in* q *with quadratic coefficients in* q' (or polynomial in q' with polynomial coefficients in q), leads to exactly the same g_2 and g_3 [18]. *Therefore, one can* *associate to* (3.2), (3.3) *and also to the elliptic curve corresponding to the orbits of* Γ *in* $\mathbb{C}\mathbf{P}_{15}$ [18], *the same Weierstrass's canonical form,* $y^2 = P_3(z) = 4z^3 - g_2 z - g_3$ [43, 46] ([1]) (see also next Sect.).

In this framework, an algebraic variety is of particular interest: the one for which the elliptic parametrization *degenerates into a rational one,* for $J = \infty$, or equivalently $\Delta = 0$. This yields candidates for algebraic criticality, and disorder conditions, appearing on the same level [1, 4]. Unfortunately, these algebraic conditions are too involved: it is necessary to concentrate on models for which factorizations of Δ occur [38].

3.3 COMMENT ON THE ISING MODEL IN A MAGNETIC FIELD. — One of the interests of the sixteen-vertex model, beside the fact that it is a generalization of the symmetric eight-vertex Baxter model, is that it has many subcases relevant for statistical mechanics on lattices [29]. In particular, the (anisotropic) nearest neighbour Ising model in a magnetic field on the square lattice, or even the checkerboard Ising model in a magnetic field (see pp. 350-354 of [29]), are subcases of the sixteen-vertex model. One can imagine that the critical manifold of the Ising model in a magnetic field (see [47, 35]) could be given by the vanishing condition of the discriminant Δ [18] restricted to the subcase of the sixteen-vertex model corresponding to the Ising model in a magnetic field. In fact, the straight approach of section (3.2) *actually fails, since equations* (3.2) and (3.3) *degenerate.* They factorize in p and p' (resp. q and q') : $F(p) \cdot G(p') = 0$. Conversely, such a factorization of the biquadratic equations means that the sixteen-vertex model is equivalent to a checkerboard Ising model in a magnetic field. Some more sophisticated approach should be introduced to cope with such "singular" cases.

Other cases where the approach of section (3.2) seems to fail, correspond to subcases of the sixteen-vertex model for which the discriminant Δ is already equal to zero. It will be shown in section (5.2), on the example of the six-vertex model (which can be seen as a critical subcase of the symmetric eight-vertex model, see pp. 271-272 of [4]), how singularities inside already critical subvarieties may occur.

4. Analysis of the biquadratic equations.

Let us illustrate the analysis of the biquadratic equations (3.2) and (3.3) for particular admissible patterns for which factorizations of Δ occur. Consider the particular forms $\Phi_1(x,y)$ and $\Phi_2(x,y)$ of the biquadratic equations (3.2) and (3.3) for model (2.2) (which contains the Baxter model as a subcase). The biquadratic equation (3.2) reads:

$$\begin{aligned}\Phi_1(x,y) &= e_1\left(x^2y^2+1\right) + e_2\left(xy^2+x\right) + e_3\left(x^2y+y\right) + e_4\left(x^2+y^2\right) + e_6\ xy \\ &= x^2\ A_x(y) + 2xB_x(y) + C_x(y) \\ &= y^2\ A_y(x) + 2yB_y(x) + C_y(x)\end{aligned} \tag{4.1}$$

where x and y denote here p and p'. The other biquadratic equation (3.3) reads $\Phi_2(x,\ y) = 0$, where x and y denote q and q', and where the e_i's are replaced by the f_i's. The "canonical equation", $\Phi_1(x,y) = 0$, implies (see [43, 46]):

$$\frac{\mathrm{d}y}{\sqrt{P(y)}} = \pm\frac{\mathrm{d}x}{\sqrt{Q(x)}} \tag{4.2}$$

([1]) The new variable z is given by the ratio of an Hessian and of $\Phi(1,y)$.

where

$$Q(x) = B_y(x)^2 - A_y(x)C_y(x) \quad \text{and} \quad P(y) = B_x(y)^2 - A_x(y)C_x(y)$$

are two different degree four polynomials in respectively x and y, but having the *same fundamental invariants* g_2 and g_3 [18, 44, 45]. $P(y)$ reads:

$$P(y) = A_0\ y^4 + 4A_1\ y^3 + 6A_2\ y^2 + 4A_1\ y + A_0 \tag{4.3}$$

where the A_i's can be written in terms of the coefficients of $\Phi_1(x,y)$:

$$A_0 = e_2^2 - 4e_1e_4, \qquad 4A_1 = 2e_2e_6 - 4\,(e_1 + e_4)\,e_3, \qquad 6A_2 = e_6^2 + 2e_2^2 - 4\left(e_1^2 + e_3^2 + e_4^2\right)$$

The polynomial $Q(x)$ is obtained from the polynomial $P(x)$ by exchanging e_2 and e_3.

In the general case (2.2), one can find two different homographic transformations: $X = H_1(x)$ and $Y = H_2(y)$ to rewrite (4.2) as:

$$\frac{\mathrm{d}Y}{\sqrt{(1-k^2Y^2)(1-Y^2)}} = \pm\frac{\mathrm{d}X}{\sqrt{(1-k^2X^2)(1-X^2)}} \tag{4.4}$$

where k is the modulus of the elliptic functions. Then, by integration, one obtains $H_1(x)/H_2(y) = \mathrm{sn}(2u,\ k)/\mathrm{sn}(2u \pm 2\eta,\ k)$, thus parametrizing the relation between x and y. This will be seen explicitly on a particular example, in subsection (4.1).

Remarkably, for model (2.2), the analysis can be simply performed introducing the three roots r_1, r_2 and r_3 of the Weierstrass's canonical form (elliptic curve) associated to $\Phi_1(x,y)$ [43, 46]:

$$y^2 = 4z^3 - g_2z - g_3 = 4\,(z-r_1)\,(z-r_2)\,(z-r_3) \tag{4.5}$$

The invariants g_2, g_3 and Δ read in terms of the roots:

$$g_2 = -4\,(r_1r_2 + r_2r_3 + r_3r_1)\,, \qquad g_3 = 4\ r_1r_2r_3,$$
$$\Delta = g_2^3 - 27\ g_3^2 = 16\,(r_1-r_2)^2\,(r_2-r_3)^2\,(r_3-r_1)^2$$

An elementary calculation shows that the results simplify in terms of two expressions α and β defined by:

$$\alpha = A_2 - A_0$$
$$\beta = (A_0 + 4A_1 + 3A_2)\,(A_0 - 4A_1 + 3A_2)$$

or in terms of the e_i's:

$$\alpha = \frac{1}{6}\left(e_6^2 - 4e_1^2 - 4e_2^2 - 4e_3^2 - 4e_4^2 + 24e_1e_4\right)$$
$$\beta = \tfrac{1}{4}\,(e_6 + 2e_1 + 2e_2 + 2e_3 + 2e_4)\,(e_6 + 2e_1 - 2e_2 - 2e_3 + 2e_4)$$
$$(e_6 - 2e_1 + 2e_2 - 2e_3 - 2e_4)\,(e_6 - 2e_1 - 2e_2 + 2e_3 - 2e_4)$$

The roots r_1, r_2, r_3 (with $r_1 + r_2 + r_3 = 0$ as it should) read:

$$r_1 = \frac{1}{4}\left(-\alpha - \sqrt{\beta}\right), \qquad r_2 = \frac{1}{4}\left(-\alpha + \sqrt{\beta}\right), \qquad r_3 = \frac{\alpha}{2}$$

and the invariants read respectively:

$$g_2 = A_0^2 + 3A_2^2 - 4A_1^2, \qquad g_3 = (A_2 - A_0)\left(2A_1^2 - A_0A_2 - A_2^2\right),$$
$$\Delta = (A_0 + 4A_1 + 3A_2)\,(A_0 - 4A_1 + 3A_2)\left(A_0^2 - 3A_0A_2 + 2A_1^2\right)^2$$

or in terms of α and β:

$$g_2 = \frac{1}{4}\left(3\alpha^2 + \beta\right), \quad g_3 = \frac{1}{8}\left(\alpha + \sqrt{\beta}\right)\left(\alpha - \sqrt{\beta}\right)\alpha, \quad \Delta = \frac{1}{64}\left(3\alpha + \sqrt{\beta}\right)^2\left(3\alpha - \sqrt{\beta}\right)^2\beta \tag{4.6}$$

In the $e = f = g = h = 0$ limit, where the model reduces to the Baxter model, both conditions $\beta = 0$, and $3\alpha + \sqrt{\beta} = 0$, are respectively the disorder conditions [21, 20, 4] and the critical conditions [4] (see Appendix A). Both conditions read $\Delta = 0$, which means that the elliptic parametrization of the model degenerates into a rational one. Expressions of α and β are given in Appendix A, in terms of the entries of some sixteen-vertex models.

CANONICAL FORM FOR A BIQUADRATIC EQUATION. — Homographic transformations on x and y (corresponding to G_{Bethe} introduced in subsection (3.2)) can be used to reduce $\Phi_1(x,y)$, or $\Phi_2(x,y)$, to some simple canonical forms $\tilde{\Phi}_1(x,y)$ (resp. $\tilde{\Phi}_2(x,y)$). Let us introduce other expressions σ, σ', ξ, ξ' and ρ in order to write simply α and β :

$$\alpha = \frac{1}{12}\left(\sigma\sigma' + \xi\xi' - \rho^2\right), \qquad \beta = \frac{1}{4}\sigma\sigma'\xi\xi'$$

with:

$$\begin{array}{ll} \sigma = e_6 + 2e_1 + 2e_2 + 2e_3 + 2e_4, & \sigma' = e_6 + 2e_1 - 2e_2 - 2e_3 + 2e_4 \\ \xi = e_6 - 2e_1 - 2e_2 + 2e_3 - 2e_4, & \xi' = e_6 - 2e_1 + 2e_2 - 2e_3 - 2e_4 \\ \rho = 4(e_1 - e_4) & \end{array} \tag{4.7}$$

One now considers the following homographic transformations on x and y in equation (4.1):

$$x \to p_\lambda(H(x)), \qquad y \to p_\mu(H(y)) \tag{4.8}$$

where:

$$H(x) = \frac{x-1}{x+1} \quad \text{and} \quad p_\lambda(x) = \lambda \cdot x \tag{4.9}$$

The values of λ and μ are defined by:

$$\lambda^4 = \frac{\sigma'\xi'}{\sigma\xi} \quad \text{and} \quad \mu^4 = \frac{\sigma'\xi}{\sigma\xi'}$$

The homographic transformations (4.8) enable to cancel coefficients e_2 and e_3 :

$$\tilde{\Phi}(x,y) = \gamma\left(x^2y^2 + 1\right) + 2\delta\, xy - \left(x^2 + y^2\right) \tag{4.10}$$

where:

$$\delta = \frac{\rho}{\sqrt{\xi\xi'}}, \qquad \gamma = \frac{\sqrt{\sigma\sigma'}}{\sqrt{\xi\xi'}} \tag{4.11}$$

It now becomes simple to give an explicit elliptic parametrization of model (2.2). For example, for a subcase of (2.2) given by $e = f = g = h$, (see also Appendix A):

$$R = \begin{pmatrix} a & e & e & d \\ e & b & c & e \\ e & c & b & e \\ d & e & e & a \end{pmatrix} \tag{4.12}$$

one has:

$$\sigma = \sigma' = 4\ c'd', \qquad \xi = 4\ a'_1 b', \qquad \xi' = 4\ a'_2 b', \qquad \rho = 2\left(a'_1 a'_2 + b'^2 - c'^2 - d'^2\right)$$

where:

$$\begin{array}{lll} a'_1 = \dfrac{a+b+c+d+4e}{2}, & a'_2 = \dfrac{a+b+c+d-4e}{2} & \\ b' = \dfrac{a+b-c-d}{2}, & c' = \dfrac{a-b+c-d}{2}, & d' = \dfrac{a-b-c+d}{2} \end{array} \tag{4.13}$$

generalize the duality transformation of the Baxter model (see p. 205 of [4], see also below Eq. (5.4)). Then one can write:

$$\gamma = \frac{c'd'}{b'\sqrt{a'_1 a'_2}}, \qquad \delta = \frac{a'_1 a'_2 + b'^2 - c'^2 - 2'^2}{2b'\sqrt{a'_1 a'_2}} \tag{4.14}$$

which gives an elliptic parametrization of model (4.12).

$$\begin{aligned} a'_1 &= \mu\ \rho\ \mathrm{sn}(v+\eta, -k) \\ a'_2 &= \mu^{-1}\ \rho\ \mathrm{sn}(v+\eta, -k) \\ b' &= \rho\ \mathrm{sn}(v-\eta, -k) \\ c' &= \rho\ \mathrm{sn}(2\eta, -k) \\ d' &= -\rho\ k\ \mathrm{sn}(v+\eta, -k)\mathrm{sn}(v-\eta, -k)\mathrm{sn}(2\eta, -k) \end{aligned} \tag{4.15}$$

where sn, cn and dn are the Jacobian elliptic functions [48, 49] and k is given by:

$$k + \frac{1}{k} = \frac{\delta^2 - \gamma^2 - 1}{\gamma} \tag{4.16}$$

and 2 η given by (see also p. 42 of [8] or p. 143 of [9]):

$$\delta = \mathrm{cn}(2\eta, -k)\ \mathrm{dn}(2\eta, -k) \quad \text{and} \quad \gamma = -k\ \mathrm{sn}^2(2\eta, -k) \tag{4.17}$$

This can be easily generalized to model (2.2). Equations (4.13, 4.14) can be simply understood from the fact that model (2.2) is equivalent to the asymmetric eight-vertex model up to weak-graph transformations (the weak-graph transformations associated to $H(x)$, or $-H(x)$, in (4.9)). Let us note that condition $\beta = 0$, for which the parametrization reduces to a rational one, means that one of the new variables (4.13) vanishes. It is clear on the invariants (4.14) that the parametrization degenerates in such cases.

5. The Baxter model.

5.1 ELLIPTIC PARAMETRIZATION OF THE BAXTER MODEL. — An important subcase of models (2.2) and (4.12) is the symmetric eight-vertex Baxter model [4, 8]. Its R-matrix is given by:

$$\begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix} \tag{5.1}$$

Φ_1 and Φ_2 actually identify and read:

$$\Phi(x,y) = \gamma\left(x^2y^2+1\right) + 2\delta xy - \left(x^2+y^2\right) \tag{5.2}$$

with:

$$\gamma = \frac{cd}{ab} \quad \text{and} \quad \delta = \frac{a^2+b^2-c^2-d^2}{2ab} \tag{5.3}$$

which corresponds, on the coefficients of equation (4.1), to: $e_1 = \gamma$, $e_2 = e_3 = 0$, $e_4 = -1$ and $e_6 = 2\,\delta$. One has the elliptic parametrization: $x = \sqrt{-k}\ \mathrm{sn}(2u, -k)$ and $y = \sqrt{-k}\ \mathrm{sn}(2u \pm 2\eta,\ -k)$, with k, the modulus of the elliptic function, and the shift η defined by equations (4.16) and (4.17).

Transformation $k \to 1/k$ immediately pops out from equations (4.16). Another transformation plays a special role: the (low-to-high temperature) duality transformation of the Baxter model (see p. 205 of [4], see also Appendix B). It is a weak-graph transformation [19] which reads:

$$\begin{aligned} a \to a^* &= \frac{1}{2}(a+b+c+d) \quad & b \to b^* &= \frac{1}{2}(a+b-c-d) \\ c \to c^* &= \frac{1}{2}(a-b+c-d) \quad & d \to d^* &= \frac{1}{2}(a-b-c+d) \end{aligned} \tag{5.4}$$

From section (4), one can evaluate the expressions of A_0, A_1, A_2, α and β :

$$\begin{aligned} &A_0 = 4\gamma, \qquad A_1 = 0, \qquad A_2 = \frac{2}{3}\left(\delta^2-\gamma^2-1\right) = 4\left(k+\frac{1}{k}\right)\gamma \\ &\alpha = \frac{2}{3}\left(-\gamma^2+\delta^2-6\gamma-1\right) = \frac{2}{3}\left(k+\frac{1}{k}-6\right)\gamma, \\ &\sqrt{\beta} = 2(\gamma+\delta-1)(\gamma-\delta-1) = -2\left(k+\frac{1}{k}+2\right)\gamma \end{aligned}$$

The roots of the elliptic curve (4.5) can be written in terms of γ and δ as follows:

$$r_1 = \frac{1}{3}\left(-\gamma^2+\delta^2+6\gamma-1\right) = \frac{1}{3}\left(k+\frac{1}{k}+6\right)\gamma, \tag{5.5}$$

$$r_2 = \frac{2}{3}\left(-\gamma^2-\delta^2+1\right) = -\frac{2}{3}\left(k+\frac{1}{k}\right)\gamma, \tag{5.6}$$

$$r_3 = \frac{1}{3}\left(-\gamma^2+\delta^2-6\gamma-1\right) = \frac{1}{3}\left(k+\frac{1}{k}-6\right)\gamma \tag{5.7}$$

This gives the following expressions for the fundamental invariants g_2 and g_3 and the discriminant Δ :

$$\begin{aligned} g_2 &= \frac{4}{3}\left(\gamma^4+\delta^4+14\gamma^2-2\delta^2-2\gamma^2\delta^2+1\right) \\ &= \frac{4}{3}\gamma^2\left(k^2+\frac{1}{k^2}+14\right) \\ g_3 &= \frac{8}{27}\left(\gamma^2-\delta^2+1\right)\left(-\gamma^2+\delta^2+6\gamma-1\right)\left(-\gamma^2+\delta^2-6\gamma-1\right) \\ &= -\frac{8}{27}\gamma^2\left(k+\frac{1}{k}-6\right)\left(k+\frac{1}{k}\right)\left(k+\frac{1}{k}+6\right) \\ \Delta &= 256\ \gamma^2(\gamma+\delta-1)^2(\gamma-\delta-1)^2(\gamma-\delta+1)^2(\gamma+\delta+1)^2 \\ &= 256\ \gamma^6\left(k+\frac{1}{k}-2\right)\left(k+\frac{1}{k}+2\right) \end{aligned}$$

In addition to the modulus k of the elliptic functions, one can also introduce the modulus k_0, and its complementary modulus k_0', defined from the roots of the Weierstrass's canonical form (4.5):

$$k_0^2 = \frac{r_2 - r_3}{r_1 - r_3} = \frac{(\gamma+\delta+1)(\gamma-\delta+1)}{4\gamma}$$
$$k_0'^2 = 1 - k_0^2 = \frac{r_1 - r_2}{r_1 - r_3} = \frac{(\gamma+\delta-1)(-\gamma+\delta+1)}{4\gamma}$$

One notes that most of the equations of the previous section remain unchanged for model (4.12) (see also Appendix A). One also notes that k_0 is related to the above modulus k, by a Landen transformation [28] and the transformation $k \to 1/k$, as follows:

$$k_0 = \pm\frac{1}{k_l} = \pm\frac{1+k}{2\sqrt{k}} \tag{5.8}$$

One also verifies (see [4] page 246, equation (10.11.3)) that:

$$k + \frac{1}{k} - 2 = \frac{(a-b-c-d)(a-b+c+d)(a+b-c+d)(a+b+c-d)}{4abcd} \tag{5.9}$$

Remarkably, *the modulus k also satisfies:*

$$\begin{aligned} k + \frac{1}{k} + 2 &= \frac{(a-b+c-d)(a-b-c+d)(a+b-c-d)(a+b+c+d)}{4abcd} \\ &= \frac{4\, a^*b^*c^*d^*}{abcd} \end{aligned} \tag{5.10}$$

It may happen that the modulus k is not in the interval [0, 1]. One thus has to follow a "rearrangement procedure" (see p. 268 of [4]) which amounts to use the symmetries of the eight-vertex models (negate or permute the homogeneous parameters of the model a, b, c and d, or perform a duality transformation (5.4)) in order to be in the so-called "principal regime" [4], and consequently have a "rearranged" modulus k_r in the interval [0, 1]. This rearrangement procedure corresponds to symmetries of the model represented by transformations on the homogeneous parameters of the model a, b, c, d, or transformations on the modulus: the duality (5.4), the transformation which permutes k and its complementary modulus k' $\left(k \leftrightarrow k' = \sqrt{1-k^2}\right.$, imaginary Jacobi transformation), .

Three distinct cases, depending on the value of δ and γ given by (5.3), namely the ferroelectric, antiferroelectric and disordered phases, have to be distinguished in order to classifiy the various phases of the model (see for instance p. 246 of [4]). In the disordered regime the right-hand side of (5.9) is negative: k is no longer in the interval [0, 1] and has to be replaced by k_r.

5.2 RATIONAL REDUCTION FOR THE BAXTER MODEL. — The critical varieties of the symmetric eight-vertex model come from equation (5.9), with $k = +1$. Four disorder conditions also come for the Baxter model, from equation (5.10) with $k = -1$: two already known (p. 274 of [4]) and two emerging from the analysis of the (two-layer) diagonal transfer matrix [31] (see also Appendix A, Eq. (A4)). It is interesting to look in parallel at equations (5.9) and (5.10) which are both associated to a rational parametrization of the model: the critical subvarieties,

$$(a+b+c-d)(a+b-c+d)(a-b+c+d)(a-b-c-d) = 0$$
$$(\gamma+\delta+1)(\gamma-\delta+1) = 0, \quad k_0^2 = 0, \quad k_0'^2 = 1, \quad k = +1$$

and the disorder subvarieties,

$$(a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d)=0$$
$$(\gamma-\delta-1)(\gamma+\delta-1)=0,\quad k_0^2=1,\quad k_0'^2=0,\quad k=-1$$

Other rational cases can be considered: the elliptic parametrization also reduces to a rational parametrization (that is $\Delta = 0$), when one of the four parameters a, b, c, d vanishes, which yields:

$$\gamma = 0 \text{ or } \infty, \qquad k_0^2=\infty, \qquad k_0'^2=\infty, \qquad k=0 \text{ or } \infty$$

The six-vertex model [9, 29] is such a case: it is obtained from the eight-vertex model by setting $d=0$.

The six-vertex model corresponds to ferroelectric ($\delta > 1$) and antiferroelectric ($\delta < -1$) eight-vertex Baxter model in the $k \to 0$ limit. In the disordered regime, $-1 < \delta < +1$, the rearrangement procedure gives that the $d \to 0$ six-vertex limit is the $k \to 1$ limit (and not the $k \to 0$ limit as a straight limit of (5.9) would suggest, see page 271 of [4])([2]). In other words, *in the disordered regime, the six-vertex model is the condition for the eight-vertex model to be critical.* Conversely, a critical eight-vertex model can be mapped onto a "disordered" six-vertex model. This example shows that one can have critical points inside an already critical variety (the condition for the symmetric eight-vertex model to reduce to the six-vertex model, that is $d=0$ or $\gamma=0$). In this example, the critical subvarieties of an already critical variety cannot be obtained from the condition for the elliptic parametrization to reduce to a rational one ($\Delta = 0$). The critically condition for the six-vertex model actually corresponds to impose an additional condition: to restrict to the limits of this disordered regime: $\delta = 1$ or $\delta = -1$.

6. Symmetries of models with elliptic parametrization: isogenies of elliptic functions for the Baxter model.

6.1 MODULAR GROUP AND THE BAXTER MODEL: A $\mathcal{S}_3$-SYMMETRY. — From the three roots r_1, r_2 and r_3, one can define three periods ω_i, $i=1, 2, 3$ such that: $\gamma(\omega_i)=r_i$, $\gamma(\omega_i)=0$ and $\gamma''(\omega_i) = -2(r_i-r_j)(r_j-r_k)$, $((i, j, k)$ cyclic permutations of $(1, 2, 3))$, where $\gamma(x)$ is the elliptic Weierstrass function. The fundamental periods ω_1 and ω_3 define a lattice period. The ratio of the periods, $\tau = \omega_3/\omega_1$, is given by:

$$\tau = i\frac{K\left(\sqrt{1-k^2}\right)}{K(k)} = i\frac{K\left(\sqrt{\dfrac{r_1-r_2}{r_1-r_3}}\right)}{K\left(\sqrt{\dfrac{r_2-r_3}{r_1-r_3}}\right)} \tag{6.1}$$

with $K(x)$ being the elliptic integral function:

$$K(x) = \int_0^{\pi/2} \frac{\mathrm{d}\psi}{\left(1-x^2\sin^2\psi\right)^{1/2}} \tag{6.2}$$

In terms of the modulus of the elliptic functions, the modular invariant $J(\tau)$ reads:

$$J(\tau) = \frac{g_2^3}{\Delta} = \frac{4}{27}\frac{\left(k_0^4-k_0^2+1\right)^3}{k_0^4\left(k_0^2-1\right)^2} = \frac{6}{27}+\frac{2}{27}\sum_{i=1}^{6}k_{0i}^4 \tag{6.3}$$

([2]) Under the duality transformation (5.4), the six-vertex condition $d=0$ also maps onto a disorder condition $a+d=b+c$, $(d^*=0)$. This illustrates again the relations between the critical and disorder subvarieties, for which the parametrization becomes rational.

with: $k_{0i}^2 = k_0^2,\ 1/k_0^2, 1-k_0^2,\ 1/\left(1-k_0^2\right),\ k_0^2/\left(k_0^2-1\right),\ \left(k_0^2-1\right)/k_0^2,\ (i=1,\ ..,\ 6)$. The k_{0i}'s correspond to transformations generated by the elementary transformations on the modulus k_0 such as $k_0 \rightarrow 1/k_0$, or $k_0 \rightarrow k_0' = \sqrt{1-k_0^2}$ (complementary modulus). These transformations permute the three roots $r_1,\ r_2,\ r_3$. The modular invariant $J(\tau)$ is thus given as the sum over a group of transformations isomorphic to the permutation group of three elements: $\mathcal{S}_3$ (see pp. 720 and 754 of [27]). This symmetry group, and its representation as transformations on the modulus k, the modulus k_0, its complementary modulus k_0', the three roots r_1, r_2, r_3, the variables p, p', q, q' and the four homogeneous parameters a, b, c, d, are given in Appendix B. This group is closely related to the above mentioned rearrangement procedure.

These six transformations belong to a larger group of transformations of the elliptic functions: *the modular group* [27], which is a group of transformations preserving the lattice period of the elliptic functions and of course the modular invariant $J(\tau)$. This group of the transformations reads on τ :

$$\tau \longrightarrow \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$$

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ being an $SL(2,\ \mathrm{Z})$-matrix $(\alpha\delta-\beta\gamma=1)$. For instance, transformation $\tau \rightarrow \tau \pm 1$, becomes on the modulus k : $k \rightarrow \pm ik/k'$. One remarks that taking into account the (subtle) compatibility between the group Γ and the weak-graph duality group G, a modular invariant J emerges [18] (it is even invariant by the larger group $\mathrm{G_{Bethe}}$) : the modular group thus comes naturally from the analysis of symmetry groups of quite different nature.

6.2 THE LANDEN TRANSFORMATION AND THE BAXTER MODEL. — Equations (5.9) and (5.10) also underline the role played by the *Landen transformation* [28] on models for which an elliptic parametrization occurs. As far as moduli of elliptic functions are concerned, the Landen transformation amounts to barter k to $k_l = 2\sqrt{k}/(1+k)$. Using k_l and its complementary modulus $k_l' = \sqrt{1-k_l^2}$, one sees that the left-hand side of equations (5.9) and (5.10) read respectively $-4\ k_l'^2/k_l^2$ and $-4/k_l^2$. The transformation discovered by Landen [28] corresponds to the transformation $(x,\ k) \rightarrow (x_l,\ k_l)$ according to:

$$x^2k_l^2\left(1-k_l^2x_l^2\right) - k_l^4x_l^2\left(1-x_l^2\right) = 0 \tag{6.4}$$

This transformation yields the differential equality:

$$\frac{\mathrm{d}x_1}{\sqrt{(1-x_l^2)\,(1-k_l^2x_l^2)}} = \frac{\pm 1}{1+\sqrt{1-k_l^2}}\,\frac{\mathrm{d}x}{\sqrt{(1-x^2)\,(1-k^2x^2)}} \tag{6.5}$$

The Landen transformation is associated to the multiplication of τ, the ratio of the two periods of the elliptic functions, by a factor two. It does not belong to the modular group [27] $(\alpha\delta-\beta\gamma \neq 1)$. It is however a *transformation of the elliptic functions*. It is important to note that this transformation on the elliptic parametrization of our models *can actually be seen as an exact generator of the renormalization group* of the models for which the critical varieties read $k=+1$, or $J=\infty$ (the Baxter model is such a model). The iteration of the Landen transformation, or of its inverse transformation:

$$k = \frac{1-\sqrt{1-k_l^2}}{1+\sqrt{1-k_l^2}} \tag{6.6}$$

converges to the two remarkable varieties $k=0$ and the critical varieties $k=+1$. This is quite clear on τ. The iteration of the transformation $\tau \rightarrow 2\ \tau$ has two fixed points, $\tau=0$ and

$\tau = \infty$, which means that one of the two periods becomes infinite: the parametrization becomes rational. Other remarkable transformations can also be introduced and identified with (exact) generators of the renormalization group: the Legendre, the Jacobi([3]), . transformations (see p. 525 of [48]) which correspond to multiply τ by 3, 5, 7, .. These transformations are also algebraic transformations generalizing (5.8), (6.4) and (6.5).

6.3 COMPLEX MULTIPLICATION FOR THE BAXTER MODEL. — Let $F(z)$ be an elliptic function. Within the framework of symmetries and transformations of elliptic functions, it is well-known that, for every integer m, one can write $F(mz)$ as a rational function of $F(z)$. An important question is: are there some other values of m for which $F(mz)$ can be expressed as a rational function of $F(z)$? When such a situation occurs, F is said to have *complex multiplication* [28, 24-26]. This situation occurs only when τ belongs to an imaginary quadratic field (i.e. is a solution of a quadratic equation $C\tau^2 + (D-A)\tau - B = 0$, where A, B, C, D are integers, the discriminant $(D-A)^2 + 4BC$ being negative), then there exists an algebraic relation between this function and the same function where its arguments have been multiplied by λ, λ being any number in the quadratic field $\Phi(\tau)$.

The simplest examples of complex multiplication are $g_2 = 0$ (i.e. $J(\tau) = 0$, for $\tau = \omega_3/\omega_1 = e^{2\pi i/3}$) also called the "equianharmonic" case [28, 48], and $g_3 = 0$ (i.e. $J(\tau) = 1$, for $\tau = i$) also called the "lemniscatic" case [28, 48]. In these two cases the lattice period has clearly additional symmetries (invariance under the rotation of $2\pi/3$, $\pi/2$, .). Of course, one can try to find systematically other cases for which complex multiplication occurs, imposing $J(\tau)$ to be an algebraic integer [50]([4]). This is quite involved: let us restrict here, only to the two conditions $g_2 = 0$ and $g_3 = 0$, and write explicitly, for the Baxter model, these algebraic subvarieties for which additional symmetries occur for the elliptic functions (and hopefully for the physics of the model!).

1. $g_3 = 0$: The "lemniscatic" cases (see pp. 658-662 of [28] and p. 524 of [48]).

$$\gamma^2 - \delta^2 + 1 = 0$$
$$\sum_i x_i^4 - \sum_{i \neq j} x_i^2 x_j^2 = 0$$
$$k_0^2 = \frac{1}{2}, \quad k_0'^2 = \frac{1}{2}, \quad \tau = i, \quad J(\tau) = 1, \quad k = \pm i$$

For simplicity the notations $x_i = \{a,\ b,\ c,\ d\}$, $(i = 1,\ .\ ,\ 4)$ have been introduced.

$$\gamma^2 - \delta^2 \pm 6\gamma + 1 = 0$$
$$\sum_i x_i^4 - \sum_{i \neq j} x_i^2 x_j^2 \mp 24 x_1 x_2 x_3 x_4 = 0$$
$$k_0^2 = \frac{1}{2} \mp \frac{3}{2}, \quad k_0'^2 = \frac{1}{2} \pm \frac{3}{2}, \quad \tau = i, \quad J(\tau) = 1, \quad k = \pm 3 \mp 2\,\epsilon\sqrt{2}$$

where $\epsilon = \pm 1$.

([3]) Not to be confused with the previously mentioned imaginary Jacobi transformation $\tau \rightarrow -1/\tau$.

([4]) When complex multiplication occurs, the modular invariant $J(\tau)$ is necessarily an *algebraic integer* (i.e. a complex number satisfying an algebraic equation with rational integral coefficients, the coefficient of the highest degree term being equal to 1).

2. $g_2 = 0$: The "equianharmonic" cases (see p. 652 of [28]).

$$\gamma^2 - \delta^2 + 1 \pm 2i\sqrt{3\gamma} = 0$$

$$\sum_i x_i^4 - \sum_{i \neq j} x_i^2 x_j^2 \mp 8i\sqrt{3} x_1 x_2 x_3 x_4 = 0$$

$$k_0^2 = \mathrm{e}^{\pm i\pi/3}, \quad k_0'^2 = \mathrm{e}^{\mp i\pi/3}, \quad \tau = \mathrm{e}^{2i\pi/3}, \quad J(\tau) = 0, \quad k = -\left(2\epsilon \pm \sqrt{3}\right) i$$

where $\epsilon = \pm 1$.

These results can straightforwardly be generalized to more general models (see Eq. (4.6) with Appendix A).

7. Conclusion.

Symmetries of different nature occur on lattice models of statistical mechanics. These symmetries and their relations can be understood on simple examples of two-dimensional vertex-models for which elliptic parametrization occurs. These symmetries have been analyzed here for sixteen-vertex models, in the light of effective algebraic geometry. Heuristically, one has to distinguish the "gauge-like" weak-graph transformations (which emphasize the irrelevant parameters to be "gauged-away"), the symmetry group Γ generated by the inversion relations on the model, which correspond to highly non-trivial transformations. Γ underlines some "spectral parameter" on elliptic curves, enabling the construction of the Bethe Ansatz for the model, or the calculation of the partition function using the inversion trick [4, 10, 11]. In the framework of elliptic parametrization, other symmetries can be considered: the modular group, and more generally, the *transformations of elliptic functions (isogenies)* such as the Landen transformation (which do not preserve the modular invariant): this last symmetry identifies with an exact transformation of the renormalization group of the models for which the critical varieties read $k = +1$, or $J = \infty$. These various symmetries have to be compatible. The compatibility between the weak-graph transformations and the group Γ is a little bit subtle and involved, and has been detailed elsewhere [18]. The compatibility between Γ and the modular group, or more general transformations of elliptic functions, can be understood simply when one remarks that Γ actually leaves invariant the modulus of the elliptic function, or the modular invariant.

Some symmetries are at the cross-roads of these different symmetries: for instance the duality (5.4) and, more generally, the group of subsection (6.1) isomorphic to $\mathcal{S}_3$ (analyzed in Appendix B), can be seen as weak-graph transformations, but can also be seen to play a role in the modular invariant (see Eq. (6.3)), in the group Γ and in the rearrangement procedure described in section (5.1).

We think that the distinction between such different kinds of symmetries (with their subtle relations) is a fruitful approach to analyse lattice models in statistical mechanics (seeking for exact results even in dimensions greater than two). Clearly, there is a one-to-one, correspondence between the physics of the problems (relevant or irrelevant variables, critical manifolds, renormalization group, ...) and the symmetries and transformations acting on the algebraic varieties parametrizing the models.

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Appendix A.

Values of α and β for the Baxter model and some of its generalizations.

1 - The 4×4 matrix of the symmetric eight-vertex Baxter model (four parameters) reads:

$$R = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}$$

The coefficients e_i's and f_i's of the biquadratic equations (3.2) and (3.3) read respectively:

$$e_1 = f_1 = cd, \ e_2 = f_2 = e_3 = f_3 = 0, \ \ e_4 = f_4 = -ab, \ \ e_6 = f_6 = a^2 + b^2 - c^2 - d^2$$

The two expressions α and β introduced in section (4) read:

$$6\alpha = -2a^2b^2 - 2a^2d^2 - 2a^2c^2 - 2b^2d^2 - 2b^2c^2 - 2d^2c^2 + a^4 + b^4 + d^4 + c^4 - 24\, abcd$$
$$4\beta = (a - d + c - b)^2(a + b - c - b)^2(a - d + b - c)^2(a + d + b + c)^2$$

One verifies immediately that:

$$3\alpha - \sqrt{\beta} = -16\, abcd$$
$$3\alpha + \sqrt{\beta} = (a + b + c - d)(a - b - c - d)(a + b - c + d)(a - b + c + d)$$
$$\alpha - \sqrt{\beta} = -\frac{1}{3}\,(-2a^2b^2 - 2a^2d^2 - 2a^2c^2 - 2b^2d^2 - 2b^2c^2 - 2d^2c^2 + a^4 + b^4 + d^4 + c^4 + 24\, abcd)$$
$$\alpha + \sqrt{\beta} = \frac{2}{3}\,(-2a^2b^2 - 2a^2d^2 - 2a^2c^2 - 2b^2d^2 - 2b^2c^2 - 2d^2c^2 + a^4 + b^4 + d^4 + c^4)$$

Equation (3) $\alpha + \sqrt{\beta} = 0$ corresponds to the critical varieties of the model.

2 - Let us consider the five parameter-dependent vertex-model given by the following 4×4 R-matrix:

$$R = \begin{pmatrix} a & e & e & d \\ e & b & c & e \\ e & c & b & e \\ d & e & e & a \end{pmatrix}$$

It is a subcase of model (2.2) for $e = f = g = h$. The coefficients e_i's and f_i's in equations (3.2) and (3.3) read respectively:

$$e_1 = f_1 = cd - e^2, \quad e_2 = f_2 = -e_3 = -f_3 = -e(a + b - c - d)$$
$$e_4 = f_4 = e^2 - ab, \quad e_6 = f_6 = a^2 + b^2 - c^2 - d^2$$

α and β of section (4) read:

$$\begin{aligned} 6\alpha = {} & 16\, e^2ac + 16\, e^2ad + 16\, be^2c + 16\, be^2d + 16\, ce^2d + 16\, e^2ab - 8\, e^2a^2 \\ & - 8e^2b^2 - 8e^2d^2 - 8e^2c^2 - 2a^2b^2 - 2a^2d^2 - 2a^2c^2 - 2b^2d^2 - 2b^2c^2 - 2d^2c^2 \\ & - 32\, e^4 + a^4 + b^4 + d^4 + c^4 - 24\, abcd \end{aligned} \tag{A1}$$
$$4\beta = (a - d + c - b)^2(a + d - c - b)^2(a - d + b - c)^2(a + d + b + c - 4e)\,(a + c + b + d + 4e)$$

One also verifies that:

$$9\alpha^2 - \beta = -16\left(ac - c^2 - 2e^2 + cd + cb\right)\left(da - 2e^2 - d^2 + cd + db\right)\left(ab - 2e^2 - b^2 + cb + db\right)\left(a^2 - ac + 2e^2 - da - ab\right) \tag{A2}$$

The vanishing of (A2) gives good candidates for the critical varieties of these models. The first four factors of the vanishing condition of β (Eq. (A1)) are disorder conditions, as can be seen directly on a disorder criterion [31, 38] (such a local disorder criterion corresponds to equation (3.1) for $p = p'$ and $q = q'$). The local disorder conditions [31] actually read (see also [38]):

$$R(u \otimes v) = \lambda \cdot u \otimes v \tag{A3}$$

and:

$$R^2(u \otimes v) = \mu \cdot u \otimes v \tag{A4}$$

with:

$$u = \begin{pmatrix} 1 \\ p \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ q \end{pmatrix}$$

Equation (A.3) gives $a + d = b + c$, for $p = q = \pm 1$, and $a - d = \pm(b - c)$, for $p = -q = \pm 1$.

3 - Let us consider the following eight-parameter (arrow-reversal invariant) vertex-model given by the 4×4 R-matrix (2.2):

$$R = \begin{pmatrix} a & e & f & d \\ g & b & c & h \\ h & c & b & g \\ d & f & e & a \end{pmatrix}$$

The coefficients e_i's and f_i's in equations (3.2) and (3.3) read respectively:

$$\begin{array}{lll} e_1 = cd - hf, & e_2 = ce + gd - bf - ha, & e_3 = af + hb - ed - cg, \\ e_4 = ge - ab, & e_6 = a^2 + b^2 + h^2 + f^2 - e^2 - d^2 - c^2 - g^2, & \\ f_1 = cd - ge, & f_2 = cf + hd - ga - be, & f_3 = ae + gb - fd - ch, \\ f_4 = hf - ab, & f_6 = a^2 + b^2 - h^2 - f^2 + e^2 - d^2 - c^2 + g^2 & \end{array}$$

α and β then read:

$$\begin{aligned} 6\alpha = & \left(a^2 + b^2 - c^2 - d^2 - e^2 + f^2 - g^2 + h^2\right)^2 - 4(bf + ha - ce - gd)^2 \\ & - 4(hf - cd)^2 - 4(af + hb - ed - cg)^2 - 4(ab - ge)^2 + 24(hf - cd)(ab - ge) \\ 4\beta = & (a + g - e - d + c - h + f - b)(a - g + e + d - c - h + f - b) \\ & (a + d - f - e - c - b + g + h)(a - d - f + e + c - b - g + h) \\ & (a - d + e + g + b - f - h - c)(a + h - c + f + b - d - e - g) \\ & (a + d - e - g + b - f - h + c)(a + h + c + f + b + d + e + g) \end{aligned}$$

The first six factors of the vanishing condition of β correspond to disorder conditions, as can be checked directly (Eq. (A3) and (A4) with $p, q = \pm 1$).

Appendix B.

The Baxter model and its biquadratic form.

Let us consider the biquadratic equation (3.2) for the Baxter model (which identifies with Eq. (3.3) for this model):

$$\Phi(x, y \,;\, \gamma_u,\, \delta_u,\, u) = \gamma_u \left(x^2 y^2 + 1\right) + 2\delta_u xy - u\left(x^2 + y^2\right) \tag{B1}$$

where:

$$\gamma_u = cd, \qquad \delta_u = \frac{a^2 + b^2 - c^2 - d^2}{2}, \qquad u = ab \tag{B2}$$

The modulus k_0, given by the anharmonic ratio of the three roots (5.5) and of the point at infinity reads:

$$k_0^2 = \{r_2,\ r_3;\ r_1,\ \infty\} = \frac{(\gamma_u + \delta_u + u)(\gamma_u - \delta_u + u)}{4\gamma_u u}$$

Two generators Λ_1, Λ_2 of the action of the group introduced in subsection (6.1), isomorphic to the permutation group σ_3, have the following representation on the modulus k_0 :

$$\Lambda_1\left(k_0^2\right) = \frac{k_0^2}{k_0^2 - 1}, \qquad \Lambda_2\left(k_0^2\right) = 1 - k_0^2$$

with $\Lambda_1^2 = \Lambda_2^2 = \mathbf{1}$, $\Lambda_1\Lambda_2\Lambda_1 = \Lambda_2\Lambda_1\Lambda_2$ where $\mathbf{1}$ is the identity transformation.

Their associated homographic transformations on x (or y) read:

$$p_1(x) = \frac{x-1}{x+1}, \qquad p_2(x) = ix$$

Let us give the representation of this order six group as transformations on γ_u, δ_u, u, r_1, r_2, r_3, k_0^2, $k_0'^2$, k, a, b, c, d, with k_l denoting the Landen modulus $\left(k_l = 2\sqrt{k}/(1+k)\right)$:

$\mathbf{1}$	p_1	p_2	$p_1 \cdot p_2$	$p_2 \cdot p_1$	$p_1 \cdot p_2 \cdot p_1$
$x \to x$	$x \to \frac{x-1}{x+1}$	$x \to ix$	$x \to \frac{x+i}{x-i}$	$x \to i\left(\frac{x-1}{x+1}\right)$	$x \to i\left(\frac{x+i}{x-i}\right)$
$2\,\gamma_u$	$\gamma_u + \delta_u - u$	$-2\,\gamma_u$	$-\gamma_u - \delta_u + u$	$-\gamma_u + \delta_u - u$	$\gamma_u - \delta_u + u$
$2\,\delta_u$	$2(\gamma_u + u)$	$2\,\delta_u$	$2(\gamma_u + u)$	$2(-\gamma_u + u)$	$2(-\gamma_u + u)$
$2\,u$	$-\gamma_u + \delta_u + u$	$2\,u$	$-\gamma_u + \delta_u + u$	$\gamma_u + \delta_u + u$	$\gamma_u + \delta_u + u$
r_1	r_1	r_3	r_2	r_3	r_2
r_2	r_3	r_2	r_3	r_1	r_1
r_3	r_2	r_1	r_1	r_2	r_3
k_0^2	$\frac{k_0^2}{k_0^2 - 1}$	$1 - k_0^2$	$\frac{1}{1 - k_0^2}$	$\frac{k_0^2 - 1}{k_0^2}$	$\frac{1}{k_0^2}$
$k_0'^2$	$\frac{1}{k_0'^2}$	$1 - k_0'^2$	$\frac{k_0'^2 - 1}{k_0'^2}$	$\frac{1}{1 - k_0'^2}$	$\frac{k_0'^2}{k_0'^2 - 1}$
k	$\frac{1 + k_l}{1 - k_l}$	$-k$	$-\frac{1 + k_l}{1 - k_l}$	$1 + 2ik_l - 2k_l^2$	$-1 - 2ik_l - 2k_l^2$
$2a$	$a + b + c + d$	$2a$	$a + b + c + d$	$a + b - c + d$	$a + b - c + d$
$2b$	$a + b - c - d$	$2b$	$a + b - c - d$	$a + b + c - d$	$a + b + c - d$
$2c$	$a - b + c - d$	$-2c$	$-a + b - c + d$	$a - b - c - d$	$-a + b + c + d$
$2d$	$a - b - c + d$	$2d$	$a - b - c + d$	$a - b + c + d$	$a - b + c + d$

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