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Disorder solutions for Ising and Potts models with a field

M T Jaekel† and J M Maillard‡

† Laboratoire de Physique Théorique, de l'Ecole Normale Supérieure, §24 rue Lhomond, 75231 Paris Cedex 05, France

‡ Laboratoire de Physique Théorique et Hautes Energies,|| 4 Place Jussieu, 75230 Paris Cedex 05, France

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Abstract. The exact expressions of the disorder varieties and the corresponding values taken by the partition functions of the checkerboard Ising and Potts models with a magnetic field are obtained. Different specialisations and extensions of these exact results are examined.

1. Introduction

In the framework of two-dimensional Ising and Potts models, few exact solutions (with an analytical expression for the partition function) are known, when the interactions include a magnetic field (for a review, see Wu 1982). Remarkable examples were provided in the case of anisotropic models on a triangular lattice, by Verhagen (1976) and Rujan (1984) for Ising and Potts models respectively. These solutions belong to the class of the so-called disorder solutions (Stephenson 1970, Welberry and Galbraith 1973, Enting 1977, 1978). A local criterion to derive them in a simple way has also been given recently (Jaekel and Maillard 1985). Using the latter for the anisotropic Ising and Potts models with a magnetic field, we shall obtain the disorder varieties and the corresponding (rational) values of the partition functions for the checkerboard lattice. By means of an appropriate limit, these solutions will also provide those associated with the triangular lattice.

When the number of spin states is greater than two, the case of the Potts model appears to be more restrictive (leading to a disorder variety of codimension two) than that of the Ising model (disorder variety of codimension one). Hence, we shall first try to give, as long as possible, a common treatment to both models, but when coming to the analytical evaluation of the disorder varieties and of the partition functions, we shall have to deal with each model separately. A second part will be devoted to the study of the symmetry properties of the solution and its various limits.

2. Derivation of the disorder solutions

The partition function per site Z of the checkerboard Ising or Potts model with a

§ Laboratoire Propre du Centre National de la Recherche Scientifique, associé à l'Ecole Normale Supérieure et à l'Université de Paris Sud.

|| Laboratoire associé au Centre National de la Recherche Scientifique et à l'Université Pierre et Marie Curie.

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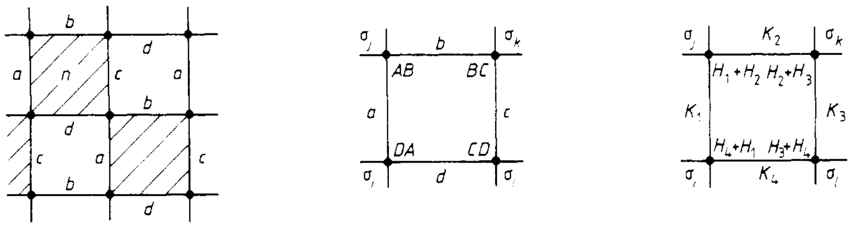
magnetic field can be written

$$Z^{2N} = \sum_{\{\sigma\}} \prod_{n=1}^{N} w_n(\sigma)$$

where σ denotes the different spins, n an elementary generating cell and w_n its associated Boltzmann weight. (In the following, we shall make use of standard conventions, as indicated in the figure; a and c (resp. K_1 and K_3) alternatively stand for nearest-neighbour interactions along columns, and b and d (resp. K_2 and K_4) along rows.)



The associated Boltzmann weight for the Potts model is given by

$$w_n(\sigma) = a^{\delta_{\sigma_i\sigma_j}} b^{\delta_{\sigma_j\sigma_k}} c^{\delta_{\sigma_k\sigma_l}} d^{\delta_{\sigma_l\sigma_i}} (DA)^{\delta_{\sigma_i 0}} (AB)^{\delta_{\sigma_j 0}} (BC)^{\delta_{\sigma_k 0}} (CD)^{\delta_{\sigma_l 0}}$$

($\sigma_i, \sigma_j, \sigma_k, \sigma_l \in \mathbb{Z}_q$, $z = ABCD$; the partition function depends only on a, b, c, d and z) and for the Ising model

$$w_n(\sigma) = \exp(K_1\sigma_i\sigma_j + K_2\sigma_j\sigma_k + K_3\sigma_k\sigma_l + K_4\sigma_l\sigma_i)$$
$$\times \exp[(H_4+H_1)\sigma_i + (H_1+H_2)\sigma_j + (H_2+H_3)\sigma_k + (H_3+H_4)\sigma_l]$$

($\sigma_i, \sigma_j, \sigma_k, \sigma_l = \pm 1$, $H = H_1 + H_2 + H_3 + H_4$; the partition function depends only on the K_i's and H). Note that these standard conventions imply different definitions of the zero of energy, so that the Ising model can be recovered from the Potts model at $q = 2$, but only after an appropriate rescaling of the Boltzmann weight. One will also notice that, *a priori*, the only symmetries of the partition functions which follow from their definitions are those of the square ($a \leftrightarrow c$, $b \leftrightarrow d$ and $a \to b \to c \to d \to a$, or $K_1 \leftrightarrow K_3$, $K_2 \leftrightarrow K_4$, $K_1 \to K_2 \to K_3 \to K_4 \to K_1$).

The local criterion (Jaekel and Maillard 1985) mentioned in the introduction can be stated as follows: when the following condition is satisfied

$$\sum_{\sigma_k, \sigma_j} w_n(\sigma_i, \sigma_j, \sigma_k, \sigma_l) = \lambda, \qquad (1)$$

independent of σ_i, σ_l ($\lambda(a, b, c, d, z)$ for the Potts model; $\lambda(K_1, K_2, K_3, K_4, H)$ for the Ising model, the partition function per site is then given by

$$Z = \lambda^{1/2}. \qquad (2)$$

The criterion just defined naturally singles out one of the four interactions (d or K_4), But, of course, one can also introduce the similar criteria which single out the other interactions and thus obtain four different disorder solutions.

In order to make explicit the equations implied by (1) on the parameters of the model, it appears convenient to introduce $q\times q$ matrices M_q, the elements of which are given by

$$M_q(a, A)_{\sigma_i\sigma_j} = A^{\delta\sigma_i 0} a^{\delta\sigma_i\sigma_j} A^{\delta\sigma_j 0}.$$

Equation (1) can be rewritten as

$$M_q(a, A)M_q(b, B)M_q(c, C) = \lambda M_q(1/d, 1/D)$$

or, using the identity: $(x-1)(1-q-x)M_q^{-1}(x, X) = M_q(2-q-x, 1/X)$

$$M_q(a, A)M_q(b, B)M_q(c, C)M_q(\bar{d}, D) = \lambda(\bar{d}-1)(1-q-\bar{d})\mathbb{1}_q \tag{3}$$

where $\bar{d}$ is equal to $2-q-1/d$ and $\mathbb{1}_q$ denotes the $q\times q$ identity matrix. In the case of the Ising model ($q=2$), because of a different definition of the Boltzmann weight, one has

$$M_2(a, A)M_2(b, B)M_2(c, C)M_2(\bar{d}, D) = -\mathrm{i}\lambda ABCD(abc\bar{d})^{1/2}(\bar{d}-1/\bar{d})\mathbb{1}_2 \tag{4}$$

with

$$\begin{aligned} &a=\mathrm{e}^{2K_1}, && b=\mathrm{e}^{2K_2}, && c=\mathrm{e}^{2K_3}, && \bar{d}=\mathrm{e}^{\mathrm{i}\pi-2K_4},\\ &A=\mathrm{e}^{2H_1}, && B=\mathrm{e}^{2H_2}, && C=\mathrm{e}^{2H_3}, && D=\mathrm{e}^{2H_4}. \end{aligned} \tag{5}$$

In the $q>2$ case, the matrix $M_q(a, A)$ can be reduced by conjugation to the product of a diagonal matrix $(a-1)\mathbb{1}_{q-2}$ and of a 2×2 matrix. (Here, we make use of a discrete Fourier transform over $\mathbb{Z}_{q-1}$, which takes into account the identical roles played by the $(q-1)$ spin states different from zero):

$$m(a, A) = \begin{bmatrix} aA^2 & (q-1)^{1/2}A \\ (q-1)^{1/2}A & a+q-2 \end{bmatrix}.$$

Therefore equation (3) is equivalent to the following system of equations

$$m(a, A)m(b, B)m(c, C)m(\bar{d}, D) = \lambda(\bar{d}-1)(1-q-\bar{d})\mathbb{1}_2 \tag{6}$$

and (the relation on the diagonal part leading to a unique equation):

$$\lambda = \frac{(a-1)(b-1)(c-1)}{1-q-\bar{d}}. \tag{7}$$

(Then, the case of the Potts model ($q>2$) differs from the Ising one ($q=2$) by the presence of an additional equation: (7).) In any case, the value of λ (and therefore of the partition function per site) is extracted from the matrix equations (4) or (6) by taking the determinant ($\det m(a, A) = (a-1)(a+q-1)A^2$); for the Potts model ($q>2$)

$$\lambda^2 = z^2\frac{(a-1)(a+q-1)(b-1)(b+q-1)(c-1)(c+q-1)}{(1-1/d)(1-q-1/d)} \tag{8}$$

and for the Ising model ($q=2$) ($\det M_2(a, A) = (a^2-1)A^2$)

$$\lambda^2 = -\frac{(a-1/a)(b-1/b)(c-1/c)}{(d-1/d)}. \tag{9}$$

The equations of the disorder variety are obtained through elimination of A, B, C, D (with the constraint $z = ABCD$) in the following matrix equation

$$
\begin{aligned}
& m(a, A)m(b, B) = \Lambda m(\bar{d}, D)^{-1} m(c, C)^{-1} \\
& \Lambda = \lambda(\bar{d} - 1)(1 - q - \bar{d}), \qquad \text{for } q > 2 \\
& \Lambda = -\mathrm{i}\lambda z(abc\bar{d})^{1/2}(\bar{d} - 1/\bar{d}), \qquad \text{for } q = 2.
\end{aligned} \tag{10}
$$

One remarks that, in this equation, the equality of the off-diagonal terms is provided by the irrelevant ratio A/B (or C/D) and by the equations (8) or (9). The equality of the diagonal elements (that depend only on AB and CD) then leads to the desired (unique) equation

In the Ising case, this can be given in the following form

$$
\begin{aligned}
16abc\bar{d}(a^2-1)(b^2-1)(c^2-1)(\bar{d}^2-1)(z-1)^2[abc\bar{d}(z+1)^2 - 4z] \\
= z^2 f(a, b, c, \bar{d}) g(a, b, c, \bar{d}) g(a, c, \bar{d}, b) g(a, \bar{d}, b, c)
\end{aligned} \tag{11}
$$

with

$$
f(a, b, c, \bar{d}) = (a-1)(b-1)(c-1)(\bar{d}-1) - (a+1)(b+1)(c+1)(\bar{d}+1)
$$

and

$$
g(a, b, c, \bar{d}) = (a-1)(b-1)(c+1)(\bar{d}+1) - (a+1)(b+1)(c-1)(\bar{d}-1).
$$

Thus, to sum up, on the variety of codimension one defined by equations (5) and (11), the partition function per site is given by (2) and (9). In the Potts case ($q > 2$), the disorder variety is characterised by two equations: one is obtained by eliminating λ between equations (7) and (8), and expresses the magnetic field in terms of the interactions of the model

$$
z^2 = \frac{a-1}{a+q-1} \frac{b-1}{b+q-1} \frac{c-1}{c+q-1} \frac{\bar{d}-1}{\bar{d}+q-1} \tag{12}
$$

and the other one can be made independent of z by substituting the z value (12) into the unique equation derived from the matricial equation (10):

$$
\lambda_{\mathrm{D}}(\lambda_{\mathrm{C}}^2 \lambda_{\mathrm{D}} + 2\mu\lambda_{\mathrm{C}} + q^2 \lambda_{\mathrm{D}}) = 0 \tag{13}
$$

where

$$
\mu = (a-1)(b-1)(c-1)(\bar{d}-1) + (a+q-1)(b+q-1)(c+q-1)(\bar{d}+q-1)
$$

$$
-q\lambda_{\mathrm{D}} = (a-1)(b-1)(c-1)(\bar{d}-1) - (a+q-1)(b+q-1)(c+q-1)(\bar{d}+q-1)
$$

($\lambda_{\mathrm{D}} = 0$ gives the disorder variety of the model without a field (Baxter 1984))

$$
\lambda_{\mathrm{C}} = \lambda_{\mathrm{D}} - q[a + b + c + \bar{d} + 2(q-2)]
$$

(and $\lambda_{\mathrm{C}} = 0$ gives the critical variety of the model without a field (see for instance Hintermann *et al* 1981)). Restricted to this disorder variety, of codimension two, the partition function is given by (2) and (7).

3. Symmetry and different limits

One merit of the previous matricial translation of the local criterion is to treat the four interaction couplings on an equal footing (see equations (3), (4) for instance) once d

is replaced by $\bar{d}$. Curiously, the expressions of the disorder varieties appear to be completely symmetric under all permutations of the four variables a, b, c and $\bar{d}$ (see equations (11), (12), (13)). Technically, this unexpected result can be understood in the following way: there exist changes of the local fields H_i (A, B, C, D), which map the problem onto an equivalent one, where two of the coupling constants K_i (a, b, c, $\bar{d}$) have been permuted, while the value of the total magnetic field remains unchanged (z fixed)

$$m(a, A)m(b, B) = m(b, B')m(a, A')$$

with $AB = A'B'$. As a consequence, and because equations (7) and (9) are also symmetric under the permutations of a, b, c, the disorder solutions finally exhibit a complete S_3 symmetry. As already remarked at the beginning of the previous section, one can change the coupling constant that is distinguished in the expression of the local criterion (here d), and then obtain a set of disorder solutions which is invariant under the S_4 symmetry (all permutations of a, b, c, d). Arguments have been given elsewhere that support the idea that this last symmetry could be valid in general, for the whole partition function of the model with a magnetic field, and not only when it is restricted to the disorder varieties (Jaekel *et al* 1985).

One advantage of dealing with an anisotropic model on the checkerboard lattice is that it also provides simultaneously solutions for various kinds of other lattices or models. For instance, the $c \to \infty (K_3 \to \infty)$ limit allows one to recover the disorder solution for a model on a triangular lattice (for the Ising model see, for instance, Jaekel and Maillard (1985) with the following substitutions: $t_1 = (a-1)/(a+1)$, $t_2 = (b-1)/(b+1)$, $-1/t_4 = (\bar{d}-1)/(\bar{d}+1)$, and the change of H into $2H$). For the Potts model ($q > 2$), the condition (13) factorises in this limit into: $\lambda_D^2 \lambda_C^2 = 0$, i.e. the union of the already known disorder variety (Rujan 1984) ($\lambda_D = 0$ and $z = 1$) and of the critical variety (Baxter *et al* 1978) of the same model, but without a field

$$ab + a\bar{d} + b\bar{d} + (q-2)(a+b+\bar{d}) + (q-1)(q-3) = 0$$

$$\Leftrightarrow abd - (a+b+d) - (q-2) = 0 \qquad (14)$$

and z given by

$$z^2 = \frac{a-1}{a+q-1} \frac{b-1}{b+q-1} \frac{d(1-q)-1}{d-1}.$$

On this variety, the partition function is given by

$$Z = \frac{(a-1)(b-1)}{(1/d-1)}.$$

In fact, as is shown by equation (13), for the checkerboard Potts model, the disorder variety has two (in general) irreducible components: one being the already known disorder variety for the model without a field (Baxter 1984) ($\lambda_D = 0$ and $z = 1$), and the other one being of a quite high degree

$$\lambda_C^2 \lambda_D + 2\mu\lambda_C + q^2 \lambda_D = 0. \qquad (15)$$

Let us just remark that this last variety shares a common intersection with both the disorder and the critical varieties ($\lambda_D = 0$, $\lambda_C = 0$) of the checkerboard Potts model without a field. Besides the already mentioned triangular lattice, there are still other cases when (15) can be reduced. The honeycomb lattice limit ($c \to 1$) is one of them ($K_3 \to 0$):

$$\lambda_D(\lambda_C + q)^2 = 0$$

and another one is given by the limit $c \to 1 - q$ ($K_3 \to i\pi$):

$$\lambda_D(\lambda_C - q)^2 = 0.$$

Moreover, other examples of disorder solutions for Potts (or Ising) models with a magnetic field can also be obtained by means of the same technique. We shall only mention the use of the solution on a triangular lattice to deduce two others. One corresponds to the checkerboard Potts model, but now with an inhomogeneous field, acting on only one half of the spins (those of types σ_i and σ_k in the figure). The disorder variety is still of codimension two: one of its two equations identifies with the criticality condition for the checkerboard Potts model without a field:

$$\lambda_C = 0 \Leftrightarrow (q-1)(q-3) = abcd - (q-2)(a+b+c+d) - (ab+ac+ad+bc+bd+cd).$$

The other equation still gives the expression of the field in terms of the four coupling constants

$$z = \frac{a-1}{a+q-1}\,\frac{b-1}{b+q-1}\,\frac{c-1}{c+q-1}\,\frac{\bar{d}-1}{\bar{d}+q-1}.$$

Restricted to this disorder variety, the partition function is given by (2) and (7). Another one corresponds to the anisotropic Kagomé-Potts model, with interactions a, b, d along the three different directions, and with a magnetic field (the exponential of which is given by z). One gets a disorder variety of codimension two again: one equation identifies with the criticality condition for the triangular Potts model (14), while the other expresses the magnetic field as a rather complicated function of a, b and d. The expression of the partition function on this disorder variety is then

$$Z = q^{1/3}\left(\frac{(a-1)(b-1)}{(1/d-1)}\right)^{2/3}.$$

4. Conclusion

One merit of the disorder solutions is to exhibit, rather simply, some features of the models. In particular, the S_4 symmetry (which, when restricted to each of the disorder solutions, reduces to an S_3 symmetry), that appears in the Potts model without a field can be confirmed by other exact solutions (at criticality and for $q=2$, $q \to 0$) and expansions (large q, high temperature ...). The presence of a magnetic field seems to preserve this unexpected S_4 symmetry of the checkerboard Potts model.

The same criterion (3) is easily generalised to other discrete spin models, with a magnetic field, but it is then more difficult to put into evidence the S_3 symmetry, on other models than the scalar Potts model. This, and also the algebraically simple character of the disorder solutions, emphasise the remarkable analytical structure of the partition function of the Potts model, even in the presence of a magnetic field.

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