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LETTER TO THE EDITOR

Susceptibility of the checkerboard Ising model

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Abstract. We show that at the disorder variety, the n-point correlation functions of the checkerboard Potts model have a simple causal structure, and decay as simple exponentials for space-like separations. An exact expression for the susceptibility in the Ising case is obtained. The result is additional evidence in favour of the conjectured S_4 symmetry of the checkerboard Potts model in the presence of a finite field.

The study of equilibrium statistical models restricted to disorder varieties (Verhagen 1976, Enting 1977, Peschel and Emery 1981, Dhar 1983, Rujan 1984, Jaekel and Maillard 1984, Baxter 1984) has been a topic of much interest in recent years. The disorder surfaces are special algebraic varieties (point sets satisfying one or more algebraic equations in their coordinates) in the parameter space of the Boltzmann weights of the model, for which the partition function can be calculated quite easily, and has a simple algebraic structure. On these varieties (called disorder varieties hereafter), the configurations of the d-dimensional model may be considered as time histories of the probabilistic evolution of a $(d-1)$-dimensional model, and thus these models are also interesting in the context of non-equilibrium statistical mechanics.

For two-dimensional models with reflection symmetry in disorder varieties, it is known that intra-row correlation functions have a simple exponential form (see Stephenson (1970) for the triangular Ising model and Baxter (1984) for the checkerboard Potts model). In this letter, we extend these results and show that correlation functions at disorder varieties have a simple causal structure, and the correlation functions are simple for space-like separations. For two-dimensional models, the correlation functions for space-like separations are simple exponentials.

The simple form of correlation functions at the disorder variety suggests the development of a systematic perturbation theory around these remarkable varieties. As a first step towards this goal, we have calculated the susceptibility of the Ising model on the checkerboard lattice (which includes as a special case the triangular lattice) restricted to the disorder variety. We note that the exact expression for susceptibility is not known for any of the two-dimensional *ferromagnetic* Ising models. It has been calculated exactly only for some very special models, i.e. the super-exchange model (Fisher 1960) and a triangular Ising model with some one, two and three spin couplings (Enting 1977).

The expression for susceptibility has an unexpected S_3 symmetry, which is the remainder of a more general S_4 symmetry conjectured to hold for the checkerboard

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Potts model for arbitrary q and finite fields (Jaekel *et al* 1985). Our calculation is thus additional rather strong evidence in favour of the conjectured symmetry.

Using the local criterion of Jaekel and Maillard (1985) we first show that the correlation functions of the checkerboard Potts model at the disorder variety have a simple exponential behaviour with distance for space-like separations (see also Enting (1978)).

Consider the scalar q-state Potts model on the checkerboard lattice. The Boltzmann weight of a small square with boundary spins α, β, γ and δ is $W(\alpha, \beta, \gamma, \delta)$ (figure 1) given by

$$W(\alpha, \beta, \delta, \gamma) = \exp(K_1 \delta_{\alpha,\beta} + K_2 \delta_{\beta,\gamma} + K_3 \delta_{\gamma,\delta} + K_4 \delta_{\delta,\alpha}).$$

The Boltzmann weight of a configuration of spins is defined as the product of Boltzmann weights of all the black squares of the checkerboard. Of course, equivalently we may define it as the product of weights of all the white squares (this corresponds to an interchange of K_1 with K_3 and K_2 with K_4).

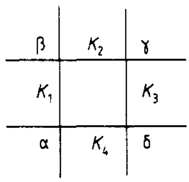

Figure 1. The nearest-neighbour couplings for the spins α, β, γ, δ at the four corners of an elementary black square of the checkerboard lattice.

The local criterion of Jaekel and Maillard is as follows: if the Boltzmann weights W are such that

$$\sum_{\beta,\gamma} W(\alpha, \beta, \gamma, \delta) = \lambda \tag{1}$$

independent of the choice of α and δ, then the partition function per site is $\lambda^{1/2}$. To derive this result, we need only note that for a finite lattice with particular boundary conditions, integration over the spins in the topmost row gives rise to a factor λ for each black square, and the same boundary conditions are reproduced for the next topmost row. Since the partition function is insensitive to the choice of boundary conditions, the result follows. For the checkerboard Potts model, the disorder condition (1) simplifies to

$$\frac{e^{-K_4} - 1}{e^{-K_4} + q - 1} = \frac{e^{K_1} - 1}{e^{K_1} + q - 1} \, \frac{e^{K_2} - 1}{e^{K_2} + q - 1} \, \frac{e^{K_3} - 1}{e^{K_3} + q - 1}$$

and the partition function per site $\lambda^{1/2}$ is given by

$$\lambda^{1/2} = \left(\frac{(e^{K_1} - 1)(e^{K_2} - 1)(e^{K_3} - 1) e^{K_4}}{1 - e^{K_4}} \right)^{1/2}.$$

We define a space-like line on the checkerboard lattice as a sequence of steps in which each horizontal step is directed rightwards, and each vertical step (upwards or downwards) is preceded by a horizontal step on a K_2 bond. A set of sites is called space-like separated if there exists a space-like line going through all of them.

To find the correlation functions along a space-like line, we integrate from the top row of spins downwards recursively 'eating away' the black squares, and from the bottom layer upwards by integrating over white squares. In this way all sites of the lattice are eaten away, except for the ones on the line. Clearly, all correlation functions for space-like separated points are the same as those on a line (i.e. *one-dimensional*). For the Potts model, these are known to have a simple product form and can be written down by inspection. For example, the two-point function $(q\langle\delta_{\sigma_0,\sigma_R}\rangle-1)/(q-1)$ is just the product $t_1^{n_1}t_2^{n_2}t_3^{n_3}\bar{t}_4^{n_4}$, where n_1, n_2, n_3 and n_4 are just the numbers of bonds of type 1, 2, 3 and 4, respectively, in the space-like path connecting the two points, and $t_i=(e^{K_i}-1)(e^{K_i}+q-1)^{-1}$ for $i=1,2,3$, and $\bar{t}_4=(e^{-K_4}-1)(e^{-K_4}+q-1)^{-1}$. The effective coupling constant for bonds of the type 4 is $(-K_4)$ and is reversed in sign because a K_4 bond on the space-like path appears in the integration of the black squares above, as well as the white squares below.

This 'causal' structure of n-point functions at the disorder variety is clearly generalisable to higher dimensions, and is related to the fact that these models are equivalent to stochastic evolution models with local interactions. The simple exponential fall-off is specific to two-dimensional models.

Calculation of correlation functions for non-space-like separations is not so simple. Integrating over as many spins as possible from above as well as below, we are still left with a number of unintegrated squares (figure 2). While for small separations these may be evaluated by brute force, an analytical solution for arbitrary separations is not possible. We are thus unable to calculate the susceptibility for the general Potts model. In the Ising case, the problem simplifies and an exact calculation for the susceptibility can be done.

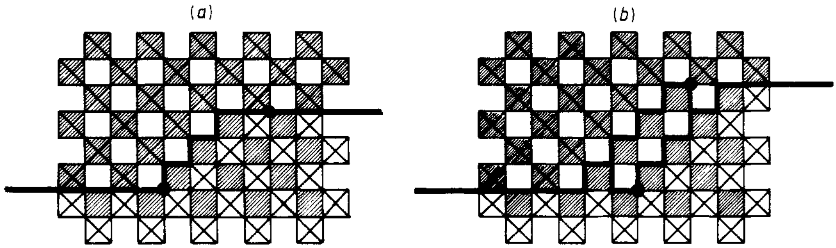

Figure 2. Calculation of two-spin correlations at the disorder variety. The squares that can be integrated over trivially are crossed out. (a) For space-like separations, only spins on the dark line remain unintegrated. (b) For non-space-like separations the spins inside the region enclosed inside the dark lines also remain unintegrated.

The disorder condition for the checkerboard Ising model is given by

$$t_1t_2t_3+t_4=0 \tag{2}$$

where $t_i=\tanh K_i (i=1,\ldots,4)$.

The sites of the checkerboard lattice can be divided into two sublattices, to be called the even and odd sublattices. At zero field each lattice has zero magnetisation. Let $h\,G^{\text{odd}}(\boldsymbol{R})$ be the magnetisation induced at the site $(\boldsymbol{\alpha}+\boldsymbol{R})$ if an infinitesimal field h is imposed at an odd site $\boldsymbol{\alpha}$. The correlation function $G^{\text{even}}(\boldsymbol{R})$ is defined similarly. By symmetry

$$G^{\text{odd}}(\boldsymbol{R})=G^{\text{even}}(-\boldsymbol{R}). \tag{3}$$

The susceptibility χ is given by the relation

$$\chi = \sum_{\boldsymbol{R}} G^{\text{odd}}(\boldsymbol{R}) = \sum_{\boldsymbol{R}} G^{\text{even}}(\boldsymbol{R}).$$

Let $\mathcal{M}^{\text{odd}}(Y)$ and $\mathcal{M}^{\text{even}}(Y)$ be the induced magnetisation at the odd and even sites with ordinate Y, respectively, when an infinitesimal field h is introduced at all sites on the X axis (ordinate 0):

$$\mathcal{M}^{\alpha}(Y) = h \sum_{m=-\infty}^{+\infty} [G^{\text{even}}(2m, Y) + G^{\text{odd}}(2m-1, Y)] \tag{4}$$

where α is even or odd according to whether Y is even or odd. A similar equation holds for $\mathcal{M}^{\alpha'}(Y)(\alpha' \neq \alpha)$.

Using equation (4) we get for small fields

$$\chi = \frac{1}{2h} \sum_{Y=-\infty}^{+\infty} [\mathcal{M}^{\text{odd}}(Y) + \mathcal{M}^{\text{even}}(Y)]. \tag{5}$$

The calculation of $\mathcal{M}^{\text{odd}}(Y)$ and $\mathcal{M}^{\text{even}}(Y)$ is straightforward. Eating away the black squares above the row with ordinate Y gives a factor λ independent of the configuration of spins at row Y. Hence $\langle\beta\rangle$ and $\langle\gamma\rangle$ are the same as their value if all the black squares above the row containing spins β and γ were absent. Let $Z(\alpha, \delta)$ be the restricted partition of the lattice excluding the interactions between the spins α, β, γ and δ, with spins α and δ held fixed, and integrating over all other spins. From the definition of equilibrium averages we get

$$\langle\beta\rangle = \left(\sum_{\alpha,\beta,\gamma,\delta} \beta W(\alpha, \beta, \gamma, \delta) Z(\alpha, \delta)\right)\left(\sum_{\alpha,\beta,\gamma,\delta} W(\alpha, \beta, \gamma, \delta) Z(\alpha, \delta)\right)^{-1} \tag{6}$$

where

$$W(\alpha, \beta, \gamma, \delta) = (1 + t_1\alpha\beta)(1 + t_2\beta\gamma)(1 + t_3\gamma\delta)(1 + t_4\alpha\delta) \tag{7}$$

and α, β, γ and δ are Ising variables taking values ± 1. Using the relations

$$\sum_{\beta,\gamma} \beta W(\alpha, \beta, \gamma, \delta) = 4[(t_1 + t_2 t_3 t_4)\alpha + (t_2 t_3 + t_1 t_4)\delta]$$

and

$$\sum_{\alpha,\beta,\gamma,\delta} W(\alpha, \beta, \gamma, \delta) Z(\alpha, \delta) = 4(1 + t_1 t_2 t_3 t_4) Z(\alpha, \delta)$$

we get

$$\langle\beta\rangle = [(t_1 + t_2 t_3 t_4)\langle\alpha\rangle + (t_2 t_3 + t_1 t_4)\langle\delta\rangle](1 + t_1 t_2 t_3 t_4)^{-1}. \tag{8}$$

The expression for $\langle\gamma\rangle$ in terms of $\langle\alpha\rangle$ and $\langle\delta\rangle$ is written down by symmetry. This shows that $\mathcal{M}^{\text{odd}}(Y)$ and $\mathcal{M}^{\text{even}}(Y)$ are linearly related to $\mathcal{M}^{\text{odd}}(Y-1)$ and $\mathcal{M}^{\text{even}}(Y-1)$. Writing in a matrix form ($Y \geqslant 1$)

$$\begin{bmatrix} \mathcal{M}^{\text{odd}}(Y) \\ \mathcal{M}^{\text{even}}(Y) \end{bmatrix} = T \begin{bmatrix} \mathcal{M}^{\text{odd}}(Y-1) \\ \mathcal{M}^{\text{even}}(Y-1) \end{bmatrix} \tag{9}$$

where T is a 2×2 matrix:

$$(1 + t_1 t_2 t_3 t_4)^{-1} \begin{bmatrix} t_1 t_4 + t_2 t_3 & t_1 + t_2 t_3 t_4 \\ t_3 + t_1 t_2 t_4 & t_3 t_4 + t_1 t_2 \end{bmatrix}. \tag{10}$$

From equations (9) and (5) we get

$$\chi = \left(\frac{1}{2h}\right)(1, 1)\left(\mathbb{1} + \sum_{n=1}^{\infty} 2T^n\right)\begin{bmatrix}\mathcal{M}^{\text{odd}}(0)\\ \mathcal{M}^{\text{even}}(0)\end{bmatrix}. \tag{11}$$

It is easily shown that

$$\mathcal{M}^{\text{odd}}(0) = \mathcal{M}^{\text{even}}(0) = h(1+t_2)(1-t_4)(1-t_2t_4)^{-1}.$$

Substituting in equation (11) we get finally

$$\chi = \frac{(1+t_2)(1+t_4)}{2(1-t_2t_4)}(1, 1)\left(\frac{1+T}{1-T}\right)\begin{pmatrix}1\\1\end{pmatrix} \tag{12}$$

whose T is given by equation (10).

The expression for χ is clearly invariant under the interchange of t_1 and t_3. However, there is a strong evidence that the partition function of the checkerboard Potts model for any odd and even line in the presence of a finite field has an unexpected S_4 symmetry with respect to any permutation of the four variables t_1, t_2, t_3 and t_4 (Jaekel *et al* 1985). If this conjectured symmetry is exact χ should be invariant under arbitrary permutations of t_1, t_2 and t_3 (since t_4 is singled out by the disorder criterion). This symmetry is far from obvious by inspection of equations (10) and (12). However, a complete evaluation of the matrix element in equation (12) gives a rather compact expression for χ (we used the symbolic manipulation program REDUCE (Hearn 1984) to handle the rather tedious algebra) on the disorder variety

$$\chi = \frac{(1+t_1)(1+t_2)(1+t_3)(1+t_1t_2t_3)}{(1-t_1t_2)(1-t_2t_3)(1-t_1t_3)} \tag{13}$$

when

$$t_1t_2t_3 + t_4 = 0. \tag{14}$$

The result is manifestly symmetric under all permutations of t_1, t_2 and t_3. As a check on the calculation, we have verified that χ agrees with the high temperature expansion for the susceptibility of the checkerboard Potts model (Syozi and Naya 1960) up to seventh order in t_i.

A disorder solution for the checkerboard Ising model can also be obtained in the presence of a finite field (Jaekel and Maillard 1985). This does not allow a direct calculation of the susceptibility as one is obliged to take derivatives only along the disorder surface. Knowing the derivatives in a non-tangential direction (say, along K_4) the calculation of susceptibility is, in principle, possible, but is much more complicated than the treatment above due to the complicated equation of a disorder surface in the presence of a finite field.

The unexpected simplicity and symmetry of the expression for susceptibility is additional strong evidence in favour of the conjectured S_4 symmetry of the partition function of the checkerboard Ising model in the presence of a finite magnetic field. In the absence of an external field, Bhazanov and Stroganov (1985) have shown that it is related to a 'hidden' SL(2, R) symmetry of the vertex version of the model. In the presence of a finite field the symmetry is even more subtle, and deserves further study.

The simple structure of the n-point correlation functions should help in developing a systematic perturbation theory about the disorder solutions. We showed that the expression for susceptibility for the two-dimensional Ising model at the disorder point is a simple rational expression in the high temperature variables. Does a similar result hold for the checkerboard Potts model?

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