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DEFORMATIONS OF DYNAMICS ASSOCIATED TO THE CHIRAL POTTS MODEL

M.P. Bellon[1], J-M. Maillard, G. Rollet, C-M. Viallet

Abstract.

We describe deformations of non-linear (birational) representations of discrete groups generated by involutions, having their origin in the theory of the symmetric five-state Potts model. One of the deformation parameters can be seen as the number q of states of a chiral Potts models. This analogy becomes exact when q is a Fermat number. We analyze the stability of the corresponding dynamics, with a particular attention to orbits of finite order.

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Postal address: Laboratoire de Physique Théorique et des Hautes Energies Université de Paris VI et VII, URA280 Tour 16, 1er étage, boîte 126. 4 Place Jussieu/ F–75252 PARIS Cedex 05

[1] Present address: Laboratoire de Physique Théorique ENSLAPP, Ecole Normale Supérieure de Lyon, 46 Allée d'Italie, F-69364 Lyon Cedex 07, FRANCE

1 Introduction

In previous publications [1, 2, 3, 4, 5], we have presented a class of *birational transformations* in projective space, which are symmetries of the Yang-Baxter equations (or star-triangle relation and their higher dimensional generalizations) and also symmetries for the model. These transformations are generated by involutions which represent the *inversion relations* of lattice models of statistical mechanics [6, 7, 8]. They amount to taking inverse of matrices whose entries belong to the parameter space.

This construction has a priori a certain rigidity: it provides with explicit examples of birational automorphisms of $\mathbb{C}\mathbf{P}_N$ for any N, with *integer* coefficients. Some of these mappings have non-trivial properties : their orbits are dense in non trivial algebraic subvarieties of $\mathbb{C}\mathbf{P}_N$.

We study here some deformations of the transformations appearing in the symmetric five-state Potts model, for which the parameter space identifies with projective space $\mathbb{C}\mathbf{P}_2$. We introduce families of deformations, depending on respectively four, two and one parameters. For the single–parameter family, this parameter q can be seen heuristically as the number of colours of q-state chiral Potts models, and the orbits under iteration of the transformations *lie on algebraic curves* (this is no longer the case for the two (or four) parameter families of deformations we produce). We thus have a *q-deformation* of the linear pencil of elliptic curves foliating the parameter space $\mathbb{C}\mathbf{P}_2$ of the symmetric five state Potts model. Among the orbits, the ones where the group has a *finite order* representation pop out immediately, and we describe some of them.

2 Construction

Let us consider q-state spin models with nearest neighbour interactions on a square lattice. The Boltzmann weights $w(\sigma_i, \sigma_j)$ for an oriented bond $\langle ij \rangle$ can be seen as the entries m_{ij} of a $q \times q$ matrix. The matrix of Boltzmann weights is a $q \times q$ matrix M with (complex) homogeneous entries m_{ij}. Choosing a specific model means fixing q and imposing constraints on the Boltzmann weights.

2.1 The inversion relation

Two distinct inverses act on the matrix of Boltzmann weights: the matrix inverse I and the Hadamard (element by element) inverse J. Let us write down the inversion relations [9, 7] :

$$\sum_{\sigma} w(\sigma_i, \sigma) \cdot I(w)(\sigma, \sigma_j) = \mu \, \delta_{\sigma_i \sigma_j}, \quad (1)$$

$$w(\sigma_i, \sigma_j) \cdot J(w)(\sigma_i, \sigma_j) = 1. \quad (2)$$

where $\delta_{\sigma_i \sigma_j}$ denotes the usual Kronecker delta. This gives the following action on the matrix M:

$$I \ : \quad M \longrightarrow M^{-1} \quad (3)$$

$$J \ : \quad m_{ij} \longrightarrow 1/m_{ij} \quad (4)$$

In the inversion relations of two dimensional lattice models of Statistical Mechanics [7, 8, 9, 10, 11], one acts on the Boltzmann weights for vertical bonds with I (resp. J) and on the horizontal bonds with J (resp. I).

The two involutions I and J generate an infinite discrete group Γ isomorphic to the infinite dihedral group $\mathbb{Z}_2 \ltimes \mathbb{Z}$. The $\mathbb{Z}$ part of Γ is generated by IJ. In the parameter space of the model, that is to say some projective space $\mathbb{C}\mathbf{P}_{N-1}$ (N homogeneous parameters), I and J are (bi)rational involutions. They give a *non-linear representation* of this group by an infinite set of birational transformations [1]. It may happen that the action of Γ on specific points yields finite orbits (the representation of Γ identifies with the p-dihedral group $\mathbb{Z}_2 \ltimes \mathbb{Z}_p$). One only wants to consider constraints between the Boltzmann weights preserved by these two inversions.

We choose to restrict ourselves to constraints on the matrix of Boltzmann weights, of the form $m_{ij} = m_{kl}$ for a number of pairs of indices. Such constraints are automatically preserved by J. They amount to giving a partition of the set of the entries of M such that all elements of a given part are set equal. Only a limited number of partitions give a pattern such that the matrix inverse I will preserve it. We call these patterns *admissible* patterns [12].

On these remaining parameters $x_0, x_1, \ldots, x_{N-1}$, (that is to say a point in $\mathbb{C}\mathbf{P}_{N-1}$), the action of the Hadamard inverse J is simply $x_k \longrightarrow 1/x_k$. The action of the matrix inverse I is $x_k \longrightarrow i_k(x_0, x_1, \ldots, x_{N-1})$, where the i_k are polynomials with integer coefficients in the x_j's.

The matrix of Boltzmann weights for the q-state chiral Potts model is the general cyclic $q \times q$ matrix:

$$M = \begin{pmatrix} x_0 & x_1 & x_2 & & \dots & x_{q-1} \\ x_{q-1} & x_0 & x_1 & x_2 & \dots & x_{q-2} \\ x_{q-2} & x_{q-1} & x_0 & x_1 & & \\ & x_{q-2} & x_{q-1} & x_0 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & x_1 \\ x_1 & x_2 & \dots & \dots & x_{q-1} & x_0 \end{pmatrix}. \tag{5}$$

For many models of statistical mechanics on lattices, the matrix of Boltzmann weights is a stochastic matrix ($\sum_j m_{ij}$ =independent of i). We have remarked that this is the case of our "admissible patterns" for the classifications we have performed [1, 12]). One remarks that our matrix algebra identifies with Bose-Mesner algebra occurring in algebraic combinatorics [13, 14, 15].

2.2 Orbits

When the iteration of IJ leads to algebraic curves, there exist an algebraic invariant Δ such that the curve's equation reads $\Delta = cste$ [1]. As seen in [16], there is an elliptic uniformization [17], making IJ a mere translation $\theta \to \theta + \lambda$ of the uniformizing (spectral) parameter θ. The situation is that one has translations on the circle S^1 with a shift not commensurate to the circumference (irrational 'rotation number'). The generic orbits are thus dense in the (algebraic) curves.

2.3 Symmetric five-state Potts model

The symmetric five-state Potts model [4] corresponds to matrix (5) for $q = 5$ when it is a symmetric one. From the very construction of the two involutions I and J, they are birational transformations with *integer* coefficients. It is possible to introduce deformations of our mappings. We will concentrate here on deformations of the mappings associated with the symmetric five-state chiral Potts model. The transformations I and J read [3], in terms of the inhomogeneous variables $u = x_1/x_0$, and $v = x_2/x_0$:

$$I: \quad u \;\rightarrow\; \frac{-u + uv - u^2 + v^2}{1 + u + v - vu - v^2 - u^2} \qquad v \rightarrow \frac{-v + uv - v^2 + u^2}{1 + u + v - vu - v^2 - u^2} \tag{6}$$

$$J: \quad u \rightarrow 1/u, \qquad v \rightarrow 1/v. \tag{7}$$

In the examples developed in [1], there are collineations intertwining the two involutions (matrix inverse I and element by element inverse J). They generalize the Kramers-Wannier duality [1, 2]. This property is not too surprising since in $\mathbb{C}\mathbf{P}_2$, the Noether theorem [18] proves that every birational automorphism of the plane can be represented as a product of quadratic transformations and projective transformations. The birational transformations in $\mathbb{C}\mathbf{P}_n$, $n > 2$, are much more complicated.

The collineation intertwining I et J for this model is an involution and reads:

$$\begin{aligned} D: \qquad u &\rightarrow \frac{1 + (\omega + \omega^4)u + (\omega^2 + \omega^3)v}{1 + 2\,u + 2\,v} \quad , \\ v &\rightarrow \frac{1 + (\omega^2 + \omega^3)u + (\omega + \omega^4)v}{1 + 2\,u + 2\,v} \end{aligned} \tag{8}$$

with $\omega = \exp(2i\pi/5)$.

2.4 Invariants

The localization of the orbits of Γ on curves, or more generally non trivial subvarieties of the space of parameters, rather than as "clouds" of points, is the sign of the existence of *algebraic* invariants of the group Γ [1]. Equating these invariants to some constant gives the equations of the algebraic varieties. For the symmetric five-state Potts model a linear pencil of algebraic curves (elliptic curves) emerges in the study of the orbits of the group Γ generated by I and J [1]. The rational expression

$$\Delta = \frac{(u^2 + v^2 + 3uv)(u-1)(v-1)}{(u-v)^2(2 + 3\,u + 3\,v + 2\,uv)}, \tag{9}$$

is invariant by I and J. The curves in the pencil are given by $\Delta = const.$ This can be seen on the orbits of Γ [1], and verified explicitly by a direct calculation. One has a linear pencil of curves, with intersection points at which the invariant Δ is undetermined. This means that our mappings are defined in $\mathbb{C}\mathbf{P}_2$ *minus* some points.

One sees that Γ is an *infinite set of automorphisms* of the subvarieties. Therefore, if the subvarieties are curves, they are necessarily of genus 0 or 1 [17]. A detailed analysis to be found in parallel publications [16] shows that one has pencils of curves which are generically of genus 1. For a *finite number* of values of Δ, the curve factorizes in a number of genus 0 components with a rational parametrization. It is clearly the case for the line $u = v$, associated with the standard scalar Potts limit, which is stable by Γ and for which one can introduce a rational parametrization to represent Γ [19]. Actually, in this model, this line $u = v$ comes together with an hyperbola of equation:

$$2uv + 3(u + v) + 2 = 0. \tag{10}$$

Numerically, the line $u = v$ is not stable and the iterations of IJ escape to the hyperbola at the points $u = v = \frac{1}{2}(-3 \pm \sqrt{5})$ [2]. One has genus zero curves *only* for the following values of Δ: ∞, $-1/8$, 0, and $\frac{1}{2}(11 \pm 5\sqrt{5})$.

Notice that the points of the curve $\Delta = 1/2$ have a finite orbit of order 6 under Γ. More generally, subvarieties made out of points having finite orbits are automatically stable. For example the integrability variety of Au-Yang et al. is such a variety [20, 11]. More examples are easy to find explicitly by writing the condition $(IJ)^r = identity$, for arbitrary integer r. One guesses that all these finite orbit subvarieties belong to the linear pencil. The curves $\Delta = \frac{1}{2}(11 \pm 5\sqrt{5})$ decompose into the hyperbolae [4]:

$$(u^2 - v)(\omega^{\pm 2} + \omega^{\pm 3}) - u(1 - v) = 0, \tag{11}$$
$$(v^2 - u)(\omega^{\pm 2} + \omega^{\pm 3}) - v(1 - u) = 0, \tag{12}$$

with $\omega = \exp(2i\pi/5)$. Acting on the points of these curves, IJ is of order five and the action of Γ gives finite orbits. Also notice that Δ is invariant by the collineation (8) which is nothing but the Kramers-Wannier duality transformations, that is to say the Fourier transform in $\mathbb{Z}_5$ [21, 22]).

2.5 Deformations of the birational transformations

We may look for more general collineations by not demanding that they are involutive. We request that they send $(0, 0)$ onto $(1, 1)$ since this constraint has a natural physical meaning: it is the existence of a mapping from the (high-temperature) decoupling limit where the matrix (5) tends to the constant matrix, to the (low-temperature) limit where this matrix tends to the identity matrix. One thus gets a four parameter family of collineations which are generically of infinite order.

They read

$$u \rightarrow \frac{1 - v + A(u-v)}{1 + Cu + Dv}$$
$$and \qquad v \rightarrow \frac{1 - u + B(u-v)}{1 + Cu + Dv} \tag{13}$$

The collineation associated to spin models must be such that one recovers the duality transformation for the standard scalar Potts model $u = v$. This gives a further constraint on C and D :

$$C + D = q - 1 \tag{14}$$

The previous birational transformations can be deformed by breaking the symmetry between the two (inhomogeneous) variables u and v. The collineations which are the duality transformations of some q-state nearest neighbour spin model [22], satisfy the following conditions : they are *involutions or transformations of order four* [20, 11] and they map the point $(u, v) = (0, 0)$ onto the point $(1, 1)$. This last constraint corresponds to the low-to-high temperature correspondence (it maps the identity $q \times q$ matrix onto the matrix were all the entries are equal to one which corresponds to the decoupling of the spins). These constraints are quite severe. There is actually a two-parameter family of such collineations $D_{Q,R}$:

$$D_{Q,R}: \quad u \rightarrow \widehat{u} = \frac{1 - (u+v)/2 + (Q+R)(u-v)/2}{1 + (Q^2-1)(u+v)/2 - R(Q-1)(u-v)/2}$$
$$v \rightarrow \widehat{v} = \frac{1 - (u+v)/2 - (Q-R)(u-v)/2}{1 + (Q^2-1)(u+v)/2 - R(Q-1)(u-v)/2} \tag{15}$$

It is straightforward to look at the orbits of $I_{Q,R}J = D_{Q,R}JD_{Q,R}J$ and see that these orbits generically *do not preserve* algebraic curves.

However various lines are preserved: the line $u = v$ and the set of lines $u = 1$, $v = 1$, $\widehat{u} = 1$, $\widehat{v} = 1$.

3 q-generalizations

The duality transformation for such general q-state chiral Potts model read on the q homogeneous parameters of the model [20, 11]

$$x_i \rightarrow \widehat{x}_j = \sum_{\alpha=0}^{q-1} \omega^{\alpha j}.x_\alpha \tag{16}$$

Let us restrict the model by imposing equalities between the x_i's, $i \neq 0$, such that these equalities are invariant under the duality transformation (16) and thus under the group Γ (they are automatically invariant by the Hadamard inverse J).

In terms of the remaining inhomogeneous variables $y_i = x_i/x_0$, $i = i_1, \ldots, i_{N-1}$, transformation (16) is a collineation. Let us concentrate on the cases where there are only two inhomogeneous variables denoted u and v. It then reads :

$$\begin{aligned} D: \qquad u \quad &\rightarrow \quad \widehat{u} = \frac{1 + \alpha u + \beta v}{1 + (q-1)(u+v)/2} \\ v \quad &\rightarrow \quad \widehat{v} = \frac{1 + \alpha v + \beta u}{1 + (q-1)(u+v)/2}, \end{aligned} \tag{17}$$

where α is a sum of power of q^{th} root of unity, β being the sum of the other power of q^{th} root of unity, α and β satisfy $\alpha + \beta = -1$.

The "duality" transformation (16) is a transformation *of order four* [20, 11]. This implies the following relation on α and β:

$$-\,4\,\alpha\beta = \pm q - 1$$

It is an involution if and only if the $q \times q$ matrix (5) is symmetric ($x_i = x_{q-i}$) which read on α and β : $-4\alpha\beta = q - 1$ and gives the exact expression of α or β :

$$\alpha = \frac{-1 \pm \sqrt{q}}{2}, \quad \beta = \frac{-1 \mp \sqrt{q}}{2} \tag{18}$$

Transformation (17) with α and β given by (18) correspond to the $R \rightarrow 0$ limit of (15) with $Q^2 = q$.

It is an interesting question to see for which value of the integer q (different from $q = 5$) the values $\frac{1}{2}(-1 \pm \sqrt{q})$ can be realized as sum of q^{th} root of unity.

This question will not be addressed here. However recalling Gauss, Kummer and Vandermonde's ideas on cyclotomic polynomials one can see for instance that $q = 17$ is such a number. We have the following identities involving 17^{th} root of unity:

$$\omega^{3^0} + \omega^{3^2} + \omega^{3^4} + \omega^{3^6} + \ldots + \omega^{3^{14}} = \frac{1}{2}(-1 + \sqrt{17})/2, \qquad (19)$$

$$\omega^{3^1} + \omega^{3^3} + \omega^{3^5} + \omega^{3^7} + \ldots + \omega^{3^{15}} = \frac{1}{2}(-1 - \sqrt{17})/2. \qquad (20)$$

These special values of q correspond to *Fermat numbers*[2] (integers of the form $2^{2^r} + 1$) or products of such numbers up to some power of 2. This underlines some particular values of q for the q-state nearest neighbour Potts models: $q = 3, 5, 17, 257, 65537\ldots$

Let us consider the group Γ generated by the two involutions $I = DJD$ and J, where D is given by (17). The orbit of Γ lie remarkably on an algebraic curve (see figure 1).

These algebraic curves are given by the equation Δ =constant, where Δ reads :

$$\Delta = \frac{(u-1)(v-1)((q-1)(u^2+v^2) + 2(q+1)uv)}{(2 + (q-2)(u+v) + 2uv)(u-v)^2} \qquad (21)$$

These algebraic curves form a *linear pencil of elliptic curves*, which generalize the linear pencil of elliptic curves already obtained for the symmetric five-state Potts model [1](and equation (9)). One notes that Δ is invariant by D given by (17) and (18). These curves are generically elliptic curves except for a *finite number* of values of Δ. $\Delta = 0$ is clearly made out of genus zero curves. The vanishing of Δ corresponds to four lines $u = 1$, $v = 1$, $u = c_q v$ (which is nothing but $\widehat{u} - 1 = 0$) and $v = c_q u$, where:

$$c_q = \frac{-(q+1) + 2\sqrt{q}}{q-1}. \qquad (22)$$

[2]Fermat numbers are closely connected with problem of geometric construction of regular polygons by means of ruler and compass. For the construction of a regular n-gon to be possible it is necessary, and sufficient, that the representation of n as a product of prime numbers takes the form $n = 2^N p_1 \ldots p_k$ where $N \geq 0$, $p_1, \ldots, p_k$ are all different prime numbers of the form $2^M + 1$. This geometric construction is in one-to-one correspondence with the resolution of the equation $x^q - 1 = 0$ (the q-gon) in terms of a "tower" of roots of quadratic equations corresponding to the subsums of q^{th} root of unity similar to (19) and (20)

The line $u = v$, which corresponds to the standard scalar Potts model, is known to have a rational parametrization [19]. The hyperbola appearing in the denominator of Δ is also a genus zero curve. Note that it is not only Γ invariant but that is also (globally) invariant by the duality transformation (17). $\Delta = -1/2$ also correspond a curve of genus zero (two lines and one hyperbola) *remarkably independent of the value of* q :

$$(u+v)(u+v-2)(2uv-u-v) = 0 \tag{23}$$

The exhaustive list of values of Δ for which the curve is of genus zero is completed with the values:

$$\Delta = -2\frac{1 \mp \sqrt{q}}{(2 \mp \sqrt{q})^2} \tag{24}$$

the equation of the curve being

$$(u+v-v^2-uv)\sqrt{q} \pm (v^2-u-3v+3uv) = 0 \tag{25}$$

and of course the equation obtained exchanging u and v.

Eq. (25) generalize the two equations already obtained for the symmetric five-state Potts model [1] (equation (11)). These equations were finite orbits of IJ for $q = 5$ ($(IJ)^5 = e$). This is no longer the case for arbitrary q : the orbits of the points of equation (25) are of infinite order. On the other hand, finite orbits for the group Γ generated by transformations (17) can easily be exhibited for arbitrary q. For instance the following two polynomials are respectively the curves of the points of order three and four:

$$8\,uv(v-1)(u-1) - (u+v)(u-v)^2(q-1) = 0 \tag{26}$$

and

$$\begin{aligned}(u^3vq - 4\,qu^2v + 2\,v^2qu^2 + qu^2 + 2\,quv - 4\,quv^2 + uv^3q + qv^2 - v^2 \\ -6\,uv + 8\,uv^2 - uv^3 - u^2 + 8\,u^2v - 6\,u^2v^2 - u^3v) \\ (-4\,uv - 2\,u^2 - 2\,v^2 + u^3 - 2\,u^3q + u^3q^2 - 4\,u^2v^2 + 7\,u^2v + 7\,uv^2 \\ +2\,quv^2 + 2\,qu^2v - 4\,quv + 2\,qu^2 + 2\,qv^2 - 2\,uv^3 - 2\,u^3v + 2\,u^3vq \\ +2\,uv^3q - 4\,v^2qu^2 + v^3 - 2\,v^3q + v^3q^2 - uq^2v^2 - vq^2u^2) \;=\; 0\end{aligned} \tag{27}$$

The curve of the points of order 3 is the curve $\Delta = \frac{1}{2}(q-1)$, and the points of order 4 are on the product of the curves $\Delta = -(q-1)/(q-2)$ and

$\Delta = \frac{1}{2}(q^2 - 1)$. The equation for the points of order 5 is the product of two polynomials of degree 6 in u and v, of degree 3 in q and each of them is the sum of 94 monomials. They can be seen to be the product of the equations of two curves of the pencil, with conjugated values of Δ :

$$\Delta = -\frac{1}{4}(q-1)(q^2 - q + 2 \pm q\sqrt{q^2 - 2q + 5}), \tag{28}$$

$$\Delta = -\frac{1}{2}\frac{(1 \pm \sqrt{q})(1 \mp \sqrt{q})^2)}{q - 1 \pm \sqrt{q}}. \tag{29}$$

For $q = 5$ the polynomial corresponding to the values of Δ (28) factorizes and one recovers equations (25) :

$$\begin{aligned} P(u, v, q) = \quad & (u^4 - u^3 - u^2 + u^3 v + uv - u^2 v^2 - uv^2 + v^2) \\ & (v^4 - v^3 - v^2 + v^3 u + uv - u^2 v^2 - u^2 v + u^2) \end{aligned} \tag{30}$$

The study of the points of order 6 yields also a product of curves from the pencil, with the 4 values of Δ:

$$\Delta = -\frac{1}{2}\frac{(q-1)(q+3)}{q-3}, \tag{31}$$

and the three solutions of the following order 3 polynomial:

$$x^3 - \frac{1}{2}(q-1)(q^3 - 3q^2 + 3q + 3)x^2 - \frac{3}{4}(q^2 - 2q - 1)(q-1)^2 x - \frac{1}{8}(3q+1)(q-1)^3. \tag{32}$$

Remark: the other one-parameter dependent collineation (17) of order *four* corresponds to:

$$\alpha = (-1 \pm i\sqrt{q})/2 \quad \text{and} \quad \beta = (-1 \mp i\sqrt{q})/2. \tag{33}$$

However the group Γ generated by the two involutions J and $I = DJD^{-1}$ where D is given by (17) for α and β given by (33), do not lead to a linear pencil of curves. Again one can ask the question to see for which value of the integer q the values (33) can be realized as sum of q^{th} root of unity. Actually $q = 7$ is such an integer with $\alpha = \omega + \omega^2 + \omega^4$, $\beta = \omega^3 + \omega^5 + \omega^6$, where $\omega^7 = 1$. This model has been introduced elsewhere (it is referred to model $\mathbb{P}_4$ in [1, 2]). The orbits of Γ for this model are quite chaotic (see figure 4 in [1] and figure 6 in [2]).

4 Generalizations and outlook.

All this analysis on birational transformations on $\mathbb{C}\mathbf{P}_2$ can be generalized straightforwardly to $\mathbb{C}\mathbf{P}_n$ ($n \geq 3$). Our transformations are generated by involutions in $\mathbb{C}\mathbf{P}_n$. In the most general framework an exhaustive classification of the involutions in $\mathbb{C}\mathbf{P}_2$ has been performed by Bertini [23, 24].

For involutions in $\mathbb{C}\mathbf{P}_n$ the situation is much more involved: there is no longer Noether-Castelnuovo theorem imposing each Cremona transformation to be a finite product of quadratic transformations [25].

It will be necessary to restrict the analysis to particular classes of birational transformations. A guide could be their links with statistical mechanics on lattices and more precisely integrable models of nearest neighbour interaction spin models. We think in particular that an interesting example is the group of transformations in $\mathbb{C}\mathbf{P}_3$ generated by the Hadamard inverse J and $I = DJD$, where D is an involutive collineation mapping the point $(0, 0, 0)$ in $(1, 1, 1)$.

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