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New integrable cases of a Cremona transformation: a finite-order orbits analysis

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Abstract

We analyse the properties of a particular birational mapping of two variables (Cremona transformation) depending on two free parameters (ε and α), associated with the action of a discrete group of non-linear (birational) transformations on the entries of a $q \times q$ matrix. This mapping originates from the analysis of birational transformations obtained from very simple algebraic calculations, namely taking the inverse of $q \times q$ matrices and permuting some of the entries of these matrices. It has been seen to yield weak chaos and integrability. We have found new integrable cases of this Cremona transformation, corresponding to the values of $\alpha = 0$ when $\varepsilon = \frac{1}{2}, \frac{1}{3}, +1$, besides the already known values $\varepsilon = 0$ and $\varepsilon = -1$, and also *arbitrary* α when $\varepsilon = 0$. For these cases, one has a foliation of the parameter space in elliptic curves. We give the equations of these elliptic curves. Based on this very example we show how one can find these integrability cases of the Cremona transformation and actually integrate it using a method based on the systematic study of the finite-order conditions of the Cremona transformation. The method is shown to be efficient and straightforward. The various integrability cases are revisited using many different representations of this very mapping (birational transformations, recursion in one variable, ...).

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1. Introduction

In previous papers, birational representations of infinite discrete symmetry groups generated by *involutions*, having their origin in the theory of exactly solvable models in lattice statistical mechanics [1–6] have been analysed. These involutions correspond, respectively, to two kinds of transformations on $q \times q$ matrices: the inversion of the $q \times q$ matrix and a permutation [2] of the entries of the matrix (corresponding to the parameter space of the model).

The set of such transformations is very large, as large as the number of permutations of q^2 elements: we have thus restricted ourselves in previous publications [7–10] to elementary transpositions and shown that this restricted subset of mappings falls in six classes [7–9] for $q \geqslant 4$. For $q = 4$ three of these six classes (denoted I, II and III) are integrable [3] mappings, their iterations giving *algebraic elliptic curves* [11]. The three other classes, even when the mappings are not integrable, do present remarkable properties: their iterations lie on (transcendental) curves for most of the initial points. These mappings exhibit many of the well-known chaotic features of discrete dynamical systems. However, for class IV one even has an integrable subcase (on some codimension-one algebraic variety) with again algebraic elliptic curves. One has also associated with these mappings in CP_{q^2-1} (entries of $q \times q$ matrices) a hierarchy of non-linear recursions bearing on *a single variable* which enables to cross-check the numerical and analytical analysis [8].

We will specifically consider in this paper the birational transformations of $q^2 - 1$ variables of class IV [7,8]. This class, though it is not generically integrable, is quite regular (very weak chaos) and actually exhibit two integrable subcases. It has thus been called "almost integrable" [7].

The corresponding birational transformations on $q^2 - 1$ variables (q arbitrary) can actually be associated with birational transformations in a plane (Cremona transformations [12]). Based on this very example, we will show how one can find the integrability cases of this Cremona transformation and actually integrate it. We will use a method based on the systematic study of the *finite-order conditions* of the Cremona transformations.

In a forthcoming publication, we will analyse these Cremona transformations, beyond the integrability cases, concentrating again on the analysis of the cycles of the transformations.

[2] Most of the time, the permutations considered in [7–10] are involutive.

[3] Only class I is integrable for *arbitrary value of q* [9].

2. Recalls

To set up the notations, let us consider the following $q \times q$ matrix:

$$R_q = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & \cdots \\ m_{21} & m_{22} & m_{23} & m_{24} & \cdots \\ m_{31} & m_{32} & m_{33} & m_{34} & \cdots \\ m_{41} & m_{42} & m_{43} & m_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} . \tag{2.1}$$

Let us introduce the following transformations, the matrix inverse $\widehat{I}$ and the homogeneous matrix inverse I:

$$\widehat{I}: \quad R_q \to R_q^{-1} \quad \text{and} \quad I: \quad R_q \to R_q^{-1} \cdot \det(R_q) . \tag{2.2}$$

The homogeneous inverse I is a *polynomial* transformation on each of the entries m_{ij} which associates to each m_{ij} its corresponding cofactor. In the following, t will denote an arbitrary transposition of two entries of the $q \times q$ matrix, and t_{ij-kl} will denote the transposition exchanging m_{ij} and m_{kl}. The two transformations t and $\widehat{I}$ are *involutions*, whereas the homogeneous inverse verifies $I^2 = (\det(R_q))^{q-2}\, \mathcal{I}d$, where $\mathcal{I}d$ denotes the identity transformation. We also introduce the (generically *infinite order*) transformations $K = t \cdot I$ and $\widehat{K} = t \cdot \widehat{I}$. Note that K is a (homogeneous) *polynomial* transformation on the entries m_{ij}, while transformation $\widehat{K}$ is clearly a *rational* transformation on the entries m_{ij}. In fact, $\widehat{K}$ is a *birational* transformation since its inverse transformation is $\widehat{I} \cdot t$ which is also a rational transformation. In the following, we will also consider transformation $\widehat{K}^2$ for itself (see Section 2.2.2). It will be denoted $\widehat{k}$.

2.1. Class I

The most remarkable example corresponds to the birational transformations of class I [7,9]. One representative of such class is, for example, permutation t_{12-21}. The corresponding mappings are integrable and present *factorization properties* for the iteration of the homogeneous transformation K as well as (integrable) recursions on some homogeneous polynomials.

Let us first consider the successive matrices obtained by iteration of the homogeneous transformation K on a generic $q \times q$ matrix R_q (see (2.1)) and the determinants of these various matrices:

$$M_0 = R_q, \qquad M_1 = K(M_0), \qquad f_1 = \det(M_0) . \tag{2.3}$$

Remarkably, the determinant of matrix M_1 factorizes enabling us to introduce a homogeneous polynomial f_2:

$$f_2 = \frac{\det(M_1)}{f_1^{q-3}} . \tag{2.4}$$

Again, f_1^{q-4} also factorizes in all the entries of the matrix $K(M_1)$, leading to introduce a new "reduced"matrix M_2:

$$M_2 = \frac{K(M_1)}{f_1^{q-4}}\,. \tag{2.5}$$

In fact, similar factorization properties are true at any order. Generally, for $n \geqslant 1$ and $q \geqslant 4$, one has[4]

$$M_{n+3} = \frac{K(M_{n+2})}{f_n^{q-2} f_{n+1}^2 f_{n+2}^{q-4}}, \qquad f_{n+3} = \frac{\det(M_{n+2})}{f_n^{q-1} f_{n+1}^3 f_{n+2}^{q-3}}\,, \tag{2.6}$$

and the following relation *independent of* q:

$$\widehat{K}(M_{n+2}) = \frac{K(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{f_n f_{n+1} f_{n+2} f_{n+3}}\,. \tag{2.7}$$

One important consequence of these factorizations is to introduce the *homogeneous* polynomials f_n. These polynomials *do verify, independently of* q, *a whole hierarchy of non-linear recursion relations* [9] such as

$$\frac{f_n f_{n+3}^2 - f_{n+4} f_{n+1}^2}{f_{n-1} f_{n+3} f_{n+4} - f_n f_{n+1} f_{n+5}} = \frac{f_{n-1} f_{n+2}^2 - f_{n+3} f_n^2}{f_{n-2} f_{n+2} f_{n+3} - f_{n-1} f_n f_{n+4}} \tag{2.8}$$

or, for instance, among many others:

$$\frac{f_{n+1} f_{n+4}^2 f_{n+5} - f_{n+2} f_{n+3}^2 f_{n+6}}{f_{n+2}^2 f_{n+3} f_{n+7} - f_n f_{n+4} f_{n+5}^2} = \frac{f_{n+2} f_{n+5}^2 f_{n+6} - f_{n+3} f_{n+4}^2 f_{n+7}}{f_{n+3}^2 f_{n+4} f_{n+8} - f_{n+1} f_{n+5} f_{n+6}^2}\,. \tag{2.9}$$

Let us introduce here variables [9,8] corresponding to the iteration of the *inhomogeneous* transformation $\widehat{K}$:

$$x_n = \det(\widehat{K}^n(M_0))\det(\widehat{K}^{n+1}(M_0))\,. \tag{2.10}$$

The x_n's also satisfy recursion relations, for instance,

$$R_1: \quad \frac{x_{n+1} - 1}{x_n x_{n+1} x_{n+2} - 1} = \frac{x_n - 1}{x_{n-1} x_n x_{n-1} - 1} \cdot x_{n-1} x_{n+1}\,. \tag{2.11}$$

Relation R_1 is actually equivalent to

$$R_2: \quad \frac{x_{n+2} - 1}{x_{n+1} x_{n+3} - 1} = \frac{x_{n+1} - 1}{x_n x_{n+2} - 1} \cdot x_n x_{n+2}^2\,. \tag{2.12}$$

These factorizations and recursion relations were shown in [9] *to hold true for arbitrary* $q \times q$*-matrix for permutations of class I.*

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[4] Because of factorizations (2.6) one can see that the iteration of the homogeneous transformation K yields a *polynomial growth* of the complexity of the calculations: the degrees of the determinants of matrices M_n's, as well as the degrees of the polynomials f_n's are *quadratic expressions of* n [8,9].

2.2. Class IV

Another interesting class of birational transformations, called "class IV", also emerged in such study. It also exhibits recursions on the x_n's and integrability, but not generically as this is the case for class I. It can thus be seen as an excellent "laboratory" to analyse the "frontier" between integrability and chaos. A typical representative of this class is given by permutation t_{12-32}.

2.2.1. Factorization properties

The factorizations corresponding to the iterations of transformation K detailed in Section 2.1 (see Eqs. (2.4)–(2.7)) for class I, now read for class IV, for arbitrary n:

$$
\begin{aligned}
\det(M_n) &= f_{n+1}\cdot(f_n^{q-2}\cdot f_{n-1}\cdot f_{n-2}^{q-1}\cdot f_{n-3}^2)\\
&\quad\cdot(f_{n-4}^{q-2}\cdot f_{n-5}\cdot f_{n-6}^{q-1}\cdot f_{n-7}^2)\cdots f_1^{\delta_n}\,,\\
K(M_n) &= M_{n+1}\cdot(f_n^{q-3}\cdot f_{n-2}^{q-2}\cdot f_{n-3})\cdot(f_{n-4}^{q-3}\cdot f_{n-6}^{q-2}\cdot f_{n-7})\\
&\quad\cdot(f_{n-8}^{q-3}\cdot f_{n-10}^{q-2}\cdot f_{n-11})\cdots f_1^{\zeta_n}\,,
\end{aligned}
\tag{2.13}
$$

where δ_n and ζ_n depend on the truncation. These factorizations have a periodicity with period four. One notes that the following factorization *independent of q* occurs, which is different from relation (2.7):

$$
\widehat{K}(M_n)=\frac{K(M_n)}{\det(M_n)}=\frac{M_{n+1}}{f_1 f_2\cdots f_n f_{n+1}}\,. \tag{2.14}
$$

Remarkably, the polynomials f_n, for class IV, not only satisfy this additional factorization relations but *actually satisfy, for arbitrary q, exact relations*, as, for example,

$$
\begin{aligned}
&\frac{(f_{n+2}-f_{n-1}f_{n+1})}{(f_n-f_{n-3}f_{n-1})}\cdot\frac{f_{n-6}f_{n-10}f_{n-14}\cdots}{f_{n-4}f_{n-8}f_{n-12}\cdots}\\
&\quad=\frac{f_n(f_{n-1}f_{n-5}f_{n-9}\cdots)-(f_{n+1}f_{n-3}f_{n-7}\cdots)}{f_{n-2}(f_{n-3}f_{n-7}f_{n-11}\cdots)-(f_{n-1}f_{n-5}f_{n-9}\cdots)}\,.
\end{aligned}
\tag{2.15}
$$

Though, one does not have recursions on the f_n's but "pseudo-recursions" such as (2.15), the previous variables x_n's (see (2.10)), *remarkably satisfy again a hierarchy of very simple recursions* [7–9]. As for class I, the recursions on the x_n's are *independent of q* [8]. The most simple one reads

$$
\frac{x_{n+3}-1}{x_{n+2}x_{n+4}-1}=\frac{x_{n+1}-1}{x_n x_{n+2}-1}\cdot x_n x_{n+3}\,. \tag{2.16}
$$

For completeness, let us remark that, for the simplest case of 3×3 matrices, there actually exists *non-involutive permutations $\mathcal{P}$* yielding recursion (2.16) for $\widehat{K}=\mathcal{P}\cdot\widehat{I}$, as, for example,

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \longrightarrow \begin{pmatrix} m_{11} & m_{13} & m_{12} \\ m_{31} & m_{33} & m_{32} \\ m_{22} & m_{23} & m_{21} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} m_{22} & m_{21} & m_{23} \\ m_{33} & m_{31} & m_{32} \\ m_{13} & m_{11} & m_{12} \end{pmatrix} . \tag{2.17}$$

The first permutation is the product of a 4-cycle and of two involutions. The second one is a transformation of order six, product of a 6-cycle and of a 3-cycle.

2.2.2. *Class IV as a mapping of two variables*

Studying the iteration of $\widehat{K}$ in the (q^2-1)-dimensional space CP_{q^2-1} (corresponding to the entries of $q \times q$ matrices), one can show that the associated orbits actually belong to remarkable two-dimensional subvarieties, namely *planes* [7,8]. This can be easily seen since one has, for any value of n, the following relation between matrix M_0 and its *even* iterates:

$$\widehat{K}^{2n}(M_0) = a_0 \cdot M_0 + a_1 \cdot \widehat{K}^2(M_0) + a_2 \cdot \widehat{K}^4(M_0) \tag{2.18}$$

showing that the orbits of $\widehat{K}^{2n}$ lie in *planes* (depending non-trivially on the initial matrix M_0, that is, of a point in a (q^2-1)-dimensional space). Note that this property *is also valid for the two non-involutive permutations* (2.17).

More precisely, for transposition t_{12-32}, one can recursively show [7,8] that the successive iterates of $\widehat{K}^2$ on a generic (initial) matrix M_0, can be written in the following way:

$$\widehat{K}^{2n}(M_0) = \frac{1}{x_0 x_2 \ldots x_{2n-2}} \cdot (M_0 + a_n F + b_n P) , \tag{2.19}$$

where matrix P denotes the constant $q \times q$ matrix with entries $P[1,2]=1$, $P[3,2]=-1$, $P[i,j]=0$ for $(i,j) \neq (1,2)$ or $(3,2)$, and F denotes a $q \times q$ matrix, quadratic in the entries of matrix M_0 ($F[1,1]=m_{21}m_{13}-m_{11}m_{23}$, ...): *$F$ does depend on M_0, but not on the order n of the iteration.* In other words, all the iterates of $\widehat{K}^2$ *lie in a plane which depends on the initial matrix* M_0. This plane is led by two vectors, namely a *fixed vector* P and another one F, *depending on the initial matrix*. Note that, for the first non-involutive permutation (2.17), one also has a simple constant matrix P, namely $P[3,1]=1$, $P[3,3]=-1$, $P[i,j]=0$ for $(i,j) \neq (3,1)$ or $(3,3)$.

Inside these planes, the orbits look like curves for many of the trajectories (see [7]). From recursions (2.16) one may have the "prejudice" that the orbits of transformation $\widehat{K}^2$ in CP_{q^2-1} should be curves. In fact, it has been shown in [7,8] that, in some domain of the parameter space CP_{q^2-1}, these orbits are no longer (transcendental) *curves* but may become chaotic set of points.

These calculations amounts to considering transformation $\widehat{k}=\widehat{K}^2$ as a (birational) transformation in two variables (a,b). In fact, recalling the x_n's (determinants of the iterates of M_0), one can also represent, and analyse, transformation $\widehat{k}$ as recursions on the x_n's (see (2.16)). At first sight it is not completely obvious that the integrability of $\widehat{k}$ (seen, for instance, as a birational transformation in the two variables (a,b)) should automatically yield an integrability of the recursions on the x_n's since a determinant does not contain all the "informations" on the entries of the matrices.

Let us consider the variables x_n's defined by (2.10) or more precisely the homogeneous variables q_n's: $x_n = q_{n+2}/q_n$. One then has $q_{2n} = x_0 x_2 \dots x_{2n-2} \cdot q_0$ and also $q_{2n+1} = x_1 x_3 \dots x_{2n-1} \cdot q_1$. Clearly, q_0 and q_1 are two arbitrary homogeneous quantities. From recursion (2.16) bearing on the x_n's, one gets

$$q_{n+3}q_{n+5} \cdot \frac{q_{n+6} - q_{n+2}}{(q_{n+3} - q_{n+5})} = q_{n+1}q_{n+3} \cdot \frac{q_{n+4} - q_n}{(q_{n+1} - q_{n+3})} \tag{2.20}$$

which can be partially integrated (see Eq. (8.18) in [7]) as follows:

$$q_{2n+2} + q_{2n} + \frac{\lambda_2}{q_{2n+1}} = \rho_2, \qquad q_{2n+3} + q_{2n+1} + \frac{\lambda_1}{q_{2n+2}} = \rho_1 . \tag{2.21}$$

It is worth noting that these recursions are also valid for the two non-involutive permutations (2.17). Then, one notices that recursions (2.21) can also be written, eliminating ρ_1 and ρ_2, as[5]

$$\lambda_2 = -\, q_{2n}q_{2n+1} \cdot \frac{(x_{2n}x_{2n+2} - 1)x_{2n+1}}{1 - x_{2n+1}}, \tag{2.22}$$

$$\lambda_1 = -\, q_{2n+1}q_{2n+2} \cdot \frac{(x_{2n+1}x_{2n+3} - 1)x_{2n+2}}{1 - x_{2n+2}} . \tag{2.23}$$

Let us also note that $t(F) = F$, $t(P) = -P$ and that transposition t, can simply be represented as a reflection in the (a_n, b_n)-plane: $t(a,b) \to (a, \Delta_0 - b)$. From these two representations of t and $\widehat{k} = \widehat{K}^2$ (see (2.19)) in the (a_n, b_n)-plane, one gets a representation of $\widehat{I}t\widehat{I}$, which is actually an involution. One can introduce the following change of variables (see also [8]):

$$u_n = \frac{q_0 q_1}{q_1 q_2 + q_0 q_1 + \lambda_2} \cdot \frac{q_{2n}}{q_0}, \qquad v_n = -\,\frac{q_0 q_1}{\lambda_2} \cdot \frac{q_{2n+1}}{q_1} \cdot \frac{q_{2n}}{q_0} . \tag{2.24}$$

In fact it has been shown in [8] that the change of variables $(a_n, b_n) \to (u_n, v_n)$ is (for a fixed initial matrix M_0) a *linear transformation* (see Eq. (6.31) in [8]). Thus, the integrability of $\widehat{k} = \widehat{K}^2$ in the $(q^2 - 1)$-dimensional space of the entries of $q \times q$ matrices, which corresponds to integrability of $\widehat{k}$ in the (a_n, b_n)-plane, also corresponds to the integrability of $\widehat{k}$ in the (u_n, v_n)-plane, that is, as a consequence of (2.24), to the integrability in the (q_n, q_{n+1})-plane.

The involutive transformation $\widehat{I}t\widehat{I}$ takes the remarkably simple form (*independent of any parameter*!)

$$\widehat{I}t\widehat{I}: \quad (u,v) \to (u', v') = \left(\frac{u + v - uv}{v}, \frac{u + v - uv}{u}\right) \tag{2.25}$$

[5] Note a misprint in [8]: the actual expressions of λ_1 and λ_2 have opposite values as compared to (2.22) (see (6.20) and (6.21) in [8])

and transformation t is represented as the following (*two parameters*) transformation:

$$t: \quad (u,v) \to (u, 1+\varepsilon - v + \alpha u), \tag{2.26}$$

where ε and α read

$$\varepsilon = \frac{\lambda_1 - \lambda_2}{\lambda_2}, \qquad \alpha = -\frac{(q_2(q_1+q_3)+\lambda_1)(q_1(q_0+q_2)+\lambda_2)}{q_1 q_2 \lambda_2} = -\frac{\rho_1\rho_2}{\lambda_2}. \tag{2.27}$$

Let us also recall that there does exist an *integrable* subcase of these mappings associated to class IV, corresponding to $\lambda_1 = \lambda_2$ (i.e. $\varepsilon = 0$) [7]. It yields the following integrable recursion:

$$\frac{x_{n+2}-1}{x_{n+1}x_{n+3}-1} = \frac{x_{n+1}-1}{x_n x_{n+2}-1} \cdot \frac{x_n x_{n+2}}{x_{n+1}}. \tag{2.28}$$

The corresponding q_n's actually *satisfy two biquadratic equations*[6] depending on the parity of n (see [7]). For this integrable $\varepsilon = 0$ subcase the group, generated by transformations (2.25) and (2.26), yields a foliation of the (u,v)-plane in terms of curves, which form a *linear pencil of elliptic curves.* This can be seen by noticing that, for $\varepsilon = 0$, an algebraic expression i is actually invariant under both transformations $\widehat{I}t\widehat{I}$ and t:

$$i = \frac{(1-u)\cdot(1-v)\cdot(v-\alpha u)}{u}. \tag{2.29}$$

One should also note that the $\varepsilon = -1$ case has also been seen to be integrable [8]. This $\varepsilon = -1$ case, which corresponds to $\lambda_1 = 0$, yields a *simple rational parameterization* of the iteration. This simple case will be revisited in the following.

3. Graphical approach: $\alpha = 0$

For heuristic reasons, let us consider the $\alpha = 0$ case (which happens to correspond to a *rational parameterization*, when $\varepsilon = 0$).

For some values of ε, the mapping becomes *integrable*, i.e., a generic point on the (i,v)-plane stays upon iteration always on a curve. For these values, an algebraic $\widehat{k}$-invariant exists.

If one has a *linear pencil* of curves, the algebraic invariant, for these integrable values of ε, must necessarily be of the form

$$\Delta(i,v) = \frac{P(i,v)}{Q(i,v)} = C, \tag{3.1}$$

where $P(i,v)$ and $Q(i,v)$ are algebraic expressions in terms of i and v. Let us make a few remarks. The invariant $\Delta(i,v)$ is clearly not unique: it is defined up to a

[6] It is well known that *biquadratic equations* are associated with elliptic curves [6].

homographic transformation. On the other hand, some points in the (i, v)-plane belong to all the algebraic curves (base points): the invariant is undetermined at these points, i.e., polynomials $P(i, v)$ and $Q(i, v)$ are zero simultaneously at these points.

The method: The values of ε, where integrability occurs, correspond to a foliation of the (i, v)-plane in terms of elliptic curves for a (generically) chaotic mapping. These values can be easily detected graphically. For these situations, where there clearly exists an invariant, one can look for curves which degenerate into lines, hyperbolas or parabolas. An invariant by the mapping can then be easily written using the equations of these simple curves. We will illustrate this graphical method in the following for three integrable cases corresponding to the values of ε, namely $1, \frac{1}{2}$ and $\frac{1}{3}$.

3.1. Definitions and notations: Variables i and v

From relation (2.29) taken for $\alpha = 0$, the variable u can be simply written in terms of the variable v and the algebraic i (which is not invariant since ε is not necessarily equal to 0 here):

$$u = \frac{v \cdot (1 - v)}{i + v(1 - v)} \tag{3.2}$$

and the mappings t and $\widehat{I}\, t\, \widehat{I}$ (see (2.25),(2.26)) read, respectively,

$$t\colon \quad (i, v) \to \left(i \cdot \left(1 - \frac{\varepsilon}{v}\right) \left(1 - \frac{\varepsilon}{v - 1}\right), 1 - v + \varepsilon \right), \tag{3.3}$$

$$\widehat{I}t\widehat{I}\colon \quad (i, v) \to \left(i, 1 - \frac{i}{v - 1} \right). \tag{3.4}$$

These very simple representations of the birational transformations of class IV enabled us to perform a large number of numerical calculations which confirm the analysis performed in [7]. The iterations of these transformations often yield orbits which look like curves (weak chaos [7]).

Let us now analyse in detail the mappings generated by (3.3),(3.4). One has then to study the iteration of transformations $\widehat{k} = \widehat{K}^2$ or its inverse $\widehat{k}^{-1}$:

$$\widehat{k} = \widehat{I}t\widehat{I}t\colon \quad (i, v) \to \left(i \cdot \left(1 - \frac{\varepsilon}{v}\right) \cdot \left(1 - \frac{\varepsilon}{v - 1}\right), 1 + \frac{i}{v}\left(1 - \frac{\varepsilon}{v - 1}\right) \right), \tag{3.5}$$

$$\widehat{k}^{-1}\colon \quad (i, v) \to \left(i \cdot \left(1 + \frac{\varepsilon(v - 1)}{i}\right) \cdot \left(1 - \frac{\varepsilon(v - 1)}{v - 1 - i}\right), \right.$$
$$\left. \frac{i}{v - 1} \cdot \left(1 + \frac{\varepsilon(v - 1)}{i}\right) \right).$$

3.2. Two simple integrable values $\varepsilon = 1$ and $\varepsilon = -1$

Let us first consider $\varepsilon = 1$. When looking at the orbits of $\widehat{k}$, for $\varepsilon = 1$, or rather for ε very close to 1, one sees very clearly a foliation of the parameter space in hyperbolas: $\varepsilon = 1$ is integrable. One immediately gets an algebraic invariant $\Delta(i, v)$ and the associated hyperbolas (A is a constant):

$$\Delta(i,v) = \frac{i^2 \cdot (v-1)^2}{(v^2 - v - i)^2}, \qquad i = \pm A \cdot \frac{v \cdot (v-1)}{(v - 1 \pm A)} . \tag{3.6}$$

It can be easily checked, using the mapping, that indeed $\Delta(i, v)$ is $\widehat{k}$-invariant.

One can also verify straightforwardly that $\varepsilon = -1$ is also an integrability condition [7]. One immediately gets an algebraic invariant under transformations (3.5):

$$\Delta(i,v) = \frac{(v-1)^2}{(v^2 - v - i)^2} . \tag{3.7}$$

3.3. $\varepsilon = \frac{1}{2}$

The graphical method of visualization of the orbits (in the (i, v)-plane) enables to see new values of ε yielding integrable mappings. Fig. 1(a) represents the elliptic curves corresponding to $\varepsilon = \frac{1}{2}$. It corresponds to 50 different orbits of $\widehat{k} = \widehat{K}^2$ and thus shows clearly the existence of the foliation of the (i, v)-plane in (elliptic) curves (see (3.12) below). One sees also very clearly in Fig. 1(a) the base points of this foliation.

More accurately, one detects graphically three situations where factorizations into simple curves occur. One can see in Fig. 1(a), as a consequence of these factorizations, simple curves emerging, namely a line and two hyperbolas $e_1(i, v) = 0$, three lines and one hyperbola $e_2(i, v) = 0$, and finally one parabola $e_3(i, v) = 0$:

$$e_1(i,v) = (2i + v) \cdot (2iv + v - 3i - 1) \cdot (-v^2 + v + 2iv - i) , \tag{3.8}$$

$$e_2(i,v) = i \cdot (2v - 1) \cdot (2i - v + 1) \cdot (v^2 - v + 2iv - 3i) , \tag{3.9}$$

$$e_3(i,v) = v^2 - v - i . \tag{3.10}$$

The three expressions satisfy a remarkable relation

$$e_1(i,v) - e_2(i,v) + e_3^2(i,v) = 0 . \tag{3.11}$$

From these results an algebraic invariant emerges for $\varepsilon = \frac{1}{2}$:

$$\Delta(i,v) = \frac{e_1(i,v)}{(e_3(i,v))^2} . \tag{3.12}$$

Fig. 1(b) represent the chaotic orbits corresponding to a small perturbation of this integrable foliation (namely ε =0.52). Such a small perturbation destroys the integrable foliation of Fig. 1(a).

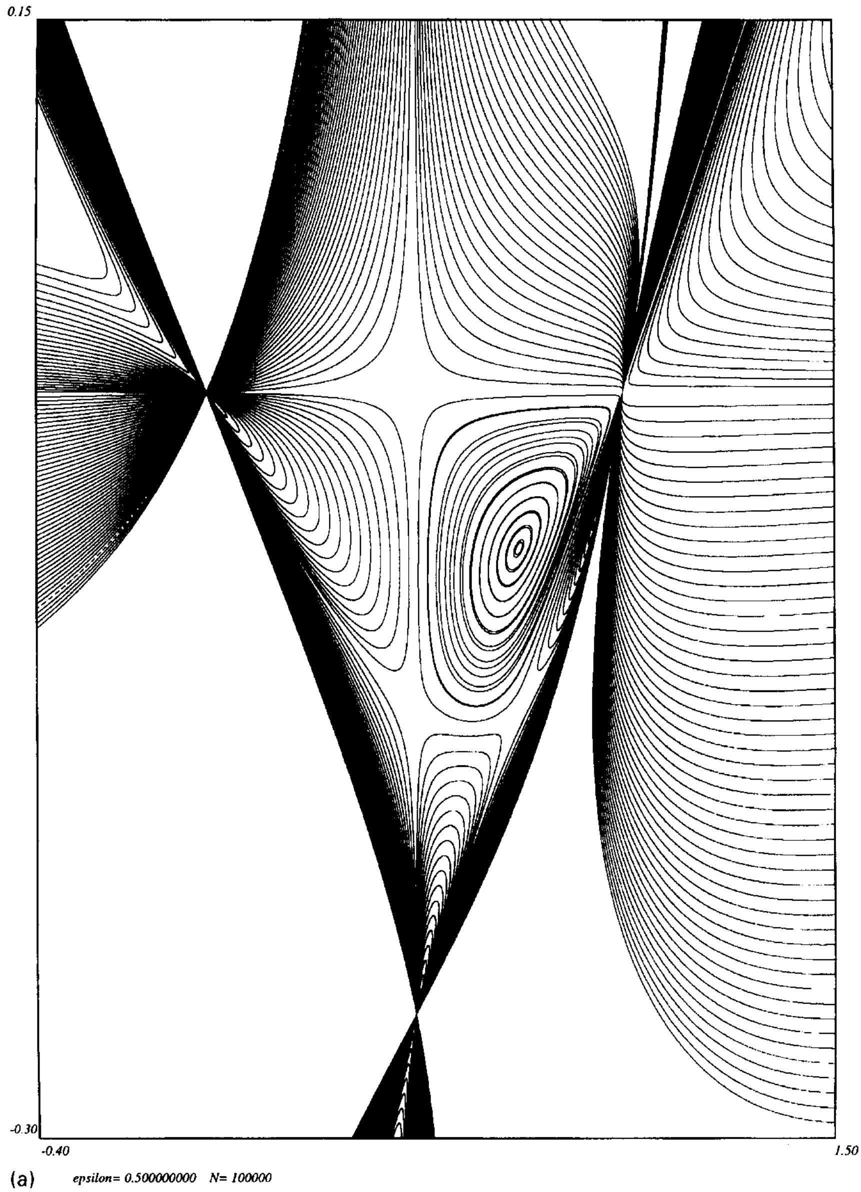

Fig. 1(a). Foliation of the (i, v)-plane: 50 orbits for $\varepsilon = \frac{1}{2}$.

3.4. $\varepsilon = \frac{1}{3}$

Similarly, $\varepsilon = \frac{1}{3}$ is another value of ε for which a foliation of the plane pops out (see Fig. 2(a)). Again the base points can clearly be seen in Fig. 2(a). One even detects graphically three situations where factorizations occur in this foliation. These situations, which correspond to the vanishing of some expression $e_n(i, v)$, are, respectively, four

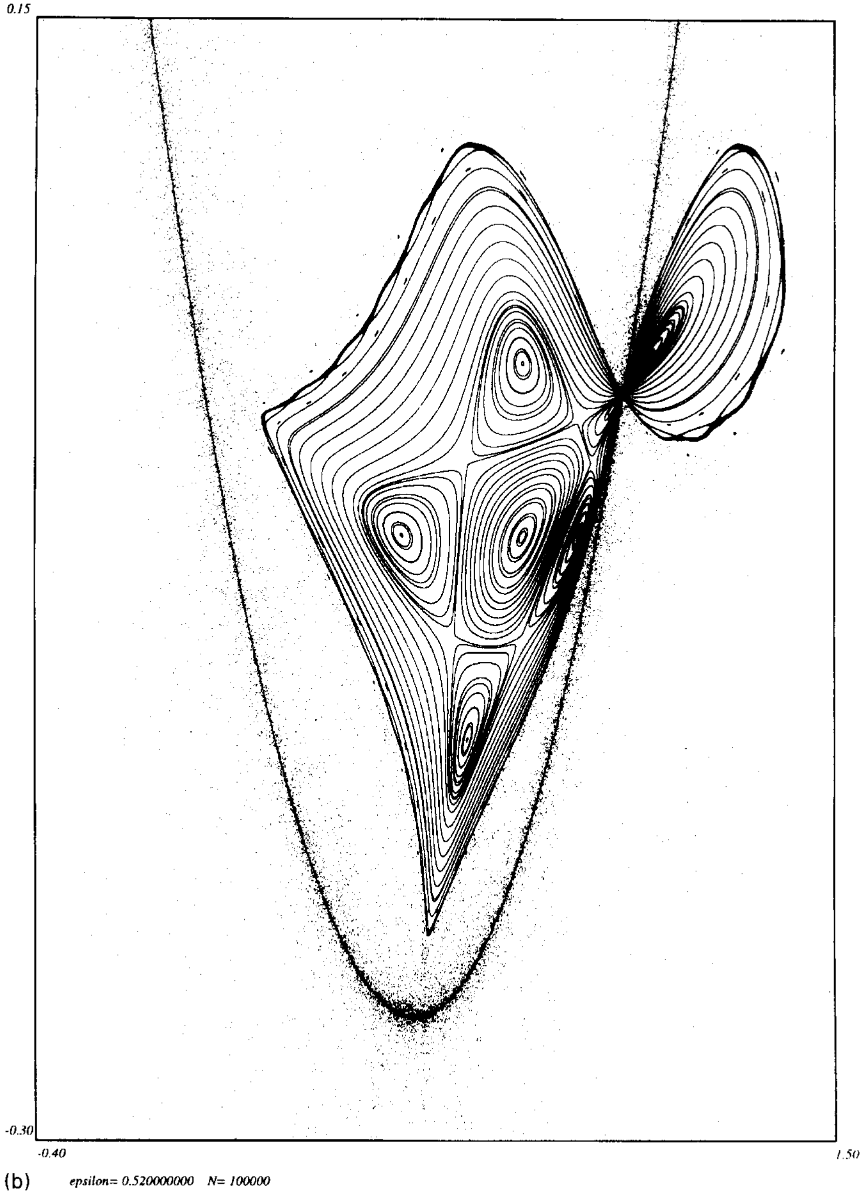

Fig. 1(b). Fifty orbits for $\varepsilon = 0.52$.

hyperbolas $e_1(i, v) = 0$, two lines and one hyperbola $e_2(i, v) = 0$, one parabola and one line $e_3(i, v) = 0$:

$$e_1(i, v) = (v^2 - 3iv - 3v + 4i + 2) \cdot (v^2 - 3iv - v + 2i) \\ \cdot (v^2 + 3iv - v - 2i) \cdot (3v^2 + 9iv - v - 12i - 2), \quad (3.13)$$

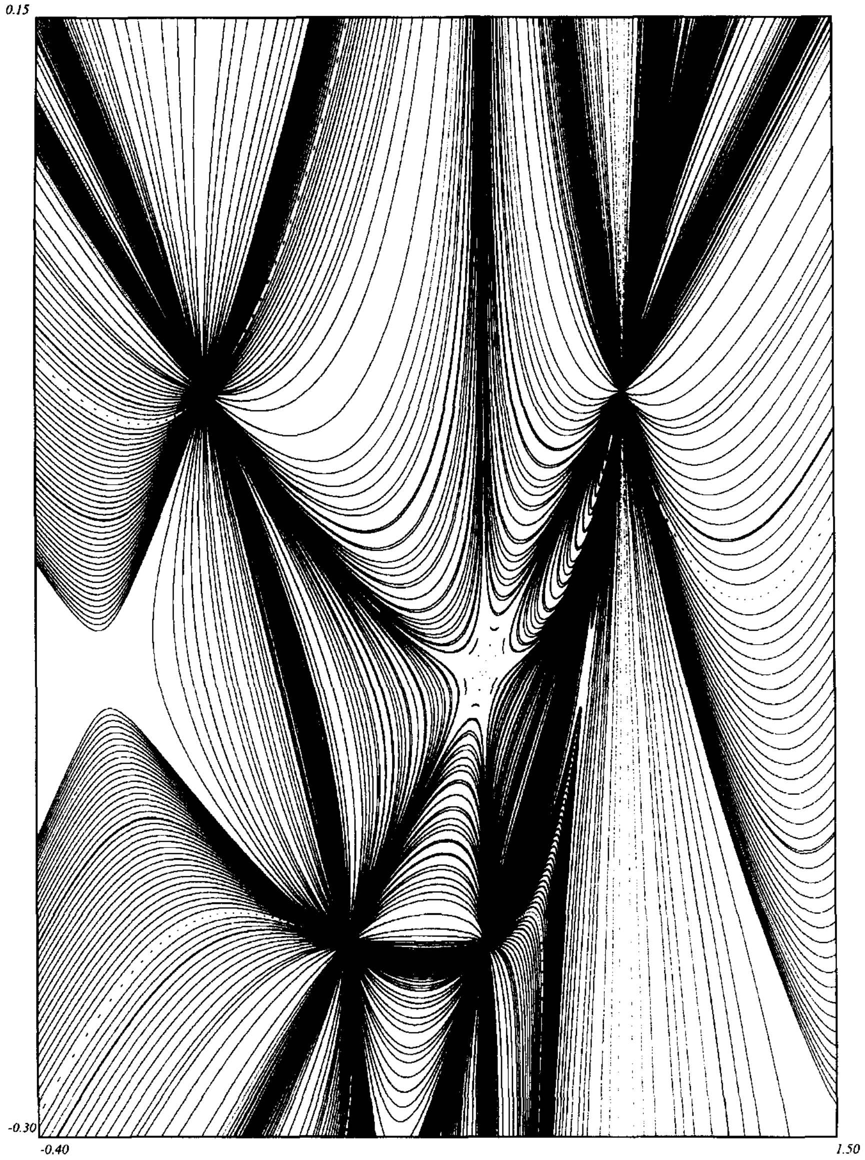

Fig. 2(a). Foliation of the (i, v)-plane: 50 orbits for $\varepsilon = \frac{1}{3}$.

$$e_2(i,v) = (3v-2)\cdot(v-3i-1)\cdot(-v^2+v-3iv+4i)\,, \tag{3.14}$$

$$e_3(i,v) = (v-1)\cdot(v^2-v-i)\,. \tag{3.15}$$

The three expressions again satisfy a remarkable relation

$$e_1(i,v) - \tfrac{1}{3}e_2^2(i,v) + \tfrac{16}{3}e_3^2(i,v) = 0\,. \tag{3.16}$$

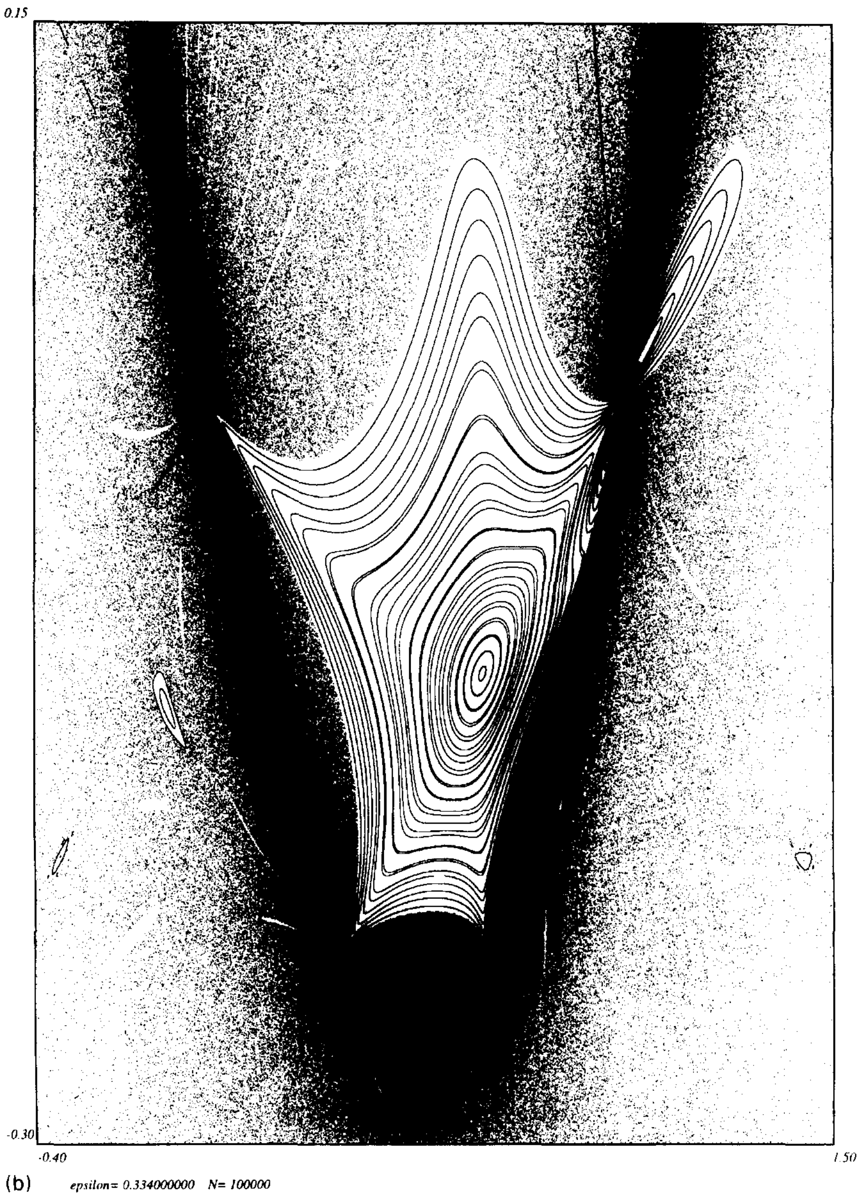

Fig. 2(b). Fifty orbits for $\varepsilon = 0.334$.

The invariant for $\varepsilon = \frac{1}{3}$, can be chosen as follows:

$$\Delta(i, v) = \frac{e_1(i, v)}{(e_3(i, v))^2} . \tag{3.17}$$

Fig. 2(a) represents the foliation of the (i, v)-plane in elliptic curves corresponding to $\varepsilon = \frac{1}{3}$. Fig. 2(b) represents the chaotic orbits corresponding to a small perturbation of ε, namely $\varepsilon = 0.334$. Again one sees that the foliation of the (i, v)-plane in elliptic curves is quite unstable.

For $\varepsilon = \frac{1}{3}$ any point (i, v) satisfying $e_2(i, v) = 0$ is a point of *order six* by the mapping. This can be easily seen by iterating a point lying on the line of Eq. $v = \frac{2}{3}$:

$$\left(i, \frac{2}{3}\right) \xrightarrow{\widehat{k}} (i, 1+3i) \xrightarrow{\widehat{k}} \left(\frac{-2+9i+81i^2}{27+81i}, \frac{8+36i}{9+27i}\right) \xrightarrow{\widehat{k}} \left(\frac{-5-27i}{27+81i}, \frac{2}{3}\right)$$
$$\xrightarrow{\widehat{k}} \left(\frac{-5-27i}{27+81i}, \frac{4}{9+27i}\right) \xrightarrow{\widehat{k}} \left(\frac{2-9i-81i^2}{-27-81i}, \frac{1-9i}{3}\right) \xrightarrow{\widehat{k}} \left(i, \frac{2}{3}\right). \tag{3.18}$$

In fact, this means that the 6-cycle points are no more isolated fixed points of $\widehat{k}^6$. A *whole curve* of order six takes place. This remark will be used in the next section to introduce a method for calculating the invariants based on the search of the *finite-order curves*. From a graphical point of view, the curve $e_2(i, v) = 0$ is a set of points of order six which should not easily be seen. However, when we scan the parameter space in order to see the foliation, we get points close to this hyperbola the orbits of which are infinite and "densify" elliptic curves very close to the hyperbola, thus enabling to "visualize" the finite-order curve $e_2(i, v) = 0$.

Let us remark that such graphical inspections cannot really be used to find exhaustively all the values of ε yielding integrability. One needs to perform analytical calculations in order to get such an exhaustive list and prove the corresponding integrability.

4. Integrable cases: Finite-order approach

Let us now use another more systematic approach to prove the integrability of the Cremona transformation for the previous values of ε, based on the study of the finite-order orbits of the mapping.

4.1. General remarks on foliations

Let us first make some general remarks in order to introduce the finite-order method. When one looks at the orbits of transformations $\widehat{k}$, integrability means that, for any point on the (i, v)-plane (for $\alpha = 0$), or on the (u, v)-plane (for arbitrary α), the mapping yields an infinite set of points which densify a curve. One thus gets a foliation of the plane parameter space $\mathscr{P}(u, v, \lambda) = 0$, where the constant λ depends on the initial point (u_0, v_0) in the iteration.

If one assumes that $\mathscr{P}(u, v, \lambda)$ is algebraic, one can expand[7] it as

$$\mathscr{P}(u, v, \lambda) = P_M(u, v) \cdot \lambda^M + P_{M-1}(u, v) \cdot \lambda^{M-1}$$
$$+ \cdots + P_{M-r}(u, v) \cdot \lambda^{M-r} + \cdots + P_1(u, v) \cdot \lambda + P_0(u, v). \tag{4.1}$$

[7] The case of a *linear pencil* of (elliptic) curves corresponds to $M = 1$: $P_1(u, v) \cdot \lambda + P_0(u, v) = 0$.

These algebraic curves have (by construction !) an infinite set of (birational) automorphisms: they are therefore elliptic curves [11]. One can thus (in principle) introduce an (elliptic) parametrization of the birational mappings on each curve $\mathscr{P}(u,v,\lambda)=0$. There exists a (spectral) parameter θ and a shift η (which depends on λ that is on the curve, but not on the point in the curve) such that the involutions $\widehat{I}$ and t are represented by reflections: $\theta \to -\theta$ and $\theta \to -\theta+\eta$ and the (generically infinite order) transformation $\widehat{k}$ amounts to performing a *translation* of η: $\theta \to \theta+\eta$. The finite-order conditions, namely $\widehat{k}^N = identity$, thus read:

$$\theta = \theta + N \cdot \eta \,. \tag{4.2}$$

These conditions just amounts to imposing that the shift η is *commensurate* with one of the two periods of the elliptic functions. *This is a condition bearing on* η, *or equivalently on* λ, *independently of the point on the curve*. The finite-order points are not isolated points: they correspond to a *whole curve*, that is *particular values* of λ. These results do not require a foliation into a *linear pencil* of elliptic curves.

This means the following: writing the condition of finite order N, $\widehat{k}^N = identity$, read two conditions, namely $U_N(u,v)=0$ and $V_N(u,v)=0$. These two conditions must factorize some curves of the foliation corresponding to some finite set of values of λ:

$$F_N(u,v) = \prod_{\alpha} \mathscr{P}(u,v,\lambda_\alpha) = 0 \,. \tag{4.3}$$

Writing systematically these conditions of finite order N, and getting the gcd of U_N and V_N, one gets therefore $F_N(u,v)$ and thus one can get the previous polynomial coefficients $P_M(u,v)$ in (4.1) from the system of equations (linear in the P_M's):

$$\begin{aligned} &P_N(u,v)\cdot\lambda_r^N + P_{N-1}(u,v)\cdot\lambda_r^{N-1} + \cdots + P_{N-M}(u,v)\cdot\lambda_r^{N-M} \\ &\quad + \cdots + P_1(u,v)\cdot\lambda_r + P_0(u,v) = F_N(u,v) \,. \end{aligned} \tag{4.4}$$

In fact, most of the time, due to factorizations in the expressions of the $F_N(u,v)$'s, it is not necessary to "accumulate" a large number of such finite-order conditions. The various factors of the $F_N(u,v)$'s do have covariance properties with a simple cofactor term (one does not have an accumulation of different independent cofactors). Simple ratio of these factors enable to get quickly an invariant.

The method: In the following, we will use the previous remarks to actually get algebraic invariants of Cremona transformations (birational transformations of two variables). The method is as follows: we will systematically write, for a given Cremona transformation $(x,y)\to(x',y')$, the finite-order conditions $\widehat{k}^N = identity$ on the two components, getting two algebraic expressions $X_N(x,y)=0$ and $Y_N(x,y)=0$. Actually, one rather writes (for computer memory reasons in the formal calculations) $\widehat{k}^{N_1} = (\widehat{k}^{-1})^{N_2}$ with $N_1+N_2=N$ and $|N_1-N_2|\leqslant 1$. One will factorize these two polynomials $X_N(x,y)$ and $Y_N(x,y)$ in order to get their gcd which will be denoted G_N. This gcd transforms under the Cremona transformation $\widehat{k}$ into itself, up to a cofactor denoted $\tilde{C}_N$. One will accumulate such gcd's and associated cofactors until one is actually able to build an algebraic invariant of $\widehat{k}$ as products and ratios of these gcd's.

Of course, considering factors of these gcd's, or simple curves emerging from a graphical analysis (see $e_1(i,v)=0$, (3.8),(3.13)) *which are not finite-order curves*, one may be able to build simpler algebraic invariants. The cofactors obtained by this method have particular algebraic properties [13,14].

4.2. Integrable candidates for ε from finite-order analysis for $\alpha = 0$: (i,v)-analysis

Let us now use similar calculations in order to seek for relations between ε and α (or just particular values of ε and α) such that the mapping becomes integrable.

Let us first concentrate on condition $\alpha=0$ for which one can benefit from the introduction of the variables i and v (see Section 3).

The finite-order conditions of order N, namely $\widehat{k}^N=$ identity, yield two conditions on i, v and ε: $I_N(i,v)=0$ and $V_N(i,v)=0$. The elimination of i, for example, between $I_N(i,v)$ and $V_N(i,v)$ factorizes, among others, the following simple expressions (*depending only on ε!*): $\varepsilon, (\varepsilon+1), (\varepsilon-1), (\varepsilon-\frac{1}{2})$ and $(\varepsilon-\frac{1}{3})$ up to $N=6$.

One remarks, from this analysis, that some particular values of ε, which are candidates for integrability, immediately pop out. It will be seen in the following that these values are indeed values of integrability of the mapping. The fact that these singled out values of ε emerge from conditions which should depend on the remaining variable i is remarkable. One cannot expect all the integrable values of ε to be obtained that way and, in principle, one has to perform several eliminations of variables in the remaining other conditions which actually mix ε and v. The calculations, which are a bit tedious, are sketched in Section A.1 of Appendix A.

All the factors in (A.2) are key ingredients to build the possible algebraic invariants of the Cremona transformation. The finite-order algebraic curves of integrable Cremona transformations are to be found among the various factors of (A.2) in Appendix A (and similar ones for $N\geqslant 7$). On the other hand, the corresponding values of ε are to be found among the various factors similar to (A.3) in Appendix A.

4.3. Invariant from finite-order analysis for $\alpha=0$: (i,v)-analysis

From the previous analysis, one gets a (*finite*) set of values of ε as possible candidate for integrability for $\alpha=0$. Let us now analyse these various values for ε and use the method of Section 4.1 to find the corresponding algebraic invariants of the mapping (3.5) in the (i,v)-plane.

Let us first remark that a simple covariant exists for arbitrary ε. If one considers, parabola $P_1=i-v^2+v=0$, one can easily verify that it transforms covariantly under $\widehat{k}$:

$$P_1 \to \tilde{C}_1 \cdot P_1 \quad \text{with } \tilde{C}_1 = -\frac{(v-1-\varepsilon)^2 \cdot i}{v^2 \cdot (v-1)^2} \tag{4.5}$$

$\varepsilon=\frac{1}{3}$: The gcd's at orders $N=4,5$ and 7 are trivial. At order six the gcd of I_6 and V_6 reads $G_6=(3v-2)\cdot(3iv-4i-v+v^2)\cdot(3i+1-v)$. It transforms under $\widehat{k}$ as follows:

$G_6 \to \tilde{C} \cdot G_6$, where the cofactor $\tilde{C}$ reads

$$\tilde{C} = \frac{(3v-4)^3 \cdot i^2}{27v^3 \cdot (v-1)^4} \, . \tag{4.6}$$

At order eight the gcd of I_8 and V_8 reads

$$\begin{aligned} G_8 = {} & 768iv + 352v^2 + 1944v^4 i^4 - 7776v^3 i^4 + 3264v^3 i - 2560v^2 i + 448 i^2 v^2 \\ & -128v^7 - 1280 i^2 v + 512 i^2 + 2264 v^4 - 1440 v^3 + 2048 v^3 i^2 - 2112 i v^4 \\ & -1696 v^5 + 24 v^8 - 3312 v^4 i^2 - 6336 i^3 v^3 + 8448 i^3 v^2 - 4864 i^3 + 832 v^5 i v \\ & +624 v^6 + 1024 i^3 + 2016 v^5 i^2 + 11232 i^4 v^2 - 6912 i^4 + 1728 v^4 i^3 - 192 i v^6 \\ & -432 v^6 i^2 + 1536 i^4 \, . \end{aligned}$$

G_8 transforms, under $\widehat{k}$, as follows: $G_8 \longrightarrow \tilde{C}^2 \cdot G_8$, where the cofactor $\tilde{C}$ is the same as for G_6, namely (4.6). One thus gets immediately an invariant by transformation $\widehat{k}$:

$$I_1 = \frac{G_8}{G_6^2} \quad \text{or the simpler } \widehat{k}\text{-invariant} \quad I_2 = \left(\frac{G_6}{P_1 \cdot (v-1)} \right)^2 . \tag{4.7}$$

Other values of ε: A similar analysis can be performed for $\varepsilon = 0, -1, \frac{1}{2}, 1$. The calculations are detailed in Section A.2 of Appendix A. The results in Appendix A confirm the graphical approach of Section 3. One can actually get the algebraic $\widehat{k}$-invariants, thus showing that these values of ε lead to the integrability of the mapping. On the contrary, one does not get any non-trivial gcd's for the other values of ε that pop out in (A.3) in Appendix A, for instance, $\varepsilon = 11$, $\varepsilon = -5$, the roots of: $4\varepsilon^4 + 35\varepsilon^3 - 83\varepsilon^2 + 65\varepsilon - 13 = 0$ and so on. Apparently, when $\alpha = 0$, the only integrable cases are $\varepsilon = \pm 1$, $\varepsilon = 0$, $\varepsilon = \frac{1}{2}$, $\varepsilon = \frac{1}{3}$.

4.4. *Invariant from finite-order analysis:* (u, v)*-analysis for arbitrary* α

Let us now consider the general analysis where α is arbitrary. In that case, one does not have an equation like (3.2) anymore and one must thus return to the original variables u and v. In term of these variables transformation $\widehat{k}$, and its inverse $\widehat{k}^{-1}$, read, respectively,

$$\widehat{k}: \quad (u, v) \to \left(\frac{1-(1-u)(v-\alpha\cdot u-\varepsilon)}{1-(v-\alpha\cdot u-\varepsilon)}, \frac{1-(1-u)(v-\alpha\cdot u-\varepsilon)}{u} \right),$$

$$\widehat{k}^{-1}: \quad (u, v) \to \left(\frac{v+u-uv}{v}, (v-\alpha\cdot u+\alpha+\varepsilon) - \frac{(v^2-\alpha u^2)}{uv} \right).$$

$\alpha = 0$ *and arbitrary* ε: Let us first assume that $\alpha = 0$. The (u, v) analysis of $\alpha = 0$ for various values of ε, namely $1, -1, \frac{1}{2}, \frac{1}{3}$, is given in Section A.3 of Appendix A. One recovers results, similar to the one obtained in Section 3, when the Cremona transformation is seen as a birational transformation in i and v.

$\varepsilon = 0$ *and arbitrary* α: Let us now assume that $\varepsilon = 0$ and apply the same method as before. For $N = 3$ one gets two conditions ($U_3 = 0$ and $V_3 = 0$):

$$U_3 = (-1+v)\cdot(-v\alpha u - \alpha u^2 + u^2 v\alpha - v + v^2 - u + uv - uv^2)$$
$$\cdot(2\,\alpha u^2 - \alpha u - 1 + v + u - 2\,uv)\,,$$
$$V_3 = (uv - v\alpha u - \alpha u^2 + u^2 v\alpha - v + v^2 - u - uv^2)\cdot(\alpha u - \alpha u^2 + 1 - v)\cdot(v - \alpha u)\,.$$

Remarkably, the gcd of these two expressions is non-trivial, *for arbitrary* α, and reads

$$G_3 = u\cdot(-v - u + uv)\cdot\alpha + (v^2 - u + uv - v - uv^2)\,.$$

This expression of u and v, which is α-dependent, is actually covariant by transformation $\widehat{k}$:

$$G_3 \to \tilde{C}\cdot G_3 \quad \text{where } \tilde{C} = \frac{1 - v + \alpha u + uv - \alpha u^2}{u\cdot(1 - v + \alpha u)}\,. \tag{4.8}$$

At order four one gets larger expressions for U_4 and V_4 which factorize and remarkably have a common factor. The gcd of U_4 and V_4, denoted G_4, reads

$$G_4 = u^2\cdot(2uv - 2u - 2v + 1)\cdot(-v - u + uv)\cdot\alpha^2$$
$$- u\cdot(u-1)\cdot(-1+v)(4uv^2 - 4v^2 - 4uv + v + u)\cdot\alpha$$
$$+ v\cdot(u-1)\cdot(-1+v)\cdot(2uv^2 - 2v^2 - 2uv + 2v + u)\,.$$

G_4 is actually covariant by transformation $\widehat{k}$: $G_4 \to \tilde{C}^2\cdot G_4$, where the cofactor $\tilde{C}$ is the same as (4.8). Therefore, for $\varepsilon = 0$, but *arbitrary* values of α, the algebraic expression

$$I_0 = \frac{G_4}{G_3^2} \tag{4.9}$$

is actually invariant under the iteration of the infinite-order transformation $\widehat{k}$. One thus has an integrable birational mapping (*depending on one continuous parameter* α) yielding a foliation of the (u, v) plane by a *linear pencil of elliptic curves* given by Eq. (4.9).

In the following, it will be seen that one can actually associate to this mapping two recursions of drastically different nature, *bearing on a single variable*: one amounts to a simple change of variable (see (5.1) in Section 5 below), and the other one amounts to introducing the x_n's (determinants of the iterates of R_q previously introduced, see (2.10)).

5. A recursion on a single variable for $\alpha = 0$

Let us consider $\alpha = 0$ for arbitrary ε. The variable i (see (2.29)) is an invariant for $\varepsilon = 0$. For arbitrary ε it has, a priori, no special property. However, if one restricts to

$\alpha = 0$, one can actually write u and v as a function of i and v (see (3.2)). We will use this property to describe the $\alpha = 0$ (and $\varepsilon \neq 0$) situation. Paradoxically, one uses the variable i, introduced for arbitrary α and $\varepsilon = 0$, to describe the $\alpha = 0$ for arbitrary ε situation.

One can choose, instead of (i, v), two other variables (s, r) defined by

$$s = \frac{i}{v} \cdot \left(1 - \frac{\varepsilon}{v - 1}\right), \qquad r = v - \varepsilon \,. \tag{5.1}$$

Then, the action of $\widehat{k}$ on (s, r) becomes

$$\widehat{k}: \quad (s, r) \rightarrow \left(\frac{r \cdot (s - \varepsilon)}{(s + 1)}, s + (1 - \varepsilon) \right). \tag{5.2}$$

The interest of these new variables (s, r) is that one can easily obtain a *recursion* bearing on a *single variable* (s or r):

$$s_{n+2} = \frac{(s_{n+1} - \varepsilon)(s_n - \varepsilon + 1)}{(s_{n+1} + 1)} \quad \text{or} \quad r_{n+2} = (1 - \varepsilon) + \frac{(r_n - \varepsilon)(r_{n+1} - \varepsilon)}{(r_{n+1} + \varepsilon)} \tag{5.3}$$

showing that the quantity $r_{n+1} - s_n = 1 - \varepsilon$ is independent of the iteration.

Let us analyse recursion (5.3). Any recursion of "length two" (i.e. one gets s_{n+2} as a function of s_{n+1} and s_n) can be seen as a rational transformation on two variables:

$$s_{n+2} = F(s_n, s_{n+1}) \quad \text{becomes } (s, t) \rightarrow (t, F(s, t)) \,. \tag{5.4}$$

Recursion (5.3) is straightforwardly associated to a new rational transformation $\tilde{k}$ and one verifies immediately that this rational transformation is birational. Transformation $\tilde{k}$ and its inverse $\tilde{k}^{-1}$ read, respectively,

$$\tilde{k}: \quad (s, t) \rightarrow \left(t, \frac{(t - \varepsilon) \cdot (s + 1 - \varepsilon)}{t + 1}\right), \tag{5.5}$$

$$\tilde{k}^{-1}: \quad (s, t) \rightarrow \left(\frac{st + t - s + s\varepsilon + \varepsilon - \varepsilon^2}{s - \varepsilon}, s\right).$$

In order to integrate recursion (5.3) one can use the previous approach based on the systematic analysis of the finite-order curves of the birational transformation (5.5). Since the change of variables from (u, v) to s_n (or r_n) defined in (5.1) is only defined for $\alpha = 0$, all the calculations in the following correspond to $\alpha = 0$ and to the "integrable" values of ε, namely $\varepsilon = 1, -1, 0, \frac{1}{2}, \frac{1}{3}$.

Let us remark that, for *any value of* ε, the line $D_1 = \varepsilon + t - s$ is covariant by the action of transformation $\tilde{k}$:

$$\tilde{k}: \quad D_1 \rightarrow \tilde{C} \cdot D_1 \quad \text{with cofactor } \tilde{C} = \frac{\varepsilon - t}{t + 1} \,. \tag{5.6}$$

Let us now revisit the integrability cases $\varepsilon = 1, -1, 0, \frac{1}{2}, \frac{1}{3}$ for this very recursion (5.3) or, equivalently, the (new) associated Cremona transformation (5.5).

Integration of the recursion for $\varepsilon = \frac{1}{3}$. Let us first consider $\varepsilon = \frac{1}{3}$, and use the method of finite-order orbits analysis. Then, condition $\tilde{k}^N =$ identity gives two conditions $S_N = 0$ and $T_N = 0$. Their gcd's are trivial for $N = 3, 4, 5, 7, \ldots$ One gets the first *non-trivial gcd's* G_N for $N = 6$ and $N = 8$, namely,

$$G_6 = (3s+1)\cdot(3t+1)\cdot(9ts+9t+1-3s)$$

and

$$\begin{aligned} G_8 &= 2187t^4s^4 + 5832t^4s^3 + 5346t^4s^2 + 1944t^4s + 243t^4 + 1944t^3s^3 + 450t^2 \\ &\quad + 3240t^3s^2 + 1512t^3s + 216t^3 - 486s^4t^2 - 648t^2s^3 + 324t^2s^2 + 360t^2s \\ &\quad - 216ts^3 - 72ts^2 - 744ts + 264t + 27s^4 + 378s^2 - 256s + 43\,. \end{aligned}$$

These expressions transform as follows under $\tilde{k}$: $(G_6, G_8) \rightarrow (-\tilde{C}\cdot G_6, \tilde{C}^2\cdot G_8)$ where $\tilde{C}$ is cofactor (5.6) taken for $\varepsilon = \frac{1}{3}$. One thus gets the $\tilde{k}$-invariants

$$I_1 = \frac{G_8}{G_6^2} \quad \text{or more simply} \quad I_2 = \left(\frac{G_6}{D_1}\right)^2 . \tag{5.7}$$

Let us remark that these different expressions for the invariant are related to a large set of identities on the covariants G_n's. If one calculates G_{10}, one sees that it is not independent of G_6 and G_8, namely $G_{10} - 16\cdot G_8^2 = -6\cdot(2\cdot G_8 - G_6^2)^2$. In fact, there are an infinite number of such relations. Most of the time they are consequences of the foliation of the two-dimensional parameter space into a linear pencil of elliptic curves: an infinite set of finite-order curves, like $G_N = 0$, corresponds to algebraic values of the invariant, thus yielding many non-trivial relations between the G_N's.

Other values of ε: Similar calculations are performed in Appendix B for the other integrable values of ε, namely $\varepsilon = \frac{1}{2}, 1, -1, 0$. Clearly, one recovers (as it should) similar algebraic $\tilde{k}$-invariants associated with integrability.

6. Expression of ϵ and α in terms of entries of $q \times q$ matrices: Recursions in the x_n's

The purpose of this section is to see if the integrability seen for $\widehat{k} = \widehat{K}^2$ in CP_{q^2-1}, that is, in the (u, v)-plane (or in the (i, v)-plane when $\alpha = 0$), can also be seen on the determinants of the iterates of $\widehat{K}$.

6.1. Recursions in the x_n's: Parity discussion

Recalling Eqs. (2.21) and (2.27) one can express ε (related to the ratio of λ_1 and λ_2) and α as a function of the x_n's defined in (2.10) and thus as a function of the entries of the $q \times q$ matrix M_0.

Denoting L the ratio λ_1/λ_2 one has the following results with $n=0,2,4,\dots$:

$$\varepsilon_n=\frac{\lambda_1}{\lambda_2}-1 \quad \text{with} \quad \frac{\lambda_1}{\lambda_2}=\frac{x_n\cdot x_{n+2}}{x_{n+1}}\cdot\frac{(x_{n+1}x_{n+3}-1)\cdot(x_{n+1}-1)}{(x_nx_{n+2}-1)\cdot(x_{n+2}-1)}\,, \tag{6.1}$$

that is (with $n=0,2,4,\dots$):

$$x_{n+3}=\frac{x_n\cdot x_{n+2}\cdot(x_{n+1}-1)+x_{n+1}\cdot(x_{n+2}-1)\cdot(x_n\cdot x_{n+2}-1)\cdot L}{x_n\cdot x_{n+1}\cdot x_{n+2}\cdot(x_{n+1}-1)}\,. \tag{6.2}$$

Let us also remark that one gets at the next order

$$\varepsilon_{n+1}=\frac{\lambda_2}{\lambda_1}-1 \quad \text{with} \quad \frac{\lambda_2}{\lambda_1}=\frac{x_{n+1}\cdot x_{n+3}}{x_{n+2}}\cdot\frac{(x_{n+2}x_{n+4}-1)\cdot(x_{n+2}-1)}{(x_{n+1}x_{n+3}-1)\cdot(x_{n+3}-1)}\,, \tag{6.3}$$

that is (with $n=0,2,4,\dots$)

$$x_{n+4}=\frac{x_{n+1}\cdot x_{n+3}\cdot(x_{n+2}-1)\cdot L+x_{n+2}\cdot(x_{n+3}-1)\cdot(x_{n+1}\cdot x_{n+3}-1)}{x_{n+1}\cdot x_{n+2}\cdot x_{n+3}\cdot(x_{n+2}-1)\cdot L}\,. \tag{6.4}$$

One thus has $\varepsilon_0=\varepsilon_2=\varepsilon_4=\cdots$ and $\varepsilon_1=\varepsilon_3=\varepsilon_5=\cdots$. The fact that one "jumps" from n to $n+2$ is a consequence of the fact that $\widehat{k}=\widehat{K}^2$ is singled out. The elimination of the ratio λ_2/λ_1 between (6.1) and (6.3) clearly gives (2.16). Let us introduce α_n:

$$\alpha_n=\frac{(1+x_{n+1}-x_{n+2}x_{n+1}-x_{n+1}x_{n+2}x_{n+3})(1+x_n-x_nx_{n+1}-x_nx_{n+1}x_{n+2})}{x_{n+1}\cdot(x_nx_{n+2}-1)\cdot(x_{n+2}-1)}\,.$$

One should note that the successive values of α_n's are such that $\alpha_n=\alpha_{n+2}$, i.e. $\alpha_0=\alpha_2=\cdots$, and $\alpha_1=\alpha_3=\cdots$. One can however have invariant expressions, independently of the parity of n, namely,

$$\left(\frac{\varepsilon_n}{\alpha_{n+1}}\right)^2=\left(\frac{\varepsilon_{n+1}}{\alpha_{n+2}}\right)^2, \quad n=0,2,4,\dots \tag{6.5}$$

This is a straight consequence of $\alpha=-\rho_1\cdot\rho_2/\lambda_2$ (see (2.27)). The expressions of α and ε in (2.26) correspond in fact to $\alpha=\alpha_0=\alpha_2=\cdots$ and $\varepsilon=\varepsilon_0=\varepsilon_2=\cdots$.

6.2. *The two $\alpha=0$ conditions*

The vanishing condition of α yields for n even or odd,

$$x_{n+2}=\frac{1+x_n-x_nx_{n+1}}{x_n\cdot x_{n+1}}\,. \tag{6.6}$$

One could analyse this very recursion (6.6) for itself (assuming that n is not of a fixed parity) using, for instance, the method developed in this paper to integrate it (if integrable!) i.e. by associating a Cremona transformation (see (5.4)). One verifies that this mapping is not integrable for itself: it has to be considered *coupled with another recursion* on the x_n's.

In fact, condition (6.6) valid for $n=0$ or $n=1$, presents some remarkable compability with the iteration of $\widehat{K}$ and, furthermore, one can verify that, if it is valid for $n=0$, it will be verified *for any n* even and, on the contrary, if it is valid for $n=1$ it will be verified *for any n* odd. There are thus two branches

$$B_1: \quad 1+x_n-x_nx_{n+1}-x_nx_{n+1}x_{n+2}=0, \quad n=0,2,4,\ldots, \tag{6.7}$$

$$B_2: \quad 1+x_{n+1}-x_{n+2}x_{n+1}-x_{n+1}x_{n+2}x_{n+3}=0, \quad n=0,2,4,\ldots. \tag{6.8}$$

Therefore, these two conditions should be seen as a part of a system recursions. Condition (6.7) (or condition (6.8)) is actually compatible with the system of recursions (6.1) and (6.3) for *arbitrary value of the ratio* λ_1/λ_2. One thus has to consider two situations corresponding to the two branches of $\alpha=0$, namely B_1 and B_2.

6.2.1. The first branch B_1: non-integrable

Let us first consider the first branch B_1. The system of recursions (6.2), (6.4) and (6.7) is a *compatible* system. One can actually verify that (6.4) can be deduced from (6.2) and (6.7). One can replace this compatible system of recursions ((6.2), (6.4) and (6.7) taken only for *n even*) by

$$x_{n+2}=\frac{1+x_n-x_nx_{n+1}}{x_n\cdot x_{n+1}}, \tag{6.9}$$

$$x_{n+3}=\frac{(x_n+1)\cdot(-1-x_n+2x_nx_{n+1})\cdot L+x_n\cdot(1+x_n-x_nx_{n+1})}{(1+x_n-x_nx_{n+1})\cdot x_n\cdot x_{n+1}},$$

where $n=0,2,4,\ldots$ and where L denote the ratio of λ_1 and λ_2 that is $1+\varepsilon$. Of course, this interesting compatibility property (valid for arbitrary value of the ratio L that is arbitrary value of ε) *is not sufficient to imply integrability.*

Coming back to system (6.9), one can use the method developed in this paper to integrate it (if integrable!). For this purpose let us again associate, to this system of recursion, its corresponding Cremona transformation $\widehat{k}$ and its inverse

$$\widehat{k}: \quad (x,y)\to\left(\frac{1+x-xy}{xy},\frac{(x+1)\cdot(-x+2xy-1)\cdot L+(1+x-xy)\cdot x}{(1+x-xy)\cdot xy}\right),$$

$$\widehat{k}^{-1}: \quad (x,y)\to\left(\frac{-L+xy+Lx^2}{L+x^2-xy-Lx^2+x},\frac{x}{xy+L\cdot x^2-1}\right).$$

For the different integrable values of ε (or L), one gets that the successive gcd's of the X_N and Y_N conditions (corresponding to writing $\widehat{k}^N=identity$ on the two (x,y) coordinates) are just simple functions of x for $N=3$, 4, 5, 6, 7, 8, namely x and $1+x$. This suggests that this very system of recursions (6.9) is *not integrable.* Examples of 3×3-matrices corresponding to branch B_1 are given in Section D.2 of Appendix D. The graphical analysis of their iterates confirm this non-integrability.

6.2.2. The second branch B_2: Integrable

Let us consider the second branch B_2. One also has a compatible system of recursions and one can replace the system of recursion (6.2), (6.4) and (6.8) by

$$x_{n+3} = \frac{1 + x_{n+1} - x_{n+1}x_{n+2}}{x_{n+2} \cdot x_{n+1}},$$

$$x_{n+3} = \frac{(x_n \cdot x_{n+2} - 1) \cdot (x_{n+2} - 1) \cdot L}{x_n \cdot x_{n+2} \cdot (x_{n+1} - 1)} + \frac{1}{x_{n+1}}, \tag{6.10}$$

where $n = 0, 2, 4, \ldots$ It can also be rewritten eliminating x_{n+3} between the two last equations and x_{n+2} from the first equation (6.10):

$$x_{n+2} = \frac{x_n \cdot (1 - x_{n+1}^2) + x_{n+1} \cdot L}{x_n \cdot x_{n+1} \cdot L},$$

$$x_{n+3} = \frac{(x_n + x_n \cdot x_{n+1} - x_{n+1}) \cdot L + x_n \cdot (x_{n+1}^2 - 1)}{x_n \cdot (1 - x_{n+1}^2) + x_{n+1} \cdot L}. \tag{6.11}$$

One can also associate to (6.11) the following Cremona transformation $\widehat{k}$ and its inverse $\widehat{k}^{-1}$:

$$\widehat{k}: \quad (x, y) \to \left(\frac{x - xy^2 + Ly}{yLx}, \frac{Lx + yLx - x + xy^2 - Ly}{x - xy^2 + Ly} \right), \tag{6.12}$$

$$\widehat{k}^{-1}: \quad (x, y) \to \left(\frac{(yx - 1 + x)L}{x \cdot (yLx - L + Lx - xy^2 + 2y - 2yx + 2 - x)}, \frac{1}{yx - 1 + x} \right).$$

Let us assume that $L = \frac{4}{3}$ (i.e. $\varepsilon = \frac{1}{3}$) in the system of recursion (6.11). The successive gcd's G_N read

$$G_4 = 1 + y, \qquad G_6 = 3(y - 1) \cdot (y + 1) \cdot (2yx - x + 3xy^2 - 4y) \cdot (yx - 2 + x),$$

$$G_8 = (y + 1) \cdot \widehat{G}_8,$$

with

$$\begin{aligned} \widehat{G}_8 = {} & 64yx + 24x^2 + 520x^2y^2 - 344x^2y^4 + 256xy^3 + 128y^2 - 384xy^2 \\ & - 256y^3 + 544y^3x^2 + 80yx^3 - 432y^3x^3 - 64y^2x^3 - 256y^4x^3 \\ & - 16y^5x^3 - 16x^2y^5 + 384xy^4 + 128y^4 - 320y^5x + 60y^6x^4 + 72y^7x^4 \\ & - 144x^3y^7 + 312x^2y^6 + 27x^4y^8 - 192x^3y^6 - 20x^4y^2 + 3x^4 + 58x^4y^4 \\ & - 8x^4y + 24x^4y^3 + 40x^4y^5 - 16yx^2. \end{aligned}$$

The covariants are, respectively,

$$\tilde{C}_4 = \frac{4x}{3x - 3xy^2 + 4y}, \qquad \tilde{C}_6 = -\frac{\tilde{C}_4{}^3}{x^2 \cdot y}, \qquad \tilde{C}_8 = \frac{\tilde{C}_4{}^5}{x^4 \cdot y^2}$$

satisfying $\tilde{C}_4 \cdot \tilde{C}_5 = \tilde{C}_6{}^2$, yielding the $\widehat{k}$-invariant: $I_0 = (G_8 \cdot G_4)/G_6^2$.

Similar calculations for $\varepsilon = \frac{1}{2}, 0, -1$ are given in Appendix C.

The integrability seen for $\widehat{k} = \widehat{K}^2$ in the (u, v)-plane, or in the (i, v)-plane, *can actually be also seen on the determinants of the iterates* of $\widehat{k} = \widehat{K}^2$.

Note that these calculations are also valid for the *non-involutive* permutation (2.17), thus providing integrability cases for these quite non-trivial examples of birational transformations.

Let us remark that all these calculations can be revisited on the original birational transformations $\widehat{k} = \widehat{K}^2$ bearing on the entries of $q \times q$ matrices. Sections D.1 and D.2 of Appendix D provide, respectively, a "dictionary" and some examples of the $\alpha = 0$ cases for 3×3 matrices.

6.3. *Iteration of* 3×3 *matrices: Branch* B_2

For 3×3 matrices the $\alpha = 0$ conditions can be written explicitly (see also Section D.1 of Appendix D). Let us denote the initial matrix and its entries as follows:

$$M_0 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} . \tag{6.13}$$

Let us assume that the permutation of entries b and h represents the permutation t of class IV (see Section 2.2.1). Matrix M_0 belongs to the branch B_2 if one of these two conditions is satisfied:

$$ae + fh - ei - bd + bf - ce - dh + eg = 0 \quad \text{and} \quad di - fg + af - cd = 0 .$$

For $di - fg + af - cd = 0$ the ratio λ_1/λ_2 simplifies and reads

$$\frac{\lambda_1}{\lambda_2} = \frac{4ed(af - cd)}{(f - d)(efg + ged - fbd - fhd - eaf - bd^2 - hd^2 + 2ecd + ead)} . \tag{6.14}$$

For $-ei + fh + ae - bd + bf - ce - dh + eg = 0$ the ratio λ_1/λ_2 simplifies and reads

$$\frac{\lambda_1}{\lambda_2} = \frac{(bf + fh - bd + 2ae - dh - 2ce) \cdot (ae - dh - bd + ge)}{(2afh + 2a^2e - 2abd + 2abf - 2ace - 2adh + 2age - 2egc)e} . \tag{6.15}$$

The conditions for branch B_1 are given in Section D.1 of Appendix D.

Let us, for instance, consider the integrable value $\varepsilon = \frac{1}{2}$. Section D.2 of Appendix D gives an example of such a matrix (see (D.7)) corresponding to branch B_2 and $\varepsilon = \frac{1}{2}$. One can revisit the *factorization scheme* (similar to the one detailed for the birational mappings of class IV detailed in Section 2.2.1, see (2.13)) in this very integrable case. One actually see, similarly to a phenomenon already seen in the $\varepsilon = 0$ integrable case [7], that the (generic) factorization scheme (2.13)) is modified (occurrence of additional factorizations) yielding a *polynomial growth*[8] *of the calculations* [7]. The factorization scheme reads

$$f_1 = \det(M_0), \qquad M_1 = K(M_0), \qquad f_2 = \frac{\det(M_1)}{f_1},$$
$$M_2 = K(M_1), \qquad f_3 = \frac{\det(M_2)}{f_1^2 \cdot f_2},$$
$$M_3 = \frac{K(M_2)}{f_1}, \qquad f_4 = \frac{\det(M_3)}{f_1 \cdot f_2 \cdot f_3}, \qquad M_4 = K(M_3), \qquad f_5 = \frac{\det(M_4)}{f_1^2 \cdot f_2^2 \cdot f_3^2 \cdot f_4},$$
$$M_5 = \frac{K(M_4)}{f_1 \cdot f_2 \cdot f_3}, \qquad f_6 = \frac{\det(M_5)}{f_1^2 \cdot f_2^2 \cdot f_3 \cdot f_4 \cdot f_5}, \qquad M_6 = \frac{K(M_5)}{f_1 \cdot f_2}, \cdots \tag{6.16}$$

Note that relation (2.14) is still valid for this new factorization scheme. When compared to the factorization scheme of class IV (see (2.13)), one sees that, already for f_3 and M_3, an extra term factorizes namely f_1. This term makes the whole difference between integrability and non-integrability. The successive degrees of the (homogeneous) polynomials f_n's read, respectively, for $f_1, f_2, \ldots$: $3, 3, 3, 6, 6, 6, 9, 9, 12, 12, 18, 15, 24, 21, 30, \ldots$ The factorisation scheme and the associated ($\varepsilon = \frac{1}{2}$ integrable) recursions on the f_n's are not very simple as a consequence of the fact that the branch B_2 is associated with a *system* of two recursions (see (6.10)).

7. Conclusion

We have first analysed graphically a particular Cremona transformation depending on two continuous parameters ε and α. This graphical method shows curves, *globally invariant*, for which the Cremona transformation is of infinite order (the orbit of $\widehat{k}$ densifies an algebraic curve). The gcd's of the finite-order conditions yield algebraic expressions which enable to get quickly the foliation of the integrable Cremona transformations. When integrable the vanishing of these gcd's happen to be particular curves of the foliation. In fact, the graphical method enables to see both types of curves (the finite- and the infinite-order ones).

We provided explicit and simple examples (Cremona transformation associated with class IV) in order to show how a general method based on the analysis of the *finite-order conditions* enables to actually integrate Cremona transformations.

The integrability cases of this Cremona transformation have been revisited using various representations (Cremona transformation in (i, v), in (u, v), recursion in one variable, Cremona transformations associated with this recursion in one variable, recursion in the determinant of iterated matrices and Cremona transformation associated with this last recursion). Fortunately, the integrability cases of these various representations

[8] The relation between polynomial growth of the calculations and integrability has been discussed in detail in [8,10,15].

match. We have also obtained an algebraic invariant (4.9) of Cremona transformations depending on one continuous parameter α.

In fact, all these calculations are not specific of the birational transformation of class IV. They can be worked out on *any Cremona transformation* (birational transformation in two variables). Actually, Appendix E shows another example of foliation of the plane in algebraic elliptic curves obtained for an example of birational symmetries of the parameter space of a six-state chiral Potts model [1,16].

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Appendix A. Finite-order analysis for $\alpha = 0$ and various ϵ's

A.1. Integrable candidates for ε: (i, v)-analysis for $\alpha = 0$

In order to get the possible "integrable" values of ε, let us first eliminate ε in the two equations corresponding to $\widehat{k}^N = identity$. For instance, writing $\widehat{k}^6 = identity$ yields

$$I_6 = -\varepsilon \cdot (i + 1 - 2v + v^2) \cdot \widehat{I_6}\,, \tag{A.1}$$

where $\widehat{I_6}$ is a polynomial in i and v, and an expression for V_6 that is too large to be reproduced here. After performing the elimination of ε between $\widehat{I_6}$ and V_6, one obtains

$$\begin{aligned} R_0 &= i \cdot v^{10} \cdot (v+2) \cdot (3v-2) \cdot (v-2)^3 (v-1)^8 \cdot (i - v^2 + v)^5 \cdot (-v + 1 + i)^5 \\ &\quad \times (iv - 2i - 2 + 5v - 5v^2 + 2v^3) \cdot (3iv - 4i - v + v^2)(i + 3 - 5v + 2v^2) \\ &\quad \times (i - v + v^2)(i^2 + 4i - 5iv + 2v + 2v^2 i - 3v^2 + v^3) \cdot (-1 + i + v)^3 \cdots\,, \end{aligned} \tag{A.2}$$

where the $\cdots$ denote eight other factors. Let us just consider, to illustrate the method, one of the factors of R_0, namely

$$C_1 = iv - 2i - 2 + 5v - 5v^2 + 2v^3 = 0\,.$$

The elimination of i between C_1 and V_6 gives $R_1 = -(v-1)^4(-\varepsilon - 1 + 2v) \cdot \tilde{R}_1$ where

$$\begin{aligned} \tilde{R}_1 &= -232v^4 - 32v^2 + 136v^3 + 204v^5 - 93v^6 + 160\varepsilon^2 v - 489v^4\varepsilon + 216v^3\varepsilon^2 \\ &\quad - 24v^2\varepsilon^3 + 18v^7 - 16\varepsilon + 219\varepsilon v^5 - 54\varepsilon v^6 - 32\varepsilon^2 + 144\varepsilon v - 122v^4 e^2 - 16\varepsilon^3 \end{aligned}$$

$$+ 42\varepsilon^2 v^5 + 6v^7\varepsilon + 3\varepsilon^3 v^5 + 16\varepsilon^3 v^3 - 9\varepsilon^3 v^4 + 16\varepsilon^3 v - 440\varepsilon v^2 + 624\varepsilon v^3 - 240\varepsilon^2 v^2 - 9\varepsilon^2 v^6 .$$

The elimination of i between C_1 and $\widehat{I_6}$ gives $R_2 = 2(v-1)^9 \cdot (-\varepsilon - 1 + 2v) \cdot \tilde{R}_2$ where

$$\begin{aligned}\tilde{R}_2 = {} & 64 + 10668v^4 - 640v + 2800v^2 - 6992v^3 - 9296v^5 + 2745v^6 - 9491v^9 \\ & + 2007v^8 + 1679v^7 + 12207v^{10} - 1408\varepsilon^4 v + 64\varepsilon + 128\varepsilon^6 + 48v^9\varepsilon^5 + 832v^2\varepsilon^6 \\ & + 22896\varepsilon v^5 - 8301\varepsilon v^6 - 256\varepsilon^2 - 57v^{10}\varepsilon^4 - 9v^8\varepsilon^6 + 348v^{12}\varepsilon^2 + \cdots ,\end{aligned}$$

and $\cdots$ denote 62 other monomials of ε and v. The elimination of v between $\tilde{R}_1$ and $\tilde{R}_2$ gives a polynomial in ε which contains many factors:

$$\begin{aligned}C_0 = {} & (\varepsilon + 7) \cdot (2\varepsilon - 1) \cdot (3\varepsilon^2 - 10\varepsilon + 11) \cdot (\varepsilon - 1)^{11} \cdot (\varepsilon + 1)^{20} \\ & \times (4\varepsilon^6 + 44\varepsilon^5 - 23\varepsilon^4 - 697\varepsilon^3 + 1431\varepsilon^2 - 711\varepsilon + 104) \\ & \cdot (36\varepsilon^9 + 186\varepsilon^8 - 1268\varepsilon^7 + 2234\varepsilon^6 - 1522\varepsilon^5 + 257\varepsilon^4 + 118\varepsilon^3 - 32\varepsilon^2 - 1) \\ & \cdot (12\varepsilon^2 - 5\varepsilon + 1) \cdot (4\varepsilon^4 + 35\varepsilon^3 - 83\varepsilon^2 + 65\varepsilon - 13) \cdot (-2 + 3\varepsilon)^2 \cdot (-1 + 3\varepsilon)^6 .\end{aligned} \tag{A.3}$$

Of course, when one performs such sequence of eliminations one get many "spurious" candidates: this set of possible values of ε have to be reintroduced in the original equations to see the ones really yielding a common factorization for I_6 and V_6. In fact, this set of possible values of ε just correspond to factor C_1 in R_0 and one has to perform similar calculations for every other factor in R_0.

For higher values of the integer N the two conditions, for which one wants to see a common factor to factorize, become quickly quite "large", and the procedure to get the common factor requires to calculate and combine many resultants. The calculations become quickly very large.

A.2. Finite-order analysis for $\alpha = 0$ and various ε's: (i, v)-analysis

Let us consider the Cremona transformation (3.5) in i and v for $\alpha = 0$ and for $\varepsilon = \frac{1}{2}, 1$.

$\varepsilon = \frac{1}{2}$: The calculations of Section 4.3 can be performed for $\varepsilon = \frac{1}{2}$. The first non-trivial gcd's are obtained for $N = 5, 7$ and 8:

$$G_5 = v^2 - v + 2iv - 3i, \qquad G_8 = (v-1) \cdot (2v-3), \qquad G_7 = G_8 \cdot \tilde{G}_7 ,$$

with

$$\tilde{G}_7 = 8i^3 v^2 - 16i^3 v + 6i^3 + 6i^2 - 9i^2 v + 7iv + 6i^2 v^2 - 10v^2 i + 3v^2 + 5v^3 i - 6v^3 - 2iv^4 + 3v^4$$

transforming under $\widehat{k}$ as $G_i \to \tilde{C}_i \cdot G_i$ with

$$\tilde{C}_5 = \frac{i \cdot (2v-3)^2(-v+1+2i)}{4v \cdot (v-1)^2 \cdot (v^2-v+2iv-3i)}, \qquad \tilde{C}_8 = \frac{(-v^2+v+2iv-3i)i}{2(v-1)^3 v^2}$$

$$\tilde{C}_7 = \frac{(2v-3)^4 \cdot (-v^2+v+2iv-3i) \cdot i^3}{32v^6 \cdot (v-1)^7}$$

yielding the $\widehat{k}$-invariant

$$I_0 = \frac{G_7}{G_8 \cdot P_1^2} = \frac{\tilde{G}_7}{(i-v^2+v)^2} \quad \text{where } P_1 = i - v^2 + v \text{ (see (4.5))}$$

$\varepsilon = 1$: The first non-trivial gcd's are for $N = 4, 6, 8$:

$$G_4 = i \cdot (v-1), \quad G_6 = i \cdot (3i^2v^2 - 6i^2v + 4i^2 + 2iv - 2v^2i + v^4 + v^2 - 2v^3),$$

$$G_8 = G_4 \cdot (i^2v^2 - 2i^2v + 2i^2 - 2v^2i + 2iv + v^4 - 2v^3 + v^2)$$

transforming under $\widehat{k}$ as $G_i \to \tilde{C}_i \cdot G_i$ with

$$\tilde{C}_4 = \frac{i \cdot (v-2)^2}{v^2 \cdot (v-1)^2}, \qquad \tilde{C}_6 = \frac{(v-2)^5 \cdot i^2}{(v-1)^4 \cdot v^5} \quad \text{and} \quad \tilde{C}_8 = \tilde{C}_4^3 .$$

Therefore one gets a first $\widehat{k}$-invariant by considering the ratio $I_0 = G_8/G_4^3$ or more simply using parabola P_1: $I_1 = G_4/P_1$.

$\varepsilon = 0$: For $\varepsilon = 0$, one remarks that, since i is invariant, the transformation amounts to performing a homographic transformation on v (the parameters of the homographic transformation depending on i). The previous analysis is pointless: the invariant is already known (namely i).

A.3. Finite-order analysis for $\alpha = 0$ and various ε's: (u, v)-analysis

$\alpha = 0$ and $\varepsilon = \frac{1}{3}$: Let us assume here that $\varepsilon = \frac{1}{3}$. The first non-trivial gcd's are obtained for $N = 6$ and 8:

$$G_6 = (-2+3v) \cdot (-3u + 3uv + 4 - 3v) \cdot (3uv - u - 3v),$$

$$\begin{aligned} G_8 = {} & -9984uv^2 + 1536v^2 + 512u^2 + 1024uv + 20\,160u^2v^2 + 32\,256uv^3 \\ & + 12\,528u^4v^4 - 54\,144u^2v^3 - 3328u^2v + 39\,168u^3v^3 - 6912v^3 - 256u^3 \\ & + 1944v^6 - 46\,656uv^4 + 72\,144u^2v^4 - 16\,320u^3v^2 + 11\,232v^4 - 7776u^4v^5 \\ & - 49\,248u^3v^4 + 3328u^3v - 7776v^5 + 31\,104uv^5 - 46\,656u^2v^5 + 96u^4 \\ & - 1056u^4v - 10\,368u^4v^3 + 31\,104u^3v^5 + 1944u^4v^6 - 7776u^3v^6 - 7776uv^6 \\ & + 11\,664u^2v^6 + 4632u^4v^2 . \end{aligned}$$

These two covariants G_6 and G_8 transform under $\widehat{k}$ with the cofactors:

$$\tilde{C}_6 = \frac{4 - 3v + 3uv - u}{(-4+3v) \cdot u} \quad \text{and} \quad \tilde{C}_8 = \tilde{C}_6^2 ,$$

yielding the $\widehat{k}$-invariant $I_0 = G_8/G_6^2$.

$\alpha = 0$ *and* $\varepsilon = \frac{1}{2}$: Let us assume here that $\varepsilon = \frac{1}{2}$. The first non-trivial gcd's are deduced for $N = 7$ and 8:

$$\begin{aligned} G_7 = {} & 2u^3 - 12u^3 v + 26u^3 v^2 - 24u^3 v^3 + 8u^3 v^4 - 24u^2 v^4 + 72u^2 v^3 - 76u^2 v^2 \\ & + 24v^3 + 33u^2 v - 5u^2 + 24uv^4 - 72uv^3 + 72uv^2 - 27uv + 6u - 8v^4 \\ & - 22v^2 + 6v , \\ G_8 = {} & 144uv^2 - 44v^2 - 54uv + 3u + 12v - 10u^2 - 152u^2 v^2 + 66u^2 v + 48uv^4 \\ & - 48u^2 v^4 - 144uv^3 + 144u^2 v^3 - 48u^3 v^3 + 52u^3 v^2 - 24u^3 v + 48v^3 + 4u^3 \\ & - 16v^4 + 16u^3 v^4 . \end{aligned}$$

These two covariants G_7 and G_8 transform under $\widehat{k}$ with the same cofactor:

$$\tilde{C} = \frac{3 - 2v + 2uv - u}{(3 - 2v) \cdot u}$$

yielding the $\widehat{k}$-invariant $I_0 = G_8/G_7$.

$\alpha = 0$ *and* $\varepsilon = 1$: Let us now assume that $\varepsilon = 1$. The first non-trivial gcd's are for $N = 4, 6$ and 8:

$$\begin{aligned} G_4 &= u \cdot (1 - v) \cdot (1 - u) , \\ G_6 &= 3v^2 - 6uv^2 + 3u^2 v^2 - 6v + 12uv - 6u^2 v + 4 - 6u + 3u^2 , \\ G_8 &= G_4 \cdot (u^2 v^2 - 2u^2 v + u^2 + 4uv - 2uv^2 - 2u + v^2 - 2v + 2) . \end{aligned}$$

Note that G_6 is immediately an *invariant*, while the two covariants G_4 and G_8 transform under $\widehat{k}$ with the same cofactor $\tilde{C}$:

$$\tilde{C} = \frac{2 - v + uv - u}{(-2 + v)u}$$

yielding the $\widehat{k}$-invariant $I_0 = G_8/G_4 = \frac{1}{3}(2 + G_6)$.

$\alpha = 0$ *and* $\varepsilon = 0$: Let us assume $\varepsilon = 0$. The first non-trivial gcd's are deduced for $N = 3$ and 4:

$$G_3 = -v^2 + uv^2 - uv + v + u, \qquad G_4 = 2uv^2 - 2v^2 - 2uv + 2v + u .$$

These two covariants G_3 and G_4 transform under $\widehat{k}$, with the same cofactor:

$$\tilde{C} = -\frac{1 - v + uv}{(1 - v) \cdot u} \quad \text{yielding the } \widehat{k}\text{-invariant } I_0 = G_4/G_3 . \tag{A.4}$$

Appendix B. Recursion in a single variable for $\epsilon = \frac{1}{2}, 1, -1, 0$

Let us consider the recursion in a single variable defined by (5.3) for the "integrable" values of ε ($\varepsilon = \frac{1}{2}, 1, -1, 0$).

Recursion for $\varepsilon = \frac{1}{2}$. Let us now consider $\varepsilon = \frac{1}{2}$ and use the same analysis as before. One gets the first *non-trivial gcd's* G_N for $N = 7$ and 8, namely,

$$G_7 = 32t^3s^3 + 48t^3s^2 + 16t^3s + 24t^2s^2 + 12t^2s + 12t^2$$

$$- 8ts^3 - 22ts + 12t + 12s^2 - 12s + 3\,,$$

$$G_8 = 64t^3s^3 - 16ts^3 - 12s^2 + 12s - 3 + 28ts - 12t + 48t^2s^2$$

$$+ 24t^2s - 12t^2 + 96t^3s^2 + 32t^3s\,.$$

These two expressions G_7, G_8 transform under $\tilde{k}$ with the same cofactor:

$$\tilde{C} = \left(\frac{2t-1}{2t+2}\right)^2,$$

that is, the square of (5.6). One thus gets the $\tilde{k}$-invariant $I_0 = G_8/G_7$ or more simply, using D_1 (see (5.6)): $I_1 = G_7/D_1^2$.

Recursion for $\varepsilon = 1$: For $\varepsilon = 1$ the first non-trivial gcd's are obtained for $N = 4$ and 6, namely, $G_4 = s \cdot t$ and $G_6 = 3t^2s^2 - 2st + s^2 - 2s + t^2 + 2t + 1$. These expressions transform under $\tilde{k}$ by the cofactors $\tilde{C}_4 = (t-1)/(t+1)$ and $\tilde{C}_6 = \tilde{C}_4^2$. One thus gets the $\tilde{k}$-invariants

$$I_0 = \frac{G_6}{G_4^2} \quad \text{or more simply} \quad I_1 = \left(\frac{G_4}{D_1}\right)^2.$$

Recursion for $\varepsilon = -1$: Let us consider $\varepsilon = -1$. In these variables, transformation $\tilde{k}^2$, for $\varepsilon = -1$, is quite trivial (translation): $\tilde{k}^2(s,t) = (s+2, t+2)$. All the gcd's are trivial. The foliation of the (s,t) plane corresponds to the lines: $s - t = \text{constant}$.

Recursion for $\varepsilon = 0$: For $\varepsilon = 0$, one gets the first *non-trivial gcd's* G_N for $N = 3, 4$, namely, $G_3 = (st + t + 1) \cdot (t - s)$ and $G_4 = (2st + 2t + 1) \cdot (t - s)$. These expressions transform under $\tilde{k}$ with the same cofactor $\tilde{C}$, yielding a $\tilde{k}$-invariant I_0 or more simply I_1. They read, respectively,

$$\tilde{C} = \frac{-t}{t+1}, \qquad I_0 = \frac{G_4}{G_3}, \qquad I_1 = \frac{G_3}{D_1}\,.$$

Appendix C. Branch B_2 for $\epsilon = \frac{1}{2}, 0$

Let us consider the birational mapping (6.12), associated with the system of the two compatible recursions (6.11).

Let us consider $L=\frac{3}{2}$ (i.e. $\varepsilon=\frac{1}{2}$). The successive gcd's G_N, read, respectively, $G_4=1+y$ and

$$G_7=-15yx+12x+18y-45y^2+72xy^2-2x^2+2x^2y^2-42x^2y^4$$
$$-77y^3x^2+18y^3-2yx^3+2y^3x^3-10y^2x^3+75xy^3+27yx^2$$
$$+16y^4x^3+8y^5x^3+8x^2y^5-24xy^4+2x^3\,,$$
$$G_8=(y+1)\cdot\widehat{G}_8\,,$$

with

$$\widehat{G}_8=51yx-3x-18y+45y^2-18xy^2+8x^2-14x^2y^2-39xy^3$$
$$-15yx^2+29y^3x^2-18y^3+2yx^3-6y^3x^3+33xy^4-x^3$$
$$+4y^2x^3-7y^4x^3+4y^5x^3+4x^3y^6-20x^2y^5\,.$$

The calculation of the corresponding cofactors yield an invariant I_0. They read, respectively,

$$\tilde{C}_4=\frac{3x}{2x-2xy^2+3y},\qquad \tilde{C}_7=\frac{\tilde{C}_4^3}{x^2\cdot y},\qquad \tilde{C}_8=\frac{\tilde{C}_4^4}{x^3\cdot y},\qquad I_0=\frac{G_8}{G_4\cdot G_7}\,.$$

Let us assume $L=1$ (i.e. $\varepsilon=0$) in the system of recursion (6.11). The successive gcd's G_N are given by

$$G_3=(yx-1)\cdot(yx+x+y),\qquad G_4=(y+1)\cdot(yx-1)\cdot(-x+xy^2-2y)\,,$$
$$G_5=(x^2y^4+3y^3x^2+2x^2y^2-yx^2-x^2-3yx-4xy^2-y^2-xy^3)\cdot(yx-1)\,,$$
$$G_6=(y+1)\cdot(yx-1)\cdot(2xy^2+yx-x-3y)\cdot(yx+x+y)\,.$$

The cofactors read

$$\tilde{C}_3=\frac{1}{yx\cdot(x-xy^2+y)},\quad \tilde{C}_4=\tilde{C}_3^2\cdot x^2\cdot y,\quad \tilde{C}_5=\tilde{C}_3^2\cdot x^2,\quad \tilde{C}_6=\tilde{C}_3^3\cdot x^3\cdot y$$

yielding the $\widehat{k}$-invariant $I_0=G_6\cdot G_3/G_4\cdot G_5$ or more simply, $I_1=G_5\cdot(x\cdot y-1)/G_3^2$.

Appendix D. 3 × 3 matrices

D.1. Branch B_1 in terms of entries of 3 × 3 matrices

We use the notations of Section 6.3 and assume that the permutation of entries b and h represent the permutation t of class IV. Branch B_1, reads (besides a singular condition of non invertibility of M_0, $\det(M_0)=0$ and an algebraic condition which does not survive for higher values of n):

$$(a - i - c + g) \cdot (d + f) = 0 . \tag{D.1}$$

In the $d = -f$ case the ratio λ_1/λ_2 simplifies and reads

$$\frac{\lambda_1}{\lambda_2} = \frac{(a + i - g - c)(a + i + c + g)}{4ai - 4cg} \tag{D.2}$$

which yields, for instance, the following conditions for $\varepsilon = 0$ and $\varepsilon = -1$:

$$\begin{aligned} &\varepsilon = 0 \rightarrow (a - i - c + g) \cdot (a - g + c - i) = 0 , \\ &\varepsilon = -1 \rightarrow (a + i - g - c) \cdot (a + i + c + g) = 0 . \end{aligned} \tag{D.3}$$

Quadratic relations among the entries of M_0 hold for $\varepsilon = \frac{1}{2}, \frac{1}{3}, 1$.

In the $a = i + c - g$ case the ratio λ_1/λ_2 simplifies and reads

$$\frac{\lambda_1}{\lambda_2} = -\frac{2e \cdot (i + c)}{bf + fh + bd - 2ei - 2ce + dh} \tag{D.4}$$

which yields the following conditions for $\varepsilon = 0$ and $\varepsilon = -1$:

$$\varepsilon = 0 \rightarrow (b + h) \cdot (d + f) = 0, \qquad \varepsilon = -1 \rightarrow e \cdot (i + c) = 0 .$$

D.2. Some examples

Branch B_1: If one plots the iteration of $\widehat{k} = \widehat{K}^2$, or of the associated Cremona transformation in (s,t) (see (5.5)), for the various "integrable" values of ε, one gets quite chaotic orbits for the successive (s,t) or (s_n, s_{n+1}) for branch B_1. This can be seen iterating with $\widehat{k} = \widehat{K}^2$ the 3×3 initial matrices:

$$M_0 = \begin{bmatrix} -262 & 3 & -258 \\ 5 & -1 & 7 \\ -2 & 8 & -6 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -40 & 3 & -36 \\ 5 & -1 & 2 \\ -2 & 1 & -6 \end{bmatrix}$$

which correspond to branch B_1 and, respectively, $\varepsilon = \frac{1}{3}$ and $\varepsilon = \frac{1}{2}$ (and more precisely $a = c + i - g$ see Section D.1).

The simplicity of $\varepsilon = -1$ can be seen on matrix

$$M_0(m) = \begin{bmatrix} 2 & 3 & 6 \\ 5 & -1 + 32 \cdot m & 2 \\ -2 & 1 - 32 \cdot m & -6 \end{bmatrix} \tag{D.5}$$

which also corresponds to branch B_1. In fact, matrix (D.5) gives $x_n = 1$ for n even, $x_n = 13$ for $n = 2, 6, 10, 14, \ldots$ and $x_n = \frac{1}{13}$ for $n = 4, 8, 12, 16, \ldots$ This transformation of order four in the x_n's, is however a simple translation on the matrices $M_0(m) : m \rightarrow m + 1$.

Branch B_2: The following matrix $M_0(h)$ depending on one parameter h, corresponds to branch B_2:

$$M_0 = \begin{bmatrix} -40 & 3 & -7 \\ 13 & 1 & 2 \\ -5 & h & -11h-71 \end{bmatrix}. \tag{D.6}$$

$\varepsilon = \frac{1}{3}$ is then equivalent to $h = -\frac{19}{3}$ or $h = -\frac{12}{13}$.

An example of matrix associated with branch B_2, iterated in the factorization scheme (6.16) of Section 6.3 is, for instance,

$$M_0(f) = \begin{bmatrix} -255428 + 2304f & 515 & 317 \\ 111 & 1 & f \\ \lambda & 1789 & \mu \end{bmatrix}, \tag{D.7}$$

where $\lambda = -242\,701\,372 - 1\,765\,520\,640f + 15\,925\,248f^2$, and: $\mu = -243\,212\,861 - 1\,765\,516\,032f + 15\,925\,248f^2$.

Appendix E. Analysis of birational symmetries of a six-state chiral Potts model

Let us consider a six-state chiral Potts model [1,16]. It is known that there exists an infinite group of birational symmetries of the two-dimensional parameter space [1,16]. This group is, for instance, generated by an (infinite-order) birational transformation $\widehat{K}$ (or its inverse $\widehat{K}^{-1}$):

$$\widehat{K}: \quad (x,y) \to \left(\frac{1+x+2y-x^2-2xy-y^2}{2y^2-x^2-x}, \frac{1+x+2y-x^2-2xy-y^2}{x^2+xy-y^2-y} \right),$$

$$\widehat{K}^{-1}: \quad (x,y) \to \left(\frac{2x^2-y^2x-y^2}{2x^2y+y^2x+y^2x^2-x^2-2xy-y^2}, \frac{-(x^2+x^2y-xy-y^2)}{2x^2y+y^2x+y^2x^2-x^2-2xy-y^2} \right).$$

These birational transformations are *integrable* and have the following invariant [1,16]:

$$B = \frac{(2x^3+x^2+2x^2y-2xy-y^2x-2y^2)\cdot(x-y^2)^2}{(x+y)^4\cdot(x-1)\cdot(1-y)^2}. \tag{E.1}$$

Using the previously detailed systematic method one gets

$$\begin{aligned} G_3 &= x - y^2, \quad G_4 = (y-1)\cdot(x-1)\cdot(x+y)^2, \\ G_5 &= -8y^2x^2 - y^4 - 3y^6 + 3x^5 + 2y^5 - 6x^3y - 4xy^3 + x^5y^2 + 4x^4y^3 \\ &\quad - 13x^4y^2 - 2x^5y + 8x^3y^4 - 20x^3y^3 + 11x^3y^2 + 6y^5x^2 \\ &\quad - 11y^4x^2 + 8x^4y + 20x^2y^3 - 8y^5x + 13y^4x, \end{aligned}$$

$$
\begin{aligned}
G_6 &= (y-1)\cdot(x+y)\cdot(2x^3+x^2+2x^2y-2xy-y^2x-2y^2)\cdot(x-y^2)^2,\\
G_7 &= -48x^5y^3+18x^7y+y^4x^{10}-170x^8y^2-60x^9y+40x^2y^7-184x^2y^8\\
&\quad +40x^6y^3-6xy^8-70xy^9+8y^7x+14y^6x^2-14y^5x^3-59x^4y^4-164x^7y^3\\
&\quad +18x^8y+2x^6y^4-22x^5y^5-156x^4y^6+117x^3y^6-224x^3y^7-164x^3y^9\\
&\quad +300x^3y^8+246x^8y^3+2x^4y^8+90x^4y^7-8x^{10}y^2+24x^{10}y\\
&\quad -130x^5y^6-9x^{10}\\
&\quad -70x^9y^3+115x^9y^2-184x^8y^4+300x^7y^4-224x^7y^5\\
&\quad +81x^5y^4-8y^{10}-4y^9\\
&\quad -156x^6y^6-22x^5y^7+90x^6y^5+122x^4y^5-170x^2y^{10}+246x^2y^9-60xy^{11}\\
&\quad +115xy^{10}+9y^{12}x+18y^{11}x^2+9y^{10}x^3-6y^4x^9\\
&\quad +40y^5x^8+117y^6x^7-9y^{12}\\
&\quad +122y^7x^6+9y^{12}x^2+18y^{11}x^3+8y^5x^9+14y^6x^8+9x^8+9x^7y^2+24y^{11}\\
&\quad -14y^7x^7+40x^4y^9-4x^{10}y^3+81x^5y^8-59y^8x^6-48y^9x^5+9x^9+y^8
\end{aligned}
$$

with covariants

$$
C_3=\frac{\alpha\cdot\beta\cdot\gamma}{\lambda\cdot\mu^2},\qquad C_4=-\frac{\alpha^2\cdot\beta^2\cdot\gamma^2}{\lambda^3\cdot\mu^3},\qquad C_5=\frac{\alpha^3\cdot\beta^3\cdot\gamma^4\cdot\delta}{\lambda^5\cdot\mu^6},
$$

$$
C_6=\frac{\alpha^3\cdot\beta^4\cdot\gamma^5\cdot\delta^2}{\lambda^6\cdot\mu^8},\qquad C_7=\frac{\alpha^6\cdot\beta^6\cdot\gamma^8\cdot\delta^2}{\lambda^{10}\cdot\mu^{12}},
$$

with

$$
\alpha=y-1,\qquad \beta=3y+2x+1,\qquad \gamma=1+x+2y-x^2-2xy-y^2,
$$

$$
\delta=x-1,\qquad \lambda=x^2+x-2y^2,\qquad \mu=x^2+xy-y^2-y\,.
$$

From these covariants it is straightforward to see that it was necessary to go up to order seven to get the algebraic $\widehat{k}$-invariant:

$$
Q=\frac{G_7}{G_5^2}\,. \tag{E.2}
$$

The expression of this invariant is slightly more complicated than (E.1) but it is related to (E.2) by a very simple (rational) relation

$$
Q=\frac{1-7B+B^2}{(B+1)^2}=1-\frac{9\cdot B}{(1+B)^2}\,. \tag{E.3}
$$

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