28 July 1997

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 232 (1997) 211-216

Classical limit for a 3D lattice spin model

I.G. Korepanov [a,b,1], J.-M. Maillard [c,b,2], S.M. Sergeev [d,b,3]

[a] *University of Technology, Chelyabinsk 454080, Russian Federation*
[b] *Centre Émile Borel, Institut Henri Poincaré, 11 rue P. et M. Curie, 75231 Paris Cedex 05, France*
[c] *LPTHE, Tour 16, 1er etage, Boite 126, 4 place Jussieu, 75252 Paris Cedex 05, France*
[d] *Branch Institute for Nuclear Physics, Protvino 142284, Russian Federation*

Received 2 January 1997; revised manuscript received 8 May 1997; accepted for publication 14 May 1997
Communicated by A.P. Fordy

Abstract

The quasiclassical limit of the integrable statistical 3D lattice model known as the Zamolodchikov–Bazhanov–Baxter model is considered. We obtain a classical equation of motion for the scalar field, defined on the cubic lattice in 2 + 1-dimensional space-time, and show that it can be seen as a generalization of the Miwa equations. © 1997 Elsevier Science B.V.

PACS: 05.50; 02.10; 02.20
Keywords: Tetrahedron equations; Hirota–Miwa equations; Miwa equations; Semi-classical limit; 2 + 1 integrability

1. Introduction

There are now many integrable models in mathematical physics with one spatial and one temporal dimension, like the KdV, sine-Gordon equations etc. These models are relatively well studied both in their classical and quantum formulations [1]. The next natural step on the way to "realistic" 3 + 1-dimensional models must be the study of integrable models with two spatial dimensions. Much information is already available on such models as the Kadomtsev–Petviashvili equations [2] and other classical equations. On the other hand, there exists a quantum 2 + 1-dimensional model based on Zamolodchikov's solution of the tetrahedron equations, the Zamolodchikov–Bazhanov–Baxter model [3-5]. Many of its integrable structures and properties have still to be understood: one can, for instance, mention the problem of generalizing the Bethe ansatz for such 3D models. Moreover, no connection has yet been established between this very model and any classical integrable model in 2 + 1 dimensions, however the well-known correspondence between classical transfer matrices and quantum Hamiltonian has been studied by Baxter and Quispel: they have actually derived a two-dimensional quantum Hamiltonian commuting with the layer-to-layer transfer matrix of the three-dimensional Zamolodchikov–Bazhanov–Baxter model [6].

In fact, there are (at least) two types of three-dimensional models: classical models and quantum models. One should, however, recall the paper by Kashaev and Reshetikhin [7], where a relation between the quantum affine Toda theory and the Hirota–

[1] E-mail: igor@prima.tu-chel.ac.ru.
[2] E-mail: maillard@lpthe.jussieu.fr.
[3] E-mail: sergeev@mx.ihep.su.

PII S0375-9601(97)00394-0

Miwa equation has been established[4] [9]. There is another approach linking the tetrahedron equation and classical dynamical systems: it amounts to introducing a classical "functional[5] tetrahedron equation". One should also recall the paper by Kashaev where discrete three-dimensional equations are shown to be associated with the (local) Yang–Baxter relations: Kashaev in particular shows how one can construct certain operator solutions to the tetrahedron equations [12]. For now, the approach we will develop here seems to be very different from that of the paper by Kashaev [12], and the possible interconnections between these two approaches remain to be clarified.

The aim of this paper is to obtain a quasiclassical limit of the Zamolodchikov–Bazhanov–Baxter lattice model [3–5]. The main idea is to regard the limit when the number of colors N of the model tends to infinity as the usual limit for the $1/\hbar$ expansion in the quasiclassical approach ($N \sim 1/\hbar$). In such a way we will obtain a model in which field variables are defined on the cubic lattice and satisfy a discrete equation which turns out to be a generalization of the Miwa[6] equations [9].

The paper is organized as follows. The definition of the Zamolodchikov–Bazhanov–Baxter model is recalled in Section 2. In Section 3, a $1/\hbar$ derivation of the equation of motion for the classical field is suggested. Section 4 shows how these equations of motion can be reduced, under some restriction, to the Miwa equations. A brief discussion is given in Section 5.

[4] In fact, the Hirota–Miwa equations could also be called McCoy–Perk–Wu equations [8]: they were actually also discovered in the framework of the (two or n-point) correlation functions of the anisotropic two-dimensional Ising model by McCoy, Perk and Wu. These equations were seen as a (double) discrete generalization of the Painlevé equations.

[5] More details can be found in Ref. [10,11].

[6] The Miwa equation (discrete B-KP) has four terms instead of three in the Hirota–Miwa equation (discrete A-KP). The Miwa equation can be written $\sum_{j=1}^{4} \alpha_j \cdot \tau(n+e_j) \cdot \tau(n-e_j) = 0$, where $\sum_{j=1}^{4} e_j = 0$ and $\alpha_4 = \alpha_1 \cdot \alpha_2 \cdot \alpha_3$. The α_j's are some constants that can be arbitrarily changed by multiplying τ by an exponential of quadratic form in n. In a suitable continuous limit it reduces to the B-KP equation, while it is itself equivalent *to the whole* B-KP hierarchy (see for instance Refs. [9]). In a particular limit (namely $\alpha_j = \epsilon \cdot \beta_j$, $\epsilon \longrightarrow 0$, β_j being fixed) the Miwa equation reduces to Hirota's three-term equation associated with the discrete Toda system (see for instance Ref. [13]): $\sum_{j=1}^{3} \beta_j \cdot \tau(n+e_j) \cdot \tau(n-e_j) = 0$.

2. The Zamolodchikov–Bazhanov–Baxter model

The Zamolodchikov–Bazhanov–Baxter (ZBB) model [3–5] is a statistical model defined on a three-dimensional cubic lattice. To each elementary cube of the lattice a Boltzmann weight function is assigned. It depends on eight spins located at the corners of the cube, and some extra parameters called "spectral parameters". To each configuration of the lattice spins the product of all the corresponding Boltzmann weights is assigned. The partition function of the model is the sum of such products over all spin configurations. The integrability of the model follows from the existence of a commutative set of layer-to-layer transfer matrices [4,14,15].

In order to describe the Boltzmann weight function of the ZBB model, let us introduce some auxiliary functions and fix the notations. Such notations were already used in Ref. [16].

First, let us introduce a positive integer N, called *the number of colors*. The spin variables, like spin a in the following formula, takes integer values modulo N. Let p be a complex number. Then introduce the function

$$w(p|a) = \prod_{s=1}^{a} \frac{\Delta(p)}{1 - p\omega^s}, \qquad (1)$$

where

$$\omega = \exp(2\pi i/N), \quad \Delta(p)^N = 1 - p^N. \qquad (2)$$

The Boltzmann weight, assigned to an elementary cube of the lattice, is

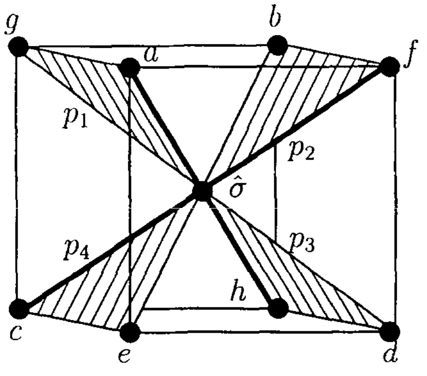

Fig. 1. Elementary cube of the lattice.

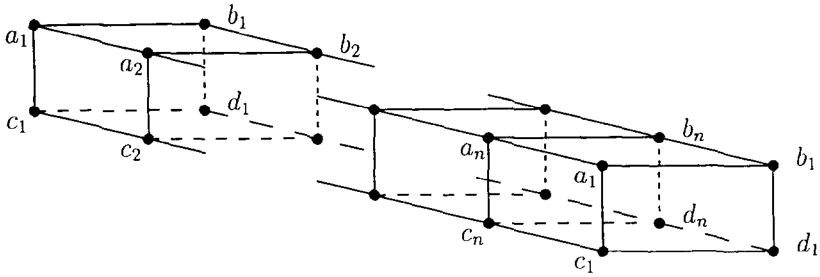

Fig. 2. A "pile" formed by a line of n cubes in the front-to-back direction of the lattice with periodic boundary.

$$W(a|e,f,g|b,c,d|h) = \sum_{\hat\sigma} \frac{w(p_3|d-h+\hat\sigma)w(p_1|a-g+\hat\sigma)}{w(p_4|e-c+\hat\sigma)w(p_2|f-b+\hat\sigma)} \times \omega^{\hat\sigma(c+b-g-h)}. \tag{3}$$

The integrability of the model is satisfied [4] provided

$$p_1 p_3 = \omega \cdot p_2 p_4. \tag{4}$$

The summation over the dummy central spin $\hat\sigma$ is shown in Fig. 1.

Let us now consider a projection of the cubic lattice in the front-to-back direction as shown in Fig. 2. Each "pile", formed by n cubes in the front-to-back direction, will be considered as a weight for a two-dimensional model with quite involved spin interactions (see Ref. [4]),

$$\mathcal{W}(A,B,C,D) = \prod_k W(a_{k+1}|c_{k+1},b_{k+1},a_k|b_k,c_k,d_{k+1}|d_{k+1}) \tag{5}$$

where A (B, C, D) denotes the sequence of spins $A = \{a_k\}\ldots$ Here k corresponds to a labeling of the layers (see Fig. 2, see also Ref. [4] for more details).

Each $\mathcal{W}$ takes into account the "hidden" summation over the "dummy" variables $\hat\sigma_k$. It is useful to consider those spins as the very spins of a 2D lattice by choosing the sub-orbit in the space of the spin states of the type

$$\sum \hat\sigma_k = 0 \tag{6}$$

for each $\mathcal{W}$. This trick does not change the partition function [4]. With this restriction, the model is nothing but the generalized chiral Potts model [17].

One can introduce internal spins σ_k obeying

$$\hat\sigma_k = \sigma_k - \sigma_{k+1}, \tag{7}$$

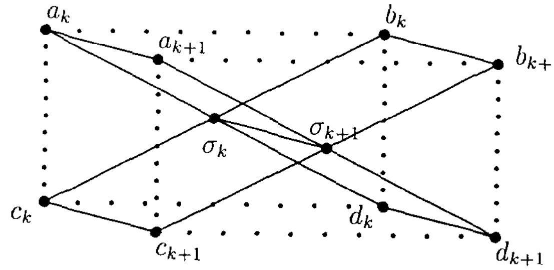

Fig. 3. Modified Boltzmann weight.

so that relation (6) holds automatically (see Fig. 3). Therefore, the elementary cube on the kth position depends on $(a,b,c,d,\sigma)_k$ and $(a,b,c,d,\sigma)_{k+1}$, as shown in Fig. 3.

3. Quasiclassical limit

Let us now discuss how one can obtain a "classical" version of the ZBB model.

In a quantum field theory, described in terms of a path integral, the partition function is usually defined as

$$\mathcal{Z} = \int \mathcal{D}\phi(x) \exp\left(\frac{1}{\hbar} S([\phi(x)]\right). \tag{8}$$

A solution to the classical equation of motion

$$\frac{\delta S}{\delta\phi(x)}([\phi]) = 0 \tag{9}$$

is the saddle point configuration for the path integral (8).

In order to obtain an equation like (9) for the ZBB model, let us apply some simple heuristic considerations. Namely, when N tends to infinity, let us formally perform the following changes,

$$\frac{N}{2\pi} = \frac{1}{\hbar}, \quad a = \frac{\overline{a}}{\hbar}, \ldots, \quad \sum_a = \int_0^{2\pi} \frac{\delta\overline{a}}{\hbar}. \tag{10}$$

Then

$$w(p|a) = \exp\left(-\frac{1}{\hbar} \int_0^{\overline{a}} (1 - pe^{i\phi})\delta\phi \right) [1 + O(\hbar)], \tag{11}$$

where $|p| < 1$. Leaving only the leading terms in the $1/\hbar$ expansion, one obtains the following form for the partition function,

$$\mathcal{Z} \to \left(\prod \int \delta\,\text{spin} \right) \exp\left(\frac{S[\text{spins}]}{\hbar} \right), \tag{12}$$

and one can now apply Eq. (9) to (12).

Differentiating "the action" $S[\text{spins}]$ with respect to the (continuous) spins, one gets the "classical equations of motion". In fact there are two different types of equations of motion, corresponding respectively to the derivatives over $\overline{a}$-type spins and over $\overline{\sigma}$-type spins. The $\partial/\partial\overline{\sigma}_k$ derivative, being applied to the action corresponding to (5), gives

$$\frac{(1 - g_1(k)^{-1}) \cdot (1 - g_2(k)^{-1})}{(1 - g_2(k)^{-1}) \cdot (1 - g_4(k)^{-1})} = \frac{(1 - g_1(k+1)) \cdot (1 - g_3(k+1))}{(1 - g_2(k+1) \cdot (1 - g_4(k+1))} \tag{13}$$

where a function $g(k)$, assigned to the *links* of the 2D lattice, is defined as

$$g_1(k) = p_1 \cdot \exp[i(\overline{a}_k - \overline{a}_{k-1} - \overline{\sigma}_k + \overline{\sigma}_{k-1})], \tag{14}$$

g_1 being the link between $\overline{a}$ and $\overline{\sigma}$ spins as shown in Fig. 4, and similarly for g_2, g_3, g_4, according to Fig. 4, with the "*white*" $\overline{\sigma}$-type spin inside.

The indices of the g's always correspond to the indices of the former parameters p, as can be seen in Fig. 5. In some formulae below it is implied that the four links belong to a common vertex, as in Fig. 4, while in others it is implied that they surround a plaquette, as in Fig. 6.

In order to obtain the derivative of the action $S[\text{spins}]$ with respect to a *black* (i.e. a-type) spin, one has to consider eight Boltzmann weights surrounding this spin. The equation of motion reads

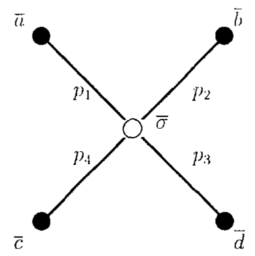

Fig. 4. Element of the 2D lattice, white $\overline{\sigma}$ inside.

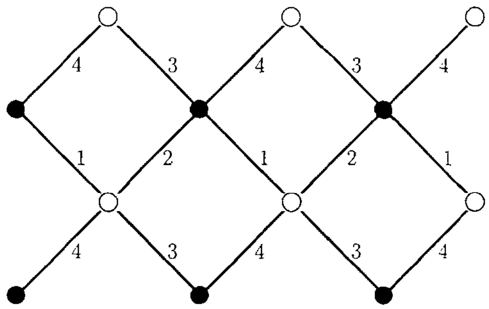

Fig. 5. One 2D layer.

$$\frac{(1 - g_1(k)) \cdot (1 - g_2(k))}{(1 - g_2(k)) \cdot (1 - g_4(k))} = \frac{(1 - g_1(k+1)^{-1}) \cdot (1 - g_3(k+1)^{-1})}{(1 - g_2(k+1)^{-1}) \cdot (1 - g_4(k+1)^{-1})}, \tag{15}$$

where the $g(k)$'s are assigned to the corresponding links, surrounding the *black* spin.

According to definition (14) of the g's there exists, in each front-to-back layer and for each *plaquette*, an identity

$$g_1(k) \cdot g_3(k) = g_2(k) \cdot g_4(k). \tag{16}$$

The field variables have been assigned to the *links* of the 2D lattice with *black* and *white* vertices so that

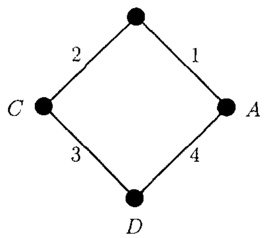

Fig. 6. Vertex variables.

Eqs. (13) and (15), together with condition (16), define the evolution of these variables.

It can be well suited to avoid the "segregation" of vertices, replacing somehow the two equations (13) and (15) by a single one. For this purpose let us introduce the variables f_n, instead of the g_n, as follows,

$$f_3 = g_3, \quad f_4 = g_4, \quad f_2 = g_2^{-1}, \quad f_1 = g_1^{-1}. \tag{17}$$

Then the difference between *white* and *black* vertices disappears, and one gets for the surrounding of each *vertex*

$$\frac{f_2}{f_1} \cdot \frac{(1-f_1)\cdot(1-f_3)}{(1-f_2)\cdot(1-f_4)} = \frac{f_4'}{f_3'} \cdot \frac{(1-f_1')\cdot(1-f_3')}{(1-f_2')\cdot(1-f_4')}, \tag{18}$$

where the f_n's correspond to the kth front-to-back layer, and the f_n''s to the $k+1$th layer. For each *plaquette* one then has the following condition,

$$\frac{f_2}{f_1} = \frac{f_4}{f_3}. \tag{19}$$

4. Connection with Miwa equations

Let us now establish a relation between Eq. (18), together with condition (19), on the one hand, and some known equation in discrete space-time, on the other. It will be shown that the system (18, 19) is, to some extent, more general than the well-known Miwa equations [9].

First of all, let us introduce the new variables $\varphi_{\dots}$, instead of the $f_{\dots}$'s, so that (19) holds automatically. The φ's will not be associated with the links, but with the *vertices* of the lattice. Let each f, corresponding to some link, be the *ratio* of the φ's corresponding, respectively, to the *bottom end* and *upper end* of the link, for example (see Fig. 6),

$$f_1 = \frac{\varphi_A}{\varphi_B}, \quad f_2 = \frac{\varphi_C}{\varphi_B},$$
$$f_3 = \frac{\varphi_D}{\varphi_C}, \quad f_4 = \frac{\varphi_D}{\varphi_A}. \tag{20}$$

Then, the equation for the φ's reads (see Fig. 7 for notations)

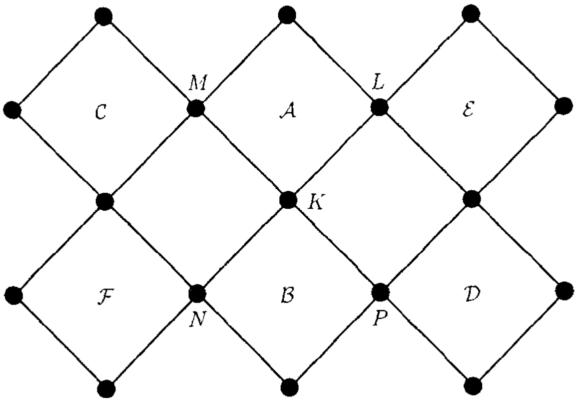

Fig. 7. Location of the variables ϕ and ψ.

$$\frac{\varphi_L\cdot\varphi_N}{\varphi_M\cdot\varphi_P} \cdot \frac{(\varphi_K-\varphi_M)\cdot(\varphi_P-\varphi_K)}{(\varphi_K-\varphi_L)\cdot(\varphi_N-\varphi_K)} = \frac{(\varphi_K'-\varphi_M')\cdot(\varphi_P'-\varphi_K')}{(\varphi_K'-\varphi_L')\cdot(\varphi_N'-\varphi_K')} \tag{21}$$

Again the notation φ' corresponds to the $k+1$th layer.

The second step is a little bit more complicated. Instead of solving Eq. (21) in its full generality, we are going to introduce some *ansatz* which reduces the number of variables by half. Namely, let us introduce new variables $\psi_{\dots}$ belonging to *a half of the plaquettes*, as depicted in Fig. 7.

Then let us set each φ be equal to the product of two neighboring ψ's, regardless of whether they are situated "horizontally" or "vertically". For instance

$$\varphi_K = \psi_A \cdot \psi_{\mathcal{B}}, \qquad \varphi_L = \psi_A \cdot \psi_{\mathcal{E}}, \quad \dots \tag{22}$$

It will be seen, from the calculations below, that this ansatz does not lead to any contradiction: the ansatz is actually compatible with the equations of motion.

Now the equation for the ψ's obtained from (21) and (22) reads

$$\frac{\psi_{\mathcal{E}}\cdot\psi_{\mathcal{F}}}{\psi_{\mathcal{C}}\cdot\psi_{\mathcal{D}}} \cdot \frac{(\psi_{\mathcal{B}}-\psi_{\mathcal{C}})\cdot(\psi_{\mathcal{D}}-\psi_{A})}{(\psi_{\mathcal{B}}-\psi_{\mathcal{E}})\cdot(\psi_{\mathcal{F}}-\psi_{A})} = \frac{(\psi_{\mathcal{B}}'-\psi_{\mathcal{C}}')\cdot(\psi_{\mathcal{D}}'-\psi_{A}')}{(\psi_{\mathcal{B}}'-\psi_{\mathcal{E}}')\cdot(\psi_{\mathcal{F}}'-\psi_{A}')}. \tag{23}$$

The remarkable fact is that both sides of (23) are products of the corresponding sides of the equation

$$\frac{\psi_{\mathcal{E}}\cdot\psi_{\mathcal{B}}}{\psi_{A}\cdot\psi_{\mathcal{D}}} \cdot \frac{(\psi_{\mathcal{D}}-\psi_{A})}{(\psi_{\mathcal{B}}-\psi_{\mathcal{E}})} = \frac{(\psi_{\mathcal{D}}'-\psi_{A}')}{(\psi_{\mathcal{B}}'-\psi_{\mathcal{E}}')} \tag{24}$$

with the same equation, where the indices are changed as follows,

$$\mathcal{E}\to\mathcal{A},\quad \mathcal{D}\to\mathcal{B},\quad \mathcal{A}\to\mathcal{C},\quad \mathcal{B}\to\mathcal{F}.$$

The last step is the substitution

$$\psi=\frac{X'}{X}, \tag{25}$$

yielding

$$\frac{X'_{\mathcal{A}}X_{\mathcal{D}}-X'_{\mathcal{D}}X_{\mathcal{A}}}{X'_{\mathcal{E}}X_{\mathcal{B}}-X'_{\mathcal{B}}X_{\mathcal{E}}}=\frac{X''_{\mathcal{A}}X'_{\mathcal{D}}-X''_{\mathcal{D}}X'_{\mathcal{A}}}{X''_{\mathcal{E}}X'_{\mathcal{B}}-X''_{\mathcal{B}}X'_{\mathcal{E}}}. \tag{26}$$

X' and X'' correspond respectively to the $k+1$th and $k+2$th layers. Here the r.h.s. is obviously the same as l.h.s., but "shifted" by a unit of time. So, the l.h.s. of (26) is an *integral of motion*. Still, it can depend on spatial coordinates, but if one sets it to be equal to a constant λ (independent of the spatial coordinates), one obtains the well-known Miwa equation[7] [9]

$$X'_{\mathcal{A}}X_{\mathcal{D}}-X'_{\mathcal{D}}X_{\mathcal{A}}-\lambda\cdot X'_{\mathcal{E}}X_{\mathcal{B}}+\lambda\cdot X'_{\mathcal{B}}X_{\mathcal{E}}=0. \tag{27}$$

5. Conclusion

In the quasiclassical limit we have changed the discrete spin variables into continuous (complex) variables $g(k)$. The calculation of the "continual integral" (8) in the quasiclassical limit takes place in a neighborhood of some "trajectory" which is a stationary point of the functional S. This stationarity condition is formulated as a *local* equation, and that is exactly what could be seen as the equation of motion for the classical model corresponding to the quantum ZBB model.

The deduced classical equation of motion seems to be a new "integrable" equation in $2+1$-dimensional space-time. Actually, we managed to establish a connection between this new equation and the Miwa equations by using the ansatz (22), which reduces the number of variables by half and is compatible with the time evolution.

[7] Relation (27) is the Miwa equation $\sum_{j=1}^{4}\alpha_j\cdot\tau(n+e_j)\cdot\tau(n-e_j)=0$ (where $\sum_{j=1}^{4}e_j=0$ and $\alpha_4=\alpha_1\alpha_2\alpha_3$) with $\alpha_4=\lambda$, $\alpha_3=-\lambda$, $\alpha_2=-1$, $\alpha_1=1$.

Acknowledgement

This work has been performed in the Centre Émile Borel during the "Integrable Semester". We would like to thank H. Au-Yang, P. Kulish, B.M. McCoy, T. Miwa, J.H.H. Perk, C.M. Viallet and F.Y. Wu for many fruitful discussions on 3D integrability. I.K. has been supported by the Steklov-CNRS joint program and a grant of the Russian Foundation for Fundamental Research, project 96-01-00708. S.S. has been supported by a grant of the Ministère de l'Enseignement Supérieur et de la Recherche and by the INTAS-ICFPM grant 93-2492-ext.

References

[1] V.E. Zakharov, What is Integrability?, Springer Verlag Series in Non-linear Dynamics (1991);
L.D. Faddeev, Integrable Models in 1 + 1 Dimensional Quantum Field Theory, Les Houches Lectures, 1982 (North-Holland, Amsterdam, 1984);
L.D. Faddeev and L.A. Takhtajan, Hamiltonian methods in the theory of solitons (Springer, Heidelberg, 1986).
[2] B.B. Kadomtsev and B.I. Petviashvili, Sov. Phys. Dokl. 15 (1970) 539.
[3] A.B. Zamolodchikov, Commun Math. Phys. 79 (1981) 489.
[4] V.V. Bazhanov and R.J. Baxter, J. Stat. Phys. 69 (1992) 453.
[5] V.V. Bazhanov and R.J. Baxter, J. Stat. Phys. 71 (1992) 839.
[6] R.J. Baxter and G.R.W. Quispel, J. Stat. Phys. 58 (1990) 411.
[7] R.M. Kashaev and N.Yu. Reshetikhin, Affine Toda field theory as a 3-dimensional integrable system, Berkeley preprint (1995).
[8] B.M. McCoy, J.H.H. Perk and T.T. Wu, Phys. Rev. Lett. 46 (1981) 757;
J.H.H. Perk and H.W. Capel, Physica A 89 (1977) 265;
J.H.H. Perk, Phys. Lett. A 79 (1990) 1.
[9] T. Miwa, Proc. Japan Acad. 58A (1982) 9;
E. Date, M. Jimbo and T. Miwa, J. Phys. Soc. Japan 52 (1983) 766.
[10] V.V. Bazhanov, S.M. Sergeev and V.V. Mangazeev, Quantum dilogarithm and the tetrahedron equation, preprint IHEP, 95-141 (1995), to be published.
[11] S.M. Sergeev and R.M. Kashaev, On pentagon, ten-term, and tetrahedron relations, preprint ENSLAPP-L-611/96, submitted to Commun. Math. Phys. (1996).
[12] R.M. Kashaev, Lett. Math. Phys. 35 (1996) 389.
[13] R. Hirota, J. Phys. Soc. Japan 50 (1981) 3785.
[14] V.V. Bazhanov and Yu.G. Stroganov, Teor. Mat. Fiz. 52 (1982) 105.
[15] R.J. Baxter, Physica D 18 (1986) 321.
[16] V.V. Mangazeev, R.M. Kashaev and Yu.G. Stroganov, Int. J. Mod. Phys. A 8 (1993) 587.
[17] V.V. Mangazeev, V.V. Bazhanov, R.M. Kashaev and Yu.G. Stroganov, Commun. Math. Phys. 138 (1991) 393.