

High-order Fuchsian equations for the square lattice Ising model: $\chi^{(6)}$

S Boukraa¹, S Hassani², I Jensen³, J-M Maillard⁴ and N Zenine²

¹ LPTHIRM and Département d'Aéronautique, Université de Blida, Algeria

² Centre de Recherche Nucléaire d'Alger, 2 Bd. Frantz Fanon, BP 399, 16000 Alger, Algeria

³ ARC Centre of Excellence for Mathematics and Statistics of Complex Systems, Department of Mathematics and Statistics, The University of Melbourne, Victoria 3010, Australia

⁴ LPTMC, Université de Paris 6, Tour 24, 4ème étage, case 121, 4 Place Jussieu, 75252 Paris Cedex 05, France

E-mail: boukraa@mail.univ-blida.dz, I.Jensen@ms.unimelb.edu.au, maillard@lptmc.jussieu.fr and njzenine@yahoo.com

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Abstract

This paper deals with $\tilde{\chi}^{(6)}$, the six-particle contribution to the magnetic susceptibility of the square lattice Ising model. We have generated, modulo a prime, series coefficients for $\tilde{\chi}^{(6)}$. The length of the series is sufficient to produce the corresponding Fuchsian linear differential equation (modulo a prime). We obtain the Fuchsian linear differential equation that annihilates the 'depleted' series $\Phi^{(6)} = \tilde{\chi}^{(6)} - \frac{2}{3}\tilde{\chi}^{(4)} + \frac{2}{45}\tilde{\chi}^{(2)}$. The factorization of the corresponding differential operator is performed using a method of factorization modulo a prime, introduced in a previous paper. The 'depleted' differential operator is shown to have a structure similar to the corresponding operator for $\tilde{\chi}^{(5)}$. It splits into factors of smaller orders, with the left-most factor of order 6 being equivalent to the symmetric fifth power of the linear differential operator corresponding to the elliptic integral E . The right-most factor has a direct sum structure, and using series calculated modulo several primes, all the factors in the direct sum have been reconstructed in exact arithmetics.

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1. Introduction and recalls

The magnetic susceptibility (high temperature χ_+ and low temperature χ_-) of the square lattice Ising model is given by [1]

$$\chi_+(w) = \sum \chi^{(2n+1)}(w) = \frac{1}{s} \cdot (1 - s^4)^{\frac{1}{4}} \cdot \sum \tilde{\chi}^{(2n+1)}(w) \tag{1}$$

and

$$\chi_-(w) = \sum \chi^{(2n)}(w) = (1 - 1/s^4)^{\frac{1}{4}} \cdot \sum \tilde{\chi}^{(2n)}(w). \tag{2}$$

in terms of the self-dual temperature variable $w = \frac{1}{2}s/(1 + s^2)$, with $s = \sinh(2J/kT)$. The n -particle contributions $\tilde{\chi}^{(n)}$ are given by $(n - 1)$ -dimensional integrals [2–5],

$$\tilde{\chi}^{(n)}(w) = \frac{1}{n!} \cdot \left(\prod_{j=1}^{n-1} \int_0^{2\pi} \frac{d\phi_j}{2\pi} \right) \left(\prod_{j=1}^n y_j \right) \cdot R^{(n)} \cdot (G^{(n)})^2, \tag{3}$$

where⁵

$$G^{(n)} = \prod_{1 \leq i < j \leq n} h_{ij}, \quad h_{ij} = \frac{2 \sin((\phi_i - \phi_j)/2) \cdot \sqrt{x_i x_j}}{1 - x_i x_j}, \tag{4}$$

and

$$R^{(n)} = \frac{1 + \prod_{i=1}^n x_i}{1 - \prod_{i=1}^n x_i}, \tag{5}$$

with

$$x_i = \frac{2w}{1 - 2w \cos(\phi_i) + \sqrt{(1 - 2w \cos(\phi_i))^2 - 4w^2}}, \tag{6}$$

$$y_i = \frac{2w}{\sqrt{(1 - 2w \cos(\phi_i))^2 - 4w^2}}, \quad \sum_{j=1}^n \phi_j = 0. \tag{7}$$

As n grows, the series generation in the variable w of the integrals (3) becomes very time consuming. In [6] calculations modulo a prime were performed on simplified integrals $\Phi_H^{(n)}$ and this work demonstrated that most of the pertinent information (singularities, critical exponents, etc) can be obtained from linear ODEs known modulo a prime corresponding to the integrals $\Phi_H^{(n)}$. In order to go beyond $\tilde{\chi}^{(4)}$ this strategy was adopted previously for the 5-particle contribution $\tilde{\chi}^{(5)}$ [7, 8] and here for the 6-particle contribution $\tilde{\chi}^{(6)}$.

In a previous paper [7] massive computer calculations were performed on $\tilde{\chi}^{(5)}$, $\tilde{\chi}^{(6)}$ and χ (in exact arithmetics and/or modulo a prime). These calculations confirmed previously conjectured singularities for the linear ODEs of the $\tilde{\chi}^{(n)}$'s as well as their critical exponents, and shed some light on important physical problems such as the existence of a natural boundary for the susceptibility of the square Ising model and the subtle resummation of logarithmic behaviours of the n -particle contributions $\tilde{\chi}^{(n)}$ to give rise to the power laws of the full susceptibility χ . As far as $\chi^{(5)}$ is concerned, the linear ODE for $\tilde{\chi}^{(5)}$ was found modulo a single prime [7] and it is of minimal order 33.

⁵ The Fermionic term $G^{(n)}$ has several representations [3].

1.1. Results on $\tilde{\chi}^{(5)}$

In [8] the linear differential operator for $\tilde{\chi}^{(5)}$ was carefully analysed. In particular it was found that the minimal-order linear differential operator for $\tilde{\chi}^{(5)}$ can be reduced to a minimal-order linear differential operator L_{29} of order 29 for the linear combination

$$\Phi^{(5)} = \tilde{\chi}^{(5)} - \frac{1}{2} \tilde{\chi}^{(3)} + \frac{1}{120} \tilde{\chi}^{(1)}. \quad (8)$$

We shall use the term ‘depleted’ series for a series obtained by subtracting from $\tilde{\chi}^{(n)}$ a definite amount of the lower n -particle contributions $\tilde{\chi}^{(n-2k)}$, $k = 1, 2, \dots$, as in (8), such that the differential operator annihilating the depleted series is of lower order. Since the depleted series is annihilated by an ODE of lower order, it follows that in the ODE for the original series, we must have the occurrence of a direct sum structure. It was found [8] that the linear differential operator, L_{29} , can be factorized as a product of an order 5, an order 12, an order 1 and an order 11 linear differential operator

$$L_{29} = L_5 \cdot L_{12} \cdot \tilde{L}_1 \cdot L_{11}, \quad (9)$$

where the order 11 linear differential operator has a direct-sum decomposition:

$$L_{11} = (Z_2 \cdot N_1) \oplus V_2 \oplus (F_3 \cdot F_2 \cdot L_1^s). \quad (10)$$

Z_2 is a second-order operator also occurring in the factorization of the linear differential operator [9] associated with $\tilde{\chi}^{(3)}$ and it corresponds to a *modular form of weight one* [10]. V_2 is a second-order operator equivalent to the second-order operator associated with $\tilde{\chi}^{(2)}$ (or equivalently to the complete elliptic integral E). F_2 and F_3 are remarkable second- and third-order *globally nilpotent* linear differential operators, respectively [8, 10]. The first-order linear differential operator \tilde{L}_1 quite remarkably has a *polynomial solution*. The fifth-order linear differential operator L_5 was shown to be equivalent to the symmetric fourth power of (the second-order operator) L_E corresponding to the complete elliptic integral E . The complete and detailed analysis of L_{12} , the order 12 operator in (9), is beyond the scope of our current computational resources (see [8] for details).

It is important to note that these factorization results are exact and have been obtained from series and ODEs obtained modulo a *single prime*. For the reconstruction in exact arithmetics of the factors occurring in the differential operator L_{11} , we had to obtain the series and ODEs for more than one prime. The length of the series necessary to obtain the underlying ODE is initially unknown, except perhaps for some rough estimates. Once the first non-minimal-order ODEs have been obtained modulo a prime, the *minimum length* of the series necessary to obtain non-minimal-order ODEs for any other primes is *known exactly*. This knowledge comes from a relation we reported in [7] and that we called the ‘ODE formula’. Beyond understanding the terms occurring in the ‘ODE formula’ and the light they shed on the ODEs underlying the problem, the formula has been of most importance in terms of gains in the computational effort. For instance, we initially generated, modulo a prime, 10 000 terms for $\tilde{\chi}^{(5)}$ and we found that we can obtain non-minimal ODEs using only some 7400 terms, while non-minimal-order ODEs for $\Phi^{(5)}$ can be obtained using some 6200 terms, representing a great reduction in the required computational effort.

1.2. The ODE formula

Let us denote by Q the order of the ODE we are looking for and by D the degree of the polynomials in front of the derivatives (we write the ODE in the homogeneous derivative $x \frac{d}{dx}$). We must then have $(Q + 1)(D + 1)$ terms in the series in order to determine the

unknown polynomial coefficients. If an ODE exists, it appears that the number of terms actually necessary for the ODE to be obtained is given by

$$N = (Q + 1)(D + 1) - f, \tag{11}$$

where f is a positive integer and indicates the number of ODE solutions to the linear system of equations for the polynomial coefficients.

From empirical observation, we have seen [7] that N is also given, linearly in terms of Q and D , by

$$N = d \cdot Q + q \cdot D - C. \tag{12}$$

While Q and D are the order and the degree, respectively, of *any non-minimal-order* ODE that we choose to look for, the parameters d , q and C depend on the series we are working with. In all the cases we have considered, we have found that q is the order of the *minimal-order* ODE and d is the number of singularities (counted with multiplicity) excluding any apparent singularities and the singular point $x = 0$. The parameter C was shown in [8] to be in an exact relationship with the degree D_{app} of the apparent polynomial of the minimal-order ODE:

$$D_{app} = (d - 1)(q - 1) - C - 1. \tag{13}$$

Note that there are many ODEs that annihilate a given series. Among all these ODEs, there is a unique one of minimal order. In our calculations we have seen that it is easier to produce ODEs, which are not of minimal order [11], in the sense that fewer terms are needed to obtain these ODEs compared to what is required to obtain the minimal-order ODE. Even more important for computational purposes, there is a non-minimal-order ODE that requires the minimum number of terms in order to be obtained.

Next we demonstrate how we use the ODE formula to optimize our calculations, i.e. generate just the necessary number of terms in the series. From (11)–(13), the parameter D is given as

$$D = d - 1 + \frac{D_{app} + f}{Q - q + 1} \tag{14}$$

and this must be a positive integer. The parameters f and Q are integers with the constraints $f \geq 1$ and $Q \geq q$. It is a simple calculation to run through the integers f and Q resulting in a positive integer D . For each such triplet (Q_0, D_0, f_0) the number $N_0 = (Q_0 + 1)(D_0 + 1) - f_0$ is the number of terms in the series required to obtain f_0 ODEs of order Q_0 and degree D_0 . Among all these N_0 there is a minimum. We call the corresponding ODE the ‘optimal ODE’. To obtain the ODE for other primes, it is thus only necessary to generate the minimum number of series terms.

For instance, for $\tilde{\chi}^{(5)}$, the ODE formula reads

$$N = 72 \cdot Q + 33 \cdot D - 887 = (Q + 1)(D + 1) - f. \tag{15}$$

The optimal ODE, i.e. the ODE that requires the minimum number of terms in the series, has the triplet $(Q_0, D_0, f_0) = (56, 129, 8)$ which corresponds to the minimum number $N_0 = 7402$. Note that the minimal-order ODE has the triplet $(33, 1456, 1)$ and requires 49 537 series terms.

The minimum number of terms N_0 is implicitly given by the ODE formula (12). Plugging the parameter D given in (14) in $N = (Q + 1)(D + 1) - f$, one obtains

$$N = (Q + 1)d + D_{app} + \frac{(D_{app} + f)q}{Q - q + 1}. \tag{16}$$

We can view N as a *continuous* function of Q and f and we find that it has two extremums when $dN/dQ = 0$. For the positive extremum one has

$$Q_0 = q - 1 + \frac{1}{d} \sqrt{(D_{\text{app}} + f)qd}, \tag{17}$$

$$D_0 = d - 1 + \frac{1}{q} \sqrt{(D_{\text{app}} + f)qd}, \tag{18}$$

$$N_0 = qd + D_{\text{app}} + 2\sqrt{(D_{\text{app}} + f)qd}. \tag{19}$$

For the example of $\tilde{\chi}^{(5)}$ considered above, one obtains (with $f = 1$)

$$Q_0 \simeq 57.20, \quad D_0 \simeq 125.97, \quad N_0 \simeq 7388.09. \tag{20}$$

The gain in the number of terms is already very significant for $Q = q + 1$ and can be measured by the discrete derivative of the hyperbola $N(Q)$ given in (16). Since we should compute over the integers, it is easier to compute the difference of $(D + 1)(Q + 1)$ evaluated at the points $Q = q$ and $Q = q + 1$. At the order $Q = q$, from (14) one obtains $D(Q = q) = d - 1 + D_{\text{app}} + f_1$, where f_1 is a positive integer. At the order $Q = q + 1$, one has $D(Q = q + 1) = d - 1 + (D_{\text{app}} + f_2)/2$, where f_2 is a positive integer with the same parity as D_{app} . The gain in the number of terms is

$$\Delta N(q, q + 1) = -d + qf_1 + \frac{1}{2}(D_{\text{app}} - f_2)q + (f_1 - f_2). \tag{21}$$

For $\chi^{(5)}$, and with the values $f_1 = 1$ and $f_2 = 2$ (since $D_{\text{app}} = 1384$ is even), the ‘saving’ in the number of terms is 22 736 to be compared with the 49 537 terms needed to obtain the minimal-order ODE (i.e. $Q = q$). As Q increases, one approaches the minimum of the hyperbola (16) which is $N_0 = 7388.09$ (with $f = 1$). Over the integers the minimum is 7402 obtained with $f = 8$. This process can be repeated by computing $\Delta N(q, q + 2)$ and in this case $D_{\text{app}} + f_3$ should be multiple of 3.

As can be seen from the ‘discrete’ derivative (21), the degree of the apparent polynomial is crucial. For ODEs with no apparent singularities the minimal-order ODE is the optimal ODE. In this case, the hyperbola $N(Q)$ can still have a minimum that is not in the integers.

Note that we may define a *minimal-degree* ODE, i.e. the ODE that has $D = d$ meaning that there is no singularities other than the ‘true’ singularities of the minimal-order ODE (no apparent and no spurious singularities⁶). The order of this *minimal-degree* ODE is (see (14))

$$Q = q + D_{\text{app}} + f - 1, \tag{22}$$

giving for $\tilde{\chi}^{(5)}$, the order $Q = 1417$ and 103 513 as the number of terms (the minimum f being 1). Note that this *minimal-degree* ODE is useless for our computational purposes.

In this paper all of these types of modular calculations and approaches have been applied to $\tilde{\chi}^{(6)}$. Section 2 shows the computational details (timing, etc) for the generation of the first series and the first ODEs, modulo a prime, from which we infer the optimal length of the series to be generated for other primes. In section 3, we report on the ODE annihilating $\tilde{\chi}^{(6)}$ and on the ODE annihilating the corresponding ‘depleted’ series. The singularities and local exponents confirm the results obtained from a diff-Padé analysis, given in a previous paper [7]. In section 4, the programme of factorization developed for $\tilde{\chi}^{(5)}$ is used to factorize as far as possible the differential operator corresponding to the ODE of $\tilde{\chi}^{(6)}$. We will see that our conjecture [8, 11] on the factorization structure of the $\tilde{\chi}^{(n)}$ holds for $n = 6$. Some right factors in the differential operator for $\tilde{\chi}^{(6)}$ are obtained in exact arithmetics. Section 5 is the conclusion.

⁶ If we denote by L_q the *minimal-order* differential operator, the *non-minimal-order* differential operator $L_{Q,D}$ (with $Q > q$ and $D > d$) has $D-d$ singularities which are spurious with respect to L_q . The spurious singularities are the ones of the operator L_{Q-q} occurring in the factorization $L_{Q,D} = L_{Q-q} \cdot L_q$.

Table 1. Summary of results for various series. The last three columns are the data for the optimal ODE. The $\Phi_H^{(n)}$ series are the model integrals [6].

Series	$N = d \cdot Q + q \cdot D - C$	Q_0	D_0	$(Q_0 + 1)(D_0 + 1)$
$\tilde{\chi}^{(1)}$	$1Q + 1D + 1$	1	1	4
$\tilde{\chi}^{(2)}$	$1Q + 2D + 1$	2	1	6
$\tilde{\chi}^{(3)}$	$12Q + 7D - 37$	11	17	216
$\tilde{\chi}^{(4)}$	$7Q + 10D - 36$	15	9	160
$\tilde{\chi}^{(5)}$	$72Q + 33D - 887$	56	129	7410
$\tilde{\chi}^{(6)}$	$43Q + 52D - 1121$	84	73	6290
$6\tilde{\chi}^{(3)} - \tilde{\chi}^{(1)}$	$12Q + 6D - 26$	10	17	198
$6\tilde{\chi}^{(4)} - 2\tilde{\chi}^{(2)}$	$6Q + 8D - 17$	13	8	126
$6\tilde{\chi}^{(5)} - 3\tilde{\chi}^{(3)}$	$68Q + 30D - 732$	52	120	6413
$6\tilde{\chi}^{(6)} - 4\tilde{\chi}^{(4)}$	$40Q + 48D - 945$	80	66	5427
$\Phi_H^{(3)}$	$10Q + 5D - 21$	8	13	126
$\Phi_H^{(4)}$	$5Q + 6D - 12$	9	6	70
$\Phi_H^{(5)}$	$45Q + 17D - 277$	28	80	2349
$\Phi_H^{(6)}$	$26Q + 27D - 342$	48	39	1960
$\Phi_H^{(7)}$	$145Q + 49D - 1943$	92	257	23994

2. The series of $\tilde{\chi}^{(6)}$ modulo a prime

As shown in [7] the calculation of a series for $\tilde{\chi}^{(6)}$ is a problem with computational complexity $O(N^4 \ln N)$. Note that $\tilde{\chi}^{(2n)}$ is an even function in w and we therefore generally work with a series in the variable $x = w^2$, though the series for $\tilde{\chi}^{(6)}$ is still calculated in the w variable. In table 1 we have listed a summary of results for formula (11) for various series with new results for $\tilde{\chi}^{(6)}$ added. In [7] we gave a rough estimate of the number of terms required to obtain the ODE for $\tilde{\chi}^{(6)}$ and thought this beyond our computational resources. However, upon closer inspection of table 1 one observes that the minimum number of terms required to find the ODE in x for $\tilde{\chi}^{(2n)}$, $n = 1, 2$, (or $\Phi_H^{(2n)}$) is always smaller than the number of terms required for $\tilde{\chi}^{(2n-1)}$ (or $\Phi_H^{(2n-1)}$). This also holds for the combination $6\tilde{\chi}^{(n)} - (n - 2)\tilde{\chi}^{(n-2)}$. It is reasonable to expect that this would be true for $\tilde{\chi}^{(6)}$ as well. In particular this would mean that the number of terms required to find the ODE for $6\tilde{\chi}^{(6)} - 4\tilde{\chi}^{(4)}$ should be smaller than the 6400 or so terms needed to find the ODE for $6\tilde{\chi}^{(5)} - 3\tilde{\chi}^{(3)}$. There is of course no way of knowing whether or not this line of reasoning is correct. In particular we would have liked to further reduce the number of terms to be calculated (one can for instance note that the number of terms required to find the optimal ODE for $\tilde{\chi}^{(2n)}$ or $\Phi_H^{(2n)}$ is some 10–20% less than the number of terms required to find the optimal ODE for $\tilde{\chi}^{(2n-1)}$ or $\Phi_H^{(2n-1)}$, respectively), but since finding the ODE for the first time is a hit-or-miss proposition we naturally wanted to ensure, to the greatest extent possible, that we had enough terms to find the ODE for $6\tilde{\chi}^{(6)} - 4\tilde{\chi}^{(4)}$. For this reason it was decided to generate a series to order 6500 in x (13 000 in w) for $\tilde{\chi}^{(6)}$ with the firm hope that this would suffice to find the optimal ODE for at least $6\tilde{\chi}^{(6)} - 4\tilde{\chi}^{(4)}$ (in fact it is also enough terms to find the optimal ODE for $\tilde{\chi}^{(6)}$ itself).

In [7] the calculation of $\tilde{\chi}^{(5)}$ to 10 000 terms required some 17 000 CPU hours on an SGI Altrix cluster with 1.6 GHz Itanium2 processors. Given that the algorithms for $\tilde{\chi}^{(5)}$ and $\tilde{\chi}^{(6)}$ have the same computational complexity this would indicate that the time required to calculate the series for $\tilde{\chi}^{(6)}$ to 13 000 terms in w would be at least 50 000 CPU hours (the algorithm for

$\tilde{\chi}^{(6)}$ has a slightly larger pre-factor than that for $\tilde{\chi}^{(5)}$. In fact it turned out that almost 65 000 CPU hours were required and this calculation was performed over a 6 month period.

The series to order 6500 was calculated modulo the prime 32 749. As in [8] we want to factorize various differential operators and reconstruct the right-most factors exactly using the results from several primes. We thus need to reduce as much as possible the length of the series by identifying some right factors.

As we detail in the following section the optimal ODE for $\tilde{\chi}^{(6)}$ can be obtained with less than 6300 terms while the optimal ODE for the combination $6\tilde{\chi}^{(6)} - 4\tilde{\chi}^{(4)}$ requires ‘just’ over 5400 terms. Furthermore, we find (using our series modulo a single prime) that $\tilde{\chi}^{(2)}$ is a solution of this ODE and that one can simplify further by considering the linear combination $\Phi^{(6)} = \tilde{\chi}^{(6)} - \frac{2}{3}\tilde{\chi}^{(4)} + \frac{2}{45}\tilde{\chi}^{(2)}$ whose optimal ODE requires a little more than 5100 terms.

The ODE for $\tilde{\chi}^{(6)}$ has $\frac{d^2}{dx^2}$ as the lower derivative, meaning that $c_1 + c_2x$ is a solution (c_1 and c_2 are constants). Checking that $c_1 + c_2x$ is still a solution of the ODE for $\Phi^{(6)}$ and producing the series $\frac{d^2}{dx^2}\Phi^{(6)}(x)$, we arrive at a series whose minimal ODE requires a little less than 5000 terms.

We therefore calculated a further two series to order 5000 modulo the primes 32 719 and 32 717. These calculations required an additional 45 000 hours of CPU time. Using the factorization procedure detailed in section 4, we found a factor of order 3, X_3 , which right divides the differential operator for $\frac{d^2}{dx^2}\Phi^{(6)}(x)$, and we managed to reconstruct X_3 in exact arithmetic using three primes⁷. Applying X_3 , that is form the series $X_3(\frac{d^2}{dx^2}\Phi^{(6)})$, results in a series whose optimal ODE requires less than 4800 terms. At about the same time as these developments took place a new system was installed by National Computational Infrastructure (NCI) whose National Facility provides the national peak computing facility for Australian researchers. This new system is an SGI XE cluster using quad-core 3.0 GHz Intel Harpertown cpus. Our code runs about 40% faster (takes about 0.6 times the time) on this facility when compared to the Altix cluster and a calculation of a series to order 4800 takes about 11 000 CPU hours per prime. We calculated series to this order for further six primes, namely 32 713, 32 707, 32 693, 32 687, 32 653 and 32 647 (some of these calculations were performed on the facilities of the Victorian Partnership for Advanced Computing using a cluster with AMD Barcelona 2.3 GHz quad-core processors).

3. Fuchsian differential equation for $\tilde{\chi}^{(6)}$

From the $\tilde{\chi}^{(6)}$ series modulo a prime, we obtained various ODEs which have the ODE formula

$$N = 43Q + 52D - 1121 = (Q + 1)(D + 1) - f, \tag{23}$$

thus showing that the ODE for $\tilde{\chi}^{(6)}$ is of minimal order 52. We denote by L_{52} the corresponding linear differential operator.

The polynomial in front of the highest derivative and carrying the singularities of L_{52} (i.e. the ODE of $\tilde{\chi}^{(6)}$) reads

$$(1 - 16x)^{30} \cdot (1 - 25x) \cdot (1 - 9x) \cdot (1 - x) \cdot (1 - 4x)^5 \cdot (1 - 8x) \\ \times (1 - x + 16x^2) \cdot (1 - 10x + 29x^2) \cdot P_{\text{app}}, \tag{24}$$

where P_{app} is a polynomial whose roots are apparent singularities. Even though we have not computed the minimal-order ODE, from (13), we can infer that the degree of P_{app} is $D_{\text{app}} = 1020$. All the singularities agree with the ones found in [7] from a diff-Padé analysis

⁷ X_3 is equivalent to the differential operator L_3 given in this paper.

and we have confirmation that $(1 - 8x)$ is the *only* singularity not predicted [6] by the simplified integrals $\Phi_H^{(6)}$.

Furthermore, using the exact (modulo a prime) ODE, we can confirm the local exponents computed from a diff-Padé analysis in [7] for all singularities except those at $x = 0, 1/16$ and $x = \infty$, which are correct but incomplete. The complete set of local exponents⁸ at these latter points read

$$\begin{aligned} x = 0, \quad \rho &= -1, -1/2, 0^3, 1/2, 1^5, 3/2, 2^5, 3^4, 4^4, 5^4, 6^4, 7^3, 8^3, 9^3, 10^3, 11, \dots, 17, \\ x = 1/16, \quad \rho &= -2, -7/4, -3/2, -5/4, -1^3, -1/2, 0^6, 1/2, 1^4, 2^4, 3^3, 4^3, \\ &\quad 5^3, 6^2, 7^2, 8^2, 9^2, 10, \dots, 21, \\ x = \infty, \quad \rho &= -1^2, -1/2, 0^3, 1/2^6, 1^2, 3/2^5, 2, 5/2^3, 3, 7/2^3, 4, 9/2^2, \\ &\quad 11/2^2, 13/2^2, 15/2^2, 17/2^2, 19/2^2, 21/2^2, 23/2^2, 25/2, 27/2, \\ &\quad 29/2, 31/2, 33/2, 35/2, 37/2, 19. \end{aligned}$$

Having obtained the ODE formula (23), one can see that the minimal-order ODE requires 56 391 terms (plug $Q = q = 52, d = 43, D_{\text{app}} = 1020$ and $f = 1$ into (16)). And it is a simple calculation (see the paragraph after (14)) to obtain the number of terms necessary for the optimal ODE. This corresponds to $Q = 84, D = 73, f = 3$ and $N = 6287$ terms. If we had to produce the optimal ODE for $\chi^{(6)}$ for other primes it is 6290 series coefficients that should be generated.

As mentioned in the previous section, our conjecture that $\tilde{\chi}^{(n)}$ satisfy (with $\alpha_{n-2} = (n - 2)/6$)

$$\tilde{\chi}^{(n)} = \alpha_{n-2} \cdot \tilde{\chi}^{(n-2)} + \beta_{n-4} \cdot \tilde{\chi}^{(n-4)} + \dots + \Phi^{(n)}, \tag{25}$$

is also verified. For the series

$$\Phi^{(6)} = \tilde{\chi}^{(6)} - \frac{2}{3}\tilde{\chi}^{(4)} + \frac{2}{45}\tilde{\chi}^{(2)}, \tag{26}$$

we obtain non-minimal-order ODEs from which we infer the ODE formula

$$39Q + 46D - 861 = (Q + 1)(D + 1) - f, \tag{27}$$

showing that the minimal order is 46 with an apparent polynomial (see (13)) of degree $D_{\text{app}} = 848$. The minimal-order ODE for $\Phi^{(6)}$ requires the generation of 41 736 coefficients series, while the optimal ODE requires 5120 terms corresponding to $Q = 79, D = 63$. It is interesting to see that the required number of terms decreases sharply from 41 736 (for the minimal-order ODE $Q = q = 46$) to 22 272 for the non-minimal-order ODE $Q = q + 1 = 47$. The gain $\Delta N(46, 47) = 19 464$ terms is given by (21) for $q = 46, d = 39, f_1 = 1$ and $f_2 = 2$ since $D_{\text{app}} = 848$ is even. The gain is $\Delta N(46, 48) = 25 958$.

Denoting by L_{46} the differential operator corresponding to $\Phi^{(6)}$ and recalling [11] the differential operator L_{10} corresponding to $\tilde{\chi}^{(4)}$, one sees from (26) that the differential operator for $\tilde{\chi}^{(6)}$ has the ‘direct sum⁹ decomposition’

$$L_{52} = L_{10} \oplus L_{46}. \tag{28}$$

The sum of the orders of the differential operators L_{46} and L_{10} is larger than 52, indicating that a common factor, namely an order 4 differential operator, occurs on the right of both L_{46} and L_{10} . The solutions of this order 4 ODE have been given in equations (31)–(33) and equation (43) of [11]. The differential operator (that we denote by $L_4^{(4)}$) is given in equation (42) of [11] as a product of four order 1 differential operators. Since the expressions for these differential

⁸ The notation is 0^3 for 0, 0, 0 and $9/2^2$ for $9/2, 9/2$, etc.

⁹ Recall [11] that the differential operator for $\tilde{\chi}^{(2)}$ is a factor in the direct sum of L_{10} .

operators were not written in [11], we give, for the sake of completeness, in appendix A the full factorization of the differential operator $L_4^{(4)}$.

Furthermore, we note that in the ODE for $\tilde{\chi}^{(6)}$ the derivatives of orders 0 and 1 are missing (the corresponding differential operator has D_x^2 as the lowest derivative¹⁰). The constant and the degree 1 polynomial x are solutions of L_{52} . The constant is a solution of the common factor $L_4^{(4)}$, but the degree 1 polynomial x is not a solution of L_{10} and thus should occur in L_{46} .

We thus have an order 5 differential operator that right divides L_{46} :

$$\tilde{L}_5 = D_x^2 \oplus L_4^{(4)} = \left(D_x - \frac{1}{x} \right) \oplus L_4^{(4)}. \tag{29}$$

We now turn to the factorization modulo a prime of the differential operator L_{46} keeping in mind that \tilde{L}_5 is a right factor.

4. Factorization modulo a prime of the differential operator L_{46}

The local exponents at the singularities of the ODE of $\Phi^{(6)}$ allow us to easily track the factors carrying the various singular behaviours. What we mean is the following. The local exponents for the ODE of $\Phi^{(5)}$, at for instance $w = 0$, are all integers. Producing the series having the highest exponent, we obtain either the full ODE or a right factor. If the series with the highest exponent yields the full ODE, then in order to obtain a right factor we have to look at the ODEs corresponding to combinations of series involving both the highest and the next highest exponent as explained and done in [8].

For $\Phi^{(6)}$ and at $x = 0$, we have two types of local exponents, integer and half-integer ones. We thus have a ‘partition’ of the solutions to the full ODE. In other words, we have ‘two highest exponents’¹¹ and it is therefore more likely that we can avoid using combination series.

The ODE for $\Phi^{(6)}$ corresponding to L_{46} has at $x = 0$ the local exponents

$$\rho = -1^2, -1/2, 0^3, 1/2, 1^5, 3/2, 2^5, 3^4, 4^4, 5^4, 6^3, 7^3, 8^3, 9^2, 10^2, 11, 12, 13; \tag{30}$$

we then have two ‘highest exponents’: $\rho = 13$ and $\rho = 3/2$. This means that we can produce both the series $x^\rho(1 + \dots)$ and see whether or not either of these gives rise to a right factor. If so we may not need to resort to the combination method presented in section 4 of [8].

At the singularity $x = 1/16$, the local exponents are

$$\rho = -2, -7/4, -3/2, -5/4, -1^3, -1/2, 0^6, 1/2, 1^4, 2^3, 3^3, 4^3, 5^2, 6^2, 7^2, 8, 9, \dots, 19$$

and we have three ‘highest exponents’: $\rho = -5/4$, $\rho = 1/2$ and $\rho = 19$.

At the singularity $x = \infty$, there are two ‘highest exponents’: $\rho = 4$ and $\rho = 33/2$ since the local exponents are

$$\rho = -1^2, -1/2^2, 0^3, 1/2^6, 1^2, 3/2^5, 2, 5/2^2, 3, 7/2^2, 4, 9/2^2, 11/2^2, \\ 13/2^2, 15/2^2, 17/2^2, 19/2^2, 21/2, \dots, 33/2.$$

Before we proceed, we introduce the notation $L_{46} = O_{n_2} \cdot O_{n_3}$, with $46 = n_2 + n_3$, which we use to indicate that the operator L_{46} factorizes into two operators of orders n_2 and n_3 , respectively. Only when a differential operator is definitive do we give it a label other than O .

Let us begin by the conjecture [8] that L_{46} has a left-most operator of order 6 which is the symmetric fifth power of L_E . Solutions to the symmetric power of L_E are polynomials of

¹⁰ The notation D_x is $\frac{d}{dx}$.

¹¹ Note that for $\tilde{\chi}^{(5)}$, other singularities than $w = 0$ have half- and fourth-integer exponents. There was no need in [8] to use the procedure presented here.

homogeneous degrees in the elliptic integrals with the coefficients of the combination being rational. The solutions carrying the half-integer exponents should therefore be those of an operator occurring necessarily on the right of L_{46} . So from the two ‘highest exponents’ $\rho = 13$ and $\rho = 3/2$ at $x = 0$, we need only obtain the ODE of the unique series $x^{3/2}(1 + \dots)$. Indeed, acting by L_{46} on the series $x^{3/2} \cdot (1 + \dots)$ produces a series annihilated by an order 40 ODE, leading to the factorization

$$L_{46} = L_6 \cdot O_{40}. \tag{31}$$

When we shift L_{46} to $x = 1/16$ and act on $t^{-5/4} \cdot (1 + \dots)$, with $t = x - 1/16$, we obtain an order 5 ODE leading to

$$L_{46} = O_{41} \cdot O_5. \tag{32}$$

Shifting L_{46} to $x = 1/16$ and acting on $t^{1/2} \cdot (1 + \dots)$, with $t = x - 1/16$, produces

$$L_{46} = O_{36} \cdot O_{10}. \tag{33}$$

Shifting L_{46} to $x = \infty$ and acting on $t^4 \cdot (1 + \dots)$, with $t = 1/x$, gives

$$L_{46} = O_{33} \cdot O_{13}. \tag{34}$$

Some factors are common to these three factorizations. Shifting the ODE back to $x = 0$ and carrying out our factorization procedure [8], one obtains (some final labelling is given)

$$L_{46} = O_{41} \cdot O_5 = O_{41} \cdot \tilde{L}_3 \cdot L_2, \tag{35}$$

$$L_{46} = O_{36} \cdot O_{10} = O_{36} \cdot O_1 \cdot L_4 \cdot \tilde{L}_3 \cdot L_2, \tag{36}$$

$$L_{46} = O_{33} \cdot O_{13} = O_{33} \cdot O_1 \cdot O_3 \cdot L_4 \cdot \tilde{L}_3 \cdot L_2. \tag{37}$$

The order 1 differential operator O_1 in the last factorization is equivalent to an order 1 differential operator occurring in \tilde{L}_5 of (29). The product $O_3 \cdot L_4 \cdot \tilde{L}_3 \cdot L_2$ can be expressed as a direct sum:

$$O_3 \cdot L_4 \cdot \tilde{L}_3 \cdot L_2 = L_3 \oplus (L_4 \cdot \tilde{L}_3 \cdot L_2). \tag{38}$$

Collecting the results given in the factorizations (31) and (37) with (38), and keeping in mind the right factor (29), one obtains

$$L_{46} = L_6 \cdot L_{23} \cdot L_{17} \tag{39}$$

with

$$L_{17} = \tilde{L}_5 \oplus L_3 \oplus (L_4 \cdot \tilde{L}_3 \cdot L_2), \tag{40}$$

$$\tilde{L}_5 = \left(D_x - \frac{1}{x} \right) \oplus L_4^{(4)}. \tag{41}$$

Having obtained all these differential operators, a final check is performed by acting on $\Phi^{(6)}$ by the corresponding ODEs in the order given in (39) and doing this we do indeed get zero.

4.1. The differential operator L_6

The sixth-order linear differential operator L_6 is the one that we conjectured [8] should annihilate a homogeneous polynomial of the complete elliptic integrals E and K of

(homogeneous) degree 5. It should then be irreducible. The local exponents at the origin of the linear ODE corresponding to L_6 are

$$x = 0, \quad \rho = -12, -11, -8, -5, -4, 0. \tag{42}$$

Plugging a generic series $\sum c_n x^n$ into the linear ODE fixes all the coefficients with the exception of the coefficient c_0 . The ‘survival’ of a single coefficient is a particular feature of an irreducible factor with one non-logarithmic solution. The differential operator L_6 being a symmetric power of L_E means that its solution is a polynomial in E and K defined as

$$K = {}_2F_1([1/2, 1/2], [1], 16x), \quad E = {}_2F_1([1/2, -1/2], [1], 16x). \tag{43}$$

The ODE corresponding to L_6 should only have singularities at $x = 0, 1/16$ and $x = \infty$, and this is indeed the case. The local exponents at $x = 1/16$ are

$$x = 1/16, \quad \rho = -48^2, -47, -44, -40, 0. \tag{44}$$

The local exponents at $x = 0$ and $x = 1/16$ suggest the following ansatz to be plugged into the linear ODE (of L_6):

$$\frac{1}{x^{12} \cdot (1 - 16x)^{48}} \cdot \sum_{i=0}^5 P_{5-i,i}(x) \cdot K^{5-i} E^i. \tag{45}$$

The polynomials $P_{5-i,i}(x)$ can be determined numerically and the solution (analytical at $x = 0$) of the ODE corresponding to L_6 is

$$\frac{1}{x^{12} \cdot (1 - 16x)^{48}} \cdot ((1 - 16x)^4 P_{5,0} \cdot K^5 + (1 - 16x)^3 P_{4,1} \cdot K^4 E + (1 - 16x)^2 P_{3,2} \cdot K^3 E^2 + (1 - 16x) P_{2,3} \cdot K^2 E^3 + P_{1,4} \cdot K E^4 + P_{0,5} \cdot E^5).$$

The polynomials $P_{5-i,i}(x)$ with coefficients known modulo a prime are of degree, respectively, 111, 112, 113, 113, 113 and 113. As conjectured the linear differential operator L_6 is thus equivalent to the symmetric fifth power of L_E .

4.2. The differential operator L_{17}

The differential operator L_{17} has in its decomposition the differential operator \tilde{L}_5 which is known exactly. The solutions of \tilde{L}_5 are the degree 1 polynomial x and the four solutions of $L_4^{(4)}$ given in [11]. As for the other factors of L_{17} , i.e. L_2, L_3, \tilde{L}_3 and L_4 , we have been able to express all of them in exact arithmetics.

To express a differential operator in exact arithmetics the straightforward approach is to rationally reconstruct the differential operator using several modulo prime calculations. However, an alternative would be to reconstruct the solutions to the differential operator if they are known. This is what we have done for L_2 and L_3 .

The singularities of the ODEs corresponding to L_2 and L_3 are *only* $x = 0, x = 1/16$ and $x = \infty$. It is therefore reasonable to assume that the solutions can be expressed as polynomials in $K(x)$ and $E(x)$.

For the ODE corresponding to L_2 , the solution (analytical at $x = 0$) written in terms of $\tilde{\chi}^{(2)}$ is

$$\text{sol}(L_2) = \left(x \frac{d}{dx} - 2 \right) \tilde{\chi}^{(2)}. \tag{46}$$

Written in this way, it is easy to recognize the coefficients in exact arithmetics with only two primes. The differential operator L_2 is thus

$$L_2 = D_x^2 - 2 \frac{(1 + 8x)}{x \cdot (1 - 16x)} D_x + \frac{4}{x \cdot (1 - 16x)}. \tag{47}$$

For the third-order differential operator L_3 , we assumed that it is equivalent to a symmetric square of L_E . Indeed, the solution (analytical at $x = 0$) is written also in terms of $(\tilde{\chi}^{(2)})^2$ and appears as

$$\text{sol}(L_3) = \frac{1}{x} \cdot \left(x(1 - 16x)^2(16x - 3) \cdot \frac{d^2}{dx^2} + (1 - 16x)(64x^2 - 44x + 9) \cdot \frac{d}{dx} - 8(1 - 8x)(16x + 9) \right) (\tilde{\chi}^{(2)})^2. \tag{48}$$

Here also, two primes are more than sufficient to recognize the coefficients. The differential operator L_3 , in exact arithmetics, reads

$$L_3 = D_x^3 + \frac{p_2}{p_3} D_x^2 + \frac{p_1}{p_3} D_x + \frac{p_0}{p_3}, \tag{49}$$

with

$$\begin{aligned} p_3 &= x^2 \cdot (1 - 16x)^2(-81 + 1986x - 17\,056x^2 + 34\,304x^3 + 8192x^4), \\ p_2 &= 2x^2 \cdot (1 - 16x)(2247 - 46\,496x + 357\,888x^2 - 565\,248x^3 - 65\,536x^4), \\ p_1 &= 6(27 - 942x + 11\,152x^2 - 101\,632x^3 + 372\,736x^4 - 65\,536x^5), \\ p_0 &= 12(9 - 308x - 6208x^2 - 101\,376x^3 - 49\,152x^4). \end{aligned}$$

We have not been able to find the solution of the ODE corresponding to \tilde{L}_3 . The rational reconstruction has been done on the differential operator itself (see appendix B). Rationally reconstructed, the differential operator \tilde{L}_3 reads

$$\tilde{L}_3 = D_x^3 + \frac{q_2}{q_3} D_x^2 + \frac{q_1}{q_3} D_x + \frac{q_0}{q_3}, \tag{50}$$

with

$$\begin{aligned} q_3 &= x^2 \cdot (1 - 4x)(1 - 16x)^3 Q_3, \\ Q_3 &= -8 + 252x - 1678x^2 + 3607x^3 + 4352x^4, \\ q_2 &= 2x \cdot (1 - 16x)^2(-12 + 1172x - 30\,499x^2 + 252\,146x^3 \\ &\quad - 872\,579x^4 + 770\,128x^5 + 1183\,744x^6), \\ q_1 &= 4(1 - 16x)(6 + 185x - 28\,373x^2 + 689\,440x^3 - 5128\,290x^4 \\ &\quad + 16\,119\,599x^5 - 13\,139\,200x^6 - 17\,825\,792x^7), \\ q_0 &= 4(-294 + 9469x + 84\,480x^2 - 4652\,220x^3 + 33\,948\,640x^4 \\ &\quad - 97\,687\,536x^5 + 89\,128\,960x^7 + 74\,981\,376x^6). \end{aligned}$$

All the calculations on the previous differential operators have been done with the two primes 32 749 and 32 719. For the differential operator L_4 we need more primes. The differential operator L_4 has the form

$$\begin{aligned} L_4 &= x^3 \cdot (1 - 16x)^4 (1 - 4x) (1 - 8x) Q_3^4 P_4^{(26)} \cdot D_x^4 \\ &\quad + x^2 \cdot (1 - 16x)^3 Q_3^3 P_3^{(33)} \cdot D_x^3 + x(1 - 16x)^2 Q_3^2 P_2^{(38)} \cdot D_x^2 \\ &\quad + (1 - 16x) Q_3 P_1^{(43)} \cdot D_x + P_0^{(47)}, \end{aligned} \tag{51}$$

where Q_3 is the apparent polynomial of \tilde{L}_3 in (50) and $P_j^{(n)}$ are polynomials in x of degree n . To perform the rational reconstruction of the polynomials $P_j^{(n)}$, we had to generate the series for $\Phi^{(6)}$ for another seven primes, then obtain the optimal ODEs and factorize the differential operators L_{46} for each prime. After the rational reconstruction was completed successfully the resulting differential operator L_4 was checked against the local exponents and the conditions

on the apparent singularities. The polynomials $P_j^{(n)}$ are given in exact arithmetics in appendix C.

Note that we have also checked that these rationally reconstructed differential operators are globally nilpotent as they should be.

4.3. The differential operator L_{23}

The differential operator L_{23} has the ODE formula

$$21Q + 23D + 1360 = (Q + 1)(D + 1) - f, \tag{52}$$

and at $x = 0$, the local exponents read

$$\rho = -25, -24, -23^2, -22^2, -21^2, -20^2, -19, -18, -17^2, \\ -16, 1, 2, 3, 4, 5, 6, -47/2, -45/2.$$

We can use the same method as before in order to factorize L_{23} . By producing the series with the highest local exponents $\rho = 6$ and $\rho = -45/2$, we obtained the full ODE for each series, i.e. an ODE formula compatible with the minimal order 23.

The singularities of the linear ODE corresponding to L_{23} are (besides $x = 0$)

$$(1 - 16x)(1 - 4x)(1 - x)(1 - 9x)(1 - 25x)(1 - 10x + 29x^2)(1 - x + 16x^2).$$

We may then shift the ODE corresponding to L_{23} to a singular point other than $x = 0$, produce the series of the highest exponent and see whether this gives an ODE of order less than 23. At $x = 1/16$, the series of the highest exponent $\rho = 11$ produced the full ODE. Likewise, at other points and exponents such as ($x = 1/4, \rho = -41/2$), ($x = 1/9, \rho = -47/2$), ($x = 1/25, \rho = -63/2$), ($x = 1, \rho = -47/2$) and ($x = \infty, \rho = -38, -47/2$), the series give rise to the full ODE.

Next we show how the *local structure* of solutions appears around $x = 0$. We introduce the notation $[x^\rho]$ to mean that the series begins as $x^\rho \cdot (\text{const} + \dots)$. The results of our computations are the following. Two sets of five solutions can be written as (with $k = 1, 2$)

$$[x^k] \ln(x)^4 + [x^{-21}] \ln(x)^3 + [x^{-22}] \ln(x)^2 + [x^{-23}] \ln(x) + [x^{-25}], \\ [x^k] \ln(x)^3 + [x^{-21}] \ln(x)^2 + [x^{-22}] \ln(x) + [x^{-24}], \\ [x^k] \ln(x)^2 + [x^{-21}] \ln(x) + [x^{-24}], \\ [x^k] \ln(x) + [x^{-21}] \quad \text{and} \quad [x^k]. \tag{53}$$

Three sets of three solutions can be written as (with $k = 3, 4, 5$)

$$[x^k] \ln(x)^2 + [x^{-21}] \ln(x) + [x^{-24}], \quad [x^k] \ln(x) + [x^{-21}] \quad \text{and} \quad [x^k]. \tag{54}$$

Two solutions can be written as

$$[x^6] \ln(x) + [x] \quad \text{and} \quad [x^6]. \tag{55}$$

Finally there are two non-logarithmic solutions behaving as $x^{-47/2} \cdot (1 + \dots)$ and $x^{-45/2} \cdot (1 + \dots)$.

Besides the series $x^\rho \cdot (1 + \dots)$ with ($\rho = 6$ and $\rho = -45/2$) that have given the full ODE, we may even try the non-ambiguous solutions such as $[x^2]$ in front of $\ln(x)^4$ and $[x^5]$ in front of $\ln(x)^2$. But these series produce the full ODE.

As is the case with the 12th-order differential operator L_{12} occurring in $\tilde{\chi}^{(5)}$, we have no final conclusion as to whether or not L_{23} is reducible, and without performing the factorization based on the combination method presented in section 4 of [8] we do not expect to be able to reach any such conclusion. The representative optimal ODE of L_{23} used in the calculations is of order 67, making the computational time obstruction more severe than what we faced with the 12th-order differential operator occurring [8] in $\tilde{\chi}^{(5)}$.

4.4. Summary

Let us now summarize our results. The linear differential operator L_{46} , corresponding to $\Phi^{(6)} = \tilde{\chi}^{(6)} - \frac{2}{3}\tilde{\chi}^{(4)} + \frac{2}{45}\tilde{\chi}^{(2)}$, can be written as

$$L_{46} = L_6 \cdot L_{23} \cdot L_{17}, \tag{56}$$

with

$$L_{17} = L_4^{(4)} \oplus \left(D_x - \frac{1}{x} \right) \oplus L_3 \oplus (L_4 \cdot \tilde{L}_3 \cdot L_2). \tag{57}$$

The order 17 linear differential operator L_{17} contains only the singularities of the linear ODE corresponding to L_{10} (the operator for $\tilde{\chi}^{(4)}$) plus the ‘new’¹² singularity $x = 1/8$. The singularity $x = 1/8$ occurs only in the fourth-order linear differential operator L_4 . The third-order differential operator \tilde{L}_3 is *responsible for the $\rho = -5/4, \rho = -7/4$ singular behaviour* around the (anti-)ferromagnetic point $x = 1/16$.

Comparing the results of $\tilde{\chi}^{(6)}$ with those of $\tilde{\chi}^{(3)}, \tilde{\chi}^{(4)}$ and $\tilde{\chi}^{(5)}$ we note that our conjecture still holds: for a given $\tilde{\chi}^{(n)}$ there is an order n differential operator equivalent to the $(n - 1)$ th symmetric power of L_E on the left of the depleted differential operators, corresponding to the linear combinations $\tilde{\chi}^{(3)} - \frac{1}{6}\tilde{\chi}^{(1)}, \tilde{\chi}^{(4)} - \frac{2}{6}\tilde{\chi}^{(2)}, \tilde{\chi}^{(5)} - \frac{3}{6}\tilde{\chi}^{(3)} + \frac{1}{120}\tilde{\chi}^{(1)}$ and now $\tilde{\chi}^{(6)} - \frac{4}{6}\tilde{\chi}^{(4)} + \frac{2}{45}\tilde{\chi}^{(2)}$.

For a given $\tilde{\chi}^{(n)}$ and once the ‘contributions’ of lower terms ($\tilde{\chi}^{(n-2k)}, k = 1, 2, \dots$) have been subtracted, the ODE of the ‘depleted’ series still contains some factors occurring in the ODE of the lower terms ($\tilde{\chi}^{(n-2k)}$). For $\tilde{\chi}^{(5)}$, we have that the differential operator $Z_2 \cdot N_1$, which occurs in the ODE of $\tilde{\chi}^{(3)}$, continues to be a right factor in the ODE of $\tilde{\chi}^{(5)} - \frac{3}{6}\tilde{\chi}^{(3)} + \frac{1}{120}\tilde{\chi}^{(1)}$. For $\tilde{\chi}^{(6)}$, we have that the differential operator $L_4^{(4)}$, which occurs in the ODE of $\tilde{\chi}^{(4)}$, continues to be a right factor in the ODE of $\tilde{\chi}^{(6)} - \frac{4}{6}\tilde{\chi}^{(4)} + \frac{2}{45}\tilde{\chi}^{(2)}$.

As was the case for $\tilde{\chi}^{(5)}$ with the differential operators of orders 2 and 3 (F_2 and F_3), we similarly have for $\tilde{\chi}^{(6)}$ the emergence of two differential operators of orders 3 and 4 (\tilde{L}_3 and L_4), which are globally nilpotent and for which we have no solutions. We may imagine that all these ODEs have solutions in terms (of symmetric power) of hypergeometric functions (with pull-back) as we succeeded to show [10] for Z_2 . Providing these solutions in terms of modular forms is clearly a challenge.

Similarly to the 12th-order differential operator L_{12} occurring in $\tilde{\chi}^{(5)}$, we faced with the differential operator L_{23} the same obstruction to its potential factorization, namely prohibitive computational times.

5. Conclusion

We have calculated, modulo a prime, a long series for the six-particle contribution $\tilde{\chi}^{(6)}$ to the magnetic susceptibility of the square lattice Ising model. This series has been used to obtain the Fuschian differential equation that annihilates $\tilde{\chi}^{(6)}$.

The method of factorization [8] previously used for $\tilde{\chi}^{(5)}$ is applied to the differential operator L_{52} of $\tilde{\chi}^{(6)}$. With the ODE known modulo a single prime, we have been able to go, as far as the computational resources allow, in the factorization of the corresponding differential operator.

We have found several remarkable results. The factorization structure of L_{52} generalizes what we have found for the linear differential operators of $\tilde{\chi}^{(3)}, \tilde{\chi}^{(4)}$ and $\tilde{\chi}^{(5)}$. In particular, we found in $\tilde{\chi}^{(6)}$ the occurrence of the term $\tilde{\chi}^{(4)}$ but also the lower term $\tilde{\chi}^{(2)}$, leading to the

¹² It is ‘new’ with respect to what we obtained from the $\Phi_H^{(6)}$ integrals [6] and our Landau singularity analysis [7].

differential operator L_{46} corresponding to the ‘depleted’ series $\Phi^{(6)} = \tilde{\chi}^{(6)} - \frac{2}{3}\tilde{\chi}^{(4)} + \frac{2}{45}\tilde{\chi}^{(2)}$. The left-most factor L_6 of L_{46} is a sixth-order operator equivalent to the symmetric fifth power of the second-order operator L_E corresponding to complete elliptic integrals of the first (or second) kind. We expect that this happens for all $\tilde{\chi}^{(n)}$, i.e. we conjecture the occurrence in $\tilde{\chi}^{(n)}$ of terms proportional to $\tilde{\chi}^{(n-2k)}$ meaning a direct sum structure, and the occurrence of an n th-order differential operator that left divides the differential operator corresponding to the ‘depleted’ series (25) of $\tilde{\chi}^{(n)}$.

Some right factors of small order appear in the factorization of L_{46} . We have used the previously reported ‘ODE formula’ to optimize our calculations. We have generated other series of the *minimum number of terms*, modulo eight other primes, and have obtained the corresponding ODEs and the corresponding factorizations. These nine factorizations have been used to perform a rational reconstruction and obtain in exact arithmetics the right factors occurring in L_{46} .

Our analysis is lacking the factorization of L_{23} for which, and similarly to L_{12} occurring in $\tilde{\chi}^{(5)}$, we have no conclusion on whether they are reducible. Even if these differential operators are known in exact arithmetics, their factorization remains a challenge for the methods implemented in various packages of symbolic calculation.

The massive calculations performed on $\tilde{\chi}^{(5)}$ and $\tilde{\chi}^{(6)}$ are at the limit of our computational resources and the next step, namely $\tilde{\chi}^{(7)}$ and/or $\tilde{\chi}^{(8)}$, seems to be really out of reach. A motivation for obtaining these very high order Fuchsian operators is to understand hidden mathematical structures from the factors of these operators. In this respect, the main results we have obtained on $\tilde{\chi}^{(6)}$ are the order 3 and 4 operators (\tilde{L}_3 and L_4) that we succeeded to get in exact arithmetics and which are waiting for an *elliptic curve* mathematical interpretation. Providing a mathematical interpretation for all these differential operators in terms of modular forms is clearly our next challenge.

The series and differential operators studied in this paper can be found in [12].

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Appendix A. The order 4 differential operator $L_4^{(4)}$

The order 4 differential operator $L_4^{(4)}$ is a right factor in L_{10} the differential operator for $\tilde{\chi}^{(4)}$. It is a product of an order 1 differential operator and an order 3 differential operator that can be written as a direct sum:

$$L_4^{(4)} = L_{1,3}^{(4)} \cdot (L_{1,2}^{(4)} \oplus L_{1,1}^{(4)} \oplus D_x). \tag{A.1}$$

In terms of the variable $x = w^2$ they are

$$L_{1,1}^{(4)} = D_x + \frac{768x^2}{(1 - 16x)(1 - 24x + 96x^2)}, \tag{A.2}$$

$$L_{1,2}^{(4)} = D_x + \frac{1 + 384x^2 + 2048x^3}{2x \cdot (1 - 16x)(1 - 48x + 128x^2)}, \tag{A.3}$$

and

$$L_{1,3}^{(4)} = D_x + 2 \frac{P_0}{p_1} \tag{A.4}$$

with

$$p_1 = x \cdot (1 - 16x)(1 - 4x)(80x + 7)(-7 + 96x - 1152x^2 + 10\,240x^3),$$

$$p_0 = 65\,536\,000x^6 - 36\,536\,320x^5 + 481\,280x^4 + 254\,592x^3 - 24\,800x^2 + 2149x - 49.$$

Appendix B. Reconstruction in exact arithmetics of the differential operator \tilde{L}_3

The ODE corresponding to \tilde{L}_3 appears as (where the singularities are easily recognized)

$$\begin{aligned} \tilde{L}_3 = & x^3 \cdot \left(x - \frac{1}{16}\right)^3 \left(x - \frac{1}{4}\right) P_3 \cdot D_x^3 + x^2 \cdot \left(x - \frac{1}{16}\right)^2 P_2 \cdot D_x^2 \\ & + x \cdot \left(x - \frac{1}{16}\right) P_1 \cdot D_x + x \cdot P_0. \end{aligned} \tag{B.1}$$

The polynomials P_3, \dots, P_0 are of degrees, respectively, 4, 6, 7 and 7 in x . We have 27 coefficients (not counting the overall one) to reconstruct. For easy labelling, these polynomials are denoted as (P_3 is the polynomial whose roots are apparent singularities)

$$P_3 = x^4 + \sum_{k=0}^3 a_k x^k, \quad P_2 = \sum_{k=0}^6 b_k x^k, \quad P_1 = \sum_{k=0}^7 c_k x^k, \quad P_0 = \sum_{k=0}^7 d_k x^k.$$

The indicial exponents obtained with both ODEs (with the two primes 32 749 and 32 719) are

$$\begin{aligned} x = 0, & \quad \rho = -2, 0, 2, \\ x = \infty, & \quad \rho = 1, 2, 5/2, \\ x = 1/16, & \quad \rho = -15/4, -13/4, -1, \\ x = 1/4, & \quad \rho = 0, 1, 7/2, \\ P_3(\alpha) = 0, & \quad \rho = 0, 1, 3. \end{aligned}$$

The ODE corresponding to the almost generic \tilde{L}_3 should give the above indicial exponents, and this leads to some conditions on the unknown coefficients a_k, b_k, c_k and d_k . The order of the ODE being 3, we obtain for each singularity a maximum of three conditions. This is a maximum because some exponents are by construction automatically satisfied. For instance, at $x = 1/4$, we obtain only one condition related to the exponent $\rho = 7/2$.

At the singularity $x = 0$, the indicial equation of \tilde{L}_3 gives $\rho = 0$ as a root automatically satisfied and a polynomial in ρ^2 depending on some of the unknown coefficients of \tilde{L}_3 . By requiring $\rho = -2$ and $\rho = 2$ as roots of this polynomial, we obtain

$$b_0 = \frac{3}{64}a_0, \quad c_0 = \frac{3}{1024}a_0. \tag{B.2}$$

With these values assigned, we require that $\rho = 1, 2, 5/2$ be roots of the indicial equation at the singularity $x = \infty$. One then gets

$$b_6 = \frac{17}{2}, \quad c_7 = 16, \quad d_7 = 5. \tag{B.3}$$

Similarly, the indicial equations evaluated at the local exponents for the singularities $x = 1/16$ and $x = 1/4$ give four equations, fixing, e.g., the coefficients b_4, b_5, c_6 and d_6 in terms of other coefficients.

Next we turn to the apparent singularities. These are given by the roots of P_3 . Calling α a root of P_3 (with unknown a_k), the indicial equation appears with $\rho = 0$ and $\rho = 1$

as automatically satisfied roots. Requiring $\rho = 3$ as the root of the indicial equation gives a polynomial in α of degree 3. Zeroing each term gives 22 solutions. Discarding all the solutions where a coefficient from \tilde{L}_3 is zero, one is left with five solutions. From these solutions, there is only one solution which is acceptable, because it matches with the actual values of the coefficients known in prime. This fixes three coefficients in terms of the others.

At this point, we have fixed 12 coefficients among the 27 using only the knowledge about the local exponents. The condition on the local exponents at the apparent singularities is only necessary; the sufficient condition is the absence of logarithmic solutions around the singularity $x = \alpha$.

The conditions on the non-occurrence of logarithmic solutions at the apparent singularities can be imposed either by requiring the conditions of equation (A.8) in [7] to be fulfilled or equivalently by zeroing the coefficients in front of the log's in the formal solutions of \tilde{L}_3 at α . With a generic apparent polynomial, the calculations can be cumbersome. So let us fix some coefficients.

One finds that the ratio $-2a_1/a_0$ appears with both primes 32 749 and 32 719 as the number 63. Also for both primes one obtains $4a_2/a_0 = 839$, $-8a_3/a_0 = 3607$ and $2^{14}d_0/a_0 = 147$.

Furthermore, one may compute the (analytical at $x = 0$) series at both primes in the hope that some coefficients will be 'simple' enough to be recognized. The series with the prime 32 749 gives

$$x^2 + 48x^3 + 1527x^4 + 7541x^5 + 3199x^6 + \dots, \tag{B.4}$$

while with the prime 32 719 it reads

$$x^2 + 48x^3 + 1527x^4 + 7571x^5 + 4069x^6 + \dots. \tag{B.5}$$

We note that the same values occur at orders 3 and 4. These numbers are therefore likely to be exact. Also the difference between the values at order 5 is a multiple of the difference 32 749 – 32 719, and similarly at order 6. It is easy to 'guess' these values as, respectively, 48, 1527, 40 290 and 952 920. Comparing with the series solution of \tilde{L}_3 fixes four coefficients.

We then have 12 coefficients fixed exactly and nine coefficients fixed by reconstruction. The formal solutions of \tilde{L}_3 at the apparent singularity α give two logarithmic solutions, with leading term, each

$$C\alpha^k(x - \alpha)^3 \ln(x - \alpha), \quad k = 0, \dots, 3, \tag{B.6}$$

where C depends on the remaining non-fixed coefficients of \tilde{L}_3 . We then have eight (nonlinear) equations for six unknowns to solve. This can be done by rational reconstruction and check.

Appendix C. The differential operator L_4 in exact arithmetics

The degree n polynomials $P_j^{(n)}(x)$ occurring in the differential operator L_4 read

$$\begin{aligned} P_4^{(26)} = & 28\,000 - 7854\,000x + 873\,083\,400x^2 - 54\,037\,012\,120x^3 + 2099\,285\,510\,560x^4 \\ & - 52\,582\,954\,690\,298x^5 + 766\,418\,384\,173\,454x^6 - 1305\,110\,830\,870\,633x^7 \\ & - 251\,264\,549\,473\,230\,968x^8 + 7727\,889\,974\,481\,947\,660x^9 \\ & - 148\,605\,250\,583\,921\,845\,896x^{10} + 2252\,938\,824\,290\,334\,087\,840x^{11} \\ & - 29\,645\,475\,671\,183\,771\,992\,224x^{12} + 354\,446\,803\,792\,968\,575\,565\,792x^{13} \end{aligned}$$

$$\begin{aligned}
& - 3850\,023\,960\,384\,577\,768\,909\,952x^{14} + 36\,761\,552\,740\,911\,534\,545\,901\,568x^{15} \\
& - 296\,338\,746\,597\,146\,803\,591\,135\,232x^{16} \\
& + 1953\,967\,934\,450\,852\,091\,348\,254\,720x^{17} \\
& - 10\,332\,892\,566\,359\,614\,848\,157\,876\,224x^{18} \\
& + 43\,345\,424\,617\,004\,971\,574\,289\,235\,968x^{19} \\
& - 142\,807\,225\,508\,285\,034\,141\,616\,963\,584x^{20} \\
& + 359\,505\,820\,412\,663\,945\,726\,355\,570\,688x^{21} \\
& - 636\,026\,962\,079\,787\,427\,490\,890\,252\,288x^{22} \\
& + 616\,797\,192\,523\,902\,897\,669\,611\,192\,320x^{23} \\
& + 45\,081\,769\,872\,830\,521\,912\,080\,728\,064x^{24} \\
& - 724\,445\,324\,775\,545\,659\,452\,335\,063\,040x^{25} \\
& + 521\,686\,412\,421\,099\,571\,093\,753\,036\,800x^{26}, \\
P_3^{(33)} = & -4480\,000 + 1569\,568\,000x - 238\,072\,889\,600x^2 + 21\,281\,848\,471\,520x^3 \\
& - 1268\,595\,101\,537\,120x^4 + 53\,555\,230\,610\,961\,720x^5 \\
& - 1640\,958\,998\,875\,092\,768x^6 \\
& + 36\,032\,181\,180\,727\,162\,732x^7 - 511\,428\,562\,675\,996\,247\,108x^8 \\
& + 1919\,885\,419\,260\,765\,103\,140x^9 + 129\,005\,457\,127\,386\,313\,373\,184x^{10} \\
& - 4541\,113\,747\,259\,527\,374\,959\,592x^{11} + 96\,035\,689\,755\,227\,434\,986\,877\,112x^{12} \\
& - 1580\,421\,468\,708\,164\,786\,235\,613\,784x^{13} \\
& + 22\,087\,897\,691\,588\,508\,601\,005\,658\,336x^{14} \\
& - 274\,269\,909\,442\,085\,751\,262\,554\,453\,856x^{15} \\
& + 3087\,338\,965\,228\,238\,905\,750\,107\,987\,648x^{16} \\
& - 31\,474\,922\,613\,166\,692\,487\,806\,647\,824\,256x^{17} \\
& + 286\,292\,076\,483\,602\,608\,978\,320\,943\,481\,344x^{18} \\
& - 2277\,952\,733\,740\,370\,146\,287\,983\,798\,312\,960x^{19} \\
& + 15\,571\,420\,858\,521\,621\,122\,719\,931\,928\,608\,768x^{20} \\
& - 90\,147\,310\,596\,750\,652\,057\,735\,905\,075\,527\,680x^{21} \\
& + 437\,037\,767\,842\,717\,994\,841\,190\,340\,774\,330\,368x^{22} \\
& - 1755\,686\,044\,559\,298\,411\,692\,425\,577\,783\,885\,824x^{23} \\
& + 5764\,646\,607\,249\,819\,312\,839\,063\,743\,970\,148\,352x^{24} \\
& - 15\,099\,644\,256\,008\,129\,321\,411\,266\,837\,095\,645\,184x^{25} \\
& + 29\,988\,658\,044\,590\,195\,137\,583\,404\,663\,925\,899\,264x^{26} \\
& - 39\,745\,090\,934\,862\,435\,542\,362\,760\,545\,321\,353\,216x^{27} \\
& + 19\,398\,167\,217\,699\,147\,074\,111\,209\,484\,113\,149\,952x^{28} \\
& + 40\,447\,257\,076\,217\,292\,533\,523\,320\,942\,836\,580\,352x^{29} \\
& - 84\,060\,042\,791\,775\,646\,091\,063\,152\,898\,350\,252\,032x^{30} \\
& + 40\,724\,840\,987\,587\,942\,318\,458\,896\,159\,738\,953\,728x^{31} \\
& + 34\,088\,801\,304\,111\,660\,197\,683\,822\,288\,919\,592\,960x^{32} \\
& - 34\,873\,025\,538\,917\,765\,121\,024\,203\,000\,119\,296\,000x^{33},
\end{aligned}$$

$$\begin{aligned}
P_2^{(38)} = & 202\,496\,000 - 84\,671\,104\,000x + 15\,961\,404\,659\,200x^2 - 1817\,819\,283\,938\,560x^3 \\
& + 141\,042\,261\,097\,575\,040x^4 - 7945\,786\,419\,559\,994\,432x^5 \\
& + 336\,970\,482\,890\,735\,391\,136x^6 \\
& - 10\,948\,102\,706\,558\,839\,518\,064x^7 + 272\,101\,251\,799\,491\,505\,044\,720x^8 \\
& - 4990\,110\,947\,182\,458\,236\,154\,960x^9 + 57\,747\,165\,172\,968\,723\,279\,034\,760x^{10} \\
& + 4251\,375\,690\,841\,730\,042\,108\,460x^{11} - 19\,664\,102\,412\,813\,111\,595\,220\,034\,000x^{12} \\
& + 586\,539\,601\,535\,060\,491\,103\,255\,831\,780x^{13} \\
& - 11\,648\,280\,832\,868\,820\,874\,871\,506\,994\,648x^{14} \\
& + 185\,168\,754\,164\,459\,407\,231\,412\,940\,918\,408x^{15} \\
& - 2524\,027\,149\,739\,792\,644\,483\,439\,537\,740\,736x^{16} \\
& + 30\,612\,264\,160\,202\,676\,427\,790\,224\,166\,656\,736x^{17} \\
& - 336\,756\,368\,430\,758\,251\,349\,398\,256\,374\,549\,440x^{18} \\
& + 3374\,928\,905\,004\,383\,352\,843\,307\,682\,288\,939\,648x^{19} \\
& - 30\,594\,829\,577\,694\,461\,795\,851\,875\,047\,251\,759\,104x^{20} \\
& + 247\,550\,135\,999\,641\,906\,395\,555\,078\,053\,550\,042\,624x^{21} \\
& - 1761\,694\,860\,791\,556\,801\,623\,390\,940\,580\,862\,476\,288x^{22} \\
& + 10\,880\,585\,300\,165\,439\,414\,813\,579\,169\,355\,207\,311\,360x^{23} \\
& - 57\,651\,463\,831\,886\,251\,900\,194\,559\,835\,893\,548\,711\,936x^{24} \\
& + 259\,270\,510\,361\,927\,197\,193\,957\,311\,877\,476\,593\,434\,624x^{25} \\
& - 977\,978\,052\,427\,489\,585\,499\,761\,822\,245\,900\,827\,754\,496x^{26} \\
& + 3043\,305\,515\,555\,663\,644\,471\,442\,318\,841\,392\,857\,612\,288x^{27} \\
& - 7593\,091\,629\,989\,468\,917\,294\,503\,828\,609\,326\,603\,304\,960x^{28} \\
& + 14\,317\,720\,902\,933\,442\,365\,662\,690\,637\,880\,059\,263\,713\,280x^{29} \\
& - 17\,337\,418\,172\,194\,871\,339\,769\,688\,830\,180\,251\,691\,646\,976x^{30} \\
& + 3546\,879\,809\,404\,692\,840\,748\,046\,019\,057\,281\,136\,590\,848x^{31} \\
& + 32\,904\,223\,733\,304\,447\,370\,184\,725\,984\,806\,679\,848\,419\,328x^{32} \\
& - 61\,070\,255\,095\,717\,193\,234\,874\,579\,385\,327\,453\,575\,577\,600x^{33} \\
& + 27\,287\,160\,011\,587\,832\,026\,533\,318\,214\,423\,160\,423\,448\,576x^{34} \\
& + 47\,538\,188\,516\,382\,446\,352\,727\,572\,349\,627\,507\,901\,726\,720x^{35} \\
& - 56\,445\,574\,686\,008\,125\,172\,119\,780\,480\,189\,438\,504\,206\,336x^{36} \\
& - 4318\,904\,703\,692\,797\,702\,055\,795\,738\,669\,690\,860\,339\,200x^{37} \\
& + 23\,615\,008\,551\,819\,589\,708\,322\,104\,774\,634\,383\,815\,475\,200x^{38}, \\
P_1^{(43)} = & -2508\,800\,000 + 1313\,872\,896\,000x - 313\,495\,056\,179\,200x^2 \\
& + 45\,402\,581\,315\,051\,520x^3 - 4502\,030\,899\,704\,432\,640x^4 \\
& + 326\,696\,241\,278\,915\,100\,672x^5 \\
& - 18\,076\,764\,858\,722\,283\,537\,408x^6 + 782\,686\,127\,310\,817\,603\,163\,904x^7 \\
& - 26\,913\,654\,199\,485\,748\,976\,447\,296x^8 + 737\,895\,426\,074\,343\,351\,817\,982\,240x^9 \\
& - 15\,941\,906\,513\,987\,915\,790\,530\,627\,104x^{10} \\
& + 258\,773\,331\,815\,879\,690\,900\,773\,968\,400x^{11}
\end{aligned}$$

$$\begin{aligned}
 & - 2607\,962\,306\,360\,230\,233\,373\,492\,782\,176x^{12} \\
 & - 5952\,736\,815\,704\,779\,243\,433\,578\,988\,240x^{13} \\
 & + 986\,994\,067\,078\,072\,761\,785\,220\,512\,495\,568x^{14} \\
 & - 27\,146\,844\,884\,553\,280\,870\,530\,528\,088\,810\,192x^{15} \\
 & + 517\,853\,080\,131\,584\,647\,940\,304\,813\,906\,843\,912x^{16} \\
 & - 8012\,481\,021\,063\,135\,055\,260\,920\,360\,860\,291\,792x^{17} \\
 & + 106\,770\,798\,207\,835\,443\,855\,237\,151\,398\,845\,884\,336x^{18} \\
 & - 1266\,450\,623\,115\,899\,739\,824\,560\,105\,189\,293\,118\,336x^{19} \\
 & + 13\,633\,188\,914\,202\,542\,686\,825\,030\,295\,715\,897\,195\,712x^{20} \\
 & - 134\,327\,854\,309\,114\,583\,892\,390\,549\,684\,219\,213\,327\,360x^{21} \\
 & + 1210\,064\,015\,789\,594\,600\,623\,288\,132\,568\,732\,268\,617\,728x^{22} \\
 & - 9885\,950\,925\,613\,173\,310\,943\,030\,286\,291\,198\,265\,745\,664x^{23} \\
 & + 72\,409\,258\,384\,998\,425\,181\,940\,033\,322\,470\,418\,064\,579\,584x^{24} \\
 & - 469\,743\,224\,562\,744\,760\,675\,515\,167\,582\,702\,668\,512\,534\,528x^{25} \\
 & + 2668\,377\,435\,339\,727\,605\,940\,145\,954\,082\,991\,900\,173\,762\,560x^{26} \\
 & - 13\,130\,369\,247\,854\,115\,699\,188\,867\,418\,934\,857\,934\,469\,332\,992x^{27} \\
 & + 55\,351\,059\,606\,988\,527\,054\,614\,355\,506\,855\,140\,926\,785\,847\,296x^{28} \\
 & - 197\,223\,974\,876\,465\,329\,508\,996\,857\,363\,261\,585\,516\,738\,379\,776x^{29} \\
 & + 582\,489\,670\,248\,346\,892\,198\,679\,343\,375\,535\,443\,002\,292\,961\,280x^{30} \\
 & - 1378\,041\,300\,571\,967\,278\,991\,550\,115\,889\,940\,047\,326\,999\,478\,272x^{31} \\
 & + 2424\,920\,581\,794\,299\,143\,009\,574\,014\,342\,980\,105\,306\,252\,509\,184x^{32} \\
 & - 2497\,995\,923\,785\,565\,959\,357\,957\,923\,036\,374\,193\,256\,014\,020\,608x^{33} \\
 & - 915\,221\,239\,768\,968\,177\,447\,587\,513\,000\,776\,938\,544\,384\,966\,656x^{34} \\
 & + 8834\,509\,277\,491\,743\,877\,951\,520\,843\,172\,281\,230\,968\,074\,797\,056x^{35} \\
 & - 14\,739\,433\,233\,907\,061\,551\,935\,598\,195\,219\,565\,259\,254\,451\,404\,800x^{36} \\
 & + 5672\,247\,850\,181\,350\,880\,926\,829\,983\,674\,652\,540\,556\,997\,033\,984x^{37} \\
 & + 16\,865\,676\,565\,374\,917\,674\,763\,783\,610\,716\,646\,680\,263\,347\,142\,656x^{38} \\
 & - 23\,057\,876\,384\,647\,717\,319\,687\,179\,507\,181\,219\,615\,528\,382\,889\,984x^{39} \\
 & - 1650\,497\,416\,603\,706\,423\,024\,253\,737\,913\,692\,788\,809\,736\,912\,896x^{40} \\
 & + 17\,950\,328\,578\,610\,802\,277\,327\,697\,770\,535\,083\,294\,270\,546\,771\,968x^{41} \\
 & - 3724\,970\,001\,243\,182\,786\,619\,005\,376\,755\,715\,492\,471\,782\,768\,640x^{42} \\
 & - 5981\,413\,341\,400\,069\,058\,756\,778\,898\,532\,874\,294\,556\,360\,704\,000x^{43}, \\
 P_0^{(47)}/16 = & 58\,841\,859\,686\,400x - 123\,282\,432\,000 - 13\,099\,552\,866\,570\,240x^2 \\
 & + 1817\,269\,523\,720\,161\,280x^3 - 176\,880\,012\,691\,621\,796\,864x^4 \\
 & + 12\,880\,441\,893\,460\,632\,329\,216x^5 - 729\,910\,851\,392\,566\,766\,105\,088x^6 \\
 & + 33\,014\,141\,392\,329\,879\,832\,166\,784x^7 \\
 & - 1210\,942\,171\,584\,302\,533\,599\,014\,752x^8 \\
 & + 36\,308\,844\,474\,092\,544\,015\,578\,885\,632x^9 \\
 & - 889\,129\,088\,058\,672\,919\,373\,638\,221\,264x^{10}
 \end{aligned}$$

$$\begin{aligned}
&+ 17\,508\,381\,271\,590\,013\,090\,109\,310\,169\,040x^{11} \\
&- 263\,494\,886\,656\,617\,518\,206\,756\,373\,588\,932x^{12} \\
&+ 2493\,591\,715\,504\,400\,008\,185\,935\,972\,185\,648x^{13} \\
&+ 5772\,652\,777\,357\,046\,820\,837\,948\,335\,210\,000x^{14} \\
&- 880\,112\,646\,375\,062\,548\,999\,453\,842\,320\,020\,740x^{15} \\
&+ 23\,493\,494\,316\,713\,860\,255\,651\,067\,234\,081\,149\,257x^{16} \\
&- 438\,058\,417\,861\,614\,862\,884\,693\,737\,035\,489\,286\,345x^{17} \\
&+ 6643\,943\,767\,863\,126\,335\,261\,566\,491\,851\,505\,292\,189x^{18} \\
&- 86\,756\,699\,901\,268\,061\,114\,746\,560\,625\,886\,582\,904\,365x^{19} \\
&+ 1006\,186\,625\,234\,680\,761\,751\,520\,210\,312\,145\,980\,149\,549x^{20} \\
&- 10\,573\,522\,222\,931\,154\,420\,271\,493\,264\,607\,253\,396\,390\,520x^{21} \\
&+ 101\,894\,357\,518\,227\,690\,884\,588\,911\,318\,694\,326\,634\,020\,120x^{22} \\
&- 904\,489\,874\,897\,014\,837\,177\,389\,619\,617\,321\,458\,811\,380\,360x^{23} \\
&+ 7377\,025\,197\,157\,259\,422\,622\,822\,297\,421\,365\,236\,335\,307\,120x^{24} \\
&- 54\,857\,750\,379\,533\,672\,661\,182\,684\,179\,932\,897\,350\,723\,993\,600x^{25} \\
&+ 368\,077\,157\,510\,764\,846\,299\,472\,690\,339\,090\,755\,869\,412\,496\,960x^{26} \\
&- 2203\,473\,576\,836\,831\,766\,402\,446\,311\,571\,835\,988\,588\,370\,992\,640x^{27} \\
&+ 11\,638\,948\,368\,194\,240\,385\,082\,022\,186\,232\,592\,838\,207\,372\,154\,880x^{28} \\
&- 53\,640\,316\,561\,668\,843\,524\,196\,008\,022\,033\,100\,643\,589\,470\,191\,616x^{29} \\
&+ 213\,035\,315\,257\,870\,225\,008\,043\,406\,258\,186\,703\,365\,521\,254\,907\,904x^{30} \\
&- 717\,584\,853\,510\,007\,605\,413\,068\,883\,853\,037\,631\,249\,102\,250\,442\,752x^{31} \\
&+ 2000\,779\,126\,900\,084\,461\,641\,442\,809\,746\,125\,018\,650\,394\,950\,107\,136x^{32} \\
&- 4414\,463\,297\,786\,097\,513\,664\,235\,192\,893\,813\,927\,566\,161\,255\,333\,888x^{33} \\
&+ 6904\,891\,435\,787\,610\,921\,130\,882\,736\,916\,279\,097\,844\,736\,823\,656\,448x^{34} \\
&- 4551\,724\,050\,684\,467\,601\,081\,502\,988\,404\,586\,537\,388\,373\,763\,424\,256x^{35} \\
&- 11\,571\,837\,065\,995\,769\,727\,688\,883\,612\,933\,393\,577\,503\,693\,010\,370\,560x^{36} \\
&+ 43\,604\,071\,314\,966\,497\,936\,511\,910\,817\,544\,815\,142\,484\,611\,319\,201\,792x^{37} \\
&- 64\,046\,695\,475\,293\,378\,492\,360\,343\,847\,354\,456\,353\,947\,960\,484\,036\,608x^{38} \\
&+ 17\,229\,520\,899\,952\,062\,417\,015\,850\,756\,255\,391\,466\,062\,797\,246\,300\,160x^{39} \\
&+ 98\,913\,305\,027\,317\,465\,024\,954\,824\,787\,190\,137\,389\,180\,923\,821\,424\,640x^{40} \\
&- 145\,693\,357\,979\,556\,257\,119\,098\,588\,624\,331\,861\,246\,512\,084\,740\,472\,832x^{41} \\
&- 1743\,763\,200\,037\,842\,518\,500\,493\,452\,602\,647\,084\,799\,741\,447\,372\,800x^{42} \\
&+ 159\,968\,299\,464\,829\,816\,606\,333\,313\,819\,738\,117\,801\,053\,481\,636\,724\,736x^{43} \\
&- 68\,453\,133\,710\,464\,189\,864\,730\,237\,770\,717\,937\,227\,743\,579\,749\,744\,640x^{44} \\
&- 84\,974\,525\,390\,992\,986\,946\,108\,353\,023\,934\,616\,304\,288\,806\,232\,653\,824x^{45} \\
&+ 41\,407\,097\,440\,632\,033\,071\,894\,561\,954\,752\,886\,956\,699\,467\,613\,470\,720x^{46} \\
&+ 28\,698\,609\,854\,675\,644\,415\,679\,733\,396\,189\,051\,258\,415\,886\,630\,912\,000x^{47}.
\end{aligned}$$

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