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Degree Complexity of a Family of Birational Maps

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Abstract We compute the degree complexity of a family of birational mappings of the plane with high order singularities.

Keywords Birational mappings · Degree complexity

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1 Introduction

Birational maps have been found to arise in lattice statistical mechanics, for instance in vertex models or in spin-edge models [17, 28, 36]. These are fundamental non-linear symmetries of the parameter space that arise from natural (geometrical) symmetries of the lattice, combined with the so-called inversion relation [29–31]. In the case of the Yang-Baxter integrability, the analysis of these symmetries can lead efficiently to a parameterization of the Yang-Baxter equations in terms of selected algebraic varieties [27, 32]. More generally, beyond Yang-Baxter integrability, these birational maps have to be compatible with the phase diagram of the model [33], and, for instance, with the renormalization group. It is important to note that for non-Yang-Baxter-integrable lattice models, these birational maps can still be integrable [14]. The connection between birational mappings and lattice statistical mechanics is discussed in greater length, for instance, in [9, 15] (Maillard, unpublished manuscript). Furthermore, and far beyond lattice statistical mechanics, one even can consider these birational transformations, *per se*, since they naturally correspond to a very important class of discrete dynamical systems, namely the *reversible* [18, 34, 35] discrete dynamical systems.

One such map gives rise to a family $k_{a,b}$ of birational maps of the plane (see [2, 20–22]). Dynamical properties of this family have been studied in a number of works [1–9, 11, 17, 23]. Recall the quantity

$$\delta(k) := \lim_{n\to\infty} \left(\deg\left(k^n\right)\right)^{\frac{1}{n}},$$

which is the exponential rate of growth of the iterates of k. This is variously known as the degree complexity, the dynamical degree, or the algebraic entropy of k. When $b \neq 0$ and a is generic, $\delta(k_{a,b})$ is the largest root of the polynomial $x^3 - x^2 - 2x - 1$. When $b = 0$ and a is generic, $\delta(k_{a,0})$ is the largest root of $x^2 - x - 1$. The form of a map can change radically under birational equivalence: a simpler form for $k_{a,0}$ which was obtained in [19] made it more accessible to detailed analysis (see [10, 12]).

A basic property is that k is reversible in the sense that $k = \jmath \circ \iota$ is a composition of two involutions. In this case, j corresponds to lattice symmetry, and ι corresponds to matrix inversion. In this paper we give (a birationally equivalent version of) k as a composition of involutions in a new way. This shows how $k_{a,b}$ fits naturally into a larger family of maps. Namely, for any rational function $F(y) = P(y)/Q(y)$, we define the involutions

$$\jmath_F(x, y) = (-x + F(y), y), \quad \iota(x, y) = \left(1 - x - \frac{x-1}{y}, -y - 1 - \frac{y}{x-1}\right),$$

and the family of birational maps is given by $k_F = \jmath_F \circ \iota$. When F is constant, the family k_F is birationally equivalent to $k_{a,0}$, and when F is linear, k_F

is equivalent to $k_{a,b}$. In this paper we determine the structure and degree complexity for the maps k_F:

Theorem 1 *Let p (resp. q) denote the degree of P (resp. Q). If $p < q$, then for generic parameters $\delta(k_F) = q + 1$. Otherwise, if $p - q \geqslant 0$ is even, then for generic parameters $\delta(k_F)$ is the largest root of the polynomial $x^2 - (p+1)x - (q+1)$. If $p - q \geqslant 0$ is odd, then for generic parameters $\delta(k_F)$ is the largest root of $x^3 - px^2 - (p+q+1)x - (q+1)$.*

When k_F is not generic, the growth rate $\delta(k_F)$ decreases (i.e. $F \mapsto \delta(k_F)$ is lower semicontinuous in the Zariski topology). One of the interesting things about the family is to know which parameters are not generic as well as the corresponding values of $\delta(k_F)$ is decreased. The exceptional values of a for the family $k_{a,0}$, as well as the corresponding values of $\delta(k_{a,0})$, were found by Diller and Favre [24]. Similarly, the exceptional values of (a, b) are given in [11]. Here we look at the maximally exceptional parameters for the case where F is cubic. These are the cubic maps with the slowest degree growth and give a 2 complex parameter family of maps which are (equivalent to) automorphisms:

Theorem 2 *If $F(y) = ay^3 + ay^2 + by + 2$, $a \neq 0$, then k_F is an automorphism of a compact, complex surface $\mathcal{Z}$. Further, the degrees of k_F^n grow quadratically, and k_F is integrable.*

We will analyze the family k_F by inspecting the blowing-up and blowing-down behavior. That is, there are exceptional curves, which are mapped to points; and there are points of indeterminacy, which are blown up to curves. As was noted by Fornæss and Sibony [25], if there is an exceptional curve whose orbit lands on a point of indeterminacy, then the degree is not multiplicative: $(\deg(k_F))^n \neq \deg(k_F^n)$. The approach we use here is to replace the original domain $\mathbf{P}^2$ by a new manifold $\mathcal{X}$. That is, we find a birational map $\varphi : \mathcal{X} \to \mathbf{P}^2$, and we consider the new birational map $\tilde{k} = \varphi \circ k_F \circ \varphi^{-1}$. There is a well defined map $\tilde{k}^* : Pic(\mathcal{X}) \to Pic(\mathcal{X})$, and the point is to choose $\mathcal{X}$ so that the induced map $\tilde{k}$ satisfies $(\tilde{k}^*)^n = (\tilde{k}^n)^*$. By the birational invariance of δ (see [16] and [24]) we conclude that $\delta(k_F)$ is the spectral radius of $\tilde{k}^*$. This method has also been used by Takenawa [37–39]. The general existence of such a map $\tilde{k}$ when $\delta(k) > 1$ was shown in [24]. We comment that the construction of $\mathcal{X}$ and $\tilde{k}$ can yield further information about the dynamics of k (see, for instance, [13] and [11]).

The bulk of this paper is devoted to proving Theorem 1. After a division, we may rewrite $F(y) = a_n y^n + \cdots + a_1 y + a_0 + P(y)/Q(y)$, where $deg(P) < deg(Q)$. In our treatment below, we first do the cases where $F(y)$ is a polynomial, which is in some sense the most singular and most difficult case because it involves iterated blowups to depth n. We give general properties of the map k_F in Section 2. In Section 3, we describe the iterated blowup process in some detail. In Section 4 we carry out the blowup process to regularize k_F in the case where F is a polynomial of even degree n. What we do here is to determine the action of the induced map k_F^* on $Pic(\mathcal{X})$; and $\delta(k_F)$ is the spectral radius of k_F^*. The case n odd is distinct, and we carry it out in Section 5. In Section 6 we handle the case where $F(y) = P(y)/Q(y)$ with $deg(P) \leqslant deg(Q)$. That is, we present the blowup procedure, and we determine k_F^*. We will see in Section 6 that the blowup process for the cases $q = 0$ and $q \geqslant p$ are essentially independent, since the blowup operations are performed in different places. After Sections 2–6, it is not hard to put these separate analyses together to cover the general case. The Picard group in the general case is generated by the elements produced in the independent cases, and the induced linear transformation k_F^* maps them the same way it does when they are independent. Thus it is just a matter of bookkeeping to combine the two cases. Since Sections 2–5 and Section 6 are the two parts that need to be put together, and since it would be repetitive to do them simultaneously, we omit the details.

The exceptional cases are also of considerable interest, but many cases arise, and it is not easy to handle them efficiently, so we do not address this issue here. As an example, however, we treat in Section 7 the case where the coefficients are as non-generic as possible when $p = 3$, $q = 0$. This leads to a proof of Theorem 2, which gives a family of automorphisms for which the degrees grow quadratically.

2 The Maps

Let us set $F(z) = \sum_{j=0}^{n} a_j z^j$ with $n \geqslant 2$ and $a_n \neq 0$. The map $k = j_F \circ \iota$ is the composition of the two involutions defined above. The map $k = [k_0 : k_1 : k_2]$ is given in homogeneous coordinates as

$$k_0 = \left(x_0x_1 - x_0^2\right)^n x_2$$

$$k_1 = x_0^{n-1}(x_1 - x_0)^{n+1}(x_2 + x_0) + x_2 \sum_{j=0}^{n} a_j \left(x_0x_1 - x_0^2\right)^{n-j} \left(x_2^2 - x_0x_1 - x_1x_2\right)^j$$

$$k_2 = x_2 \left(x_0x_1 - x_0^2\right)^{n-1} \left(x_2^2 - x_0x_1 - x_1x_2\right). \tag{1}$$

Each coordinate function has degree $2n + 1$, which means that $\deg(k) = 2\deg(F) + 1$. Since the jacobian of this map is $x_0^{3n-3}(x_0 - x_1)^{3n-1}x_2^2(x_0^2 - x_0x_1 - x_1x_2)$ we have four exceptional curves :

$$C_1 := \{x_0 = 0\},\ C_2 := \{x_0 = x_1\},\ C_3 := \{x_2 = 0\},\ C_4 := \left\{-x_0^2 + x_0x_1 + x_1x_2 = 0\right\}.$$

When $n \geqslant 2$ and $a_0 \neq 2$, the exceptional hypersurfaces are mapped as:

$$k\ :\ C_4 \mapsto [1 : -1 + a_0 : 0] \in C_3 \quad \text{and} \quad C_1 \cup C_2 \cup C_3 \mapsto e_1. \tag{2}$$

The points of indeterminacy for k are

$$e_1 := [0 : 1 : 0],\ \ e_2 := [0 : 0 : 1],\ \text{ and }\ e_{01} := [1 : 1 : 0].$$

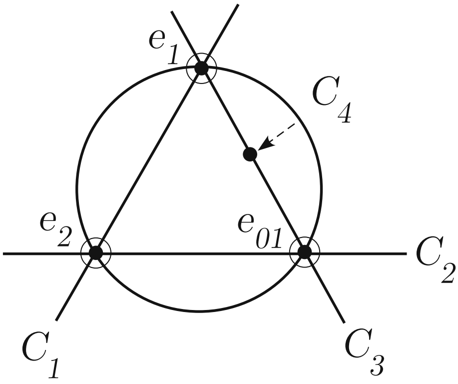


Fig. 1 $n \geqslant 2$. Exceptional curves and points of indeterminacy

Figure 1 shows the relative position of the points of indeterminacy (dots with circles around them), exceptional curves, and the critical images (big dots). The information that $C_1, C_2, C_3 \to e_1$ is not drawn for lack of space.

The sort of singularity that will be the most difficult to deal with arises from the exceptional curve $C_1 \mapsto e_1 \in C_1$. In local coordinates near e_1, this looks like

$$k[t:1:y] = \left[\frac{t^n + \cdots}{a_n(-y)^n + \cdots} : 1 : \frac{t^{n-1} + \cdots}{a_n(-y)^{n-1} + \cdots}\right]. \tag{3}$$

For this, we will perform the iterated blowups described in Section 3.

The inverse map $k^{-1} = \left[k_0^{-1} : k_1^{-1} : k_2^{-1}\right]$ is given as

$$k_0^{-1} = x_0^n x_2 \left(\check{F} - x_0^{n-1}(x_0 + x_1)\right)$$

$$k_1^{-1} = (x_0 + x_2)\left(\sum_{j=0}^{n} a_j x_0^{n-j} x_2^j - x_0^{n-1}(x_0 + x_1)\right)^2$$

$$k_2^{-1} = x_0^{n-1} x_2 \left(x_0^{n-1}\left(x_0^2 + x_0 x_1 + x_1 x_2\right) - (x_0 + x_2)\check{F}\right)$$

where $\check{F} = x_0^n F(x_2/x_0) = \sum_{j=0}^{n} a_j x_0^{n-j} x_2^j$. The jacobian for the inverse map is

$$x_0^{3n-3} x_2^2 \left(x_0^n + x_0^{n-1}x_1 - \check{F}\right)^2 \left(x_0^{n+1} - (x_0 + x_2)\left(x_0^n + x_0^{n-1}x_1 - \check{F}\right)\right)$$

The exceptional curves for k^{-1} are C'_j, $1 \leqslant j \leqslant 4$, where

$$C'_1 = C_1, \quad C'_2 := \left\{x_0^n + x_0^{n-1}x_1 - \check{F} = 0\right\}, \quad C'_3 = C_3,$$

$$C'_4 := \left\{x_0^{n+1} - (x_0 + x_2)\left(x_0^n + x_0^{n-1}x_1 - \check{F}\right) = 0\right\}.$$

$$k^{-1}: \ C'_1 \cup C'_3 \mapsto e_1, \quad C'_2 \mapsto e_2, \quad \text{and} \quad C'_4 \mapsto e_{01} \in C'_3, \tag{4}$$

3 Blowups and Local Coordinate Systems

In this section we discuss iterated blowups, and we explain the choices of local coordinates which will be useful in the sequel. Let $\pi : X \to \mathbf{C^2}$ denote the complex manifold obtained by blowing up the origin $e = (0, 0)$; the space is given by

$$X = \left\{((t, y), [\xi : \eta]) \in \mathbf{C}^2 \times \mathbf{P} \,;\; t\eta = y\xi\right\},$$

and π is projection to $\mathbf{C}^2$. Let $E := \pi^{-1}(e)$ denote the exceptional fiber over the origin, and note that π^{-1} is well defined over $\mathbf{C}^2 - e$. The closure in X of the y-axis ($\pi^{-1}(\{t = 0\} - e)$) corresponds to the hypersurface $\{\xi = 0\} \subset X$. On the complement $\{\xi \neq 0\}$ set $u = t$ and $\eta = y/t$. Then (u, η) defines a coordinate system on $X \setminus \{t = 0\}$, with a point being given by $((t, y), [1 : y/t]) = ((u, u\eta), [1 : \eta])$. We will use the notation $(u, \eta)_L$. On the set $t \neq 0$, the coordinate projection π is given in these coordinates as

$$\pi_L(u, \eta)_L = (u, u\eta) = (t, y) \in \mathbf{C^2}. \tag{5}$$

Figure 2a illustrates this blowup with emphasis on the relation between the point e and the lines $t = 0$ and $y = 0$ which contain it. The space X is drawn twice to show two choices of coordinate system; the dashed lines show where each coordinate system fails to be defined. The left hand copy of X shows the u, η-coordinate system in the complement of $t = 0$. The right hand side shows a different choice of coordinate; we would choose this coordinate system to work in a neighborhood of the point $p_1 := E \cap \{t = 0\}$.

In the u, η coordinate system (on the upper left side of Fig. 2a), the η-axis ($u = 0$) represents the exceptional fiber $E \cong \mathbf{P}^1$. The line $\gamma_\eta = \{(s, \eta)_L : s \in \mathbf{C}\}$ projects to the line $\{y = \eta t\} \subset \mathbf{C}^2$, and $(0, \eta)_L = E \cap \gamma_\eta$. It follows that $E \cap \{y = 0\} = (0, 0)_L$ in this coordinate system.

On the upper right side of Fig. 2a, we define a (ξ, v)-coordinate system on the complement of t-axis ($y = 0$):

$$\pi_R \;:\; (\xi, v)_R = (t/y, y) \to (v\xi, v) \in \mathbf{C}^2. \tag{6}$$

The exceptional fiber E is given by ξ-axis ($v = 0$). Next we blow up $p_1 = E \cap \{t = 0\} = \{\xi = v = 0\} = (0, 0)_R$. Let P_1 denote the exceptional fiber over p_1. The choice of a local coordinate system depends on the center of next blowup. Suppose the third blowup center is an intersection of two exceptional fibers $p_2 := E \cap P_1$. For this we are led to the (u, η)- coordinate system, as on the left side of Fig. 2a. Thus we have a local coordinate system on the complement of $\{t = 0\} \cup \{y = 0\}$;

$$(u_1, \eta_1)_1 = \left(t/y, y^2/t\right) \to (u_1, u_1\eta_1)_R \to \left(u_1^2\eta_1, u_1\eta_1\right) \in \mathbf{C}^2. \tag{7}$$

This (u_1, η_1)-coordinate system is defined only off the axes $(t = 0) \cup (y = 0)$; the new exceptional fiber P_1 is given by the η_1-axis.

Now we define a sequence of iterated blowups which will let us deal with the singularity (3). We start with the blowup space X as in Fig. 2b, and we continue inductively for $2 \leqslant j \leqslant n$ by setting $p_j := E \cap P_{j-1}$ and letting P_j be

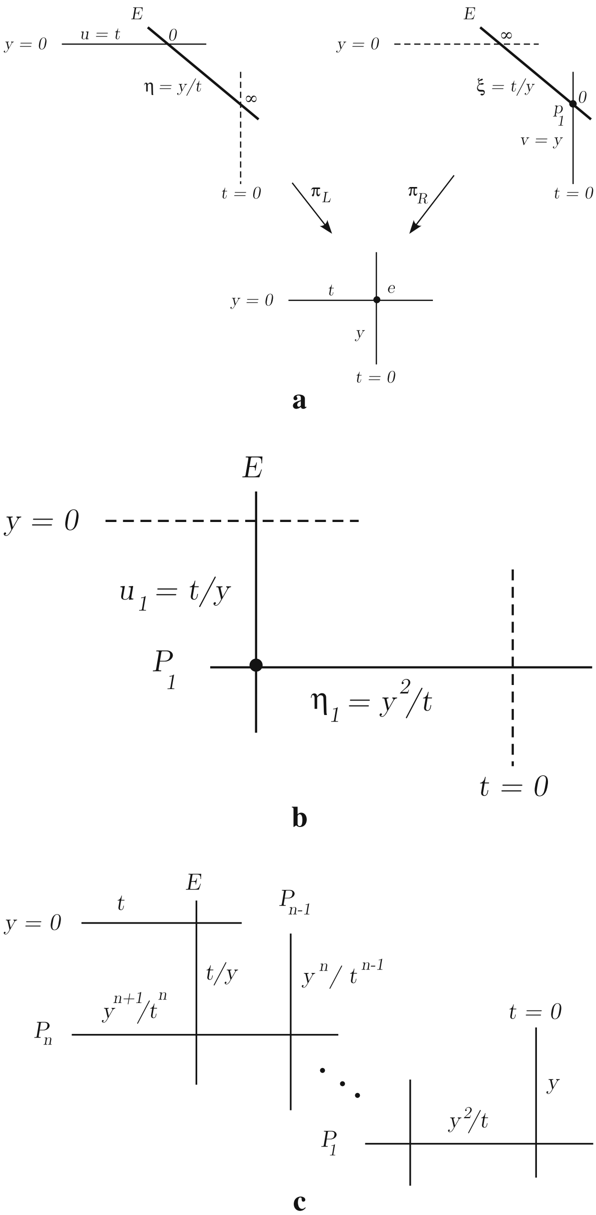


Fig. 2 **a** Two choices of local coordinate systems. **b** Blowup of p_1 in (u_1, η_1)-coordinates. **c** n-th iterated blowup

the exceptional fiber. For each $2 \leqslant j \leqslant n$, we use the left-hand coordinate system of Fig. 2a, which corresponds to (5). Thus we have the coordinate projection $\pi_j : P_j \to \mathbf{C}^2$:

$$\pi_j : (u, \eta)_j \to \left(u^{j+1}\eta, u^j\eta\right) = (t, y) \in \mathbf{C}^2, \quad \pi_j^{-1}(t, y) = (u, \eta) = \left(t/y, y^{j+1}/t^j\right). \tag{8}$$

This coordinate system is defined off of $\{y = 0\} \cup \{t = 0\} \cup P_1 \cup \cdots \cup P_{j-1}$. A point $(0, \eta = c)_j \in P_j$ is the landing point of the curve $u \mapsto (u, c)_j$ as $u \to 0$, which projects to the curve $u \mapsto (t(u) = u^{j+1}c, y(u) = u^j c) \in \mathbf{C}^2$. In Fig. 2c, the exceptional fibers P_j, $1 \leqslant j \leqslant n$ are drawn with their fiber coordinates y^{j+1}/t^j.

4 Mappings with $q = 0$ and $n =$ Even

We define a complex manifold $\pi_{\mathcal{X}} : \mathcal{X} \to \mathbf{P}^2$ by blowing up points $e_1, q, p_1, \ldots, p_{n-1}$ in the following order:

(a) blow up $e_1 = [0 : 1 : 0]$ and let E_1 denote the exceptional fiber over e_1,
(b) blow up $q := E_1 \cap C_4$ and let Q denote the exceptional fiber over q,
(c) blow up $p_1 := E_1 \cap C_1$ and let P_1 denote the exceptional fiber over p_1,
(d) blow up $p_j := E_1 \cap P_{j-1}$ with exceptional fiber P_j for $2 \leqslant j \leqslant n-1$.

The iterated blow-up of $p_1, \ldots, p_{n-1}$ is exactly the process described in Section 3, so we will use the local coordinate systems defined there. That is, in a neighborhood of Q we use a $(\xi_1, v_1) = (t^2/y, y/t)$ coordinate system. For E_1 and P_j, $1 \leqslant j \leqslant n-1$ we use local coordinate systems defined in (6–8). We use homogeneous coordinates by identifying a point $(t, y) \in \mathbf{C}^2$ with $[t : 1 : y] \in \mathbf{P}^2$. Let $k_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ denote the induced map on the complex manifold $\mathcal{X}$. In the next few lemmas, we will show that $k_{\mathcal{X}}$ maps the exceptional fibers as shown in Fig. 3.

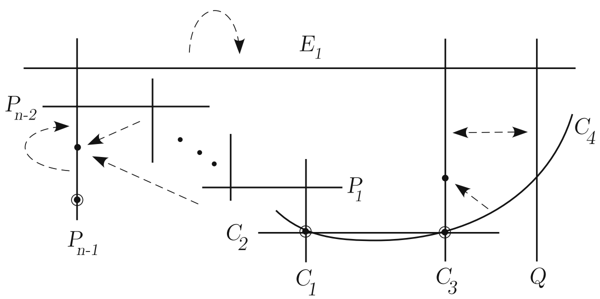
Lemma 1 *Under the induced map $k_{\mathcal{X}}$, the blowup fibers E_1 and P_{n-1} are mapped to themselves:*

$$\begin{aligned} k_{\mathcal{X}} : E_1 \ni \xi &\mapsto -\xi/(\xi+1) \in E_1 \\ P_{n-1} \ni \eta_{n-1} &\mapsto \eta_{n-1}/(1 + a_n \eta_{n-1}) \in P_{n-1}. \end{aligned} \tag{9}$$

Proof First let us work on E_1. We use the local coordinate system defined in (6), so a point in the exceptional fiber E_1 is $(\xi, 0)_R$. To see the forward image of E_1 we consider a nearby point $(\xi, v)_R \to (v\xi, v)$ with small v and we have $k_{\mathcal{X}}(\xi, 0)_R = \lim_{v \to 0} k_{\mathcal{X}}(\xi, v)_R$. By (1) we see that

$$k\,[v\xi : 1 : v] = [v\xi + \cdots : 1 + \cdots : -v(\xi+1) + \cdots]$$

Fig. 3 The space $\mathcal{X}$ and the action of $k_{\mathcal{X}}$



where we use $\cdots$ to indicate the higher order terms in v. As in Fig. 2a, the coordinate of the landing point in E_1 is given by the ratio of t- and y-coordinates. Thus we have

$$k_{\mathcal{X}}|E_1 \,:\, \xi \mapsto \lim_{v\to 0} k_0/k_2 = \lim_{v\to 0}(v\xi + \cdots)/(-v(\xi+1)+\cdots) = -\xi/(\xi+1).$$

Now we determine the behavior of $k_{\mathcal{X}}$ on P_{n-1}. A fiber point $(0,\eta_{n-1}) \in P_{n-1}$ is the landing point of the arc $u \mapsto (u,\eta_{n-1})$ as $u \to 0$. To show that $k_{\mathcal{X}}$ maps P_{n-1} to P_{n-1}, we need to evaluate:

$$\lim_{u\to 0} k_{\mathcal{X}}(u,\eta_{n-1}) = \lim_{u\to 0} \pi_{n-1}^{-1} \circ k \circ \pi_{n-1}(u,\eta_{n-1}).$$

Using the formulas for π_{n-1} and π_{n-1}^{-1} in (8), we obtained the desired limit. □

Now we may use similar calculations to show that $k_{\mathcal{X}} : P_j \to P_{n-1}$; we fix a point $(0,\eta_j) \in P_j$ and show the existence of the limit

$$\lim_{u\to 0} k_{\mathcal{X}}\big(u,\eta_j\big) = \lim_{u\to 0} \pi_{n-1}^{-1} \circ k \circ \pi_j\big(u,\eta_j\big).$$

Doing this, we find that the line C_1 and all blowup fibers P_j, $j = 1,\ldots,n-2$ are all exceptional for both $k_{\mathcal{X}}$ and $k_{\mathcal{X}}^{-1}$. And C_2 is exceptional for $k_{\mathcal{X}}$:

$$\begin{aligned} k_{\mathcal{X}} &: \; C_1, C_2, P_1, \cdots, P_{n-2} \mapsto 1/a_n \in P_{n-1} \\ k_{\mathcal{X}}^{-1} &: \; C_1, P_1, \cdots, P_{n-2} \mapsto (-1)^{n-1}/a_n \in P_{n-1} \end{aligned} \tag{10}$$

Combining (9–10) it is clear that the indeterminacy locus of $k_{\mathcal{X}}$ consists of three points

$$e_2, \quad e_{01}, \quad \text{and} \quad (-1)^{n-1}/a_n \in P_{n-1}.$$

Lemma 2 *If n is even, then the orbits of the exceptional curves $C_1, C_2, P_1, \ldots, P_{n-2}$ are disjoint from the indeterminacy locus.*

Proof By Lemma 1, the orbit of $1/a_n$ in P_{n-1} is $\{1/a_n, 1/(2a_n), 1/(3a_n), \ldots\} \subset P_{n-1}$. This is disjoint from the indeterminacy locus since it does not contain point $-1/a_n$ in P_{n-1}. □

A computation as in the proof of Lemma 1 shows that $k_{\mathcal{X}}$ maps $Q \leftrightarrow C_3$ according to:

$$\begin{aligned} k_{\mathcal{X}} : Q \ni \xi_1 &\mapsto [1 : a_0 - \xi_1 : 0] \in C_3, \\ C_3 \ni [x_0 : x_1 : 0] &\mapsto -x_1/x_0 \in Q. \end{aligned} \tag{11}$$

Lemma 3 *If $a_0 \neq 2/m$ for all $m > 0$ then the indeterminacy locus of $k_{\mathcal{X}}$ and the forward orbit of C_4 under the induced map $k_{\mathcal{X}}$ are disjoint. If $a_0 = 2/m$ for some $m > 0$, we have $k_{\mathcal{X}}^{2m-1} C_4 = e_{01}$.*

Proof Since the forward image of C_4 is $[1 : -1 + a_0 : 0] \in C_3$, using (11) we have that $k_{\mathcal{X}}^{2m-1} C_4 = [1 : ma_0 - 1 : 0] \in C_3$. Since the unique point of indeterminacy in C_3 is e_{01}, for C_4 to be mapped to a point of indeterminacy, a_0 must satisfy $ma_0 - 1 = 1$ for some $m \geqslant 0$. □

The following theorem comes directly from previous lemmas.

Theorem 3 *Suppose that n is even and $a_0 \neq 2/m$ for all integers $m \geqslant 0$. Then no orbit of an exceptional curve contains a point of indeterminacy.*

Let us recall the Picard group $Pic(\mathcal{X})$, which is the set of all divisors in $\mathcal{X}$, modulo linear equivalence, which means that $D_1 \sim D_2$ if $D_1 - D_2$ is the divisor of a rational function. $Pic(\mathbf{P}^2)$ is 1-dimensional and generated by the class of any line (hyperplane) H, and a basis of $Pic(\mathcal{X})$ is given by the class of a general hyperplane $H_{\mathcal{X}} := \pi^* H$, together with all of the blowup fibers $E_1, Q, P_1, \dots, P_{n-1}$. If r is a rational function on $\mathcal{X}$, then the pullback $k_{\mathcal{X}}^* r := r \circ k_{\mathcal{X}}$ is just the composition. To pull back a divisor, we just pull back its defining functions. This gives the pullback map $k_{\mathcal{X}}^* : Pic(\mathcal{X}) \to Pic(\mathcal{X})$. Thus from (9–10) we see that the pullback of E_1 is E_1 and the pulling back of most of basis elements are trivial, that is $k_{\mathcal{X}}^* P_j = 0$ for all $j = 1, \dots, n-2$.

Next we pull back $H_{\mathcal{X}}$. Since k has degree $2n+1$ we have $k^* H = (2n+1)H$ in $Pic(\mathbf{P}^2)$. Now we pull back by $\pi_{\mathcal{X}}^*$ to obtain:

$$(2n+1)H_{\mathcal{X}} = \pi_{\mathcal{X}}^*(2n+1)H = \pi_{\mathcal{X}}^*(k^* H). \tag{12}$$

A line is given by $\{h := \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0\}$, so $k^* H$ is the divisor defined by $h \circ k = \sum_j \alpha_j k_j$. To write this divisor as a linear combination of basis elements $H_{\mathcal{X}}, E_1, Q, P_1, \dots, P_{n-1}$, we need to check the order of vanishing of $h \circ k$ at all of these sets. Let us start with the coordinate system $\pi_{\mathcal{X}}(\xi, v) = [v\xi : 1 : v]$ near E_1, defined in Section 3. Using the expression for k given in Section 2 we see that $\alpha_0 k_0 + \alpha_1 k_1 + \alpha_2 k_2$ vanishes to order n in v. It follows that $\pi_{\mathcal{X}}^* k^* H$ vanishes at E_1 with multiplicity n. Similar computations for all other basis elements gives us $\pi_{\mathcal{X}}^* k^* H = k_{\mathcal{X}}^* H_{\mathcal{X}} + nE_1 + (n+1)Q + (n+1)\sum_j jP_j$. Combining with (12) we have

$$k_{\mathcal{X}}^* H_{\mathcal{X}} = (2n+1)H_{\mathcal{X}} - nE_1 - (n+1)Q - (n+1)\sum_{j=1}^{n-1} jP_j. \tag{13}$$

Similarly, we obtain:

$$\begin{aligned} k_{\mathcal{X}}^* : Q &\mapsto H_{\mathcal{X}} - E_1 - Q - P_1 - 2P_2 - \cdots - (n-1)P_{n-1} \\ P_{n-1} &\mapsto 2H_{\mathcal{X}} - E_1 - Q - P_1 - 2P_2 - \cdots - (n-1)P_{n-1}. \end{aligned} \tag{14}$$

Theorem 4 *$q = 0$ and $n =$ even. Suppose $F(z) = \sum_{j=1}^n a_j z^j$ is an even degree polynomial associated with J_F. If $a_0 \neq 2/m$ for any positive integer m, then the degree complexity is the largest root of the quadratic polynomial $x^2 - (n+1)x - 1$.*

Proof Since $P_1, \dots, P_{n-2}$ are mapped to 0 under the action on cohomology, it suffices to consider the action restricted to $H_{\mathcal{X}}, E_1, Q,$ and P_{n-1}.

By (13, 14) the matrix representation of $k_{\mathcal{X}}^{*}$, restricted to the ordered basis $\{H_{\mathcal{X}}, E_1, Q, P_{n-1}\}$, is

$$\begin{pmatrix} 2n+1 & 0 & 1 & 2 \\ -n & 1 & -1 & -1 \\ -n-1 & 0 & -1 & -1 \\ -n^2+1 & 0 & -n+1 & -n+1 \end{pmatrix}.$$

The characteristic polynomial is $x(x-1)(x^2-(n+1)x-1)$. □

5 Mappings with $q = 0$ and $n = $ Odd

Let us start with the space $\mathcal{X}$ from Section 4. When n is odd, we see from (10) that the image of all exceptional lines of $k_{\mathcal{X}}$ coincide with a point of indeterminacy in $p_n \in P_{n-1}$. Let $\pi_{\mathcal{Y}} : \mathcal{Y} \to \mathbf{P}^2$ be the complex manifold obtained by blowing up $\mathcal{X}$ at the point p_n, and let P_n denote the exceptional fiber over p_n. In the u_{n-1}, η_{n-1} coordinate system, p_n has coordinate $(0, 1/a_n)_{n-1}$. Thus, at P_n, we use the coordinate projection:

$$\pi_n : \mathcal{Y} \ni (u, \eta)_n \to (u^n(u\eta + 1/a_n), u^{n-1}(u\eta + 1/a_n)) \in \mathbf{C}^2.$$

Most computations in the previous section remain valid for n odd. Thus Lemma 3, (9) and (11) are still valid for the induced map $k_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}$. Under $k_{\mathcal{Y}}$ curves $C_1, C_2, P_1, \ldots, P_{n-3}$ are still exceptional:

$$\begin{aligned} k_{\mathcal{Y}} \ &: \ C_1, C_2, P_1, \ldots, P_{n-3} \mapsto -a_{n-1}/a_n^2 \in P_n \\ k_{\mathcal{Y}}^{-1} \ &: \ C_1, P_1, \ldots, P_{n-3} \mapsto (a_{n-1} - (n-1)a_n)/a_n^2 \in P_n. \end{aligned} \tag{15}$$

The blowup fibers P_n and P_{n-2} form a two cycle, $k_{\mathcal{Y}} : P_n \leftrightarrow P_{n-2}$ and P_{n-1} is mapped to itself as before. It follows that the points of indeterminacy for $k_{\mathcal{Y}}$ are e_2, e_{01} and $(a_{n-1} - (n-1)a_n)/a_n^2 \in P_n$. For all $m \geqslant 0$, we have

$$k_{\mathcal{Y}}^{2m} \ : \ P_n \ni -a_{n-1}/a_n^2 \mapsto (2ma_n - (2m+1)a_{n-1})/a_n^2 \in P_n \tag{16}$$

The induced action of k on $\mathcal{Y}$ is pictured in Fig. 4. As a consequence of (15) and (16) we have:

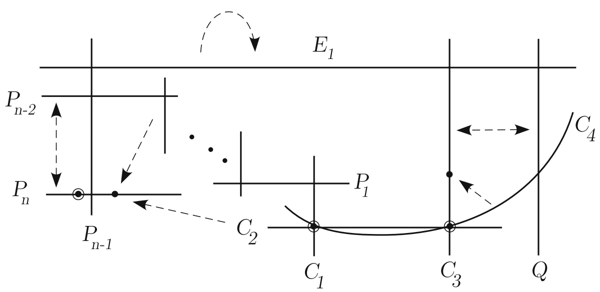


Fig. 4 The space $\mathcal{Y}$ and the action of $k_{\mathcal{Y}}$

Lemma 4 *If n is odd, and if*

$$(2m+2)a_{n-1} \neq (2m+n-1)a_n \tag{17}$$

for all $m \geqslant 0$, *then the forward orbits of* $C_1, C_2, P_1, \ldots, P_{n-3}$ *under* $k_{\mathcal{Y}}$ *do not contain any point of indeterminacy.*

Combining Lemmas 3 and 4 we have

Theorem 5 *Suppose that n is odd,* $a_0 \neq 2/m$ *for all* $m > 0$, *and* $a_{n-1} \neq (n-1)a_n/2$. *Then the forward orbits of exceptional curves do not contain any points of indeterminacy.*

To determine $k_{\mathcal{Y}}$, we use the basis $\{H_{\mathcal{Y}}, E_1, Q, P_1, \ldots, P_n\}$ for $Pic(\mathcal{Y})$. Now the exceptional lines $C_1, C_2, P_1, \ldots, P_{n-2}$ are mapped to P_n. Let $\{C_1\} \in Pic(\mathcal{Y})$ denote the class of the strict transform of C_1, i.e., the closure in $\mathcal{Y}$ of $\pi_{\mathcal{Y}}^{-1}(C_1 -$ centers of blowup). (The curve C_2 does not pass through any center of blowup, so with the same notation we have $\{C_2\} = H_{\mathcal{Y}} \in Pic(\mathcal{Y})$.) In order to write $\{C_1 = (x_0 = 0)\}$ in terms of our basis, we note first that $\pi_{\mathcal{Y}}^{-1} C_1 = C_4 \cup E_1 \cup Q \cup P_1 \cup \cdots \cup P_{n-1}$, i.e., the pullback function $x_0 \circ \pi_{\mathcal{Y}}$ vanishes on all of these curves. Thus we have to compute the multiplicities of vanishing. At P_{n-1}, for instance, we consider the (u_{n-1}, η_{n-1}) coordinate system defined in (8), and we see that $k_{\mathcal{Y}}^* x_0$ vanishes to order n at $P_{n-1} = (u_{n-1} = 0)$. Similarly we can compute the multiplicities for $E_1, Q, P_1, \ldots, P_{n-2}$ and P_n, so

$$H_{\mathcal{Y}} = \pi_{\mathcal{Y}}^* C_1 = \{C_1\} + E_1 + Q + 2P_1 + 3P_2 + \cdots + nP_{n-1} + nP_n.$$

It follows that

$$k_{\mathcal{Y}}^* P_n = \{C_1\} + \{C_2\} + \sum_{j=1}^{n-2} P_j = 2H_{\mathcal{Y}} - E_1 - Q - \sum_{j=1}^{n-2} jP_j - nP_{n-1} - nP_n.$$

For the rest of basis entries we have

$$k_{\mathcal{Y}}^* \ : \ H_{\mathcal{Y}} \mapsto (2n+1)H_{\mathcal{Y}} - nE_1 - (n+1)Q - (n+1)\sum_{j=1}^{n-1} jP_j - n^2 P_n$$

$$Q \mapsto H_{\mathcal{Y}} - E_1 - Q - P_1 - 2P_2 - \cdots - (n-1)P_{n-1} - (n-1)P_n,$$

$$E_1 \mapsto E_1, \quad P_{n-2} \mapsto P_n, \quad \text{and} \quad P_{n-1} \mapsto P_{n-1}.$$

Theorem 1: $q = 0$ and $n =$ odd. If $a_0 \neq 2/m$ for all $m > 0$, then the degree complexity is the largest root of the cubic polynomial $x^3 - nx^2 - (n+1)x - 1$.

Proof The classes of the exceptional fibers $P_1, \cdots, P_{n-3}$ are all mapped to 0, and exceptional fibers E_1 and P_{n-1} are simply interchanged. It follows that to get the spectral radius of $k_{\mathcal{Y}}^*$ we only need to consider 4×4 matrix with ordered basis $\{H_{\mathcal{X}}, Q, P_{n-2}, P_n\}$ and the spectral radius is given by the largest root of $x^3 - nx^2 - (n+1)x - 1$. ◻

6 Mappings with $p \leqslant q$

Now we consider the case $F(w) = P(w)/\prod_{\ell=1}^{q}(w - \beta_\ell)$, where the degree of P is no greater than q. In this case we have a limit $\lambda_0 = \lim_{w\to\infty} F(w)$, and $\lambda_0 \neq 0$ if $p = q$, and $\lambda_0 = 0$ if $p < q$. We see that $\mathcal{E}(j_F) = \bigcup_{\ell=1}^{q}\{y = \beta_\ell\}$. In case $p \leqslant q$, we have

$$\mathcal{E}(k_F) = C_2 \cup C_3 \cup C_4 \cup \bigcup_{\ell=1}^{q} D_\ell$$

where $D_\ell = \iota\{y = \beta_\ell\}$. This is different from the previous case (in Section 2) in several ways: (1) C_1 is no longer exceptional; (2) C_2 is mapped to e_2 instead of e_1, and (3) D_ℓ, $1 \leqslant \ell \leqslant q$, are exceptional.

As before, we start by blowing up e_1 to create an exceptional fiber E_1, and as before, C_3 maps to a point $q \in E_1$, which is indeterminate. So we also blow up q to obtain an exceptional fiber Q. Finally, we blow up the indeterminate point e_2 to create an exceptional fiber E_2. Figure 5b shows the mapping of exceptional curves under the induced map k at this stage. We note that the intersection

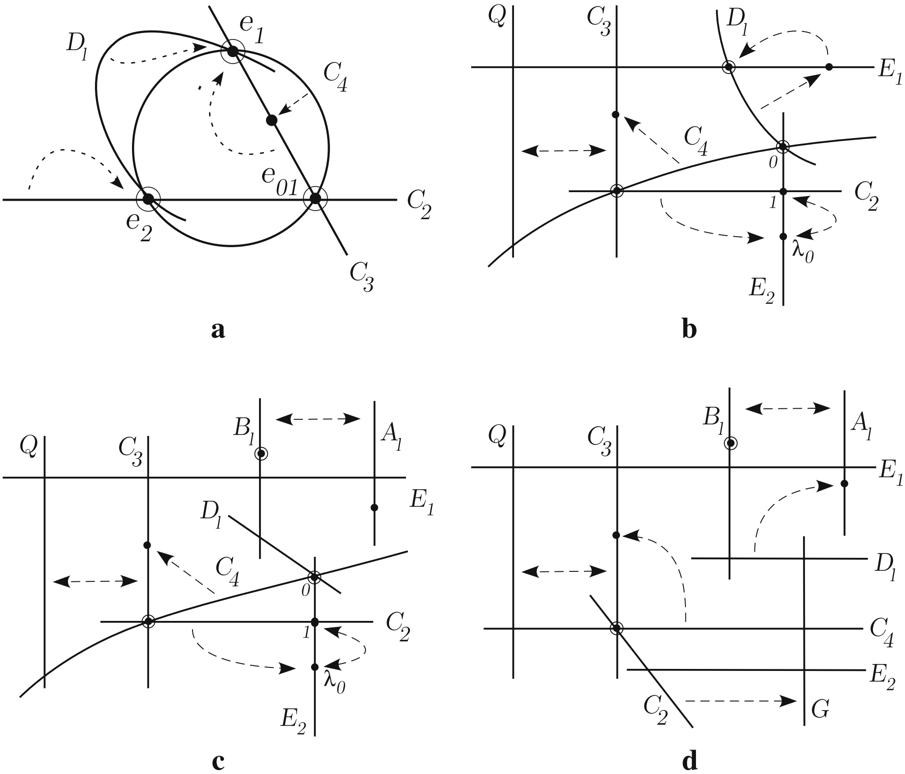


Fig. 5 **a** $p \leqslant q$. Exceptional curves and points of indeterminacy. **b** $p = q$. Mapping of the exceptional fibers after first blowups. **c** $p = q$. Mapping of the exceptional fibers at the second stage. **d** The case $p < q$

$E_2 \cap C_4 \cap D_\ell$ is indeterminate, and it corresponds to fiber coordinate equal to zero. We have drawn the case $\lambda_0 \neq 0$, in which case the exceptional image of C_2 never encounters a point of indeterminacy because $\lambda_0 \leftrightarrow 1$ is a 2-cycle. As before, we see that the orbit of C_4 never encounters the point of indeterminacy $C_2 \cap C_3 \cap C_4$ if the parameters are generic. Specifically, as in Section 4, we need $F(0) \neq 2/m$ for any positive integer m.

It remains to look at the orbits of the curves D_ℓ. If we use the coordinate system $(s, \xi) \mapsto [s : 1 : s\xi]$ for E_1, then the induced map on E_1 maps to E_1 and is the involution $\xi \mapsto -(\xi + 1)$. The image of D_ℓ is then $\xi = \beta_\ell \in E_1$, which is mapped to $-(1 + \beta_\ell)$. On the other hand,

$$k_F^{-1} : \{y = \beta_\ell\} \to \{\xi = -(\beta_\ell + 1)\} \in E_1,$$

which means that $-(\beta_\ell + 1)$ is indeterminate for k_F. So now at the second stage, we blow up the points β_ℓ and $-(\beta_\ell + 1)$ in E_1, $1 \leqslant \ell \leqslant q$. The exceptional curves for the induced map are now C_4, C_2, and D_ℓ, $1 \leqslant \ell \leqslant q$. Figure 5c shows how the exceptional fibers map at this stage. We see that for generic parameters, these orbits do not meet the indeterminacy locus.

Now we will describe the behavior of k_F^* on the Picard group in terms of the ordered basis L, E_1, Q, E_2, A_ℓ, B_ℓ, $1 \leqslant \ell \leqslant q$. We see that k_F^* acts as:

$$E_1 \leftrightarrow E_1, \quad E_2 \leftrightarrow E_2, \quad Q \to C_3, \quad B_\ell \to A_\ell \to D_\ell + B_\ell. \tag{18}$$

To make use of (18) we must write C_3 and D_ℓ in terms of our basis. In $\mathbf{P}^2$, we have $L = C_3$. Now if we move "up," taking the pullback π^* as we make the various blowups, we find that at the second stage we have

$$L = C_3 + E_1 + Q + \sum_{\ell=1}^{q} (A_\ell + B_\ell),$$

and this gives us C_3 in terms of our basis. Similarly, we start with $D_\ell = 2L$ in $\mathbf{P}^2$ since D_ℓ has degree 2. After the first stage of blowups, we have

$$D_\ell + E_1 + E_2 + Q = 2L.$$

For the second stage of blowups, we see from Fig. 5b that one of the centers of blowup (the one that produces B_ℓ) belongs to both E_1 and D_ℓ. Thus we have an "extra" B_ℓ:

$$D_\ell + B_\ell + E_1 + E_2 + \sum_{s=1}^{q} (B_s + A_s) + Q = 2L.$$

This gives D_ℓ in terms of our basis.

Finally, we need to express $k_F^* L$ in terms of our basis. We have $\{k^{-1} L\} = (2q + 3)L$ in $\mathbf{P}^2$ because k_F has degree $2q + 3$. Now when we blow up e_1, for instance, we will obtain a fiber E_1 with multiplicity. To determine the multiplicity we work in local coordinates $(s, \eta) \to [s\eta : 1 : s]$ near $E_1 = \{s = 0\}$. We write a generic line as $L = \left\{\sum a_j x_j\right\}$, so $k^{-1}\left\{\sum a_j x_j\right\} = \left\{\sum a_j k_j = 0\right\}$. In the (s, η) coordinates, we have $\sum a_j k_j[s\eta : 1 : s] = s^{q+1}\varphi(s, \eta)$, where $\varphi(0, \eta)$ is not identically zero. Thus the multiplicity of E_1 is $q + 1$, or

$$\{k_F^{-1} L\} + (q + 1) E_1 = (2q + 3) L$$

at the next level. Repeating this argument for the various blowup fibers, we find

$$\{k^{-1} L\} + (q + 1) E_1 + (q + 1) E_2 + (q + 2) Q + \\ + \sum_\ell ((q + 1) A_\ell + (q + 2) B_\ell) = (2q + 3) L, \tag{19}$$

which gives us $\{k_F^{-1} L\}$ in terms of our basis.

Using (18–19), we may write k_F^* on the Picard group, and we find that its characteristic polynomial is $x^2 - (q + 1)x - (q + 1)$. This proves Theorem 1 in the case $p = q$.

Next we consider the case where $p < q$, which means that $\lambda_0 = 0$. Thus after the second stage of blowups, we see in Fig. 5c that C_2 now maps to the point of indeterminacy $0 \in E_2$. Now we blow up $0 \in E_2$, creating a new fiber G. We find that on our new manifold, the curve C_2 is no longer exceptional and maps onto G. Thus we add G to our ordered basis in the Picard group. The action of k_F^* is changed in the following ways. First, we now have

$$E_2 \to E_2, \quad G \to C_2 = L - E_2 - G.$$

Next, there is a change in D_ℓ. Since G was obtained by blowing up the (transversal) intersection point of D_ℓ and E_2, the expression $D_\ell = 2L - E_2 - \cdots$ is changed to $D_\ell = 2L - E_2 - 2G - \cdots$. Last, we subtract an extra $(2q + 2)G$ from $k_F^*(L)$. With this new expression for k_F^*, we obtain the characteristic polynomial

$$-(x - q - 1)(x - 1)^3(x + 1)x^{2q},$$

and this proves Theorem 1 in the case $p < q$.

If $n = 1$, we have $F(w) = aw + P(w)/Q(w)$, where $deg(P) \leqslant deg(Q)$. The situation is like what we have just done in this section, with the added fact that

$$C_2 \to [0 : a : 1] \to e_2 \in I(k_F).$$

We blow up $[0 : a : 1]$ and e_2, creating new fibers M and E_2. The induced map behaves like

$$C_2 \to * \in M \leftrightarrow E_2.$$

For generic parameters, the orbit of C_2 does not encounter the indeterminacy locus. To finish the proof, now, we go back and repeat the earlier parts of Section 6.

Proof of Theorem 1 In order to prove Theorem 1 in general, we first do Section 6, which covers the cases $n = 0$ and $n - 1$. If $n \geqslant 2$, we go back to Section 4 or Section 5, according to whether n is even or odd. The associated Picard group will be larger because of the iterated blowups over the point

$0 \in E_1$. However, the fibers arising from the iterated blowup are disjoint from the blowups in Section 6 and so they still map the same as in Sections 4, 5, and the multiplicities of the pullback of a general line are the same. This gives us k_F^* in this case. Finding the characteristic polynomial in this case gives us Theorem 1.

7 Degree 3: A Family of Automorphisms

Let us consider the 2 parameter family of maps $k = \jmath_F \circ \iota$ where $F(z) = az^3 + az^2 + bz + 2$ with $a \neq 0$. We consider the complex manifold $\pi_{\mathcal{Z}} : \mathcal{Z} \to \mathbf{P}^2$ obtained by blowing up 6 points $e_2, e_{01}, p_4, p_5, p_6, r$ in the complex manifold $\mathcal{Y}$ constructed in Section 5. As we construct the blowups, we will let $E_2, E_{01}, P_4, P_5, P_6$ and R denote the exceptional fibers over $e_2, e_{01}, p_4, p_5, p_6$, and r respectively. Specifically, we blow up e_2 and e_{01} and then:

$$p_4 := -1/a \in P_3, \quad p_5 := (2-b)/a \in P_4,$$
$$p_6 := (2b-2-a)/a^2 \in P_5, \quad \text{and} \quad r := 0 \in E_2 \cap \{x_1 = 0\}.$$

We define the local coordinate system in a similar way we define local coordinates in Section 3. Using these local coordinates we can easily verify that under the induced map $k_{\mathcal{Z}}$ we have

$$C_1 \to P_4 \to C_1, \; E_2 \to P_5 \to E_2, \; C_4 \to E_{01} \to C_4', \text{ and } C_2 \to P_6 \to R \to C_2'$$

and all mappings are dominant and holomorphic.

For example, let us consider E_2. We may use coordinates w, ζ which are mapped by $\pi_{E_2} : (x, \zeta) \to [w : w\zeta : 1] \in \mathbf{P}^2$. Thus $E_2 = (w = 0)$ is given by ζ-axis in this coordinate system and by considering $\lim_{w \to 0} \pi_{P_5}^{-1} \circ k \circ \pi_{E_2}(w, \zeta)$ we find:

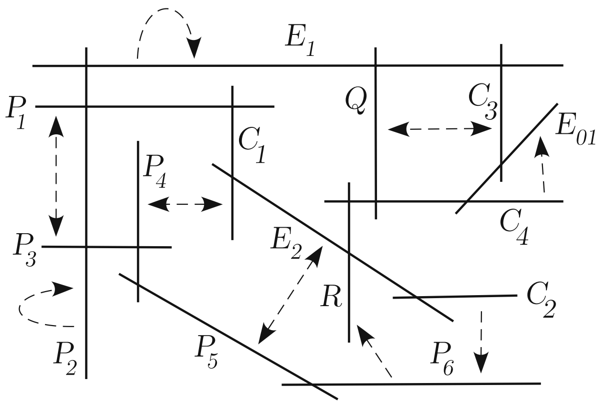
$$k_{\mathcal{Z}} \; : \; E_2 \ni \zeta \mapsto (2b - a - \zeta - 1)/a^2 \in P_5.$$

The mapping among the exceptional fibers is shown in Fig. 6. What is not shown is that $R \to C_2'$ and $E_{01} \to C_4'$

Theorem 6 *Suppose* $F(z) = az^3 + az^2 + bz + 2$ *with* $a \neq 0$. *Then the induced map* $k_{\mathcal{Z}}$ *is biholomorphic.*

Proof Since $k_{\mathcal{Z}}$ and $k_{\mathcal{Z}}^{-1}$ have no exceptional hypersurface, indeterminacy locus for $k_{\mathcal{Z}}$ is empty. It follows that $k_{\mathcal{Z}}$ is an automorphism of $\mathcal{Z}$. □

Fig. 6 The space $\mathcal{Z}$ and action of $f_{\mathcal{Z}}$



Repeating the argument in previous two sections, we have that $k^*_{\mathcal{Z}}$ acts on each basis element as follows:

$$
\begin{aligned}
H_{\mathcal{Z}} &\mapsto 7H_{\mathcal{Z}} - 3E_1 - 4P_1 - 8P_2 - 9P_3 - 10P_4 - 10P_5 - \\
&\qquad -10P_6 - 3E_2 - 6R - 4Q - 4E_{01}, \\
E_1 &\mapsto E_1, \quad P_1 \mapsto P_3 \mapsto P_1, \quad \text{and} \quad P_2 \mapsto P_2, \\
P_4 &\mapsto H_{\mathcal{Z}} - E_1 - 2P_1 - 3P_2 - 3P_3 - 3P_4 - 3P_5 - 3P_6 - E_2 - R - Q, \\
P_5 &\mapsto E_2, \quad P_6 \mapsto H_{\mathcal{Y}} - E_2 - R - E_{01}, \quad E_2 \mapsto P_5, \quad \text{and} \quad E_{01} \mapsto P_6, \\
Q &\mapsto H_{\mathcal{Z}} - E_1 - P_1 - 2P_2 - 2P_3 - 2P_4 - 2P_5 - 2P_6 - Q - E_{01}, \\
E_{01} &\mapsto 2H_{\mathcal{Z}} - E_1 - P_1 - 2P_2 - 2P_3 - 2P_4 - 2P_5 - 2P_6 - \\
&\qquad -E_2 - 2R - 2Q - E_{01}.
\end{aligned}
$$

Theorem 7 *Suppose $F(z) = az^3 + az^2 + bz + 2$ with $a \neq 0$. Then the degree of $k^n = k \circ \cdots \circ k$ grows quadratically, and k is integrable.*

Proof All the eigenvalues of the characteristic polynomial of $k^*_{\mathcal{Z}}$ have modulus one. The largest Jordan block in the matrix representation of $k^*_{\mathcal{Z}}$ is a 3×3 block corresponding to the eigenvalue 1. Thus the growth rate of the powers of the matrix is quadratic.

Integrability follows from more general results: Gizatullin [26] showed that if the growth rate is quadratic, then there is an invariant fibration by elliptic curves. In this case, we can give an explicit invariant. If we define $\phi = \phi_1/\phi_2$ to be the quotient of the following two polynomials;

$$
\begin{aligned}
\phi_1[x_0 : x_1 : x_2] &= x_0^2 x_2^2, \\
\phi_2[x_0 : x_1 : x_2] &= -2x_0^4 + 4x_0^3 x_1 - (2+a)x_0^2 x_1^2 + 2ax_1x_2^2(x_0 + x_2) - \\
&\qquad -2b\,(x_0^3 x_2 - x_0^2 x_1 x_2),
\end{aligned}
$$

then $\phi \circ k = \phi$. □

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