Annals of Combinatorics 3 (1999) 131-158

Annals of Combinatorics

Functional Relations in Lattice Statistical Mechanics, Enumerative Combinatorics, and Discrete Dynamical Systems

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Received November 11, 1998

AMS Subject Classification: 82A68, 82A69, 14E05, 14J50, 16A46, 16A24

Abstract. We recall some non-trivial, non-linear functional relations appearing in various domains of mathematics and physics, such as lattice statistical mechanics, quantum mechanics, or enumerative combinatorics. We focus, more particularly, on the analyticity properties of the solutions of these functional relations. We then consider discrete dynamical systems corresponding to birational transformations. The rational expressions for dynamical zeta functions obtained for a particular two-dimensional birational mapping, depending on two parameters, are recalled, as well as some non-trivial functional relations satisfied by these dynamical zeta functions. We finally give some functional equations corresponding to some singled out orbits of this two-dimensional birational mapping for particular values of the two parameters. This example shows that functional equations associated with curves, for real values of the variables, are actually compatible with a chaotic dynamical system.

Keywords: functional relations, inversion relations, Yang–Baxter equations, rational dynamical zeta functions, discrete dynamical systems, birational mappings, Cremona transformations, analyticity assumptions

1. Introduction

Functional equations emerge quite naturally in various domains of mathematical physics like lattice statistical mechanics, quantum mechanics, or enumerative combinatorics. They occur surprisingly in domains where one does not expect, at first sight, so much structure and constraints (anharmonic oscillator, cubic Ising model, etc.). In most cases, they are related to some deep mathematical structures, running from Rogers–Ramanujan identities to zeta functions. Let us recall briefly some miscellaneous examples which will make it clear that, despite the various domains of mathematical physics from which they originate, functional equations may present some remarkable common features.

Let us first recall in a purely *enumerative combinatorics* framework [79] some examples (given, for instance, by Polya [72, 73]) of functional equations on some associated enumeration generating functions. For instance, one can recall (see [73, p. 73]) the functional equation[1]:

$$1+\frac{x}{6}\cdot\Big(r(x)^3+3\cdot r(x)\cdot r(x^2)+r(x^3)\Big) = r(x) \tag{1.1}$$

or even a functional equation on a generating function of *two* variables (see [72, p. 44]):

$$\begin{aligned}\phi(x,y) &= 1+x\cdot\phi(x,y)\cdot\phi(x^2,y^2)\\ &\quad+\frac{xy}{6}\cdot\Big(\phi(x,y)^3-3\cdot\phi(x,y)\cdot\phi(x^2,y^2)+2\cdot\phi(x^3,y^3)\Big).\end{aligned} \tag{1.2}$$

Another simple functional equation of this type is

$$P(x) = (1-x)\cdot P(x^2). \tag{1.3}$$

This functional equation has a simple solution with a *natural frontier*[2] (which is the unit circle), namely,

$$\begin{aligned}P(x) = \prod_{n=0}^{n=\infty}(1-x^{2^n}) &= 1-x-x^2+x^3-x^4+x^5+x^6-x^7-x^8+x^9+x^{10}-x^{11}\\ &\quad+x^{12}-x^{13}-x^{14}+x^{15}-x^{16}+x^{17}+x^{18}-x^{19}+x^{20}\\ &\quad-x^{21}-x^{22}+x^{23}+x^{24}-x^{25}\cdots.\end{aligned} \tag{1.4}$$

In contrast with example (1.4), which is analytical inside the unit circle, many of such functional equations yield *divergent series* which may or may not be Borel summable [83]. In this enumerative framework, the occurrence of *infinite products* is often mentioned [72]. At this point, one can also recall the relation [39]:

$$t(x) = \sum_{n=1}^{\infty} t_n\cdot x^n = x\cdot\exp\Big(\sum_{n=1}^{\infty} t(x^n)/n\Big) = x\cdot\prod_{n=1}^{n=\infty}(1-x^n)^{-t_n}, \tag{1.5}$$

where t_n enumerates the number of rooted unlabeled trees on n vertices (see [72, p. 105]). These infinite products and their associated analytical properties have to be compared with the ones that will be recalled, as seen below, in Section 3 on dynamical zeta functions and their associated Weil product decomposition (see (3.15) below).

Thus, in the framework of enumerative combinatorics, generating functions [92] and functional equations on these generating functions occur naturally. However, one may argue that these "tree-like Polya-enumerations" often yield functional equations

[1] On a generating function counting the number of trees with k carbon atoms [73].

[2] One should not be prejudiced that a function with a natural frontier is necessarily a function with very involved analytical properties. The so-called Chazy III differential equation is an example of a differential equation having the Painlevé property and solutions with natural frontiers [40, 41].

with some "dilatation" symmetry, $x \to x^2$, possibly yielding functional equations with natural frontiers (see (1.3)), while the functional equations corresponding to graph enumerations on euclidean lattices (see (1.6) and (2.8) below) yield functional equations associated with a *shift* of some spectral parameter ($\theta \to \theta + \lambda$, or multiplicatively $x \to \lambda \cdot x$; see (1.6) and (2.12) below). It is clear that enumeration problems on trees and euclidean lattices (square, triangular, cubic, etc.; see below) are very different. One does not expect the same "growth" of the number of graphs[3], and thus, the growth of the coefficients of the associated generating functions will be quite different. Consequently, one may expect that the analytical properties of the corresponding generating functions could be very different (natural frontiers, essential singularities, Darboux singularities, or confluent singularities, in several complex variables, etc.).

Let us now recall some non-linear functional equations emerging quite naturally in *lattice statistical mechanics*. Firstly, in the solution of the hard hexagon model [20], the partition function per site has been seen to verify an exact functional relation as (with some well-suited normalizations)

$$Z(u) \cdot Z(u+\eta) = 1 + Z(u-2\eta), \tag{1.6}$$

where $Z(u)$ has some periodicity property, namely, $Z(u+5\eta) = Z(u)$. The solution of the hard hexagon model happens to be some eulerian product-like solution, and thus, functional equation (1.6) can just be interpreted as a Rogers–Ramanujan-like identity on an infinite eulerian product (see also all the Rogers–Ramanujan-like identities[4] associated with the so-called RSOS-models [9] and the related q-series [8]).

It is important to note that functional equation (1.6) is not restricted to the partition function per site of the model (largest eigenvalue of the transfer matrix) only. There actually exists a functional relation on the *transfer matrix* exactly similar to relation (1.6) (with suitable normalizations [20], just replace $Z(u)$ by $T(u)$). In this spirit, one should also mention the *fusion hierarchies* and related functional equations [93]. It has been shown that the "fused transfer matrices" satisfy, for some periodic boundary conditions, functional equations. Of particular interest are the fused transfer matrices $T^{(q,r)}(u)$ corresponding to rectangular Young tableaux of q rows and r columns, which verify a whole hierarchy of functional equations:

$$\begin{aligned} &T^{(q,r)}(u) \cdot T^{(q,r)}(u-\eta) \\ &= T^{(q+1,r)}(u) \cdot T^{(q-1,r)}(u-\eta) + T^{(q,r+1)}(u) \cdot T^{(q,r-1)}(u-\eta). \end{aligned} \tag{1.7}$$

In another context, let us mention that functional relation (1.6) actually identifies with a functional relation bearing on the *Stokes multipliers* [38, 84, 85] of the following irregular differential equation (see, for instance, [84, 85]):

$$y'' - (x^3 + \lambda) \cdot y = 0 \tag{1.8}$$

which reads

$$f(\lambda) + f(\omega\lambda) \cdot f(\lambda/\omega) = 1, \quad \text{where } \omega^5 = 1. \tag{1.9}$$

[3] The occurrence of loops in the counting of graphs in the euclidean lattices versus the absence of loops on trees is very similar to the "pruning rules" [45] in the framework of *symbolic dynamics* versus "unpruned" symbolic dynamics [45].

[4] See also the various bosonic versus fermionic representations for character formulas in [29].

Another example comes with the Jost functions of the anharmonic quantum oscillator, where Voros [91] has shown that the generating function of the spectrum of the anharmonic quartic oscillator, namely, the Fredholm determinant $\Delta(\lambda)$:

$$\Delta(\lambda) = \prod_{n=0}^{\infty}\Big(1 - \frac{\lambda}{\lambda_n}\Big), \tag{1.10}$$

where λ_n denotes the eigenvalues of the Hamiltonian, verifies the functional equation:

$$4 \cdot \Delta(\lambda) \cdot \Delta(\omega\lambda) \cdot \Delta(\omega^2\lambda) = \Delta(\lambda) + \Delta(\omega\lambda) + \Delta(\omega^2\lambda) + 1, \quad \text{where } \omega^3 = 1. \tag{1.11}$$

These functions are explicit examples of the so-called *resurgent functions* of Ecalle [47, 91].

One thus sees that similar (or exactly the *same*) functional equations in one complex variable may have simple eulerian product-like solutions (and they can be interpreted as Rogers–Ramanujan-like identities) or much more complicated functions (resurgent functions of Ecalle [47, 91], etc.) and then the interpretation corresponds to see these functional equations as one of the functional equations satisfied by zeta functions (for instance, of an anharmonic quantum oscillator [91]), the analytical structure of the zeta functions being *much more complex*.

Let us also point out some differential equations leading to "special functions" like the transcendental *Painlevé functions* [64, 70, 82]. Let us recall the fact that Painlevé differential equations can be seen as *self-similar* reductions of integrable PDE's (KdV equations, etc.) [60]. Such "self-similar" reductions can even be generalized to *discrete*-Painlevé equations. *Discrete*-Painlevé equations can be seen as (highly non-trivial discrete) reductions of discrete-KdV equations (see the work of Nijhoff et al. [51, 67–69]). The iteration of some discrete Painlevé recursions can be seen to yield *not curves*, as in integrable mappings (see, for instance, [32]) but *surfaces*. Actually, if one considers the Arnold complexity [11] of these discrete Painlevé mappings, it can be seen that one has a *polynomial growth* of the iteration calculations[5].

From these examples, it is tempting to segregate, on one side, "nice functions" often corresponding to *elliptic functions*, or to Abelian varieties (theta functions of several variables, etc.) or even corresponding to "nice" transcendental functions like the Painlevé equations [64, 70, 82], etc. and, on the other side, chaotic systems where no exact results can be found, even functional equations. The "frontier" between these two worlds could be the existence of non-linear, non-trivial functional equations such as (1.6), the question being to know to which world (the "nice" world or the "chaotic" one) analytically involved functions like the *resurgent* functions [47], like (1.11) belong to.

Considering non-linear, non-trivial functional equations as a possible "frontier" seems well-suited because this gives a large enough framework to work with (imposing the existence of differential equations, or PDE's, with an additional structure, or

[5] We think that this *polynomial growth* property [31, 33] is not a consequence of the discrete Painlevé equations themselves, but rather the underlying integrable discrete-KdV equation (existence of Lax pairs, ...) and of the associated Abelian varieties, the polynomial growth being inherited from these Abelian varieties [31].

other differential structures[6], would restrict too much the set of "nice" functions [7]). Beyond, the question of the *analyticity properties* of the solutions (if any) of functional equations is an extremely difficult one [84]: A functional equation, as in (1.9), can correspond to solutions of drastically different analytical properties, running from simple elliptic functions [20] (see (1.6)) to resurgent functions [91]. Functional equations thus provide a very interesting framework, large enough to be able to find many results and constraintfull enough to get many highly non-trivial results.

Functional equations clearly yield a lot of "structure" and constraints but they may also yield very complicated analytical properties. The analyticity properties of resurgent functions is in general so involved, even when they satisfy simple functional equations like (1.11), that one may think that a segregation between a nice world and a chaotic one is not very well-defined. In fact, at least in the framework of *dynamical systems* (and especially *discrete dynamical systems*), this separation is clear and not "fuzzy". The *growth complexity* [1, 2, 89] of the calculations (or for two-dimensional mappings, the *Arnold complexity* [11]), or the *topological entropy* [1,2,6,11], of a system are actually a well-suited way of segregating between the "nice" world (polynomial growth) and the "chaotic one". A non-zero topological entropy means that the system is chaotic [1,2] and thus, no simple analytical property should be expected at first sight.

Therefore, we will try in this paper to address, with a special emphasis on a "complexity growth point of view"[8], the previously raised questions: Does the existence of a functional equation means that the system belongs to the "nice world"[9]? Does chaos automatically mean that no functional equations can be found for the system? To which world does the *resurgent* functions of Ecalle [47, 91] belong when they actually satisfy a simple functional equation like (1.11) ?

In this paper, we will thus try to address the question of a possible "functional equation frontier" between a "nice world" and a chaotic one, and the related question of the *analyticity properties* of the solutions of functional equations as follows: We will first recall several known results of lattice statistical mechanics and graph enumerative combinatorics, which clearly belong to the "nice world" (and beyond; see (2.8), etc.). We will then consider *discrete dynamical systems* and introduce some new results [1–3] on *birational transformations*, for which exact expressions of the *dynamical zeta function* for a particular family of birational transformations of two variables, depending on two parameters, have been conjectured. This will provide a new set of functional equations with other analyticity properties (simple rationality). Finally, considering a particular case of the previous two-dimensional examples ($\alpha = 0$ and $\varepsilon = 3$), we will also provide an example of the *"non-algebraic integrability* of a reversible dynamical system of the Cremona type" as introduced by Rerikh [76]. This $\varepsilon = 3$ example will actually be seen to correspond to solutions of non-linear functional equations. This will

[6] Let us recall that finite difference functional equations are not like differential equations; even the simplest theorems of existence of solutions do not exist most of the time.

[7] We would return to the old idea of getting archives of "special functions" to solve problems.

[8] See also the rate of growth of groups and the growth of graphs and of Riemannian manifolds, for instance, in [42].

[9] The answer to this question drastically depends on the framework one considers: functional equations bearing on functions of *several* complex variables, or of *one* complex variables, etc.; see the conclusion of this paper.

allow us to address the following question: Does "transcendental integrability"[10] exist?

These last discrete dynamical examples will shed some light on the "amount" of structures, constraints, and analyticity properties that come, or may come, automatically with a functional equation. In particular, it will be seen that it is crucial to segregate between the analyticity properties of a function seen as a function of *one (or several) complex* variable(s) and the analyticity properties on this function seen as a function of one (or several) *real* variable(s).

The first part of our paper is a review of functional relations occurring in various domains of mathematical physics. Some of these results are obtained by those involved in lattice statistical mechanics, other results are known by researchers involved in enumerative combinatorics, or in field theory or particle physics, etc. A first motivation of our paper is to put together these various functional relations in order to see the emergence of some common universal features. Beyond this preliminary review, we will go a step further, in the framework of *discrete dynamical systems*, and provide some new results, in particular relations (4.12) and (4.13) in Section 4.

2. Functional Equations Emerging from the Inversion Relation

The *inversion trick* [15,87] is known to give a fantastic short cut to calculate exactly the partition functions per site of several two-dimensional lattice models (or to obtain the exact S-matrix from the unitarity relation and crossing symmetry in S-matrix theory). Basically, one has to combine two very simple functional relations:

$$S(\theta)\cdot S(-\theta) = K_{\mathrm{known}}(\theta) = F(\theta)\cdot F(-\theta), \quad \text{and:} \quad S(\theta) = S(\lambda-\theta), \qquad (2.1)$$

where $K_{\mathrm{known}}(\theta)$ denotes some known and simple expression (often explicitly of the form $F(\theta)\cdot F(-\theta)$, where $F(\theta)$ is known). Naively, as a first approximation, a simple solution of the first equation is $S(\theta) = F(\theta)$, but it does not satisfy the second equation $S(\theta) = S(\lambda-\theta)$. It is easy to satisfy the second one, writing the solution as $S(\theta) = F(\theta)\cdot F(\lambda-\theta)$, but now the first equation is no longer satisfied. Again, the first equation can be verified by dividing by some "counterterms" and so on at every step of some iterative process which will give $S(\theta)$ as an infinite product (over a group generated by the two involutions $\theta \to \lambda-\theta$ and $\theta \to -\theta$), satisfying the two functional equations. This solution is called the "minimal solution" since it corresponds to the minimal compulsory set of poles and zeroes of any solution of (2.1). Other solutions have more poles and zeroes and, more generally, more involved analytical behaviors. However, if one assumes this minimal analyticity assumption (only this compulsory set of poles and zeroes), the solution is unique.

2.1. Functional Equations Emerging from the Inversion Relation Beyond the Yang-Baxter Equations Framework

Let us first consider the *anisotropic* Ising model on a square lattice. There actually exists an *inversion relation* [15, 16] on this two-dimensional model:

$$Z(K_1, K_2)\cdot Z(-K_1, K_2+i\pi/2) = 2i\cdot \sinh(2K_2). \qquad (2.2)$$

[10] For instance, for two variables, one would have a foliation of the plane in curves that are *not algebraic*, these transcendental curves being the orbits of the iteration of some transformation [76].

Of course, one has

$$Z(K_1, K_2) = Z(K_2, K_1), \tag{2.3}$$

a consequence of the permutation symmetry of the two coupling constants. It has been shown by Baxter in [16] that the partition function per site *can actually be calculated order by order*, using some "resummed" high-temperature expansions from the functional equations (2.2) and (2.3), together with an analyticity assumption on the "resummed" high-temperature expansions, namely, that *only* $\tanh^2(K_1) = 1$ poles occur in these expansions.

For that purpose, it is convenient to define the *reduced* (high-temperature normalized) partition function per site by

$$\Lambda(t_1, t_2) = \frac{Z(K_1, K_2)}{2\cosh K_1 \cosh K_2} \quad \text{where } t_i = \tanh K_i, \quad i = 1, 2. \tag{2.4}$$

The reduced partition function then satisfies the inversion relation [16]:

$$\ln\Lambda(t_1, t_2) + \ln\Lambda(1/t_1, -t_2) = \ln(1 - t_2^2). \tag{2.5}$$

Actually, writing the (high-temperature normalized) partition function $\Lambda(t_1, t_2)$ as a resummed expansion:

$$\ln\Lambda(t_1, t_2) = \sum_{n,m} a_{n,m} \cdot t_1^{2m} t_2^{2n} = \sum_n R_n(t_1^2) \cdot t_2^{2n}. \tag{2.6}$$

Baxter [16] has shown that

$$R_n(t_1^2) = P_{2n-1}(t_1^2)/(1 - t_1^2)^{2n-1}. \tag{2.7}$$

The functions R_n are rational, with numerator and denominator of degree $2n - 1$ in t_1^2. The denominator has only a simple pole of degree $2n - 1$ at $t_1^2 = 1$ in the complex t_1^2 plane. This analyticity property (only $t_1^2 = 1$ poles) is closely related to the star-triangle integrability of the model [19].

One must point out that this is *another kind* of analyticity assumption. It is not an analyticity assumption on some complex "spectral" parameter θ corresponding to an elliptic parametrization of the model (see the minimal analyticity assumption in θ above), but on *one of the two* high temperature variables of this anisotropic model. Although these two sets of analyticity assumptions are quite different in nature, they both achieve the goal of calculating exactly the partition function per site [16, 62].

Let us consider the *anisotropic* standard scalar q-state Potts model on a square lattice. There actually exists an inversion relation[11] on this two-dimensional model [56]:

$$Z(b, c) \cdot Z(1/b, 2 - q - c) = (c - 1) \cdot (1 - q - c), \tag{2.8}$$

where b and c denote the exponential of the coupling constants of the anisotropic two-dimensional Potts model on a square lattice. Of course, one has the following simple obvious functional equation inherited from the geometrical symmetries:

$$Z(b, c) = Z(c, b). \tag{2.9}$$

[11] Even *beyond* the integrable star-triangle critical framework [56].

The anisotropic standard scalar q-state Potts model is not generically Yang–Baxter integrable (except at the first order transition point, for $q > 4$, or at the second order critical point, for $0 < q < 4$). At these critical points

$$(b-1)\cdot(c-1) = q \quad \text{or} \quad (b+1)\cdot(c+1) = 4-q, \tag{2.10}$$

the inversion trick gives the correct expression of the partition function per site [56,57].

These results can be generalized to the checkerboard standard scalar Potts model [74]. Instead of the previous two variables, one now has *four* variables: a, b, c and d. Let us introduce [56,57] a rational parametrization of the model:

$$\begin{aligned} \frac{a-1}{a+q-1} &= t\cdot\frac{u-t}{1-t^3u}, & \frac{b-1}{b+q-1} &= t\cdot\frac{v-t}{1-t^3v}, \\ \frac{c-1}{c+q-1} &= t\cdot\frac{w-t}{1-t^3w}, & \frac{d-1}{d+q-1} &= t\cdot\frac{z-t}{1-t^3z}, \end{aligned} \tag{2.11}$$

where t is one of the roots of: $t^4-(q-2)\cdot t^2+1=0$. With these notations, the critical condition of the square lattice now reads $uvwz = 1$ and the inversion relation (2.8) becomes

$$\begin{aligned} &Z(u,v,w,z)\cdot Z\Big(\frac{1}{t^2u},\frac{t^2}{v},\frac{1}{t^2w},\frac{t^2}{z}\Big) \\ &= \frac{1+t^2}{t^2}\cdot\Big(\frac{(1-u/t)\cdot(1-t^3u)}{(1-tu)^2}\cdot\frac{(1-w/t)\cdot(1-t^3w)}{(1-tw)^2}\Big)^{1/2}. \end{aligned} \tag{2.12}$$

Combined with the obvious geometrical symmetry C_{4v} of the square lattice:

$$Z(u,v,w,z) = Z(v,u,z,w) = \cdots, \tag{2.13}$$

the "minimal" solution of the partition function per site can be written as

$$\Big(Z(u,v,w,z)\Big)^2 = \frac{q}{t^2}\cdot\frac{F(u)F(1/u)}{1-tu}\cdot\frac{F(v)F(1/v)}{1-tv}\cdot\frac{F(w)F(1/w)}{1-tw}\cdot\frac{F(z)F(1/z)}{1-tz}, \tag{2.14}$$

where $F(u)$ reads

$$F(z) = \prod_{n=1}^{\infty}\frac{1-t^{4n-1}z}{1-t^{4n+1}z}. \tag{2.15}$$

Functional equations (2.12) and (2.13) are exactly similar to (2.1). It is straightforward to verify that the "minimal" solution (2.14) actually verifies the inversion relation for the checkerboard model (2.12) and the C_{4v} symmetry of the square lattice (2.13) (this solution has a larger set of symmetries, namely, it is invariant with respect to S_4, the group of permutation of the four variables a, b, c, d). This solution has all the poles singularities inherited from the right-hand side of the inversion relation (2.12) and from the action of the group generated by the inversion relation and symmetry of the square C_{4v}, and it has *only* this minimal set of singularities. It can be called the "minimal" solution since it has all the "compulsory" singularities and *only* these ones. However, this "minimal" solution happens to be the correct one *only* on the critical condition $uvwz = 1$. *Beyond this critical condition* $uvwz = 1$ (which means in fact that the model is *no longer* Yang-Baxter integrable), the inversion relation (2.12) (or (2.8)) is *still valid* [56], but the "minimal solution" *does not yield* the correct partition function per site [57]. This can be seen, quite clearly, on the large q expansion of the standard scalar q-state Potts model on an anisotropic square lattice [57], as well as on the checkerboard lattice [74]. These large q expansions enable us to see very clearly that, beyond criticality, the "true" partition function per site, which actually verifies the functional equations (2.12) and (2.13), is a *much more involved* function of the four variables than the minimal solution (2.14). What are the analyticity properties of this function? This question is raised here in *more than one* complex variable. For instance, recalling the minimal solution (2.14), one immediately sees that the logarithm of this expression is the sum of the same function of u, v, w, and z, while the large q expansion of the actual partition function per site gives an expression *mixing* these four variables. The partition function can be expressed in terms of some normalized high-temperature partition function $\ln \Lambda(u, v, w, z)$ as the following expansion [74]:

$$\begin{aligned}\ln \Lambda(u, v, w, z) &= t^2 - \frac{t^3}{2} \cdot (vwz + uwz + uvz + uvw) - \frac{t^4}{2} \\ &\quad + \frac{t^5}{2} \cdot (u + v + w + z + vwz + uwz + uvz + uvw) \\ &\quad + t^6 \cdot \left(\frac{1}{3} - \frac{1}{4} \cdot (v^2w^2z^2 + u^2w^2z^2 + u^2v^2z^2 + u^2v^2w^2)\right) + \cdots . \end{aligned} \tag{2.16}$$

Functional relations (2.12) and (2.13) mean that the partition function per site can be seen as a *generalization to several complex variables of automorphic functions* [71]. However, automorphic functions of *several complex variables* are much more complicated (multivalued) functions [58] than automorphic functions of one variable[12]. What kind of analyticity in four variables are we discovering here with expansion (2.16), mixing the variables u, v, w and z ? Does it simply amount to adding some branch cuts to the simple (infinite product of) poles and zeroes (see (2.14))? Does the "true partition function" have some kinds of complicated confluent singularities in these four variables? Does it correspond to even more complicated analytical structures? To characterize the analyticity properties of this function of *several complex variables* is a puzzling question.

In fact, there is nothing specific with two-dimensional lattice models. Let us consider the anisotropic Ising model on a *cubic* lattice. There actually exists an *inversion relation* on this three-dimensional model [59]:

$$Z(K_1, K_2, K_3) \cdot Z(-K_1, K_2 + i\pi/2, -K_3) = 2i \cdot \sinh(2K_2). \tag{2.17}$$

Of course, one has the permutation of the three coupling constants symmetry:

$$Z(K_1, K_2, K_3 = Z(K_2, K_1, K_3) = Z(K_1, K_3, K_2) = \cdots . \tag{2.18}$$

[12] See the theta-Fuchsian series of Poincaré, or simply Poincaré series [71].

One can actually check this inversion relation on this three-dimensional model through resummed high temperature expansions [54], introducing again the (high-temperature normalized) partition function per site:

$$\ln\Lambda(t_1, t_2, t_3) = \ln\Big(\frac{Z(K_1, K_2, K_3)}{2\cosh K_1 \cosh K_2 \cosh K_3}\Big) = \sum_{n,m} R_{n,m}(t_1^2)\cdot t_2^{2n}\cdot t_3^{2m}. \quad (2.19)$$

The inverse functional relation reads [59]:

$$\ln\Lambda(t_1, t_2, t_3) + \ln\Lambda(1/t_1, -t_2, -t_3) = \ln(1-t_2^2) + \ln(1-t_3^2). \quad (2.20)$$

These calculations show that, combining the functional equations (2.17) and (2.18) yield drastic constraints on the "resummed high temperature" expansions (2.19) but, unfortunately, *insufficient* to get the partition function per site of the cubic Ising model order by order [54]. These constraints however are extremely precious to build and check high temperature series [54]. In order to get, order by order, this resummed expansion (2.19), one has to "inject" at each order, some information (namely, the coefficients of $t_1^{N_1}\cdot t_2^{N_2}\cdot t_3^{N_3}$ in the standard *anisotropic* high-temperature expansion of the cubic Ising model [54], for some N_1, N_2, N_3). Instead of finding out an infinite number of coefficients to obtain the rational functions $R_{n,m}(t_1^2)$ in the resummed expansion (2.19), one just needs to provide, because of the functional equations (2.18) and (2.20), a *finite number* of coefficients at each order. The number of these coefficients, which corresponds to the "*missing information*", grows exponentially, like μ^N with the order $N = n+m$ (compare this *exponential growth* of the computing time for a *non-integrable* model, like the three-dimensional Ising model, with the *polynomial time* required for an *integrable* model; see Subsection 2.3 below). This μ is the exact equivalent of the growth complexity λ (or exponential of the topological entropy [1,2]) that we will introduce in Subsection 3.1 for discrete dynamical systems. It would be interesting to compare this "growth of the missing information", characterized by μ, for the two-dimensional non-critical Potts model and for the three-dimensional Ising model.

Performing calculations in the high-temperature variables, t_i's, is the only strategy we have at our disposal for the cubic Ising model since we do not have any "canonical foliation" of the parameter space in algebraic curves which would enable us to consider analyticity properties in *one* complex variable θ. Even when one has such a nice foliation of the parameter space in (elliptic or rational) curves, it is difficult to compare a minimal analyticity assumption in the "spectral parameter" θ and the analyticity assumption of having only $(1-t_1^2)$ poles in the resummed high temperature expansion. However, recalling the Baxter model, it has been shown [88] that one actually has only $(1-t_1^2)$ poles in the resummed expansion; it seems that Yang–Baxter integrability often provides both analyticity properties in θ and t_1, each of them being sufficient to determine accurately the partition function. However, only the inversion trick in the spectral variable θ provides a closed (infinite product) formula for the partition function per site of the Baxter model [62].

One can even write similar "inversion" relations for an anisotropic three-dimensional standard scalar q-state Potts model in a magnetic field on a cubic lattice:

$$Z(a,b,c,h)\cdot Z(1/a, 1/b, 2-q-c, 1/h) = (c-1)\cdot(1-q-c), \quad (2.21)$$

$$Z(a,b,c,h) = Z(b,a,c,h) = Z(a,c,b,h) = \cdots. \quad (2.22)$$

Again, one has the rational parametrization (2.11) but, this time, on a, b and c only. Of course, again, the resummed high temperature expansion cannot be obtained order by order from (2.21) and (2.22). One notes that, at this level, one cannot make any difference between the functional equations verified by the partition function per site on the *cubic* lattice and on the *triangular* lattice! When there is no magnetic field ($h = 1$), the functional equations (2.21) and (2.22) give a minimal solution which can be seen as a (triangular) limit of (the checkerboard) (2.14). This minimal solution is actually the correct one for the anisotropic triangular Potts model at criticality but is, of course, far from encapsulating the "complexity" of the cubic Potts model.

2.2. Inversion Trick without the Yang–Baxter Integrability: The Sixteen Vertex Model

Let us recall that the *sixteen vertex* model [61] presents a *canonical foliation* of its parameter space CP_{15} in terms of *elliptic curves*. These elliptic curves are obtained from the two inversion relations of the model. Their equations have been written down in terms of intersections of quadrics [23].

For this very model, one does not have, generically, a Yang–Baxter integrability (except when the sixteen vertex model reduces to the Baxter or a free-fermion model). However, one can certainly use the fact of having such a "canonical foliation" of the parameter space CP_{15} in terms of elliptic curves (which actually corresponds to true symmetries of the model) to introduce a canonical "spectral parameter". On each elliptic curve of CP_{15}, generated from the two inversion relations [23], one can actually write down two functional equations corresponding to these two involutions and "play" with the "inversion trick":

$$Z_E(\theta) \cdot Z_E(-\theta) = K^{(1)}_{\text{known}}(\theta) = F^{(1)}(\theta) \cdot F^{(1)}(-\theta),$$

$$Z_E(\theta) \cdot Z_E(\lambda - \theta) = K^{(2)}_{\text{known}}(\theta) = F^{(2)}(\theta) \cdot F^{(2)}(\lambda - \theta), \qquad (2.23)$$

where $K^{(1)}_{\text{known}}(\theta)$ and $K^{(2)}_{\text{known}}(\theta)$ are some known expressions (corresponding to determinants of the R-matrix).

It may well be that the "minimal" solution of the two functional equations (2.23) gives the correct partition function per site for the sixteen vertex model! It is clearly the case when one restricts to the (Yang–Baxter integrable) Baxter model [62]. More generally, the analyticity assumptions, required to show that the "minimal" solution of the two functional equations is the "correct one", are a consequence of the Yang-Baxter structure[13]. It would be tempting to compare the minimal solution, obtained for the sixteen vertex model (from the inversion trick *without* the Yang–Baxter integrability), with some expansion (weak-graph expansions [49]) for the exact partition function per site of the sixteen-vertex model, and see if they coincide[14]. If they do not, one could

[13] More precisely, the entries of the (row-to-row) transfer matrix are, for any size of the lattice, polynomial expressions of the entries of the R-matrix. If one has an elliptic foliation of the parameter space of the model (namely, the entries of the R-matrix), one can write the entries of the R-matrix as analytical (elliptic) functions of some "spectral" parameter θ. Therefore, the *entries* of the transfer matrix are analytical functions of θ, *even when* the Yang–Baxter equations are not satisfied. The analyticity of the *eigenvalues* of the transfer matrix are, however, a consequence of the Yang–Baxter equations (see, for instance, [17]).

[14] One may argue that if we were able to do that, we would recover the partition function of the two-dimensional Ising model *in a magnetic field*, which is known to be a subcase of the sixteen vertex models [61]. This is not true as the two-dimensional Ising model in a magnetic field is a *highly singular limit* of the sixteen-vertex model (rank two R matrices).

then try to characterize the analyticity properties in one complex variable θ of the "true" partition function per site. Unfortunately, weak-graph expansions of vertex models with 16 homogeneous parameters are far from being available in the literature.

2.3. Inversion Relations on the Generating Functions Corresponding to Various Graph Enumerations

Guttmann et al. [53,75] have applied this functional equation approach to study various combinatorial problems. They were able to write some functional equations on the generating functions corresponds to various problems of enumerative combinatorics. One set of functional equations just corresponds to the simple geometrical symmetries of the lattice and yields simple equations like $G(x, y) = G(y, x)$. The other corresponds to a non-trivial functional equation *exactly similar* to the inversion relations previously described. One problem here is that one finds out the existence of such inversion-like relations without being able to explain them from simple properties of local objects[15]. However, this functional equations approach, together with analytical properties like the previous ones, provides an alternative method of solution in some cases. One example is the enumeration of *staircase polygons* on a square lattice, for which Guttmann et al. were able to write the perimeter generating function:

$$P(x,y) = \sum_{n,m} p_{n,m} \cdot x^{2n} y^{2m} = \sum_{m} H_m(x^2) \cdot y^{2m}, \tag{2.24}$$

where $H_m(x^2)$ is the generating function for staircase polygons with $2m$ vertical bonds. The generating function $P(x, y)$ verifies the "inverse" functional relation:

$$P(x,y) + x^2 \cdot P(1/x, y/x) = -y^2 \tag{2.25}$$

together with the (obvious) symmetry relation $P(x, y) = P(y, x)$. One can obtain the generating function $P(x, y)$ by calculating the $H_m(x^2)$ functions, order by order, in *polynomial time*.

As previously mentioned, the occurrence of the "inverse" functional relation (2.25) is rather obscure. Where does this "hidden inverse" functional relation come from? This problem is, in fact, very similar to the occurrence of the $t \to 1/t$ functional relations on zeta functions (see below).

3. Discrete Dynamical Systems: A Two-Dimensional Birational Mapping

Birational transformations [24–28] naturally "pop out" as non-trivial, non-linear symmetries of lattice models of statistical mechanics [23, 32–35]. They are built from the so-called *inversion relations* [56, 59, 87], and from geometrical lattice symmetries.

For simplicity, we will consider a particular birational transformation which can be reduced [1] (in a quite involved way) to a *two-dimensional* mapping (see Appendix A

[15] Like the inversion relations of the R-matrices, or on the IRF Boltzmann weights [32, 62]. These *local* relations yield the inversion relation of some row-to-row, or diagonal-to-diagonal, transfer matrices and, in a last step, the inversion relation on the partition function per site [15, 62].

of [1]). More precisely, let us consider the following family of birational mappings [1,2] $k_{\alpha,\varepsilon}$ depending on two parameters (α and ε):

$$k_{\alpha,\varepsilon}: \quad (u_{n+1}, v_{n+1}) = \Big(1 - u_n + \frac{u_n}{v_n}\varepsilon + v_n - v_n/u_n + \alpha\cdot(1 - u_n + u_n/v_n)\Big). \quad (3.1)$$

As far as complexity calculations are concerned, the $\alpha = 0$ case is singled out [30] as will be seen later. In that case, a convenient change of variables [1] leads to a mapping k_ε having a very simple form:

$$k_\varepsilon: \quad (y, z) \longrightarrow \big(z + 1 - \varepsilon, y\cdot\frac{z-\varepsilon}{z+1}\big). \quad (3.2)$$

The two transformations $k_{\alpha,\varepsilon}$ and k_ε derive [33] from a transformation K_q acting on a $q\times q$ matrices M, ($q \geq 3$) such that $K_q = t\circ I$, where t permutes the two entries $M_{1,2}$ with $M_{3,2}$, and I is the homogeneous matrix inversion $I(M) = \det(M)\cdot M^{-1}$. Transformations of this type, generated by the composition of permutations of the entries and matrix inversions, are actually symmetries of the parameter space of lattice statistical models [32].

3.1. Complexity Growth

The correspondence between transformations K_q and $k_{\alpha,\varepsilon}$, more specifically between transformations K_q^2 and $k_{\alpha,\varepsilon}$, is given in [1]. It will be shown below that, beyond this correspondence, transformations K_q^2 and $k_{\alpha,\varepsilon}$ share properties concerning their "complexities". Transformation K_q is homogeneous of degree $(q-1)$ in the q^2 homogeneous entries. When performing the nth iteration, one expects, at first sight, a growth of the degree of each entries as $(q-1)^n$. It turns out that, at each step of the iteration, some factorization of all the entries occurs. The common factor can then be factorized out in each entry leading to a "reduced" matrix M_n, which is taken as the representative of the nth iterate in the projective space.

To keep track of this *growth of the calculations* (see also [65,89,90]), it is useful to define some "degree generating functions" $G(x)$:

$$G(x) = \sum_n d_n\cdot x^n, \quad (3.3)$$

where d_n is the degree of some quantity, at each iteration step (entries of the "reduced" matrices M_n's, extracted polynomials f_n's, etc.; see [32,34,35]).

Due to these factorizations, the growth of the calculations is not $(q-1)^n$ but rather λ^n, where λ will be called the "complexity growth", or simply, the "complexity". Actually, one discovers, for such birational transformations ($K_q = t\circ I$), the occurrence of a *stable factorization scheme* (see [1,31]) yielding a *rationality* of these degree generating functions with integer coefficients [33]. This growth complexity λ is the inverse of the pole of smallest modulus of any of these degree generating functions $G(x)$:

$$\log\lambda = \lim_{m\to\infty}\frac{\log d_m}{m}. \quad (3.4)$$

This rationality yields *algebraic values* for the complexity λ [33]. For $K_q = t \circ I$, where t permutes the two entries $M_{1,2}$ with $M_{3,2}$, the results show that λ is the largest root of $1+\lambda^2-\lambda^3=0$ (i.e., $1.4655... < q-1$) [33, 35].

The same calculations have also been performed on transformations $k_{\alpha,\varepsilon}$ (see (3.1)). In that case, factorizations also occur at each step, and generating functions can be calculated. These generating functions are, of course, different from the generating functions for K_q^2 (see [33]) but they have the *same* poles, and consequently, the same growth complexity λ. Thus, the complexity λ does not depend on the birational representation considered: K_q^2 for any value of q, or its reduction to two variables $k_{\alpha,\varepsilon}$.

Coming back to mapping (3.1), we have first obtained some "generic" degree generating function [1][16] for $\alpha \neq 0$:

$$G_\varepsilon^\alpha(x) = \frac{1+x^2}{1-x-x^3}. \tag{3.5}$$

The pole of smallest modulus of Equation (3.5) gives 1.46557... for the value of the complexity for the matrix transformation K_q. Secondly, we have obtained [1], for the $\alpha = 0$ case (see (3.2)), some other generating function $G_\varepsilon(x)$ (for generic values of ε):

$$G_\varepsilon(x) = \frac{1+x+x^3}{1-x^2-x^4} \tag{3.6}$$

with complexity $\lambda \simeq 1.272019\ldots$. Furthermore, for some singled out values of ε (when $\alpha = 0$), namely, $\varepsilon = 1/m$ (m is an integer ≥ 4), the generating function $G_\varepsilon(x)$ also has been obtained and seen to be a slightly modified expression [1]:

$$G_{1/m}(x) = \frac{1+x+x^3-x^{2m+1}-x^{2m+3}}{1-x^2-x^4+x^{2m+4}}, \quad \text{with } m \geq 4. \tag{3.7}$$

3.2. Rational Dynamical Zeta Functions

It is well known that the fixed points of the successive powers of a mapping are extremely important in order to understand the complexity of the phase space. A lot of work has been devoted to study these fixed points (elliptic or saddle fixed points, attractors, basin of attraction, etc.), and to analyze related concepts (stable and unstable manifolds, homoclinic points, etc.) [7]. We follow another point of view and study the generating function of the *number* of fixed points of a mapping k.

By analogy with the Riemann zeta function, Artin and Mazur [13] introduced a powerful object, the so-called *dynamical zeta function*:

$$\zeta(t) = \exp\left(\sum_{m=1}^{\infty} \#\mathrm{fix}(k^m) \cdot \frac{t^m}{m}\right), \tag{3.8}$$

where $\#\mathrm{fix}(k^m)$ denotes the number of fixed points of k^m. The generating function:

$$H(t) = \sum \#\mathrm{fix}(k^m) \cdot t^m \tag{3.9}$$

[16] Corresponding to the degree of the extracted homogeneous polynomials f_n's in the factorization schemes of K_q [1, 31].

can be deduced from the dynamical zeta function:

$$H(t) = t\frac{\mathrm{d}}{\mathrm{dt}}\Big(\log\zeta(t)\Big). \tag{3.10}$$

The (exponential of the) topological entropy h is related to the singularity of the dynamical zeta function:

$$\log h = \lim_{m\to\infty} \frac{\log(\#\mathrm{fix}(k^m))}{m}. \tag{3.11}$$

In the case of mapping (3.2), corresponding to $\alpha = 0$, this expansion coincides with the *rational* function:

$$H_\varepsilon(t) = \frac{t\cdot\left(1+t^2\right)}{(1-t^2)\,(1-t-t^2)}, \tag{3.12}$$

which corresponds to a very *simple rational* expression for the dynamical zeta function, namely,

$$\zeta_\varepsilon(t) = \frac{1-t^2}{1-t-t^2}. \tag{3.13}$$

This yields for the (exponential of the) topological entropy:

$$h \simeq 1.61803... = (1.272019...)^2.$$

The expansions of the dynamical zeta function remains unchanged [1] for all the "generic values" of ε.

When mentioning zeta functions, it is tempting to seek for simple *functional relations* relating $\zeta(t)$ and $\zeta(1/t)$. One immediately verifies that $\zeta_\varepsilon(t)$, corresponding to (3.13), satisfies two extremely simple and remarkable functional relations:

$$\zeta_\varepsilon(1/t) = \frac{\zeta_\varepsilon(t)}{2\cdot\zeta_\varepsilon(t)-1}, \quad \text{and} \quad \zeta_\varepsilon(-1/t) = \zeta_\varepsilon(t). \tag{3.14}$$

The generating function (3.12) verifies $H_\varepsilon(-1/t) = -H_\varepsilon(t)$.

An alternative way of writing the dynamical zeta functions relies on the decomposition of the fixed points into *irreducible cycles* which corresponds to the Weil conjectures [55]. Let us introduce N_r, the number of irreducible cycles of k_ε^r. One can then write the dynamical zeta function as the infinite product:

$$\zeta_\varepsilon(t) = \frac{1}{(1-t)^{N_1}}\cdot\frac{1}{(1-t^2)^{N_2}}\cdot\frac{1}{(1-t^3)^{N_3}}\cdots\frac{1}{(1-t^r)^{N_r}}\cdots. \tag{3.15}$$

The results of [10] yield $N_1 = 1, N_2 = 0, N_3 = 1, N_4 = 1, N_5 = 2, N_6 = 2, N_7 = 4, N_8 = 5, N_9 = 8, N_{10} = 11, N_{11} = 18$. One actually easily verifies that (3.13) and (3.15) have the same expansion up to order twelve with these values of the N_r's. The next N_r's should be $N_{12} = 25, N_{13} = 40, N_{14} = 58, N_{15} = 90, \ldots$:

$$\zeta_\varepsilon(t) = \frac{1-t^2}{1-t-t^2} = \frac{1}{(1-t)}\cdot\frac{1}{(1-t^3)}\cdot\frac{1}{(1-t^4)}\cdot\frac{1}{(1-t^5)^2}\cdot\frac{1}{(1-t^6)^2}\cdot\frac{1}{(1-t^7)^4}$$
$$\times\frac{1}{(1-t^8)^5}\,\frac{1}{(1-t^9)^8}\cdot\frac{1}{(1-t^{10})^{11}}\cdot\frac{1}{(1-t^{11})^{18}}\cdots.$$

For some singled out (non-generic) values of ε, namely, $\varepsilon = 1/m$, where m is an integer and ≥ 4, the number of irreducible cycles is modified yielding other expansions for the dynamical zeta function or for the associated Weil product (3.15). All these expressions are compatible with this single expression of the zeta function:

$$\zeta_{1/m}(t) = \frac{1-t^2}{1-t-t^2+t^{m+2}}. \tag{3.16}$$

Comparing the poles of (3.16) with those of the degree generating functions (3.7) (with $t = x^2$, because we are comparing K_q and k_ε associated with K_q^2), one sees that the singularities are actually the *same*. The growth complexity λ and the (exponential of the) topological entropy *identify*. We conjecture that this expression is exact, at every order of iteration and for every integer value $m \geq 4$. Again all the singularities of this expression coincide with those of the degree generating function corresponding to the Arnold complexity (see Equation (3.7)). From these results, one can conjecture an *identification* between the growth complexity λ ("asymptotic" of the Arnold complexity for two-dimensional mappings) and the (exponential of the) topological entropy.

As far as functional relations relating $\zeta(t)$ and $\zeta(1/t)$ are concerned, one immediately verifies the simple functional relation:

$$\zeta_{1/m}(1/t) = \frac{t^{m+1}\cdot\zeta_{1/m}(t)}{t^{m+1}\cdot\zeta_{1/m}(t)-\zeta_{1/m}(t)+1}. \tag{3.17}$$

Returning to dynamical zeta functions of mapping (3.1) corresponding to $\alpha \neq 0$, we have obtained the expansion of the zeta function, up to order seven, as follows:

$$\zeta_\varepsilon^\alpha(t) = 1+2t+3t^2+7t^3+15t^4+32t^5+69t^6+148t^7+\cdots, \tag{3.18}$$

thus yielding (for generic values of ε) the following possible rational expression for the dynamical zeta function:

$$\zeta_\varepsilon^\alpha(t) = \frac{(1-t^2)\cdot(1+t)}{1-t-2t^2-t^3} = \frac{(1-x^2)\cdot(1+x^2)^2}{(1-x-x^3)\cdot(1+x+x^3)}, \quad \text{with } t = x^2. \tag{3.19}$$

This new rational conjecture (3.19) corresponds to the following expression for $H(t)$:

$$H_\varepsilon^\alpha(t) = \frac{t\cdot\left(t^3+3t^2+2\right)}{(1-t^2)\cdot\left(1-t-2t^2-t^3\right)}. \tag{3.20}$$

Again one has to compare the poles of $\zeta_\varepsilon^\alpha(t)$ (see (3.19)) and those of $G_\varepsilon^\alpha(t)$ (see (3.5)). These poles again coincide, which means, at least in this example, that one has again an identification between the (asymptotic of the) Arnold complexity and the (exponential of the) topological entropy.

Introducing an "alternative" zeta function $\widehat{\zeta}_\varepsilon^\alpha(t)$:

$$\widehat{\zeta}_\varepsilon^\alpha(t) = \frac{\zeta(t)}{\zeta(t)-1} = \frac{(1-t)\cdot(1+t)}{t\cdot(1+t+t^2)}, \quad \text{with } \zeta(t) = \frac{\zeta_\varepsilon^\alpha(t)}{1+t} = \frac{(1-t^2)}{1-t-2t^2-t^3},$$

one easily verifies the simple functional relation:

$$t^2\cdot\widehat{\zeta}_\varepsilon^\alpha(t) = -\widehat{\zeta}_\varepsilon^\alpha(1/t). \tag{3.21}$$

3.3. Linear Operator Interpretation and Occurrence of Determinants

From a general point of view, rational dynamical zeta functions (see, for instance, [14, 52, 80]) are known in the literature through theorems where the dynamical systems are asked to be *hyperbolic*, or through combinatorial proofs using *symbolic dynamics* arising from Markovian partitions[17] [63] and even, far beyond these frameworks [48], for the so-called "isolated expansive sets"(see [43, 46, 48] for a definition). There also exists an explicit example of a rational dynamical zeta function in the case of *explicit linear* dynamics on the torus R^2/Z^2, deduced from an $SL(2,Z)$ matrix, namely, the cat map [7, 12] (diffeomorphisms of the torus):

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \zeta(z) = \frac{\det(1 - z \cdot B)}{\det(1 - z \cdot A)} = \frac{(1-z)^2}{1 - 3 \cdot z + z^2}. \tag{3.22}$$

For (3.22), one verifies immediately the functional equation $\zeta(z) = \zeta(1/z)$. Recalling the identity between the characteristic polynomial of a $q \times q$ matrix A and the characteristic polynomial of its inverse, yielding $P(z,A) = \det(1 - z \cdot A) = \det(A) \cdot (-z)^q \cdot P(1/z, A^{-1})$, one can see the simple functional relation $\zeta(z) = \zeta(1/z)$, as a consequence of relation:

$$P(z,A) = P(z,A^{-1}), \tag{3.23}$$

which is specific of matrix A given by (3.22). However, in general, one should not expect the "Markovian" transition matrix A to verify relation (3.23), but slightly more complicated relations (see (3.14)).

If the dynamical zeta function can be interpreted as the ratio of two characteristic polynomials of two linear operators[18] A and B, namely, $\zeta(z) = \det(1 - z \cdot B)/\det(1 - z \cdot A)$, then the number of fixed points $\#\mathrm{fix}(k^m)$ can be expressed from $\mathrm{Tr}(A^n) - \mathrm{Tr}(B^n)$. In this linear operators framework, the *rationality* of the dynamical zeta function and, therefore, the algebraicity of the topological entropy amounts to having a *finite dimensional representation* of the linear operators A and B. In the case of a rational zeta function, h, the exponential of the topological entropy is the inverse of the pole of smallest modulus. Since the number of invariant points remains unchanged under topological conjugacy (see [86] for this notion), the dynamical zeta function is also a topologically invariant function, invariant under a large set of transformations, and does not depend on a specific choice of variables. Such invariances were also noticed [1, 2] for the complexity growth λ.

At this step, one can recall various topological invariant "zeta functions" which correspond to various counting of templates, links, knots, etc. (which are thus very close to partition functions per site like the ones given in the Subsection 2.1). For instance, the twist-zeta functions (counting twist ribbons; see [50, p. 157]) can also be simply expressed in terms of the determinant $\det(1 - A(t))$ of a matrix $A(t)$ which depends on some variable t (see, for instance, the twisted matrices [50, p. 157]). The Alexander

[17] In the framework of hyperbolic systems [45].

[18] For more details on these Perron–Frobenius or Ruelle–Araki transfer operators, and other shifts on Markovian partitions in a symbolic dynamics framework, see, for instance, [36, 37, 45, 80, 81].

polynomials can also be simply written as products of *determinants* $\det(1-K_i)$, where K_i's are "linking matrices" (see [50, p. 184]).

Back to lattice statistical mechanics, and as far as the occurrence of *determinants* is concerned, one can also recall the most recent results of Baxter and Bazhanov, where they show that the partition function of the Zamolodchikov $3-D$ model [18,21,22] can actually be expressed as a *determinant* of "some" matrix. It is also worth noting that many generating functions can be expressed as determinants.

4. Divergent Series Solution of an Exact Functional Equation for $\varepsilon = 3$

Let us keep working with Cremona transformations [77,78], and, in particular, with the previous birational transformation (3.2), but for $\varepsilon = 3$. This value of ε is singled out as far as the phase portrait of transformation (3.2) is concerned; instead of a fairly chaotic phase portrait (see Figure 3 in [1]), the iteration of k_ε, for $\varepsilon = 3$, gives a *very regular phase portrait* in the (y, z) plane, especially around the fixed point of k_ε for $\varepsilon = 3$: $(y = -1, z = +1)$. These orbits seem to be curves, exactly similar to the foliation of the plane in elliptic curves (linear pencil of elliptic curves) one obtains for *integrable* mappings [30], although the $\varepsilon = 3$ value can actually be seen to correspond [4] to the *"chaotic"* complexity $\lambda = 1.618033...$ (generic complexity value for $\alpha = 0$). The question we address in this section is how to reconcile these apparently opposite facts. For this, let us first introduce a parametrization of these "curves", $(y, z) = (y(t), z(t))$, and let us consider the representation, restricted to these curves, of k_ε in terms of this parameter t.

In fact, a visualization of the curves obtained by the iteration of k_ε for $\varepsilon = 3$ singles out three curves, Γ_1, Γ_2 and Γ_3, intersecting at the fixed point of k_ε for $\varepsilon = 3$, namely, $(y, z) = (-1, 1)$. Note that each curve is *globally stable by* k_ε^3, and that k_ε and k_ε^2 map one of the three curves Γ_i onto the two others.

The appearance of these three curves Γ_i is reminiscent of the coalescence, in the $\varepsilon \to 3$ limit, of the three fixed points of k_ε^3, namely, $(y, z) = (2-\varepsilon, (\varepsilon-1)/2)$, $((1-\varepsilon)/2, \varepsilon-2)$, or $(-1, 1)$, with the fixed point of k_ε, namely, $((1-\varepsilon)/2, (\varepsilon-1)/2)$. In this limit, the triangle, made from these three *confluent* fixed points, actually corresponds to the three slopes, at $(y, z) = (-1, 1)$, of the three singled out curves Γ_i (see below).

Let us concentrate on one of these three curves, Γ_1. Since Γ_1 is globally stable by k_ε^3 but not by k_ε, one can only expect a representation of k_ε^3 (and not of k_ε), in terms of a well-suited variable t, around the fixed point of k_ε: $(y = -1, z = +1)$. Recalling transformation (3.2) yields, for $\varepsilon = 3$, the following expressions for the two y and z components of k_ε^3:

$$k_y^3 = \frac{4+12y+z-7yz-3z^2+yz^2}{1-3y+z+yz},$$

$$k_z^3 = \frac{(yz-2-3y-2z)\cdot(3+15y-8yz-3z^2+yz^2)}{(7+3y+4z-4yz-3z^2+yz^2)\cdot(1+z)}. \tag{4.1}$$

Let us now try to find the parametrization $(y(t), z(t))$ of curves Γ_i (as an expansion near the fixed point of k_ε for $\varepsilon = 3$, namely, $(y, z) = (-1, 1)$, which belong to the curves). A simple linearization of k_ε^3 around this fixed point $(y, z) = (-1, 1)$ yields the identity matrix. Therefore, one cannot have (near this fixed point) a representation of the iteration of k_ε^3 as $t \to \mu \cdot t$; it must be a shift representation $t \to \mu + t$. However, such a shift representation is not well-suited to deal with an expansion around the fixed point $(y, z) = (-1, 1)$ (the fixed point would correspond to $t = \infty$). We must represent the shift, associated with the action of k_ε, as $t \to t/(1+t)$, that is, $1/t \to 1/t + 1$. Let us then write, using this last shift representation, that one of the (three) curves, Γ_i, is actually invariant under k_ε^3:

$$k_y^3(t) = y\left(\frac{t}{1+t}\right), \quad k_z^3(t) = z\left(\frac{t}{1+t}\right). \tag{4.2}$$

These two equations when solved give, order by order, three solutions. One solution corresponds to the following expansion, depending on an *only one* parameter σ, for $y(t)$ and $z(t)$:

$$\begin{aligned} y(\sigma, t) = & -1 + \frac{2}{3} \cdot t - \sigma \cdot t^2 - \left(\frac{10}{81} - \frac{3}{2}\sigma^2\right) \cdot t^3 + \left(\frac{5}{729} + \frac{5}{9}\sigma - \frac{9}{4}\sigma^3\right) \cdot t^4 \\ & + \left(\frac{545}{6561} - \frac{10}{243}\sigma - \frac{5}{3}\sigma^2 + \frac{27}{8}\sigma^4\right) \cdot t^5 - \left(\frac{1085}{78732} + \frac{2725}{4374}\sigma - \frac{25}{162}\sigma^2 - \frac{25}{6}\sigma^3 \right.\\ & \left. + \frac{81}{16}\sigma^5\right) \cdot t^6 - \left(\frac{117935}{1062882} - \frac{1085}{8748}\sigma - \frac{2725}{972}\sigma^2 + \frac{25}{54}\sigma^3 + \frac{75}{8}\sigma^4 - \frac{243}{32}\sigma^6\right) \cdot t^7 \\ & + \left(\frac{73175}{2125764} + \frac{825545}{708588}\sigma - \frac{7595}{11664}\sigma^2 - \frac{19075}{1944}\sigma^3 \right.\\ & \left. + \frac{175}{144}\sigma^4 + \frac{315}{16}\sigma^5 - \frac{729}{64}\sigma^7\right) \cdot t^8 + \cdots, \end{aligned} \tag{4.3}$$

$$\begin{aligned} z(\sigma, t) = & \; 1 + \frac{2}{3} \cdot t - \left(\frac{4}{9} + \sigma\right) \cdot t^2 - \left(\frac{14}{81} - \frac{4}{3}\sigma - \frac{3}{2}\sigma^2\right) \cdot t^3 \\ & + \left(\frac{31}{729} - \frac{7}{9}\sigma - 3\sigma^2 - \frac{9}{4}\sigma^3\right) \cdot t^4 - \left(\frac{631}{6561} + \frac{62}{243}\sigma - \frac{7}{3}\sigma^2 - 6\sigma^3 - \frac{27}{8}\sigma^4\right) \cdot t^5 \\ & - \left(\frac{409}{26244} - \frac{3155}{4374}\sigma - \frac{155}{162}\sigma^2 + \frac{35}{6}\sigma^3 + \frac{45}{4}\sigma^4 + \frac{81}{16}\sigma^5\right) \cdot t^6 \\ & + \left(\frac{128683}{1062882} + \frac{409}{2916}\sigma - \frac{3155}{972}\sigma^2 - \frac{155}{54}\sigma^3 + \frac{105}{8}\sigma^4 + \frac{81}{4}\sigma^5 + \frac{243}{32}\sigma^6\right) \cdot t^7 \\ & + \left(\frac{35363}{6377292} - \frac{900781}{708588}\sigma - \frac{2863}{3888}\sigma^2 + \frac{22085}{1944}\sigma^3 + \frac{1085}{144}\sigma^4 \right.\\ & \left. - \frac{441}{16}\sigma^5 - \frac{567}{16}\sigma^6 - \frac{729}{64}\sigma^7\right) \cdot t^8 + \cdots. \end{aligned} \tag{4.4}$$

One thus obtains (at first sight) a family of curves depending on one parameter, namely σ. In fact, this *is not* a family of curves; parameter σ corresponds to a simple re-parametrization of a single curve. Considering expansions (4.3) and (4.4), one immediately verifies that

$$y(\sigma+2/3\cdot\tau,t)=y(\sigma,t/(1+\tau\cdot t)), \quad z(\sigma+2/3\cdot\tau,t)=z(\sigma,t/(1+\tau\cdot t)). \quad (4.5)$$

This parameter σ corresponds to transformation $t\rightarrow t/(1+\tau\cdot t)$, which just amounts to changing the shift corresponding to k_{ε}^3, from (a normalized value) 1 to another value τ:

$$\frac{1}{t}\longrightarrow\frac{1}{t}+\tau. \quad (4.6)$$

This means that one does not have a family of curves indexed by σ, but rather a *single curve* with a reparametrization parameter σ. Therefore, without any loss of generality, one can restrict to a specific value of σ, for instance, $\sigma=-2/9$. One then obtains

$$\begin{aligned} y = & -1+\frac{2}{3}t+\frac{2}{9}t^2-\frac{4}{81}t^3-\frac{67}{729}t^4+\frac{119}{6561}t^5+\frac{7031}{78732}t^6 \\ & -\frac{9004}{531441}t^7-\frac{498563}{3188646}t^8+\cdots \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} z = & 1+\frac{2}{3}t-\frac{2}{9}t^2-\frac{4}{81}t^3+\frac{67}{729}t^4+\frac{119}{6561}t^5-\frac{7031}{78732}t^6 \\ & -\frac{9004}{531441}t^7+\frac{498563}{3188646}t^8+\cdots. \end{aligned} \quad (4.8)$$

One remarks, for this particular value $\sigma=-2/9$, the following relation:

$$y(t)=-z(-t). \quad (4.9)$$

This is a remarkable result. It means that, in order to obtain, order by order, the parametrization of the curve, one just needs to find the expansion of an *only one* function $y(t)$ instead of two ($y(t)$ and $z(t)$). The expansion of $y(t)$ at higher orders can be found in the Appendix. This series, which is clearly a *divergent series*, seems to be Borel summable [83].

This solution corresponds to one of the three previously mentioned curves, say Γ_1. The two other solutions of (4.2) correspond to the following expansions for $y(t)$ and $z(t)$, depending on an only one parameter σ_2 or σ_3:

$$\begin{aligned} y(\sigma_2,t) = & -1+2/3\cdot t-\sigma_2\cdot t^2-\Big(\frac{10}{81}-3/2\sigma_2{}^2\Big)\cdot t^3 \\ & -\Big(\frac{5}{729}-5/9\sigma_2+9/4\sigma_2{}^3\Big)\cdot t^4+\cdots, \\ z(\sigma_2,t) = & 1-4/3t+\Big(2/9+2\sigma_2\Big)\cdot t^2+\Big(\frac{2}{81}-2/3\sigma_2-3\sigma_2{}^2\Big)\cdot t^3 \\ & -\Big(\frac{2}{81}+1/9\sigma_2-3/2\sigma_2{}^2-9/2\sigma_2{}^3\Big)\cdot t^4+\cdots, \end{aligned} \quad (4.10)$$

and

$$y(\sigma_3,t) = -1 - 4/3\cdot t - \Big(2/9 - 2\sigma_3\Big)\cdot t^2 + \Big(\frac{2}{81} + 2/3\sigma_3 - 3{\sigma_3}^2\Big)\cdot t^3$$
$$+\Big(\frac{2}{81} - 1/9\sigma_3 - 3/2{\sigma_3}^2 + 9/2{\sigma_3}^3\Big)\cdot t^4 + \cdots$$
$$z(\sigma_3,t) = 1 + 2/3\cdot t - \sigma_3\cdot t^2 - \Big(\frac{10}{81} - 3/2{\sigma_3}^2\Big)\cdot t^3$$
$$+\Big(\frac{5}{729} + 5/9\sigma_3 - 9/4{\sigma_3}^3\Big)\cdot t^4 + \cdots. \tag{4.11}$$

It is a straightforward calculation to see that these two expansions are nothing but the expansion of curve Γ_1 transformed by k_ε and k_ε^2. The parameters σ_i in (4.10) and (4.11) are also just re-parametrization parameters, like σ in (4.3) and (4.4). The slopes at $(y,z) = (-1,1)$, corresponding to the three curves Γ_i, are just the first order term in (4.7), (4.10) and (4.11), namely $(2/3, 2/3)$, $(2/3, -4/3)$, and $(-4/3, 2/3)$. They correspond exactly to the three edges of the triangle built from the three confluent fixed points of k_ε^3 when $\varepsilon \to 3$. Since the three Γ_i are on the same footing, let us restrict to Γ_1.

The expansion corresponding to Γ_1 (see (4.7), (4.8)), actually verifies the exact functional equation:

$$\Big(y(\frac{t}{1+t}) + y(-t) + 4\Big)\cdot\Big(y(-t)+3\Big)\cdot y(t)$$
$$+\Big(y(-t)-1\Big)\cdot\Big(y(\frac{t}{1+t}) - 3\cdot y(-t) - 4\Big) = 0. \tag{4.12}$$

This equation is obtained from the equality of the y-components of k_ε^3, coming respectively from (4.1) and from (4.2). Of course one can obtain another similar functional equation, deduced from the equality of the z-component of k_ε^3 in (4.1) and (4.2):

$$\Big((1-y(-t))\cdot y(\frac{-t}{1+t}) - y(t)y(-t) - 3y(t) + 2y(-t) - 2\Big)$$
$$\times\Big(y(t)y(-t)^2 + 4y(t)y(-t) - 3y(-t)^2 - 4y(-t) + 3y(t) + 7\Big)$$
$$+\Big(4y(t)y(-t) + 12y(t) - 4 + 4y(-t)\Big)$$
$$\times\Big(2y(-t) - 2 - 3y(t) - y(t)y(-t)\Big) = 0. \tag{4.13}$$

One easily verifies that the expansion of $y(t)$ at higher orders (see Appendix) is actually a solution of the two functional equations (4.12) and (4.13).

The plots of the orbits of k_ε^3 in the real (y,z)-plane give a very regular "phase portrait" which looks very much like a foliation of the plane in curves [30]. The previous expansions (4.3) and (4.4) give some "hint" on only three of these "curves". It would be interesting to perform similar calculations for the curves not including the fixed point $(-1,1)$. This remains to be done. The parametrization of at least three curves Γ_i, corresponding to divergent series (4.7), seems to exclude a parametrization in elliptic curves.

Actually, using the method well-suited for two-dimensional rational transformations introduced in [30], we have not been able to find any algebraic invariant corresponding to a possible linear pencil of this very regular foliation. This seems to correspond exactly to the notion of *non-algebraic integrability*[19] developed by Rerikh [76].

In fact, complexity growth calculations, performed for this $\varepsilon = 3$ case, do show [4] the *same* value for the complexity growth λ, namely, 1.61803..., than for the other generic values of ε. The system is *actually chaotic* for $\varepsilon = 3$ even if its restriction to the *real* (y, z)-plane is extremely regular (and then the "real" topological entropy is zero [4]). The corresponding "function", which gives "nice real curves", corresponds to a *divergent series* which is quite "monstrous" in the non-real (complex) plane. This kind of "function" is compatible with the picture of the Vague Attractor of Kolmogorov (see the VAK in [5] or the Nested KAM tori [44, p. 441]) *together* with the occurrence of a nice curve in real space. The occurrence of such divergent series solves the "paradox" of the compatibility between a regularity of the (real) phase portrait with a chaotic (complex) dynamics (complexity growth $\lambda \simeq 1.61803...$).

This example shows that simple *functional equations* yielding *curves* may be even foliation in curves, for *real values* of the variables, are actually compatible with a chaotic dynamical system.

5. Conclusion

We have seen that many functional equations, despite their occurrence in very different domains of mathematical physics, often share some common features. One feature is the key role played by some involutions[20] (namely, matrix inversion $M \to M^{-1}$, some "hidden inversion" for staircase polygons generating functions (2.25), $t \to 1/t$ for zeta functions, etc.). Another feature of these functional equations is often the possible representation of the generating functions as *determinants* (partition function of the $3-D$ Zamolodchikov model [18, 21, 22], dynamical zeta function expressed as ratio of $det(1 - t \cdot A)$, etc.).

Naively, obtaining a functional equation is so constraining that one just needs a "small piece" of additional information (analyticity properties, etc.) to obtain the function that one seeks. Enumerative combinatorics, or lattice statistical mechanics, are typical domains where the "missing information" is "small" (see (2.5), (2.8), or (2.20)). In fact, in contrast with differential equations or PDE's, functional equations allow a much larger set of solutions, these solutions being characterized by extremely complicated analytical behaviors (see [91], or the resurgent functions[21] of Ecalle [47]).

In fact, this question of the "amount of constraint", corresponding to functional equations, will receive completely different answers, according to the various frameworks one considers. It is clear that, in the most general framework, the answer to such a question is hopeless. This is the reason why, at the beginning of this paper, we restricted our preliminary review on functional equations to those associated with (gener-

[19] Non-algebraic integrability of the Chew–Low reversible dynamical system of the Cremona type has been addressed by Rerikh [76].

[20] Or even finite order transformations.

[21] After all, resurgent functions are not, in general, a very constrained set of functions. On the contrary, they form a rather "soft" framework.

ically infinite discrete) groups[22]. Even in a restricted framework of Abelian groups (up to semi-direct products by finite groups, etc.), one does need to discriminate between functional equations bearing on functions of *several* complex variables on one side, and on functions of *one* complex variable on the other. Heuristically, the solutions of functional equations bearing on functions of *several* complex variables can be understood as generalizations, to several complex variables, of automorphic functions. Unfortunately, almost nothing is known of the analyticity properties of such functions (multivalued functions with an infinite valuation; see [58]). The analyticity properties of the solutions can only be taken into account, in a proper way, when restricting to functional equations bearing on functions of *one* complex variable associated with an infinite discrete Abelian[23] group. In this last case, considering simple *discrete dynamical systems*, we have seen that it is also necessary, among the functional equation solutions, to make a clear distinction between the analyticity properties of functions of one *complex* variable and the analyticity properties of functions of one *real* variable.

Actually, discrete dynamical systems, in particular the iteration of simple birational transformations (originating from lattice statistical mechanics), are a well-suited framework to address all these questions. Roughly speaking, one naturally segregates the occurrence of functional equations in the following two situations: Either one has a *polynomial growth* of the iteration calculations ($\lambda = 1$) and the transformations considered are basically a *shift* on an Abelian variety (see [31, Figure 2]), often an elliptic curve (but one may even have Painlevé-like objects, with probably more involved "trajectories" on the Abelian varieties than a shift), or the complexity λ is greater than 1 and one has a chaotic situation. In the first polynomial growth domain, the analyticity properties are clear and the occurrence of functional equations are often closely related to relations on Abelian varieties [66].

Conversely, chaotic systems can certainly correspond to functional equations[24] but one does not expect, at first sight, any "nice functional equations" *yielding curves* for instance. In fact, as it has been seen, with the example (4.1) associated with $\varepsilon = 3$, a chaotic system does not rule out the existence of *curves* (at least in the *real domain*) associated with *an exact functional equation* (see (4.12)). For instance, the "price to pay" is just that the curves will only be defined for *real* values of the variables, the series associated with these curves, being *divergent series*.

[22] Or semigroups, e.g., $x \to x^2$, or $x \to x^3, \ldots$. Most of our examples are associated with infinite discrete groups generated by two involutions (infinite dihedral group) with an obvious shift symmetry: $u \to u + \eta$ or $\theta \to \theta + \lambda$. Examples with more than two involutions are also considered.

[23] Up to semidirect products by finite groups.

[24] For instance, if the functional equation identifies with the chaotic iteration one considers, but this yields fractal-like objects.

A. The Expansion of $y(t)$ for the $\varepsilon = 3$ Case

At higher orders:

$$\begin{aligned}
y(t) \;=\; & -1+\frac{2}{3}t+\frac{2}{9}t^2-\frac{4}{81}t^3-\frac{67}{729}t^4+\frac{119}{6561}t^5+\frac{7031}{78732}t^6-\frac{9004}{531441}t^7-\frac{498563}{3188646}t^8\\
& +\frac{4012423}{133923132}t^9+\frac{9273016087}{21695547384}t^{10}-\frac{65639286071}{781039705824}t^{11}-\frac{3919438859951}{2343119117472}t^{12}\\
& +\frac{58702493381929}{173976594472296}t^{13}+\frac{369544922982428887}{4175438267335104}t^{14}-\frac{2967839007982993309}{162842092426069056}t^{15}\\
& -\frac{10689162361376574962 99}{1758694598201545804 8}t^{16}+\frac{352141168413421096564 9}{276994399216743464256}t^{17}\\
& +\frac{873169015196420150636423}{1661966395300460785536}t^{18}-\frac{636080264571303689411111487}{572131931582183625420768}t^{19}\\
& -\frac{153262556077398234892492926167}{27462332715944814020196864}t^{20}\\
& +\frac{248638175527746765408018424108 1}{2087137286411805865534961664}t^{21}\\
& +\frac{13273657909726055261134806334059 7}{18577815406522667594322186 24}t^{22}\\
& -\frac{48093373227390994182050590046052 77}{31325475213410133328896792268 8}t^{23}\\
& -\frac{34666340109953257893099809606832084623}{31951984717678335995474728114176}t^{24}\\
& +\frac{5114577498745621063700043726382085736229 9}{21826400760646071318508786774793625 6}t^{25}\\
& +\frac{50495777020773376116907497144298407135656551}{261916809127752855822105441297523507 2}t^{26}+\cdots.
\end{aligned}\tag{A.1}$$

One remarks that the numerators of the coefficients in this expansion often factorize in fairly large prime numbers (in contrast with the denominators). For instance, the numerator of the coefficient of t^{20} factorizes into the product of 103116049, 33170617930 7969 and 4480807. The coefficient of t^{26} factorizes into the product of 5417, 183088852 209431303, and 5091366043929018731 8201. This function can thus be seen to produce large prime numbers.

One verifies easily that this expansion satisfies, order by order, the functional equations (4.12) and (4.13).

Acknowledgment. The third author would like to thank P. Lochak and J.P. Marco for illuminating discussions on dynamical systems.

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