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The anisotropic Ising correlations as elliptic integrals: duality and differential equations*

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Abstract
We present the reduction of the correlation functions of the Ising model on the anisotropic square lattice to complete elliptic integrals of the first, second and third kind, the extension of Kramers–Wannier duality to anisotropic correlation functions, and the linear differential equations for these anisotropic correlations. More precisely, we show that the anisotropic correlation functions are homogeneous polynomials of the complete elliptic integrals of the first, second and third kind. We give the exact dual transformation matching the correlation functions and the dual correlation functions. We show that the linear differential operators annihilating the general two-point correlation functions are factorized in a very simple way, in operators of decreasing orders.

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1. Introduction

The two-dimensional Ising model has been the object of penetrating investigations beginning with the loop algebraic computation of the free energy in 1944 by Onsager [1], the spinor

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(Fermionic) method of Kaufman [2] in 1949, the correlation computations of Kaufman and Onsager [3] also in 1949, the characterization as Toeplitz determinants by Montroll *et al* [4] in 1963, the asymptotic behavior for large separations of Wu [5] in 1966, the Painlevé III representation in the scaling region of T near T_c by Wu, McCoy *et al* [6] in 1976 and the Painlevé VI representation of the diagonal correlation functions by Jimbo and Miwa [7] in 1981. We know more about the correlation functions of the Ising model than any other system and these correlation functions have inspired developments ranging from conformal field theory to random matrices.

Nevertheless there are still many questions which remain unsolved. In this note we investigate the reduction of the anisotropic correlation functions to *homogeneous polynomials* in the three kinds of complete elliptic integrals, the implications of the Kramers–Wannier duality [8] and the linear differential equations satisfied by the anisotropic correlation functions.

In section 2 we review the reduction of the nearest neighbor correlation to the form obtained by Onsager [1]. This leads to an identity on complete elliptic integrals of the third kind. In section 3 we extend this complete elliptic integral representation to all correlations. In section 4 we find a representation of the Kramers–Wannier duality on the complete elliptic integral of the first, second and third kind, which actually transforms the correlation functions into the dual correlation functions. We conclude in section 5 with a discussion of the linear differential equations for the correlations.

2. The nearest neighbor correlation

The two-dimensional Ising model on a square lattice is defined by the interaction energy

$$\mathcal{E} = -\sum_{i,j} (E_v \cdot \sigma_{i,j}\, \sigma_{i+1,j} \, + E_h \cdot \sigma_{i,j}\, \sigma_{i,j+1}), \tag{1}$$

where $\sigma_{i,j} = \pm 1$ is the spin at row i and column j and the sum is over all values ± 1 for all spins in a lattice of L_v rows and L_h columns with either cylindrical or toroidal boundary conditions. The partition function on the $L_v \times L_h$ lattice at temperature T and the free energy in the thermodynamic limit, are defined as

$$Z(\beta;\, L_v,\, L_h) = \sum_{\sigma=\pm 1} \mathrm{e}^{-\beta\mathcal{E}}, \qquad F = -k_{\mathrm{B}}\, T \lim_{L_v,L_h\to\infty} \frac{1}{L_v L_h} \ln Z(\beta;\, L_v,\, L_h). \tag{2}$$

The result of this computation is the famous double integral formula for the free energy

$$\begin{aligned} -\,\beta F = \ln 2 + \frac{1}{2\,(2\pi)^2} \int_0^{2\pi} \mathrm{d}\theta_1 \cdot \int_0^{2\pi} \mathrm{d}\theta_2 \cdot\, \ln\{\cosh 2\beta E_v \cosh 2\beta E_h \\ -\sinh 2\beta E_v \cos\theta_1 - \sinh 2\beta E_h \cos\theta_2\}, \end{aligned} \tag{3}$$

where $\beta = 1/k_{\mathrm{B}} T$ (k_{B} being Boltzmann's constant). This free energy has a singularity of the form $(\beta - \beta_c)^2 \cdot \ln(\beta - \beta_c)^2$, at the critical value $\beta_c = 1/k_{\mathrm{B}} T_c$, defined by:

$$\sinh 2\beta_c E_v \cdot\, \sinh 2\beta_c E_h = \pm 1. \tag{4}$$

We begin our discussion of Ising correlation functions with the nearest neighbor row and column correlation functions which can be obtained from the free energy as

$$u = \frac{\partial \beta F}{\partial \beta} = -E_h \cdot \langle \sigma_{0,0}\sigma_{0,1} \rangle - E_v \cdot \langle \sigma_{0,0}\sigma_{1,0} \rangle. \tag{5}$$

Thus by differentiating (3) and doing one of the integrals by closing a contour on poles we find

$$C(0, 1) = \langle \sigma_{0,0}\sigma_{0,1} \rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \cdot \left[\frac{(1-\alpha_1 \mathrm{e}^{\mathrm{i}\theta})(1-\alpha_2 \mathrm{e}^{-\mathrm{i}\theta})}{(1-\alpha_1 \mathrm{e}^{-\mathrm{i}\theta})(1-\alpha_2 \mathrm{e}^{\mathrm{i}\theta})}\right]^{1/2}, \tag{6}$$

with

$$\alpha_1 = z_h \cdot \frac{1-z_v}{1+z_v} = \tanh \beta E_h \cdot \mathrm{e}^{-2\beta E_v}, \tag{7}$$

$$\alpha_2 = \frac{1}{z_h} \cdot \frac{1-z_v}{1+z_v} = \coth \beta E_h \cdot \mathrm{e}^{-2\beta E_v}, \tag{8}$$

where $z_v = \tanh \beta E_v$ and $z_h = \tanh \beta E_h$.

The result (6) can be reduced to complete elliptic integrals of the first, second and third kind defined as

$$\tilde{K}(k) = \frac{2}{\pi} \cdot \int_0^{\pi/2} \frac{\mathrm{d}\phi}{(1-k^2 \sin^2\phi)^{1/2}}, \tag{9}$$

$$\tilde{E}(k) = \frac{2}{\pi} \cdot \int_0^{\pi/2} \mathrm{d}\phi \cdot (1-k^2 \sin^2\phi)^{1/2}, \tag{10}$$

$$\tilde{\Pi}(n, k) = \frac{2}{\pi} \cdot \int_0^{\pi/2} \frac{\mathrm{d}\phi}{(1-n\sin^2\phi)(1-k^2 \sin^2\phi)^{1/2}}, \tag{11}$$

where we have chosen a normalization such that at $k = n = 0$ the three elliptic integrals are unity instead of $\pi/2$. We also introduce the *modulus of the elliptic functions* parametrizing the model:

$$k = \sinh 2\beta E_v \cdot \sinh 2\beta E_h = \frac{\alpha_2^{-1} - \alpha_1}{1 - \alpha_1\alpha_2^{-1}}, \tag{12}$$

$$k_< = \frac{1}{k} = (\sinh 2\beta E_v \sinh 2\beta E_h)^{-1} = \frac{\alpha_2 - \alpha_1}{1-\alpha_1\alpha_2}. \tag{13}$$

This reduction is carried out in [9] by two different methods which lead to two, at first sight, *different* expressions. Perhaps the most straightforward reduction for $T < T_c$ ($\alpha_2 < 1$) is to set $\mathrm{e}^{\mathrm{i}\theta} = \zeta$ in (6), and, then, use the substitution

$$\zeta = \frac{\alpha_1 \cdot (\alpha_1^{-1} - \alpha_2) + \alpha_1^{-1} \cdot (\alpha_2 - \alpha_1) \cdot \sin^2\phi}{\alpha_1^{-1} - \alpha_2 + (\alpha_2 - \alpha_1) \cdot \sin^2\phi}, \tag{14}$$

to obtain:

$$C_<(0, 1) = \frac{\alpha_1 - \alpha_1^{-1}}{\alpha_2^{-1} - \alpha_1} \cdot \{\tilde{K}(k_<) - (1 + \alpha_1\alpha_2^{-1}) \cdot \tilde{\Pi}(-\alpha_1 k_<, k_<)\}, \tag{15}$$

A similar computation in [9] for $T > T_c$ ($\alpha_2 > 1$) gives:

$$C_>(0, 1) = \frac{\alpha_1 - \alpha_1^{-1}}{1 - \alpha_1\alpha_2^{-1}} \cdot \{\tilde{K}(k) - (1 + \alpha_1\alpha_2^{-1}) \cdot \tilde{\Pi}(-\alpha_1 k, k)\}. \tag{16}$$

This form, however, is not particularly transparent and a more elaborate reduction, given in [9], gives[4] a form first obtained by Onsager [1]. Using the notation

$$s_h = \sinh 2\beta E_h, \qquad s_v = \sinh 2\beta E_v, \tag{17}$$

the result for $T < T_c$ is

$$\begin{aligned} C_<(0,\,1) &= \sqrt{1 + s_v^2}\cdot s_v\cdot\, s_h^{-2}\cdot\,\{(1 + s_h^2)\cdot\,\tilde{\Pi}(-s_v^{-2},\, s_v^{-1}s_h^{-1}) \;-\tilde{K}(s_v^{-1}\, s_h^{-1})\}\\ &= \sqrt{1 + \nu k_<}\cdot\,\{(1 + k_</\nu)\cdot\,\tilde{\Pi}(-\nu k_<,\, k_<) \;-(k_</\nu)\cdot\,\tilde{K}(k_<)\}, \end{aligned} \tag{18}$$

and for $T > T_c$

$$\begin{aligned} C_>(0,\,1) &= \sqrt{1 + s_v^2}\cdot\, s_h^{-1}\cdot\,\{(1 + s_h^2)\cdot\,\tilde{\Pi}(-s_h^2,\, s_v s_h) \;-\tilde{K}(s_v s_h)\},\\ &= \frac{1}{\nu}\cdot\,\sqrt{1 + \nu/k}\cdot\,\{(1 + \nu k)\cdot\,\tilde{\Pi}(-\nu\, k,\, k) \;-\tilde{K}(k)\}, \end{aligned} \tag{19}$$

where the anisotropy ν is defined as:

$$\nu = \frac{s_h}{s_v} = \frac{4\,\alpha_1\,\alpha_2}{(\alpha_2 \,-\alpha_1)\cdot\,(1 \,-\alpha_1\alpha_2)}. \tag{20}$$

One also has:

$$\nu\cdot\,k_< = \frac{4\alpha_1\alpha_2}{(1 - \alpha_1\alpha_2)^2} = s_v^{-2}, \qquad \nu\cdot\,k \,=\, \frac{4\alpha_1\alpha_2}{(\alpha_2 - \alpha_1)^2} = s_h^2. \tag{21}$$

We note that the high temperature correlation (19) is obtained from the low temperature correlation (18) by the substitution

$$\tilde{K}(s_v^{-1}s_h^{-1}) \longrightarrow s_v s_h\cdot\,\tilde{K}(s_v s_h), \tag{22}$$

$$\tilde{\Pi}(-s_v^{-2},\, s_v^{-1}s_h^{-1}) \longrightarrow s_v s_h\cdot\,\tilde{\Pi}(-s_h^2,\, s_v s_h). \tag{23}$$

The expressions (15) and (16) look quite different from (18) and (19). Nevertheless *they are actually equal*. Equating the two forms of the low temperature correlation we obtain an identity on elliptic integrals of the third kind:

$$\begin{aligned} &(z_h^2 z_v^2 \;+ z_h^2 \;- z_v^2 \;- 4z_v \;- 1)\cdot\,\tilde{K}(k_<)\\ &\quad - 2\cdot\,(z_v z_h + z_v - z_h + 1)\cdot\,(z_v z_h - z_v - z_h - 1)\cdot\,\tilde{\Pi}\left(-\frac{(1 - z_h^2)(1 - z_v)^2}{4z_v},\, k_<\right)\\ &\quad - (1 + z_v^2)\cdot\,(1 + z_h^2)\cdot\,\tilde{\Pi}\left(-\frac{(1 - z_v^2)^2}{z_v^2},\, k_<\right) = 0, \end{aligned} \tag{24}$$

where we have used

$$k_< = \frac{(1 - z_v^2)\cdot\,(1 - z_h^2)}{4z_v z_h}, \tag{25}$$

$$\frac{(1 - z_v^2)^2}{4z_v^2} = s_v^{-2}, \qquad \frac{(1 - z_h^2)(1 - z_v)^2}{4\,z_v} = \alpha_1\cdot\,k_<. \tag{26}$$

[4] Removing a misprinted factor of α_2 in (3.70) on page 97 of [9].

If we set $z = -\alpha_1 \cdot k_<$, we may verify that

$$-s_v^{-2} = 4 \cdot k_<^2 \cdot \frac{z \cdot (z-1) \cdot (z - k^2)}{(z^2 - k^2)^2}, \qquad (27)$$

and, thus, the identity (24) may be rewritten as:

$$\begin{aligned} & 4 \cdot (z-1) \cdot (z^2 - k_<^2) \cdot (z - k_<^2) \cdot \tilde{\Pi}(z, k_<) \\ & + (z^2 + k_<^2 - 2z) \cdot (z^2 + k_<^2 - 2k_<^2 z) \cdot \tilde{\Pi}\left(4 \cdot k_<^2 \cdot \frac{z \cdot (z-1) \cdot (z - k_<^2)}{(z^2 - k_<^2)^2}, k_<\right) \\ & - (z^2 - k_<^2) \cdot (z^2 - 2z - 2k_<^2 z + 3k_<^2) \cdot \tilde{K}(k_<) = 0. \end{aligned} \qquad (28)$$

Performing series[5] expansions of the lhs of (28) one can check that this identity is valid[6] when z and $k_<$ are small (*not necessarily real*) and such that $|z/k_<| \ll 1$. A proof of identity (28) is given in appendix A.

We note that the transformation

$$z \longrightarrow 4 \cdot k_<^2 \cdot \frac{z \cdot (z-1) \cdot (z - k_<^2)}{(z^2 - k_<^2)^2}, \qquad (29)$$

occurring in (28), is of *infinite order*. If one writes the previous *infinite order*[7] (rational) transformation (29) in terms of $k = 1/k_<$, it reads:

$$z \longrightarrow 4 \cdot \frac{z \cdot (1-z) \cdot (1 - k^2 z)}{(1 - k^2 z^2)^2}, \qquad (30)$$

where one recognizes, immediately, the doubling transformation, $\theta \to 2\,\theta$, on the square of the elliptic sinus, $z = \mathrm{sn}\,(\theta, k)^2$:

$$\begin{aligned} \mathrm{sn}\,(\theta, k)^2 \quad \longrightarrow \quad & \mathrm{sn}\,(2\,\theta, k)^2 \\ = 4 \cdot & \frac{\mathrm{sn}\,(\theta, k)^2 \cdot (1 - \mathrm{sn}\,(\theta, k)^2) \cdot (1 - k^2 \cdot \mathrm{sn}\,(\theta, k)^2)}{(1 - k^2 \cdot \mathrm{sn}(\theta, k)^4)^2}. \end{aligned} \qquad (31)$$

The interpretation of this identity as a doubling transformation suggests to recall addition formulae on the Jacobi's Zeta function [10, 11] (a logarithmic derivative of the Jacobi theta function), which is closely related to the ratio of the complete elliptic integral of the third kind by the complete elliptic integral of the first kind, like, for instance, the relation[8]

$$Z(u+a, k) = Z(u, k) + Z(a, k) - k^2 \cdot \mathrm{sn}(u, k) \cdot \mathrm{sn}(a, k) \cdot \mathrm{sn}(u+a, k). \qquad (32)$$

After some straightforward calculations one can interpret identity (28) as the $u = a$ limit of identity (32).

[5] For instance series expansions in t of the lhs of (28) with $z = (2 + 3\,\mathrm{i}) \cdot t^3$, $k_< = t^2$, and, conversely, see that *another identity* takes place when z and $k_<$ are small but $|z/k_<| \gg 1$.

[6] However, this identity addressing *two complex variables*, it is difficult to find what is precisely the domain of validity of this identity in the *two complex variables* z and $k_<$, the conditions $z = \pm\, k_<$ certainly playing some role, as can be seen on the denominator of transformation (29).

[7] Iterating N times the rational transformation (29), one gets rational transformations of the form $z \to R_N(z, k) = 4^N \cdot z \cdot P_N(z, k)/Q_N(z, k)$, where $P_N(z, k)$ and $Q_N(z, k)$ are polynomials of degree in z growing, for generic values of k, like 4^N (for $k = 1$ these degrees grow like 2^N).

[8] See, for instance, equations (64), (65) and (67) in [12], where the parameter a is *not required* to be a rational multiple of a period [12] (the rational cases: see, for instance, (69) in [12]). We thank Perk for reminding us, after completion of this work, the text following (66) in [12] discussing how the Π_1 can be reduced in rational cases.

2.1. Isotropic limit

To obtain the isotropic limit, where $\nu = 1$, we note that elliptic integrals of the third kind obey the identity [13]:

$$\tilde{\Pi}(-\nu\, k,\, k) \;\; + \tilde{\Pi}\Big(-\frac{k}{\nu},\, k\Big) = \tilde{K}(k) \;\; + \Big[(1 + \nu\, k)\Big(1 + \frac{k}{\nu}\Big)\Big]^{-1/2}. \qquad (33)$$

Thus, when $\nu = 1$, we have

$$\tilde{\Pi}(-k,\, k) = \frac{1}{2} \cdot \tilde{K}(k) \;\; + \frac{1}{2} \cdot \frac{1}{1 + k}, \qquad (34)$$

which can be used, in (18) and (19), to eliminate the elliptic integral of the third kind to find for $T < T_c$, in the isotropic limit where $s_v = s_h = s$, that

$$\begin{aligned} C_<(0,\, 1) &= (1 + k_<)^{1/2} \cdot \frac{1}{2}\{(1 - k_<) \cdot \tilde{K}(k_<) + 1\} \\ &= (1 + s^{-2}) \cdot \frac{1}{2}\{(1 - s^{-2}) \cdot \tilde{K}(s^{-2}) + 1\}, \end{aligned} \qquad (35)$$

and for $T > T_c$:

$$\begin{aligned} C_>(0,\, 1) &= (1 + 1/k)^{1/2} \cdot \frac{1}{2}\{(k - 1) \cdot \tilde{K}(k) + 1\} \\ &= (1 + s^{-2})^{1/2} \cdot \frac{1}{2}\{(s^2 - 1) \cdot \tilde{K}(s^2) + 1\}. \end{aligned} \qquad (36)$$

For $T > T_c$ as $k \to 0$ we have $C_>(0,\, 1) = k^{1/2}/2 \; + O(k^{3/2})$.

3. The general correlation function $C(M, N) = \langle \sigma_{0,0}\, \sigma_{M,N} \rangle$

There are two methods which can be used to compute the Ising correlation functions[9] $C(M, N)$: either in terms of their representation as determinants [4], or from their *quadratic recursion relations* [14–18].

3.1. Determinantal representation

The next simplest correlations to study, after the nearest neighbor, are the row $\langle \sigma_{0,0}\, \sigma_{0,N} \rangle$ and the diagonal $\langle \sigma_{0,0}\, \sigma_{N,N} \rangle$ correlation functions.

The row correlation function $\langle \sigma_{0,0}\, \sigma_{0,N} \rangle$ can be expressed as an $N \times N$ Toeplitz determinant [4, 9]

$$C(0,\, N) = \begin{vmatrix} a_0 & a_{-1} & \cdots & a_{-N+1} \\ a_1 & a_0 & \cdots & a_{-N+2} \\ \vdots & \vdots & \vdots & \\ a_{N-1} & a_{N-2} & \cdots & a_0 \end{vmatrix} \qquad (37)$$

with

$$a_n = \frac{1}{2\pi} \cdot \int_0^{2\pi} \mathrm{d}\theta \cdot \mathrm{e}^{-\mathrm{i}n\theta} \cdot w(\mathrm{e}^{\mathrm{i}\theta}), \qquad (38)$$

[9] The correlation functions of the Ising model are defined as usual, see, for instance, equation (14) in [4].

where the generating function $w(e^{i\theta})$ is

$$w(e^{i\theta}) = \left[\frac{(1 - \alpha_1 e^{i\theta})(1 - \alpha_2 e^{-i\theta})}{(1 - \alpha_1 e^{-i\theta})(1 - \alpha_2 e^{i\theta})}\right]^{1/2}, \tag{39}$$

with α_1 and α_2 given by (7) and (8). We note, when both α_1 and α_2 are real, that a_n is also real.

For the diagonal correlation $C(N, N)$ we may consider a triangular lattice by adding to the square lattice bonds of strength E_d connecting sites (M, N) with sites $(M + 1, N + 1)$, obtaining a determinental representation by using a straightline path from (0, 0) to (N, N) and then setting $E_d = 0$. This results in $C(N, N)$ being given by the determinant (37) with

$$\alpha_1 = 0, \qquad \alpha_2 = (\sinh 2\beta E_v \sinh 2\beta E_h)^{-1}. \tag{40}$$

The matrix elements of the diagonal correlation function $C(N, N)$ are directly seen to be hypergeometric functions which, in turn, are expressed as linear combinations of the elliptic integrals $\tilde{K}$ and $\tilde{E}$. For $T < T_c$

$$a_0 = \tilde{E}(k_<), \tag{41}$$

and for $T > T_c$:

$$a_0 = \frac{\tilde{E}(k)}{k} - \frac{1 - k^2}{k} \cdot \tilde{K}(k). \tag{42}$$

The matrix elements of the *row* correlation function are all expressed as linear combinations of the complete elliptic integrals $\tilde{K}(k)$, $\tilde{E}(k)$, $\tilde{\Pi}(s_v^{-2}, k)$ and $\tilde{\Pi}(z_h^2, k)$ by use of the change of variables (14) and the identity (24). For example, for $T > T_c$ with $k = s_v s_h$, we have

$$\begin{aligned} a_{>0} &= \sqrt{1 + \nu/k} \cdot \{(1 + \nu k) \cdot \tilde{\Pi}(-\nu k, k) - \tilde{K}(k)\}, \\ a_{>\mp 1} &= \frac{1}{(1 + k\nu)^{1/2} \pm 1} \cdot \{\tilde{E}(k) \\ &\quad \pm (1 + k\nu)^{1/2} \cdot [(1 + k/\nu) \cdot \tilde{\Pi}(-k\nu, k) - k/\nu \cdot \tilde{K}(k)]\}, \end{aligned} \tag{43}$$

and for $T < T_c$ $(k_< = s_v^{-1} s_h^{-1})$:

$$\begin{aligned} a_{<0} &= \sqrt{1 + \nu k_<} \cdot \{(1 + k_</\nu) \cdot \tilde{\Pi}(-\nu k_<, k_<) - (k_</\nu) \cdot \tilde{K}(k_<)\}, \\ a_{<\mp 1} &= \frac{1}{(1 + k_<\nu)^{1/2} \pm 1} \cdot \{\tilde{E}(k_<) + (k_<^2 - 1) \cdot \tilde{K}(k_<) \\ &\quad \pm (k_</\nu) \cdot (1 + \nu/k_<)^{1/2} \cdot [(1 + \nu k_<) \cdot \tilde{\Pi}(-\nu k_<, k_<) - \tilde{K}(k_<)]\}. \end{aligned} \tag{44}$$

More generally, the a_n (for $T > T_c$ or $T < T_c$) are, also, all of the form

$$a_n = \alpha_n \cdot \tilde{K} + \beta_n \cdot \tilde{E} + \gamma_n \cdot \tilde{\Pi}, \tag{45}$$

where the coefficients α_n, β_n, γ_n, are algebraic expressions of k and ν (and, in fact, rational expressions in $z_v = \tanh \beta E_v$ and $z_h = \tanh \beta E_h$, see the supplementary materials).

The general correlation function $C(M, N)$ is given in what is called a bordered Toeplitz determinant which, for $N > M$, is an $N \times N$ determinant whose matrix elements in the first M columns are the a_n of the diagonal correlations. These matrix elements contain only the complete elliptic integrals $\tilde{K}$ and $\tilde{E}$. The matrix elements of the *remaining columns* are the matrix elements of the row correlations which are *all linear combinations of the three types of* *elliptic integrals*, namely for $T > T_c$, $\tilde{K}(k)$, $\tilde{E}(k)$ and $\tilde{\Pi}(-k\nu, k)$. This generalizes the computation of the next to diagonal correlation of Au-Yang and Perk [18], but the details seem not to be in the literature (even in [19]).

3.1.1. Homogeneous polynomials. From these determinantal representations, and from the fact that the a_n entries are linear combinations of all three types of elliptic integrals, we conclude that the correlation functions $C(M, N)$, for instance for $T > T_c$, are, for $N > M$, *homogeneous polynomials* of degree N in $\tilde{K}(k)$, $\tilde{E}(k)$ and $\tilde{\Pi}(-k\nu, k)$, which contains all powers of $\tilde{E}$ and $\tilde{K}$ but only powers of $\tilde{\Pi}$ of orders less than or equal to $N - M$. For $N < M$ they are homogeneous polynomials of degree N in $\tilde{K}(k)$, $\tilde{E}(k)$ and $\tilde{\Pi}(-k/\nu, k)$, where the order in $\tilde{\Pi}(-k/\nu, k)$ is less than or equal to $M - N$. For general values of M and N, getting the exact expressions of the matrix elements of *all the remaining columns* as *linear combinations of the three types of elliptic integrals*, is quite difficult. This determinantal approach is not an efficient and practical approach to get exact expressions of correlation functions that are not diagonal, next-to-diagonal, or row correlation functions[10].

Remark 1. The fact that the Ising model correlation functions can be expressed as sums of products of three complete elliptic integrals is not specific of two point correlation functions. Recalling [20, 21] one can reduce, for even integer n, any n-point correlation function of the square lattice Ising model (or more general planar lattices) to sum of products of two points correlation functions. Consequently, any even number correlation functions can also be expressed as sums of products of three[11] complete elliptic integrals ($\tilde{K}(k)$, $\tilde{E}(k)$ and $\tilde{\Pi}(-k\nu, k)$).

Remark 2. The fact that the Ising model correlation functions can be expressed as sums of products of three complete elliptic integrals is reminiscent of the results of Boos *et al* [23] where it was shown that some correlation functions[12], associated with the eight-vertex model, can be expressed in terms of sum of products of three transcendental functions[13].

3.2. Quadratic recursion relations

The efficient way to compute the exact expressions of the anisotropic correlation functions amounts, in fact, to using the *quadratic difference equations* obtained by McCoy and Wu [14–16], and by Perk [15]. These recursion relations relate the (high-temperature) correlation functions $C(M, N)$ for $T > T_c$ to the *dual correlation* $C_d(M, N)$ for $T > T_c$. The *dual correlation function* $C_d(M, N)$ *is defined as the low temperature correlation with the replacement*:

[10] The a_n entries for the row correlation verify a *five term linear recursion*, see equation (116) in [19], see also the supplementary materials.

[11] Or homogeneous polynomials of four complete elliptic integrals, $\tilde{K}(k)$, $\tilde{E}(k)$, $\tilde{\Pi}(-k\nu, k)$ and $\tilde{\Pi}(-k/\nu, k)$, because $N > M$, as well as $N < M$, two-point correlation functions both occur in the decomposition of a general n-point correlation function.

[12] The correlation functions in [23] are ground state averages of the product of spin operators on consecutive columns on a same row of the lattice, to be compared with the traditional definition of correlations functions of lattice spin models (see, for instance, equation (14) in [4]).

[13] These three transcendental functions are three log-derivatives of a function expressed in terms of theta functions and the elliptic gamma function (see equation (2.32) in [23]). Seeing our results (homogeneous polynomials and more generally, sums of products of three complete elliptic integrals), as a straight subcase of the results in [23] for the eight-vertex model is not obvious (see the remarks in the first footnote of [23]).

$$s_v \longrightarrow \frac{1}{s_h} \qquad \text{and} \qquad s_h \longrightarrow \frac{1}{s_v}. \tag{46}$$

These difference equations read

$$s_h^2 \cdot [C_d(M, N)^2 - C_d(M, N-1) \cdot C_d(M.N+1)] \\ + [C(M, N)^2 - C(M-1, N) \cdot C(M+1, N)] = 0, \tag{47}$$

$$s_v^2 \cdot [C_d(M, N)^2 - C_d(M-1, N) \cdot C_d(M+1, N)] \\ + [C(M, N)^2 - C(M, N-1) \cdot C(M, N+1)] = 0, \tag{48}$$

$$s_v s_h \cdot [C_d(M, N) \cdot C_d(M+1, N+1) - C_d(M, N+1) \cdot (C_d(M+1, N)] \\ = C(M, N) \cdot C(M+1, N+1) - C(M, N+1) \cdot C(M+1, N), \tag{49}$$

which hold for all M and N, except $M = 0$, $N = 0$, where we have:

$$C(1, 0) = (1 + s_h^2)^{1/2} - s_h \cdot C_d(0, 1), \tag{50}$$

$$C(0, 1) = (1 + s_v^2)^{1/2} - s_v \cdot C_d(1, 0). \tag{51}$$

All the correlations may be computed from these quadratic relations using[14] the diagonal correlation functions $C(N, N)$ and the first nearest neighbor correlations $C(0, 0) = 1$, $C_d(0, 0) = 1$, $C(0, 1)$ and $C(1, 0)$ as input. Once one gets the expressions of the $C(M, N)$ and the *dual correlations* $C_d(M, N)$ from these *overdetermined*[15] set of quadratic relations, as homogeneous polynomials in $\tilde{E}(s_v s_h)$, $\tilde{K}(s_v s_h)$ and $\tilde{\Pi}(-s_h^2, s_v s_h)$, one can be confident in these exact expressions, the smallest miscalculation breaking immediately the rigid compatibility between these overdetermined set of quadratic relations.

Let us display the exact expressions of the first anisotropic correlation functions for $T > T_c$ where $k = s_v s_h$. We introduce the lighter notations:

$$\tilde{E} = \tilde{E}(s_v s_h), \qquad \tilde{K} = \tilde{K}(s_v s_h), \qquad \tilde{\Pi} = \tilde{\Pi}(-s_h^2, s_v s_h). \tag{52}$$

The first diagonal correlation functions read:

$$C(1, 1) = \frac{\tilde{E}}{s_h s_v} + \frac{s_h^2 s_v^2 - 1}{s_h s_v} \cdot \tilde{K}, \tag{53}$$

$$C(2, 2) = \frac{1}{3} \cdot \frac{5 - s_h^2 s_v^2}{s_h^2 s_v^2} \cdot \tilde{E}^2 + \frac{8}{3} \cdot \frac{s_h^2 s_v^2 - 1}{s_h^2 s_v^2} \cdot \tilde{E}\tilde{K} + \frac{(s_h^2 s_v^2 - 1)^2}{s_h^2 s_v^2} \cdot \tilde{K}^2. \tag{54}$$

The first row correlation functions read:

$$C(0, 1) = (s_v^2 + 1)^{1/2} \cdot \left(\frac{(s_h^2 + 1)}{s_h} \cdot \tilde{\Pi} - \frac{\tilde{K}}{s_h} \right), \tag{55}$$

[14] This system of (overdetermined) quadratic equations (47)–(49) has many solutions, for instance a one-parameter family of solution $C(M, N, \lambda)$ corresponding to a λ-extension of the correlation functions, associated with other 'initial' conditions for the quadratic recursions (see, for instance, equations (65), (100) in [22]). The $C(M, N)$ (here for $\lambda = 1$) are deduced, in a unique way, from these quadratic recursions with these initial conditions: the initial conditions cannot be 'arbitrary'.

[15] They can be viewed as discrete Painlevé lattice recursions.

$$C(0,\, 2) = \frac{s_h^2\, s_v^4 \;+ s_v^4 \;+ s_v^2 \;+ 1}{s_h^2} \cdot\, \tilde{K}^2 \;-\; \frac{\tilde{E}^2}{s_h^2} \;-\; 2 \cdot\, \frac{(s_v^2 \;+ 1)^2 \cdot\, (s_h^2 \;+ 1)}{s_h^2} \cdot\, \tilde{K} \cdot\, \tilde{\Pi}$$
$$+\, \frac{(s_v^2 \;+ 1) \cdot\, (s_h^2 \;+ 1) \cdot\, (s_h^2 \;+ s_v^2 \;+ 2)}{s_h^2} \cdot\, \tilde{\Pi}^2. \qquad (56)$$

The first off-diagonal, off-row correlation function reads:

$$C(1,\, 2) = (s_v^2 \;+ 1)^{1/2} \cdot \left(\frac{\tilde{E}^2}{s_h^2\, s_v} \;-\; \frac{s_h^2\, s_v^2 \;- 1}{s_h^2\, s_v} \cdot\, \tilde{K}^2 \;+\; \frac{s_h^2\, s_v^2 \;+ s_v^2 \;- 2}{s_h^2\, s_v} \cdot\, \tilde{E}\, \tilde{K} \right.$$
$$\left. -\, \frac{(s_h^2 \;+ 1)\, (s_v^2 \;- 1)}{s_h^2\, s_v} \cdot\, \tilde{E} \cdot\, \tilde{\Pi} \;+\; \frac{(s_h^2 \;+ 1)\, (s_h^2\, s_v^2 \;- 1)}{s_h^2\, s_v} \cdot\, \tilde{K} \cdot\, \tilde{\Pi} \right). \qquad (57)$$

The corresponding dual correlation functions read respectively:

$$C_d(1,\, 1) = \tilde{E}, \qquad (58)$$

$$C_d(2,\, 2) = \frac{1}{3} \cdot\, \frac{5\, s_h^2\, s_v^2 \;- 1}{s_h^2\, s_v^2} \cdot\, \tilde{E}^2 \;+\; \frac{2}{3} \cdot\, \frac{(s_h^2\, s_v^2 \;- 1)^2}{s_h^2\, s_v^2} \cdot\, \tilde{E}\, \tilde{K}$$
$$-\, \frac{1}{3} \cdot\, \frac{(s_h^2\, s_v^2 \;- 1)^2}{s_h^2\, s_v^2} \cdot\, \tilde{K}^2,$$
$$C_d(0,\, 1) = (s_h^2 \;+ 1)^{1/2} \cdot\, ((s_v^2 \;+ 1) \cdot\, \tilde{\Pi} \;-\, s_v^2 \cdot\, \tilde{K}), \qquad (59)$$

$$C_d(0,\, 2) = \frac{s_h^2\, s_v^4 \;+ 2\, s_h^2\, s_v^2 \;+ s_v^4 \;+ s_v^2 \;- 1}{s_h^2} \cdot\, \tilde{K}^2 - 2 \cdot\, \frac{s_h^2\, s_v^2 \;- 1}{s_h^2} \cdot\, \tilde{E}\, \tilde{K} - \frac{\tilde{E}^2}{s_h^2}$$
$$-\, 2 \cdot\, \frac{s_v^2 \cdot\, (s_h^2 \;+ 1)^2 \cdot\, (s_v^2 \;+ 1)}{s_h^2} \cdot\, \tilde{K} \cdot\, \tilde{\Pi}$$
$$+\, \frac{(s_v^2 \;+ 1) \cdot\, (s_h^2 \;+ 1) \cdot\, (s_h^2 \;+ s_v^2 \;+ 2\, s_h^2\, s_v^2)}{s_h^2} \cdot\, \tilde{\Pi}^2, \qquad (60)$$

$$C_d(1,\, 2) = (s_h^2 \;+ 1)^{1/2} \cdot \left(s_v^2 \cdot\, \frac{s_h^2\, s_v^2 \;- 1}{s_h^2} \cdot\, \tilde{K}^2 \;+\; \frac{s_v^2 \;- 1}{s_h^2} \cdot\, \tilde{E}\, \tilde{K} \;+\; \frac{E^2}{s_h^2} \right.$$
$$\left. +\, \frac{(s_h^2 \;- 1)\, (s_v^2 \;+ 1)}{s_h^2} \cdot\, \tilde{E} \cdot\, \tilde{\Pi} \;-\; \frac{(s_v^2 \;+ 1)\, (s_h^2\, s_v^2 \;- 1)}{s_h^2} \cdot\, \tilde{K} \cdot\, \tilde{\Pi} \right). \qquad (61)$$

Recalling that the *dual correlation functions* $C_d(M,\, N)$ are defined as the low temperature correlation with the replacement (46), one immediately deduces the corresponding exact expressions of the low-temperature correlation functions from the previous exact expressions (58)–(61), of the dual correlation functions.

Introducing some (low-temperature) notations:

$$\tilde{E}_< =\; \tilde{E}\left(\frac{1}{s_v\, s_h}\right), \qquad \tilde{K}_< = \tilde{K}\left(\frac{1}{s_v\, s_h}\right), \qquad \tilde{\Pi}_< = \tilde{\Pi}\left(-\frac{1}{s_v^2},\, \frac{1}{s_v\, s_h}\right). \qquad (62)$$

The corresponding low-temperature correlation functions are immediately deduced. One gets for instance:

$$C_<(1,\,1) = \tilde{E}_<, \tag{63}$$

$$C_<(0,\,1) = (1\,+\,s_v^{-2})^{1/2}\cdot\,\{(1\,+\,s_h^{-2})\cdot\,\tilde{\Pi}_<\,\,-\,s_h^{-2}\cdot\,\tilde{K}_<\}, \tag{64}$$

$$\begin{aligned} C_<(0,\,2) = {} & (1\,+\,s_h^{-2})\cdot\,(1\,+\,s_v^{-2})\cdot\,(2s_v^{-2}s_h^{-2}\,+\,s_v^{-2}\,+\,s_h^{-2})\cdot\,s_v^2\cdot\,\tilde{\Pi}_<^2 \\ & -\,2\cdot\,s_h^{-2}\cdot\,(s_h^{-2}+1)\cdot\,(s_v^{-2}+1)^2\,s_v^2\cdot\,\tilde{K}_<\cdot\,\tilde{\Pi}_< \\ & -\,2\cdot\,(s_v^{-2}s_h^{-2}\,-\,1)\,s_v^2\cdot\,\tilde{E}_<\cdot\,\tilde{K}_< \\ & +\,(s_v^{-2}s_h^{-2}\,+\,s_v^{-2}s_h^{-2}\,+\,s_h^{-4}\,+\,s_v^{-2}\,-\,1)\cdot\,s_v^2\cdot\,\tilde{K}_<^2\,\,-\,s_v^2\cdot\,\tilde{E}_<^2. \end{aligned} \tag{65}$$

More exact expressions for the correlation functions $C(M,\,N)$ and the dual correlation functions $C_d(M,\,N)$ are given in the supplementary materials. Of course one deduces the exact expressions for the correlation functions $C(N,\,M)$ from the exact expressions for the correlation functions $C(M,\,N)$ permuting $s_h \leftrightarrow s_v$ in the previous expressions.

Remark. The two-point correlation functions $C(M,\,N)$ and $C_d(M,\,N)$ (with $M\,<\,N$) are homogeneous polynomials in $\tilde{E}$, $\tilde{K}$ and $\tilde{\Pi}$, the coefficients of the corresponding monomials being rational functions of s_v and s_h when $N\,-\,M$ is even, and rational functions, up to an overall $(1\,+\,s_v^{-2})^{1/2}$ factor, when $N\,-\,M$ is odd. In fact for $C(M,\,N)$, the rational functions are very simple rational functions: a polynomial expression of s_v and s_h, divided by a denominator of the form $s_v^M\,s_h^N$ (see (53)–(57)).

3.2.1. Diagonal reduction. We remarked, in section 3.1, that the diagonal correlation functions $C(N,\,N)$ can be seen to correspond to the $\alpha_1 = 0$ limit of the anisotropic row correlation functions (see (39)). The condition $\alpha_1\,=\,0$ corresponds to a s_h small, s_v large limit:

$$s_h\,=\,s\,\cdot\,\lambda,\qquad s_v\,=\,\frac{s}{\lambda},\qquad \lambda\,\longrightarrow\,0,\qquad k\,=\,s_h\,s_v\,=\,s^2\,=\,\frac{1}{\alpha_2},\qquad \text{finite}. \tag{66}$$

In this limit (66), $\tilde{\Pi}\,=\,\tilde{\Pi}(-s_h^2,\,s_v s_h)$, the complete elliptic integral of the third kind (52) reads:

$$\begin{aligned} \tilde{\Pi} = {} & \tilde{K}\,\,+\,\frac{\tilde{E}\,\,-\tilde{K}}{s^2}\cdot\,\lambda^2\,+\,\frac{(s^4\,+\,2)\cdot\,\tilde{K}\,\,-\,2\cdot\,(s^4\,+\,1)\cdot\,\tilde{E}}{3\,s^4}\cdot\,\lambda^4 \\ & +\,\frac{(8\,s^8\,\,+\,7\,s^4\,\,+\,8)\cdot\,\tilde{E}\,\,\,-\,(4\,s^8\,\,+\,3\,s^4\,\,+\,8)\cdot\,\tilde{K}}{15\,s^6}\cdot\,\lambda^6\,\,+\,\,\cdots \end{aligned} \tag{67}$$

and one verifies that, in this limit, $C(0,\,1)$ given by (55) *actually reduces to* $C(1,\,1)$ given by (53) (with $s_h\,s_v\,=\,s^2$), that $C(0,\,2)$ given by (56) *actually reduces to* $C(2,\,2)$ given by (54), that $C(0,\,3)$ given in the supplementary materials, reduce to $C(3,\,3)$ also given in the supplementary materials.

Note that this verification of the reduction of the anisotropic correlations $C(0,\,N)$ to the diagonal correlations $C(N,\,N)$ *requires more and more terms* in the λ-expansion (66): up to λ^{2N} to check the $C(0,\,N)\,\,\rightarrow\,\,C(N,\,N)$ limit[16].

[16] In fact, more generally, one even has the limit $C(M,\,N)\,\,\rightarrow\,\,C(N,\,N)$ for all $M < N$ (and not just $M = 0$).

4. Duality

One can easily verify that the exact expressions (58)–(61) of some dual correlations $C_d(M, N)$ *can be obtained from the exact expressions* (53) and (55)–(57) of the corresponding (high temperature) correlations $C(M, N)$, when one performs a very simple transformation on the $+$ (high-temperature) s_h and s_v variables, and *on the complete elliptic integrals of the first, second and third kind*:

$$C_d(M, N)(s_h, \; s_v, \; \tilde{E}, \; \tilde{K}, \; \tilde{\Pi}) \\ = C(M, N)\left(\frac{1}{s_v}, \; \frac{1}{s_h}, \; \frac{\tilde{E}}{s_h \, s_v} \; + \; \frac{s_h^2 \, s_v^2 \; - 1}{s_h \, s_v} \cdot \tilde{K}, \; \; s_h \, s_v \cdot \tilde{K}, \; \; s_h \, s_v \cdot \tilde{\Pi}\right). \tag{68}$$

This result actually generalizes Ghosh's result [24] from the isotropic to the *anisotropic* case. Ghosh's result [24] gave a representation of the duality *only on the complete elliptic integral of the first and second kind* (see also equations (57) in [12]).

We sketch in appendix B a proof that the *involutive* transformation (68), can actually be seen as a representation of the *Kramers–Wannier duality* on the complete elliptic integrals of the first, second *and third kind*.

Because the matrix elements in the determinantal expressions for the correlation functions are all expressible as linear combinations of $\tilde{K}$, $\tilde{E}$ and $\tilde{\Pi}$, this representation (68) of the duality on the complete elliptic integrals of the first, second and third kind *holds for all correlations* $C(M, N)$.

4.1. Isotropic limit

The correlations in the isotropic limit have been extensively studied by Ghosh and Shrock [25–27]. In the isotropic limit the fundamental anisotropic segregation between $M > N$ and $M > N$, no longer exists, one has $C(M, N) = C(N, M)$, $s_h = s_v = s$. A consequence of identity (33) is that the complete elliptic integrals of the third kind actually reduce to the complete elliptic integral of the first kind K (see (33)):

$$\tilde{\Pi} = \frac{1}{2} \cdot \tilde{K} \; - \frac{1}{2} \cdot \frac{1}{1 \; + s^2}, \tag{69}$$

and consequently, all the previous exact expressions of the correlation functions $C(M, N)$ and dual correlation functions $C_d(M, N)$, are expressed as non-homogeneous polynomials in only $\tilde{E}$ and $\tilde{K}$. In the isotropic limit, the previous homogeneity property, is now 'hidden' in the exact expressions of the correlation functions expressed in terms of polynomials in only $\tilde{E}$ and $\tilde{K}$. For instance $C(0, 2)$ and $C_d(0, 2)$ read

$$C(0, 2) = \frac{(s^2 + 1)\,(s^2 - 1)^2}{2\,s^2} \cdot \tilde{K}^2 - \frac{\tilde{E}^2}{s^2} + \frac{s^2 \; + 1}{2\,s^2}, \tag{70}$$

$$C_d(0, 2) = - \frac{(s^2 + 1)\,(s^2 + 2)\,(s^2 - 1)^2}{2\,s^2} \cdot \tilde{K}^2 \; - \; 2 \cdot \frac{(s^2 + 1)\,(s^2 - 1)}{s^2} \cdot \tilde{E}\,\tilde{K} \\ - \frac{\tilde{E}^2}{s^2} + \frac{s^2 \; + 1}{2}, \tag{71}$$

and $C(1, 2)$ and $C_d(1, 2)$ read respectively:

$$C(1, 2) = (1 + s^2)^{1/2} \cdot \left(\frac{(s^2 + 1)(s^2 - 1)^2}{2 s^3} \cdot \tilde{K}^2 + \frac{E^2}{s^3} + \frac{(s^2 + 3)(s^2 - 1)}{2 s^3} \cdot \tilde{E} \tilde{K} + \frac{(s^2 + 1)(s^2 - 1)}{2 s^3} \cdot \tilde{K} - \frac{s^2 - 1}{s^3} \cdot \tilde{E} \right), \tag{72}$$

and

$$C_d(1, 2) = (1 + s^2)^{1/2} \cdot \left(\frac{(s^2 + 1)(s^2 - 1)^2}{2 s^2} \cdot \tilde{K}^2 + \frac{\tilde{E}^2}{s^2} + \frac{(s^2 + 3)(s^2 - 1)}{2 s^2} \cdot \tilde{E} \tilde{K} - \frac{(s^2 + 1)(s^2 - 1)}{2 s^2} \cdot \tilde{K} + \frac{s^2 - 1}{s^2} \cdot \tilde{E} \right). \tag{73}$$

As it should, one does verify, on these *non-homogeneous* exact expressions in only $\tilde{E}$ and $\tilde{K}$, that *all the dual correlation functions* $C_d(M, N)$ *can also be deduced from the correlation functions* $C(M, N)$, using the representation of the Kramers–Wannier duality (B.15), rewritten in s:

$$C_d(M, N)(s, \tilde{E}, \tilde{K}) = C(M, N)\left(\frac{1}{s}, \frac{\tilde{E}}{s^2} + \frac{s^4 - 1}{s^2} \cdot \tilde{K}, s^2 \cdot \tilde{K} \right), \tag{74}$$

which is nothing but the Ghosh's statement in [24]. This can be easily checked on (70) and (71), or (72) and (73). The fact that, when $M - N$ is odd the exact expressions of the correlation functions $C(M, N)$ and of the dual correlation functions $C_d(M, N)$ are very close (which is obvious on (72) and (73)), had already been remarked in the Shrock and Ghosh paper [25]. It corresponds to the simple identity valid *only when* $M - N$ *is odd*:

$$C_d(M, N)(s, E, K) = s \cdot C(M, N)(s, -E, -K), \tag{75}$$

which is easily checked on (72) and (73) (and of course, not on (70) and (71)). More exact expressions on the first correlation functions $C(M, N)$, and of the dual correlation functions $C_d(M, N)$, are given in a supplementary material.

5. Linear differential equations for $C(M, N)$

The diagonal correlation $C(N, N)$ was shown in [28] to satisfy a linear differential equation of order $N + 1$. For the isotropic case it was also seen in [29] that, for odd N, the row correlation $C(0, N)$ satisfies an equation of order $(N + 1)(N + 2)/2$ and for N even of order $(N + 2)^2/4$.

Here we will consider, in the anisotropic case, the general correlations $C(M, N)$ and show that they are solution of a linear (partial) differential operator of order

$$\frac{1}{2} \cdot (N - M + 1) \cdot (N + M + 2) = \frac{1}{2} \cdot (N + 1) \cdot (N + 2) - \frac{1}{2} \cdot M \cdot (M + 1). \tag{76}$$

The three functions in (52), $\tilde{\Pi} = \tilde{\Pi}(-\nu\, k,\, k)$, $\tilde{K} = \tilde{K}(k)$, $\tilde{E} = \tilde{E}(k)$, and their derivatives, form a three-dimensional vector space:

$$\frac{\partial\, \tilde{\Pi}_w}{\partial k} = -\frac{\nu \,+\, k}{2\,\nu\, k\, \cdot\, w} \cdot\, \tilde{K} \,+\, \frac{\nu\, k^2 \,+\, 2\, k \,+\, \nu}{2\,\nu\, k\, \cdot\, (1 \,-\, k^2)\, \cdot\, w} \cdot\, \tilde{E}, \tag{77}$$

$$\frac{\partial \tilde{K}}{\partial k} = \frac{\tilde{E}}{k\, \cdot\, (1 - k^2)} - \frac{1}{k} \cdot\, \tilde{K}, \tag{78}$$

$$\frac{\partial \tilde{E}}{\partial k} = \frac{1}{k} \cdot\, (\tilde{E} \,-\, \tilde{K}), \qquad \text{where:} \tag{79}$$

$$\tilde{\Pi}_w = w\, \cdot\, \tilde{\Pi} \qquad \text{with:} \qquad w = \left((1 \,+\, \nu\, k)\left(1 \,+\, \frac{k}{\nu}\right)\right)^{1/2}. \tag{80}$$

Do note that introducing $\tilde{\Pi}_w$, the product of $\tilde{\Pi}$ with w, instead of $\tilde{\Pi}$, gives in (77) a partial derivative with respect to k which is a linear combination of $\tilde{E}$ and $\tilde{K}$, with *no complete elliptic integral of the third kind* $\tilde{\Pi}$.

The general correlation function $C(M, N)$ (with $M \,<\, N$) being a homogeneous polynomial of degree N in $\tilde{E}$, $\tilde{K}$ and $\tilde{\Pi}$, it is *also* a homogeneous polynomial of degree N in $\tilde{E}$, $\tilde{K}$ and $\tilde{\Pi}_w$:

$$C(M, N) = \sum_{l=0}^{N-M} \sum_{\substack{m,n=0 \\ l+m+n=N}} f^{(0)}_{l,m,n}(\nu,\, k)\, \cdot\, \tilde{\Pi}_w^l\, \cdot\, \tilde{K}^m\, \cdot\, \tilde{E}^n, \tag{81}$$

It has $(N - M + 1)(N + M + 2)/2$ monomials $\tilde{\Pi}_w^l\, \tilde{K}^m\, \tilde{E}^n$ in the sum. It follows from (77)–(79) that all the $(N - M + 1)(N + M + 2)/2$ first partial derivatives with respect to k of $C(M, N)$ *will also be homogeneous polynomials* in $\tilde{E}$, $\tilde{K}$ and $\tilde{\Pi}_w$, each derivation decreasing the degree in $\tilde{\Pi}_w$, because of (77):

$$\frac{\partial^p C(M, N)}{\partial k^p} = \sum_{l=0}^{N-M} \sum_{\substack{m,n=0 \\ l+m+n=N}} f^{(p)}_{l,m,n}(\nu,\, k)\, \cdot\, \tilde{\Pi}_w^l\, \cdot\, \tilde{K}^m\, \cdot\, \tilde{E}^n,$$

$$\text{where:} \quad p = 1, \cdots, \frac{(N - M + 1)(N + M + 2)}{2}. \tag{82}$$

If one considers the systems of (partial) differential equations (81) and (82), one can obtain all the monomials $\tilde{\Pi}_w^l\, \tilde{K}^m\, \tilde{E}^n$ in terms of $C(M, N)$ and its $(N - M + 1)(N + M + 2)/2 \,-\, 1$ first derivatives, and deduce, from (82) with $p = (N \,-\, M \,+\, 1)(N \,+\, M \,+\, 2)/2$, that $C(M, N)$ is actually solution of a linear (partial) differential operator of order $(N - M + 1)(N + M + 2)/2$.

This order $(N - M + 1)(N + M + 2)/2$ linear differential operator is not irreducible. It *actually factors in* $N \,-\, M \,+\, 1$ *operators of decreasing orders* $N \,+\, 1$, N, ... $M \,+\, 2$, $M \,+\, 1$:

$$M_{N+1}\, \cdot\, M_N\, \cdot\, M_{N-1}\, \,\cdots\, \, M_{M+2}\, \cdot\, M_{M+1}, \tag{83}$$

It is shown in appendix C for the row correlation functions $C(0, N)$, but this can be generalized easily for general $C(M, N)$ correlations: the M_n's are *actually homomorphic to the* $(n \,-\, 1)$ th *symmetric power*[17] of the order-two operator L_K annihilating $\tilde{K}$.

[17] For $n \,=\, 1$, M_1 is an order-one linear differential operator.

In the special isotropic case, where $\nu = 1$, the operator (83) reduces to direct sums [28, 29].

6. Conclusion

We have shown that the anisotropic correlation functions $C(M, N)$ are *homogeneous polynomials* of degree N in the complete elliptic integrals $\tilde{E}$, $\tilde{K}$ and $\tilde{\Pi}$, when $M < N$. For $M > N$ the anisotropic correlation functions $C(M, N)$ are homogeneous polynomials of degree N in $\tilde{E}$, $\tilde{K}$, the complete elliptic integral of the third kind $\tilde{\Pi}(-\nu k, k)$ being now replaced by $\tilde{\Pi}(-k/\nu, k)$. This remarkable property is totally hidden[18] in the isotropic case (see equations for instance (72) and (73)). This is not the first case a nice property, an important symmetry requires a larger framework to be seen: the best example is certainly the (star-triangle) Yang-Baxter integrability of the Ising model which cannot be seen on the isotropic square model, but actually requires to consider the *anisotropic* model. We obtained the exact expressions of these homogeneous polynomial expressions from the quadratic recursions (47)–(49) of section (3.2). The homogeneous polynomial character of the solutions of such a remarkably rigid and overdetermined set of quadratic equations is worth noticing from an *integrable lattice maps* [30], or *discrete Painlevé* [36, 37] viewpoint.

We have shown that the linear differential operators annihilating the anisotropic row correlation functions $C(0, N)$, are operators of order $(N + 1)(N + 2)/2$, with a remarkable canonical factorization in a product of operators *homomorphic to the successive nth symmetric powers* of the operator annihilating $\tilde{K}$, the complete elliptic integral of the first kind, and similar results can be obtained for the general anisotropic correlation functions $C(M, N)$.

All the results given in this paper underline the fundamental role of the complete elliptic integral of the *third kind* in order to have a clean, clear-cut description of the anisotropic Ising model [31]. Along this line, we have been able to extend Ghosh's representation of the Kramers–Wannier duality [24] on the complete elliptic integral of the first and second kind, to a *representation of the Kramers–Wannier duality to complete elliptic integral of the third kind*. This involutive representation actually enables to *get the exact expressions of all the dual correlations $C_d(M, N)$ occurring in the quadratic relations, from the exact expressions of the correlations $C(M, N)$*. This representation of the Kramers–Wannier duality (68) can thus be seen as a *symmetry* of these overdetermined set of equations.

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[18] And is even easy to miss (as we did in [26, 27]), even in the anisotropic case if one uses $\tilde{\Pi}(-\nu k, k)$ for $M > N$ or $\tilde{\Pi}(-k/\nu, k)$ for $M < N$.

Appendix A. Proof of the identity on a complete elliptic integral of the third kind

As noted in section (2) the identity (28) can be seen as a consequence of addition formulae on the Jacobi's Zeta function (see (32)). We give here a perspective in terms of homomorphisms[19] of linear differential operators.

One can rewrite the identity (28) on the complete elliptic integral of the third kind in the following form:

$$A(x, y) \cdot \tilde{\Pi}(x, y) + B(x, y) \cdot \tilde{\Pi}(R(x, y), y) = \tilde{K}(y), \tag{A.1}$$

where:

$$R(x, y) = 4 \cdot y^2 \cdot \frac{x \cdot (x-1) \cdot (x - y^2)}{(x^2 - y^2)^2}, \tag{A.2}$$

$$A(x, y) = 4 \cdot \frac{(1-x) \cdot (x - y^2)}{\rho},$$
$$B(x, y) = \frac{(x^2 + y^2 - 2x) \cdot (x^2 + y^2 - 2y^2 x)}{(y^2 - x^2) \cdot \rho},$$
$$\text{where:} \qquad \rho = x^2 - 2x - 2y^2 x + 3y^2. \tag{A.3}$$

The complete elliptic integral of the third kind $\tilde{\Pi}(x, y)$ is known to be (see [28]) solution of a third order linear differential operator, which is the product of an order-one linear differential operator and the square of one order-one linear differential operator:

$$\mathcal{L}_3 = \left(\frac{\partial}{\partial x} + \frac{2xy^2 - 3x^2 + 2x - y^2}{(x-1) \cdot (y^2 - x) \cdot x}\right)^2 \cdot \left(\frac{\partial}{\partial x} + \frac{1}{2} \frac{y^2 - x^2}{(x-1) \cdot (y^2 - x) \cdot x}\right). \tag{A.4}$$

The complete elliptic integral of the third kind $\tilde{\Pi}(R(x, y), y)$ is solution of another third order linear differential operator $\mathcal{L}_3^{(R)}$, which is $\mathcal{L}_3$ pullbacked by the *infinite order transformation* $x \rightarrow R(x, y)$. The operator $\mathcal{L}_3^{(R)}$ is, of course, also a product of three order-one operators. A remarkable identity like (A.1) requires the two order-three linear differential operators $\mathcal{L}_3$ and $\mathcal{L}_3^{(R)}$ to be 'very closely related'. This is the case. These two order-three operators *are actually homomorphic*[20]. The order-three operators $\mathcal{M}_3$ and $\mathcal{N}_3$, annihilating $A(x, y) \cdot \tilde{\Pi}(x, y)$ and $B(x, y) \cdot \tilde{\Pi}(R(x, y), y)$ are simply obtained from $\mathcal{L}_3$ and $\mathcal{L}_3^{(R)}$ by conjugation by respectively $A(x, y)$ and $B(x, y)$, and are, of course, also homomorphic. The order-three operators $\mathcal{M}_3$ and $\mathcal{N}_3$ are also product of order-one linear differential operators:

$$\mathcal{M}_3 = \left(\frac{\partial}{\partial x} + \frac{\partial \ln(\rho_1)}{\partial x}\right)^2 \cdot \left(\frac{\partial}{\partial x} + \frac{1}{2} \cdot \frac{\partial \ln(\rho_2)}{\partial x}\right), \tag{A.5}$$

$$\mathcal{N}_3 = \left(\frac{\partial}{\partial x} + \frac{\partial \ln(\rho_3)}{\partial x}\right) \cdot \left(\frac{\partial}{\partial x} + \frac{\partial \ln(\rho)}{\partial x}\right) \cdot \left(\frac{\partial}{\partial x} + \frac{1}{2} \cdot \frac{\partial \ln(\rho_2)}{\partial x}\right), \tag{A.6}$$

[19] In the sense of the equivalence of linear differential operators [31] (corresponding to the 'Homomorphisms' command in Maple's DEtools package).

[20] This highly non-trivial situation of two homomorphic operators where the second one is the pullback of the first one by an infinite order rational function, has already been described in [30].

where:

$$\rho_1 = x \cdot \rho, \qquad \rho_2 = \frac{\rho^2}{(x-1)\cdot x \cdot (x - y^2)},$$
$$\rho_3 = \frac{(x-1)^2 \cdot (x - y^2) \cdot x^2 \cdot \rho}{(x^2 + y^2 - 2\,x\,y^2)\cdot (x^2 + y^2 - 2\,x)\cdot (x^2 - y^2)}. \qquad (A.7)$$

The lhs of identity (A.1) is the sum of $A(x, y) \cdot \tilde{\Pi}(x, y)$ and $B(x, y) \cdot \tilde{\Pi}(R(x, y), y)$ and is thus solution of the least common left multiple (LCLM)[21] of $\mathcal{M}_3$ and $\mathcal{N}_3$. In the identity (A.1), y remains fixed, and thus the rhs of (A.1), $\tilde{K}(y)$, is a constant with respect to the partial derivations in x. Proving identity (A.1) requires the constant functions in x to be solution of the previous LCLM. This is actually the case. This LCLM is an order-four operator, which is the product of four order-one operators, the order-one operator right divising this LCLM *being precisely* $\partial/\partial x$:

$$\mathcal{M}_3 \oplus \mathcal{N}_3 = \Big(\frac{\partial}{\partial x} + \frac{\partial \ln(\rho_4 \cdot \rho)}{\partial x}\Big) \cdot \Big(\frac{\partial}{\partial x} + \frac{\partial \ln(\rho_5 \cdot \rho)}{\partial x}\Big)$$
$$\cdot \Big(\frac{\partial}{\partial x} + \frac{1}{2} \cdot \frac{\partial \ln(\rho_6 \cdot \rho^4)}{\partial x}\Big) \cdot \frac{\partial}{\partial x}, \qquad (A.8)$$

where the exact expressions of the three rational functions of the two variables x and y, namely ρ_4, ρ_5 and ρ_6, is not important for this proof[22]. For instance ρ_6 reads:

$$\rho_6 = \frac{(x-1)\cdot x \cdot (x - y^2)}{(4\,x\,y^4 + x^4 - 6\,x^2\,y^2 - 3\,y^4 + 4\,x\,y^2)^2}. \qquad (A.9)$$

Finding the constant (in x) in the rhs of identity (A.1) is easily done taking the $x = 0$ limit of the lhs of (A.1). Since $\tilde{\Pi}(R(0, y), y) = \tilde{K}(y)$, the evaluation of the lhs of (A.1) at $x = 0$ gives the rhs of identity (A.1):

$$\frac{4}{3} \cdot \tilde{K}(y) - \frac{1}{3} \cdot \tilde{K}(y) = \tilde{K}(y). \qquad (A.10)$$

Appendix B. Representation of the Kramers–Wannier duality on the complete elliptic integrals

The Kramers–Wannier duality on the anisotropic model is well-known to correspond to $(s_h, s_v) \rightarrow (1/s_v, 1/s_h)$, namely $k \rightarrow 1/k$ when $\nu = s_h/s_v$ *remains invariant*.

B.1. Representation of the Kramers–Wannier duality on the complete elliptic integral of the third kind

Let us consider the function $\tilde{\Pi} = \tilde{\Pi}(-\nu\, k, k)$, pullbacked by $k \rightarrow 1/k$, ν *remaining invariant*:

[21] The LCLM of linear differential operators $L_1, \cdots, L_n$ is defined as the (minimal order) linear differential operator such that all solutions of $L_1, \cdots, L_n$ are solutions of this LCLM as well (see for instance [31]).

[22] What matters is the rightdivision of the LCLM by $\partial/\partial x$ which is not obvious. In contrast the rightdivision of the LCLM by $\partial/\partial x + 1/2 \cdot \partial \ln(\rho_2)/\partial x$ is obvious (see (A.5) and (A.6)).

$$\tilde{\Pi}_{\rm pull} \,=\, \tilde{\Pi}\Big(-\frac{\nu}{k},\, \frac{1}{k}\Big)\Big), \qquad \qquad (B.1)$$

$\tilde{\Pi}_{\rm pull}$, given by (B.1), is solution of an order-three linear differential operator $\mathcal{M}_3$ which also factorizes in the product of an order-two operator $\mathcal{M}_2$ and an order-one operator $\mathcal{M}_1$ where

$$\mathcal{M}_1 \,=\, \tilde{w} \cdot \, \frac{\partial}{\partial k} \cdot \, \frac{1}{\tilde{w}} \,=\, \frac{\partial}{\partial k} \,-\, \frac{(\nu^2 \,+\, 1) \cdot \, k \, + 2\,\nu}{2 \cdot \, k \cdot \, (1 \,+\, \nu\, k) \cdot \, (k \,+\, \nu)}, \qquad \qquad (B.2)$$

where $\tilde{w}$ is w in (80) with $k \,\longrightarrow\, 1/k$.

Now consider $k \cdot \, \tilde{\Pi}$, one finds that *it is also solution* of the *same order-three operator* $\mathcal{M}_3$. The order-three operator $\mathcal{M}_3$ has this solution

$$Sol \,=\, k \cdot \, \tilde{\Pi} \,=\, k \,-\, \frac{\nu}{2} \cdot \, k^2 \,\,+\, \frac{3\,\nu^2 \,\,+\, 2}{8} \cdot \, k^3 \,+\, \cdots \qquad \qquad (B.3)$$

with, of course, the solution of the algebraic solution of $\mathcal{M}_1$, namely

$$S_0 = \frac{k}{(1 \,+\, k/\nu)^{1/2} \cdot \, (1 \,+ \nu\, k)^{1/2}}$$
$$= k \,-\, \frac{1}{2} \cdot \, \frac{(\nu^2 \,+\, 1)}{\nu} \cdot \, k^2 \,\,+\, \frac{1}{8} \cdot \, \frac{(3\,\nu^4 \,+\, 2\,\nu^2 \,+\, 3)}{\nu^2} \cdot \, k^3 \,+\, \cdots \qquad \qquad (B.4)$$

together with the formal series solution

$${\rm Sol}_f \,=\, \Big({\rm Sol} \,\,-\, \frac{S_0}{2}\Big) \cdot \,\, \ln(k) \,+\, \mathcal{A}_2, \qquad \qquad (B.5)$$

where $\mathcal{A}_2$ denotes an analytical series in k:

$$\mathcal{A}_2 \,=\, \frac{1}{2} \cdot \, \frac{k}{\nu^4} \,-\, \frac{3}{8} \cdot \, \frac{k^2}{\nu^5} \,+\, \frac{1}{32} \cdot \, \frac{(13\,\nu^4 \,-\, 4\,\nu^2 \,+\, 8)}{\nu^6} \cdot \, k^3 \,+\, \cdots \qquad (B.6)$$

$\tilde{\Pi}_{\rm pull}$, given by (B.1), is *well-defined as a series expansion in* $1/k$ (low temperature expansions for the Ising model), namely k large. If one wants to understand it as a series expansion in k ($k \,\simeq\, 0$), one knows, since it is solution of an order-three linear differential operator $\mathcal{M}_3$, that it is a linear combination of S_0, Sol, ${\rm Sol}_f$ given by (B.3)–(B.5), but finding precisely which linear combination, amounts to performing some analytical continuation[23] from $k =\, \infty$ to $k =\, 0$. Note that *only* the linear combinations of S_0 and Sol *are analytical at* $k = 0$.

B.2. Representation of the Kramers–Wannier duality on the complete elliptic integral of the second kind

Let us consider similar $k \,\rightarrow\, 1/k$ duality calculations on the two simpler complete elliptic integrals $\tilde{E}$ and $\tilde{K}$. One easily finds, similarly, that

$$\tilde{E}_{\rm pull} \,=\, \tilde{E}\Big(\frac{1}{k}\Big), \qquad \qquad (B.7)$$

[23] This is typically a calculation of the *connection matrix* from $k =\, \infty$ to $k =\, 0$, see for instance [32].

and

$$\tilde{E}_d = \frac{\tilde{E}}{k} + \frac{k^2 - 1}{k} \cdot \tilde{K} \tag{B.8}$$

$$= \frac{k}{2} + \frac{k^3}{16} + \frac{3}{128} \cdot k^5 + \frac{25}{2048} \cdot k^7 + \cdots \tag{B.9}$$

are solutions of the *same second-order linear differential operator*. The other solution of this linear differential operator is the formal series

$$E_f = \tilde{E}_d \cdot \ln(k) + E_a, \tag{B.10}$$

where E_a is a holonomic Laurent series at $k = 0$:

$$E_a = -\frac{1}{k} + \frac{k}{8} + \frac{3\,k^3}{32} + \frac{21\,k^5}{512} + \frac{1115\,k^7}{49152} + \cdots \tag{B.11}$$

$\tilde{E}_{\rm pull}$, given by (B.7), is *well-defined as a series expansion in* $1/k$ (low temperature expansions for the Ising model), namely k large. If one wants to understand (B.7) as a series expansion in k ($k \simeq 0$), one knows, since it is a solution of the same second-order linear differential operator, as $\tilde{E}_d$, given by (B.8), that (B.7) is a linear combination of $\tilde{E}_d$ and of the formal series E_f given by (B.10). Of course, *only* $\tilde{E}_d$, given by (B.8), *is analytic at* $k = 0$.

B.3. Representation of the Kramers–Wannier duality on the complete elliptic integral of the second kind

Finally, one also finds, similarly, that

$$\tilde{K}_{\rm pull} = \tilde{K}\left(\frac{1}{k}\right), \tag{B.12}$$

and

$$\tilde{K}_d = k \cdot \tilde{K}, \tag{B.13}$$

are solutions of the *same second-order linear differential operator*. The other solution of this linear differential operator is the formal series

$$K_f = \tilde{K}_d \cdot \ln(k) + K_a, \tag{B.14}$$

where K_a is a holonomic series analytic at $k = 0$:

$$K_a = \frac{k^3}{4} + \frac{21\,k^5}{128} + \frac{185\,k^7}{1536} + \frac{18655\,k^9}{196608} + \cdots$$

$\tilde{K}_{\rm pull}$ given by (B.12), is *well-defined as a series expansion in* $1/k$ (low temperature expansions for the Ising model), namely k large. If one wants to understand (B.12) as a series expansion in k ($k \simeq 0$), one knows, since it is a solution of the same second-order linear differential operator, as $\tilde{K}_d$, given by (B.13), that (B.12) is a linear combination of $\tilde{K}_d$ and of the formal series K_f given by (B.14). Of course, *only* $\tilde{K}_d$, given by (B.13) *is analytic at* $k = 0$.

B.4. Representation of the Kramers–Wannier duality

It had already been written in [24] (see formulae (5)–(7) in [24]) that a representation of the Kramers–Wannier duality on $\tilde{E}$ and $\tilde{K}$ actually reads[24]:

[24] See also table 4, transformations of complete elliptic integrals, page 319, 13.8, in Bateman.

$$(k,\ \tilde{E},\ \tilde{K}) \quad \longrightarrow \quad \left(\frac{1}{k},\ \tilde{E}_d,\ \tilde{K}_d\right) = \left(\frac{1}{k},\ \frac{\tilde{E}}{k}\ +\ \frac{k^2\ -1}{k}\cdot\ \tilde{K},\ k\cdot\ \tilde{K}\right). \tag{B.15}$$

Let us denote $\tilde{\Pi}_p$ the complete elliptic integral of the third kind $\tilde{\Pi}$, where s_h and, s_v *have been permuted*, $s_h \ \leftrightarrow\ s_v$, namely $\tilde{\Pi}(-k/\nu,\ k)$.

From the previous calculations (appendix B.1) on the complete elliptic integral of the third kind, it is natural to imagine that $\tilde{\Pi}$ and $\tilde{\Pi}_p$ transform as linear combinations of (B.3) and (B.4), namely:

$$(\tilde{\Pi},\ \tilde{\Pi}_p) \quad \longrightarrow \quad (\alpha\cdot k\cdot\ \tilde{\Pi}\ +\ \beta\cdot\ S_0,\ \ \alpha\cdot k\cdot\ \tilde{\Pi}_p + \beta\cdot\ S_0). \tag{B.16}$$

One has the same coefficients α and β for $\tilde{\Pi}$ and $\tilde{\Pi}_p$, because changing $\tilde{\Pi}$ into $\tilde{\Pi}_p$, amounts to changing ν into $1/\nu$, S_0 being invariant. Recalling identity (33), a straightforward calculation shows that this identity is preserved by a representation of the duality $(k,\ K,\ \tilde{\Pi},\ \tilde{\Pi}_p)\ \rightarrow\ (1\big/k,\ k\cdot\ K,\ \tilde{\Pi}^{(d)},\ \tilde{\Pi}_p^{(d)})$, *only* for $\alpha = 1$ and $\beta = 0$.

Thus, the representation of the Kramers–Wannier duality (B.15), very simply extends to the two complete elliptic integrals of the third kind $\tilde{\Pi}$ and $\tilde{\Pi}_p$ as follows:

$$\begin{aligned}(k,\ \tilde{E},\ \tilde{K},\ \tilde{\Pi},\ \tilde{\Pi}_p) &\longrightarrow \left(\frac{1}{k},\ \tilde{E}_d,\ \tilde{K}_d,\ \tilde{\Pi}^{(d)},\ \tilde{\Pi}_p^{(d)}\right)\\ &= \left(\frac{1}{k},\ \frac{\tilde{E}}{k}\ +\ \frac{k^2\ -1}{k}\cdot\ \tilde{K},\ k\cdot\ \tilde{K},\ k\cdot\ \tilde{\Pi},\ k\cdot\ \tilde{\Pi}_p\right).\end{aligned} \tag{B.17}$$

Or, in terms of the variables $s_h,\ s_v$:

$$\begin{aligned}&(s_h,\ s_v,\ \tilde{E},\ \tilde{K},\ \tilde{\Pi},\ \tilde{\Pi}_p) \longrightarrow \left(\frac{1}{s_v},\ \frac{1}{s_h},\ E_d,\ K_d,\ \tilde{\Pi}^{(d)},\ \tilde{\Pi}_p^{(d)}\right)\\ &= \left(\frac{1}{s_v},\ \frac{1}{s_h},\ \frac{\tilde{E}}{s_h\ s_v}\ +\ \frac{s_h^2\ s_v^2\ -1}{s_h\ s_v}\cdot\ \tilde{K},\ s_h\ s_v\cdot\ \tilde{K},\ s_h\ s_v\cdot\ \tilde{\Pi},\ s_h\ s_v\cdot\ \tilde{\Pi}_p\right).\end{aligned} \tag{B.18}$$

It is straightforward to verify that *this transformation is, as it should, an involution*.

Appendix C. Linear differential operators for the anisotropic correlation functions

For simplicity we restrict here to row correlation functions $C(0,\ N)$, but everything can be easily generalized to general $C(M,\ N)$ correlation functions.

Let us introduce

$$w = (((1\ +s_h^2)\ (1\ +s_v^2)^{1/2} = \left((1\ +\nu\ k)\left(1\ +\frac{k}{\nu}\right)\right)^{1/2}. \tag{C.1}$$

and the order-one partial differential operator M_1:

$$M_1 = \frac{1}{w}\cdot\ \frac{\partial}{\partial k}\cdot\ w = \frac{\partial}{\partial k} + \frac{\nu^2\ +2\,\nu\,k\ +1}{2\cdot\ (1\ +\ \nu\ k)\cdot\ (k\ +\ \nu)}. \tag{C.2}$$

The action of the order-one operator M_1 on the complete elliptic integral of the third kind $\tilde{\Pi}$ gives a linear combination of $\tilde{E}$ and $\tilde{K}$ *without* $\tilde{\Pi}$:

$$M_1(\tilde{\Pi}) = -\frac{\tilde{K}}{2 \cdot (1 + \nu k) \cdot k} - \frac{\nu \cdot (k^2 + 1) + 2k}{2 \cdot (1 + \nu k) \cdot (k + \nu) \cdot (k^2 - 1)} \cdot \tilde{E}. \qquad (C.3)$$

More generally, let us consider a linear differential operator of order N:

$$L_N = p_N(k, \nu) \cdot \frac{\partial^N}{\partial k^N} + p_{N-1}(k, \nu) \cdot \frac{\partial^{N-1}}{\partial k^{N-1}} + p_{N-2}(k, \nu) \cdot \frac{\partial^{N-2}}{\partial k^{N-2}} + \cdots + p_1(k, \nu) \cdot \frac{\partial}{\partial k} + p_0(k, \nu), \qquad (C.4)$$

the action of its conjugate, $1/w^m \cdot L_N \cdot w^m$, on the product of a function $f(k)$ with $\tilde{\Pi}^m$, the mth power of the complete elliptic integral of the third kind, gives:

$$\left(\frac{1}{w^m} \cdot L_N \cdot w^m\right)(f(k) \cdot \tilde{\Pi}^m) = L_N(f_0(k)) \cdot \tilde{\Pi}^m + \mathcal{P}_1(E, K) \cdot \tilde{\Pi}^{m-1} + \mathcal{P}_2(E, K) \cdot \tilde{\Pi}^{m-2} + \cdots \qquad (C.5)$$

and, of course, if $f_0(k)$ is solution of the linear differential operator (C.4) one gets:

$$\left(\frac{1}{w^m} \cdot L_N \cdot w^m\right)(f_0(k) \cdot \tilde{\Pi}^m) = \mathcal{P}_1(E, K) \cdot \tilde{\Pi}^{m-1} + \mathcal{P}_2(E, K) \cdot \tilde{\Pi}^{m-2} + \cdots \qquad (C.6)$$

and, more generally:

$$\left(\frac{1}{w^m} \cdot L_N \cdot w^m\right)(f_0(k) \cdot \tilde{\Pi}^m + f_1(k) \cdot \tilde{\Pi}^{m-1} + \cdots) = \mathcal{P}_1(\tilde{E}, \tilde{K}) \cdot \tilde{\Pi}^{m-1} + \mathcal{P}_2(\tilde{E}, \tilde{K}) \cdot \tilde{\Pi}^{m-2} + \cdots. \qquad (C.7)$$

Let us now apply equation (C.7) on the row correlation functions $C(0, N)$ are homogeneous polynomials in $\tilde{E}$, $\tilde{K}$ and $\tilde{\Pi}$, with coefficients that are rational functions in k and ν (up to overall square root, $(s_\nu^2 + 1)^{1/2} = (1 + k/\nu)^{1/2}$ for N odd). The row correlation functions $C(0, N)$ are of the form

$$\rho_0(k, \nu) \cdot \tilde{\Pi}^m + \rho_1(k, \nu, E, K) \cdot \tilde{\Pi}^{m-1} + \rho_2(k, \nu, E, K) \cdot \tilde{\Pi}^{m-2} + \cdots, \qquad (C.8)$$

where the C_n are homogeneous polynomials in $\tilde{E}$ and $\tilde{K}$ of degree ν in $\tilde{E}$ and $\tilde{K}$. The first coefficient $C_0(k, \nu)$ is annihilated by a first order linear differential operator

$$M_1 = \frac{\partial}{\partial k} - \frac{\partial \ln(C_0(k, \nu))}{\partial k}, \qquad (C.9)$$

and one thus has, from (C.7), that

$$X_1 = \left(\frac{1}{w^m} \cdot M_1 \cdot w^m\right)(C(0, N)) = \mathcal{P}_1(\tilde{E}, \tilde{K}) \cdot \tilde{\Pi}^{m-1} + \mathcal{P}_2(\tilde{E}, \tilde{K}) \cdot \tilde{\Pi}^{m-2} + \cdots, \qquad (C.10)$$

where the $\mathcal{P}_n(E, K)$ are *also homogeneous polynomials in* $\tilde{E}$ *and* $\tilde{K}$ of degree n in $\tilde{E}$ and $\tilde{K}$. Let us denote L_K the second order operator that annihilates K. At the next step, since $\mathcal{P}_1(E, K)$ is a linear combination of $\tilde{E}$ and $\tilde{K}$, one knows that there exists a second order operator M_2, homomorphic to L_K, that annihilates $\mathcal{P}_1(\tilde{E}, \tilde{K})$. Consequently, using (C.7), one gets that

$$X_2 = \left(\frac{1}{w^{m-1}} \cdot M_2 \cdot w^{m-1}\right)(X_1), \tag{C.11}$$

$$= \mathcal{Q}_2(\tilde{E}, \tilde{K}) \cdot \tilde{\Pi}^{m-2} + \mathcal{Q}_2(\tilde{E}, \tilde{K}) \cdot \tilde{\Pi}^{m-3} + \cdots. \tag{C.12}$$

Again $\mathcal{Q}_2(\tilde{E}, \tilde{K})$ being a homogeneous quadratic polynomial in $\tilde{E}$ and $\tilde{K}$, there exists an order-three linear differential operator M_3, homomorphic to the symmetric square of L_K, that annihilates $\mathcal{Q}_2(\tilde{E}, \tilde{K})$, and so on ... the last operator M_{N+1}, annihilating a homogeneous polynomial in E and K of degree N in $\tilde{E}$ and $\tilde{K}$, being homomorphic to the symmetric N-th power of L_K, and of order $N + 1$.

One immediately deduces a canonical decomposition of the linear differential operator (in k, n is fixed) annihilating $C(0, N)$ in the form

$$M_{N+1} \cdot M_N \cdot M_{N-1} \cdots M_2 \cdot M_1, \tag{C.13}$$

where the M_n's are homomorphic to the $(n - 1)$th symmetric power of L_K. One recovers that the (partial) linear differential operator in k (ν is fixed) annihilating $C(0, N)$ is of order $(N + 1)\,(N + 2)/2$.

Remark C1. Similar calculations can be performed on the $C(M, N)$, *mutatis mutandis*. For instance, if one considers the exact expression for $C(1, 2)$, namely (57), one sees that there is no $\tilde{\Pi}^2$ terms, but it is of the form $\rho_1(k, \nu, \tilde{E}, \tilde{K}) \cdot \tilde{\Pi} + \rho_2(k, \nu, \tilde{E}, \tilde{K})$, therefore one will have a decomposition of the form $M_3 \cdot M_2$ and not $M_3 \cdot M_2 \cdot M_1$.

Remark C2. The previous calculations where performed in the k, ν variables, but there is *nothing specific with these variables*. One obtains similar results in s_h and s_v. For instance, seen as functions of s_h, s_v being fixed, the row correlation $C(0, 2)$ is, again, solution of a linear differential operator of order six, which factorizes as $M_3 \cdot M_2 \cdot M_1$, $C(1, 3)$ is solution of a linear differential operator of order nine, which factorizes as $M_4 \cdot M_3 \cdot M_2$, and, more generally $C(M, N)$ is solution of a linear differential operator of order $(N - N + 1)(N + M + 2)/2$, which factorizes as $M_{N+1} \cdot M_N \cdot M_{N-1} \cdots M_{M+2} \cdot M_{M+1}$.

C.1. Linear differential operators for the anisotropic correlation functions

The previous calculations can of course be performed in a similar way for the low-temperature correlations, mutatis mutandis, yielding the same order $(N + 1)\,(N + 2)/2$ for the operator for $C_<(0, N)$, with the same canonical factorization (C.13) on the corresponding operators.

For example the low-temperature correlation function $C_<(0, 1)$ is annihilated by an order-three linear differential operator $M_3 = M_2 \cdot M_1$, where setting $x = k_<$ the order-one and order-two operators M_1 and M_2 read:

$$M_1 = \frac{\partial}{\partial x} - \frac{1}{2\,(x + \nu)}, \qquad M_2 = p_2 \cdot \frac{\partial^2}{\partial x} + p_1 \cdot \frac{\partial}{\partial x} + p_0, \tag{C.14}$$

with

$$
\begin{aligned}
p_2 &= 4 \cdot x \cdot (x^2 - 1) \cdot (x + \nu)^2 \cdot (\nu x + 1)^2 (\nu x^3 + 3x^2 + 3\nu x + 1), \\
p_1 &= 4 \cdot (x + \nu) \cdot (1 + \nu x) \cdot \{3\nu^2 x^7 + (2\nu^2 + 14)\, \nu \cdot x^6 \\
&\quad + (24\nu^2 + 9) \cdot x^5 + (12\nu^2 + 18)\, \nu \cdot x^4 + (3\nu^2 + 2) \cdot x^3 \\
&\quad - 6 \cdot (\nu^2 + 1)\, \nu \cdot x^2 - (6\nu^2 + 3) \cdot x - 2\,\nu\} \\
p_0 &= 3\,\nu^3 \cdot x^8 + (8\nu^2 + 19)\, \nu^2 \cdot x^7 + (\nu^4 + 80\nu^2 + 18)\, \nu \cdot x^6 \\
&\quad + (55\nu^2 + 140)\, \nu^2 \cdot x^5 + (6\nu^4 + 123\nu^2 + 96)\, \nu \cdot x^4 \\
&\quad + (18\nu^4 + 99\nu^2 + 36) \cdot x^3 + (9\nu^4 + 10\nu^2 + 38)\, \nu \cdot x^2 \\
&\quad + (15\nu^4 - 2\nu^2 - 4) \cdot x + 8\,(\nu^2 - 1)\, \nu. \qquad \text{(C.15)}
\end{aligned}
$$

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